INVESTIGATING THE BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION: APPLICATION IN FINANCE

Ifa Bidika Waima

A Thesis Submitted to

The Department of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of

Master’s in Applied Mathematics (Differential Equation)

Office of Graduate Studies

Adama Science and Technology University

January, 2024

Adama, Ethiopia

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I hereby declare that this master thesis entitled ***"Investigating the Black-Scholes partial differential equation: applications in finance"*** is my own work and has not been submitted to any university for a similar purpose. The references used in this thesis are duly recognized by proper citations.

Name of Student	Signature	Date

Recommendation

We, the advisor(s) of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"Investigating the Black-Scholes partial differential equation: applications in finance"*** prepared under our guidance by Ifa Bidika Waima submitted in partial fulfillment of the requirements for the degree of Mater's of Science in Applied Mathematics. Therefore, we recommend the submission of a revised version of the thesis to the department following the applicable procedures..

______________________	________________	______________
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Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled **"Investigating the Black-Scholes partial differential equation: applications in finance"** and developed by Ifa Bidika Waima, hereby certifies that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

We, the undersigned, members of the Board of Reviewers of the thesis open defense by Ifa Bidika Waima have read and evaluated the thesis entitled ***"Investigating the Black-Scholes partial differential equation: applications in finance"*** and the candidate during the open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics (Differential Equation).

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Finally, approval and acceptance of the thesis proposal is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate Council (DGC) and School Graduate Committee (SGC).

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Acknowledgments

I begin by expressing my deepest gratitude to Almighty God for granting me strength, knowledge, and guidance throughout this academic journey.

I would like to express my sincere gratitude to my advisor, **Dr Tamirat Temesgen**, whose unwavering support, invaluable guidance, and encouragement have been instrumental in shaping this thesis.I extend my heartfelt appreciation to **Dr. Feyissa Kebede** for his willingness to assist me in my work.

I extended my deepest gratitude to my beloved family for their support and encouragement. Their sacrifices and unwavering belief in my abilities have been my motivation and strength throughout my work.

I would also like to acknowledge all staff at **Adama Science and Technology University** whose dedication and support provided an enriching academic environment.

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List of Acronyms

CAPM	Capital Asset Pricing Model
PDE	Partial Differential Equation
AA	Alcoa Corporation
GE	General Electric
AAL	American Airlines Group
LCID	Lucid Group
a.s	Almost Surely
ODE	Ordinary Differential Equation

Abstract

The Black-Scholes stochastic differential equation is a well-known useful model in finance. In this thesis, we present the derivations, assumptions, existence, and uniqueness of its solutions and practical applications of the Black-Scholes model. We obtained the equation through derivation and then we discussed its well-posedness. Analytical solutions of the Black-Scholes equation were obtained using the Feynman-Kac formula. From the analytical solution, we have derived put and call formula for European options. For American options, we used finite difference approximation. As an application, we used historical data of different stocks from Yahoo Finance and tested the efficiency of the Black-Scholes equation. In addition, geometric Brownian motion is applied to historical data of different periods having different market behavior using the Montecarlo simulation to predict stock prices. The result shows the effectiveness of the Black-Scholes model in stable situations despite its assumption of constant market volatility. Since Montecarlo's simulation relies on past prices, it provides a satisfactory approximation when market price behavior is stable, but its approximation diminishes in extremely volatile markets.Results show that the Black-Scholes equation can compute option prices until expiration in low-volatility markets with an efficient prediction. Future studies should give attention to develop an adjusted version of the model that is understandable while guaranteeing efficiency in high-volatility markets.

Keywords: *Black-Scholes equation, Montecarlo simulation,Feynman-Kac formula,Yahoo Finance.*

Chapter One

Introduction

1.1 Background of the Study

The Black-Scholes equation is a partial differential equation used to model dynamic of option price over time. It was first developed by Fischer Black, Robert Merton and Myron Scholes in 1970's. The Black-Scholes equation used in option pricing is expressed mathematically as(Hull, 2003)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

In this equation

- V is the price of the option as a function of time t and the underlying asset price S.
- σ stands for the volatility of the underlying asset.
- r is the risk-free interest rate.
- S denotes the price of the underlying asset.

Black-Scholes model gained global recognition when Myron Scholes and Robert Merton were awarded the Nobel Prize for Economics in 1997. The well-known Black-Scholes model provides a closed-form solution for the valuation of certain options, particularly European-style options. However, when dealing with more intricate models, closed-form solutions may be nonexistent, or computationally intensive approximate techniques become necessary (Gulen et al., 2019).

The Black-Scholes model determines the theoretical price at which an option should be traded, making it appealing to both individual and institutional investors(Jiang & Li, 2005). Today, a wide array of derivative instruments, known as contingent claims, are traded on exchanges worldwide. Initially applied to European options, the Black-Scholes equation's solution was later extended to encompass more complex and exotic derivatives (Choi & Choi, 2018).

The Black-Scholes formula continues to be the basis for determining option prices based on the length of time to expiration and the current price of the underlying asset, even though several other pricing models mostly modifications of the original Black-Scholes are currently in use.

From an innovation theory perspective, the Black-Scholes option pricing formula holds significance as it represents a paradigm shift. Before its publication, finance predominantly relied on risk-based pricing models rooted in equilibrium (neoclassical) thinking, with the Capital Asset Pricing Model (CAPM) serving as a prominent example (Heimer & Arend, 2008). The deduction of the Black-Scholes equation introduced a new paradigm, revolutionizing the field of finance. By demonstrating that investment in an asset can be hedged through the appropriate purchase or sale of options on that asset, the formula provided a valuable concept. For instance, if we anticipate a substantial short-term decrease in the value of certain stocks, we can sell a call option on the stock and await the price decline. However, if the stock price does not immediately decrease and instead moves upward, we face the risk of loss. In such a scenario, we can hedge our investment by purchasing the appropriate quantity of shares to create a risk-free portfolio for a specific period (Janková, 2018).

The Black-Scholes model is a beautiful framework for pricing options, but it commonly relies on assumptions that may not be associated with practical problems and complexities, which causes inefficiencies. Researchers attempted hard to improve the model's accuracy by continually modifying these assumptions and adding new parameters. However, these modifications may result in difficulties that make the model challenging to interpret and use.This complication is causing traders to continue with the classical model even though there are more efficient version available(Janková, n.d.).For this situation, besides developing the model which is understandable and insures efficiency, one should know in what condition they could use the classical Black-Scholes model.

1.2 Option

An option is an individually negotiated or exchange-traded contract between two parties that gives the buyer the right but not the duty to purchase or sell a certain quantity of a specified asset on or before a given date at a specified price (Petters & Xiaoying, 2016). In option contracts, the buyer will pay the agreement premium. The premium of an option contract is the amount that the buyer has to pay and the seller (or writer) receives at the time when both parties enter into the contract. The specified asset is called the underlying asset of the option. The specified date is called the expiration or maturity date of the option. The specified price is called the strike price or exercise price of the option. From the above definition of option, the contract is structured for one to pay money to have a choice in the future.

1.2.1 Option Styles

Depending on the time they can be exercised there are different types of options. There are two well-known styles of options. The first one is **American option**. American options can be exercised at any time before or on the expiration day. the second one is **European options**, which can only be exercised on the expiration day.

1.2.2 Types of Option

Depending on buying and selling rights options can be divided into two broad types.

Call Option

A call option is an option contract that grants the buyer the right to buy a specified quantity of an underlying asset on or by an expiration date T at a strike price K (Petters & Xiaoying, 2016).At expiration:

- If the value of the underlying asset (S) is greater than the Strike Price(K) Buyer makes the difference:$S-K$
- If the value of the underlying asset (S) is less than the Strike Price (K) The buyer does not exercise

Therefore the payoff for a call option is:

$$payoff = \max\{S(T)-K,0\} \tag{1.1}$$

More generally, the value of a call increases as the value of the underlying asset increases. The value of a call decreases as the value of the underlying asset decreases.

Put Option

A put option is an option contract that grants the buyer the right to sell a specified quantity of an underlying asset on or by an expiration date T at a strike price K (Petters & Xiaoying, 2016). At expiration:

- If the value of the underlying asset (S) is less than the Strike Price (K) Owner makes the difference:$K - S$
- If the value of the underlying asset (S)is greater than the Strike Price (K) owner does not exercise.

Therefore the payoff for the put option is:

$$payoff = \max\{K - S(T), 0\} \tag{1.2}$$

1.3 Statement of the Problem

Many mathematical problems have traditionally been approached using deterministic models, disregarding uncertainties. However, this approach may lead to inaccuracies and fail to accurately represent real-world scenarios(Mandelbrot & Hudson, 2007). An illustrative example is the prediction of market stock prices.

This research aims to explore the application of the renowned Black-Scholes stochastic differential equation in finance. The investigation will involve studying the derivations, assumptions, and practical applications of the Black-Scholes model. Additionally, the effectiveness of the Black-Scholes equation will be assessed by comparing its predictions with actual market option prices through computation and simulations.

The study is intended to answer the following basic questions:

- How effective the standard Black-Scholes model is in pricing options within real-world situations?
- Which market behavior makes sense for traders to use the standard Black-Scholes equation?

- Does Geometric Brownian Motion reliably predict stock prices for all market situations?

1.4 Objectives of the study

1.4.1 General Objectives

The general objective of the study is to apply the Black-Scholes stochastic differential equation in financial market and analyze its efficiency in predicting option prices.

1.4.2 Specific Objectives

The specific objectives of this study are to:

- review and comprehend the theoretical foundations and assumptions underlying the Black-Scholes stochastic differential equation.
- consider three different periods of trading, possibly one period with highly fluctuated price to examine the predicting efficiency of geometric Brownian motion.
- gather option data of stock and relevant financial variables, and utilize the Black-Scholes model to generate predicted option prices and compare the actual option price.

1.5 Significance of the Study

This study holds significant importance for various reasons:

1. Advancement in Financial Modeling: the study highlights the important role of mathematical models in the field of finance. Exploring the application of the Black-Scholes stochastic differential equation contributes to the advancement of financial modeling techniques, enabling more accurate predictions and informed decision-making.
2. Practical Benefits for Options Traders: the findings of this study provide valuable insights for options traders by facilitating the calculation of option values in different situations. Traders can utilize the Black-Scholes model to assess the worth of different options at any given time, aiding in effective trading strategies and risk management.
3. Contribution to Financial Mathematics Research: the study's outcomes and analysis add to the existing financial mathematics knowledge. It establishes a foundation for further research and exploration in stochastic differential equations, encouraging future advancements and breakthroughs in the field.

4. Resource for Researchers: this research study serves as a resource for researchers interested in conducting further studies on stochastic differential equations or related topics. Its comprehensive derivations, assumptions, and practical applications provide a valuable reference and inspire researchers to delve deeper into stochastic modeling and its implications in finance.

5. Alert for Option traders: the research may guide traders in making smart decisions, acknowledging situations where assumptions on the model might be inaccurate.

6. Increasing the knowledge of option trading in the country: the research enhance the country's understanding of option trading, potentially enabling local investors to gain knowledge and engage in trading activities.

By shedding light on the significance and potential of the Black-Scholes model, this study aims to inspire and encourage researchers to pursue studies in stochastic modeling and related areas.

Chapter Two

Literature Review

In this section, we review different studies about the Black-Scholes equation and the related methods that are directly related to the objectives of this study.

In MacBeth and Merville (1979), the relationship between Black–Scholes and other models in option pricing, as well as the relationship between the market and Black–Scholes were studied. They showed that on any given day different market prices of options written on the same underlying stock yield different values of implied variance rates, for the same option, which change through time. Their results show that the differences between the implied variance rates are systematically related to the difference between the stock price, the exercise price(strike price), and the time left to expire.

Davis (2010) studied the derivation of Black–Scholes equation using the principle "If options are correctly priced in the market, it should not be possible to make sure profits by creating portfolios of long and short positions in options and their underlying stocks".Sellamuthu (2014), used the Brownian motion approach to construct the Black–Scholes model. As with many other approaches his study is constrained to some assumptions such as the stock price is supposed to be log-normally distributed.

In Lee et al. (2016) log-normal distribution approach and stochastic calculus approach for deriving the Black-Scholes equation were studied. Here also the stock price is assumed to follow the log-normal distribution. In this approach using properties of normal distribution, log-normal distribution, and their mutual relations, the Black–Scholes model can be derived without using stochastic differential. Recently, Washburn and Mehmet (2021) used Ito's Lemma to derive the Black-Scholes model from stock prices. Their derivation was quite simple and understandable.

In the last four decades, the study of the Black-Scholes model continued and it is being modified using different methods, parametrization, initial and boundary conditions, and so on. This is guaranteed by citation on the earliest and original article published on the Black–Scholes model which has now become over forty-five thousand (45000). Janková (2018) studied the drawbacks and limitations of Black–Scholes Studies. From his study, The Black-Scholes model is limited to assumptions such as constant volatility and payment without dividends. However, since some of these assumptions are insignificant, he concluded that the original Black-Scholes formula can be regarded as a general pricing model that provides approximately accurate theoretical option contract prices. Isakov et al. (2019), studied algorithm for determining the volatility function in the Black–Scholes model.

Many researchers have worked on the solution of the Black-Scholes equation. To obtain numerical solutions and approximative analytical simulations, many scholars implemented various sorts of schemes.Uddin et al. (2013), studied the numerical solution of the Black–Scholes equation using finite difference approximation. They used the weighted average method using different weights for numerical approximations.

Jeong et al. (2018) used the finite difference method without boundary conditions for the Black–Scholes equation. the proposed explicit finite difference scheme does not use a far-field boundary condition. The basic idea of the method was that they reduced one or two computational grid points and only computed the updated numerical solution on that new grid point at each time step. They concluded that the method they proposed is very efficient and appropriate to find the solution to the Black-Scholes equation because option pricing and computation of the Greeks use the values at a couple of grid points. Similarly, Awasthi and Riyasudheen (2020) used a computationally convenient time integration scheme for the generalized Black-Scholes equation.

Edeki et al. (2015) finds the analytical solution of the Black-Scholes model for the European option using the projected differential transformation method. Fadugba et al. (2015) finds the solution of the homogeneous Black-Scholes equation for a European call option with dividend using the modified Mellin transform method. The method they proposed doesn't need a transformation and it is very fast and accurate for pricing the European option.

Manafian and Paknezhad (2017) solved the Black-Scholes equation using, the Cauchy-Euler method and the method of separation of variables. In this method, the linear partial differential equation (PDE) is transformed into the linear ordinary differential equation.

Gunawan et al. (2017), studied the accuracy of the Black-Scholes model and the dilution-adjusted Black-Scholes (DABS) model in pricing options traded in the Malaysian market. Their result implies that the dilution adjustment to the Black-Scholes model improves the accuracy in estimating the prices for options. Morales-Bañuelos et al. (2022) developed the modified version of Black-Scholes-Merton model and applied its efficiency on empirical data.

Another study on applications of the Black-Scholes option pricing (Chowdhury et al., 2020). In this study, they considered the price of the call option and put option as the buying price and selling price of stocks of frontier markets and then they used these prices to predict the close price.(Meng & Wang, 2023) compared the Black-Scholes model with Exotic option price model using empirical data. They concluded that,the theoretical price calculated according to the Black–Scholes pricing model are higher than the exotic option price with the proactive investment strategy,

Lin et al. (2021) studied the Black-Scholes equation and add modification term to improve efficiency. Their modification mainly on the volatility term in which they have added perturbation on volatility. They concluded the modified model is better than the classical one. Arraut and Lei (2023) explored the Black–Scholes equation's symmetry, stochastic volatility in the Merton–Garman equation, and how volatility affects investment decisions. The study includes fundamental properties of the Black–Scholes equation, delving into symmetries and suggesting links to quantum finance.

Kumar et al. (2023) examined the impact of interest rate fluctuations on option pricing and proposes modifications to the Black-Scholes model that consider this situations.The proposed model accommodates realistic assumptions about asset behavior and interest rate changes, yielding improved market data prediction and high modeling flexibility. However,this modifications introduced complexity and relatively need high computational level, that is potentially challenging for understanding and practical application. Klibanov et al. (2022) examines option pricing using the Black-Scholes equation, modifying it to the model that consider risk-free interest rates for Call and Put options. Although the modified model was better than the classical Black-Scholes equation in calculating option prices, the observed differences between the Black-Scholes model's and its modification's values were not significant.

From the above literature,the theory governing the Black-Scholes model has been better understood, moreover it is being modified to insure efficiency. But more research is still needed to determine how to use the model in finance in a way that ensures accuracy and understandability. This thesis tried to overview the background, definition, derivation, and solution of standard Black–Scholes model, additionally application of this model in finance and its efficiency.

Chapter Three

Research Methodology and Mathematical Preliminaries

3.1 Study Design

In this project, computational and investigation are applied.

Computation

Computation is the process of carrying out different calculations, and simulations using computers and algorithms. It entails the large-scale processing and analysis of data in order to extract valuable insights or resolve challenging issues. In this project, we have performed a computational simulation to asses data from Yahoo Finance using Python modules and libraries for the purpose of our investigation

Investigation

An investigation is a systematic and exhaustive process of looking into a problem or issue in order to get data, facts, and proof. The term is frequently linked to research and study, which aims to identify and understand the fundamental factors, concepts, or facts around a specific subject. In our case, we have made an investigation to examine the efficiency geometric Brownian motion of the in different time periods for different stocks and the efficiency of Black-Scholes model comparing with actual (last price) of option data.

3.2 Mathematical Procedures

Procedures and steps we have proceeded are given below.

- We have presented formulas, definitions and facts associated with option pricing, stock prices and random situations.
- We downloaded historical data from Yahoo Finance for selected stocks.
- We have used the geometric Brownian motion through Montecarlo simulations to predict stock prices and we have compared it with their actual prices.

- Calculated call and put prices compared to last prices for selected stocks, discussing Black-Scholes formula efficiency based on the result.

3.3 Mathematical Preliminaries

In this section, we pick out some mathematical concepts, definitions, notations, and useful formulas that are required for the development and understanding of the Black-Scholes equation. Since we are dealing with the stochastic model, relevant concepts from probability and stochastic differential equations, definitions, theorems, and examples are contained in this chapter.

3.3.1 Probability Space

Definition. *(Oksendal, 2013) If Ω is a given set, then a σ-algebra on Ω is a family $\mathscr{F}$ subsets of Ω with the following properties*

i. $\emptyset \in \mathscr{F}$,

ii. If $F \in \mathscr{F}$ *,then* $F^C \in \mathscr{F}$*, where* F^C=Ω/F *is the complement of* F *in* Ω *and*

iii. $A_1, A_2, A_3, \dots \in \mathscr{F}$ *Then* $A := \cup_{i=1}^{\infty} A_i \in \mathscr{F}$

The pair $(\Omega, \mathscr{F})$ *is called a measurable space.*

Definition. *(Oksendal, 2013) A probability measure on measurable space* $(\Omega, \mathscr{F})$ *is a function* $P : \mathscr{F} \to [0,1]$ *such that*

i . $P(\emptyset) = 0, P(\Omega) = 1$ *and*

ii . *If* $A_1, A_2, A_3, \dots \in \mathscr{F}$ *and* $\{A_i\}$ *are disjoint. Then* $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

The triple $(\Omega, \mathscr{F}, P)$ is called a probability space.

3.3.2 Expectation and variance of random variable

In finance, the terms expectation and variance stand for return and risk. A predicted average value for investment at some point in the future is known as the expected value (M. J. Evans & Rosenthal, 2004). Using expected value, investors can determine if an investment is profitable and, typically, how risky it is. Standard deviation is a statistic used in finance that reveals the historical volatility of an investment by comparing it to its rate of return.

Definition. *For a random variable X, the expected value of X is defined as*

$$E[X] = \int_{\Omega} X(\omega) dP(\omega).$$

3.3.3 The Normal Distribution

The most popular and often used distribution is the normal distribution. The normal distribution accurately approximates a wide variety of natural occurrences, making it the standard of reference for many probability problems(D'Agostino, 2017). It is the probability distribution that is symmetric about the mean, sometimes referred to as the Gaussian distribution(Lyon, 2014). It demonstrates that data that are close to the mean occur more frequently than data that are far from the mean.

Definition. *(Sahoo, 2013) A random variable X is said to follow a normal distribution with mean $\mu \in \mathbb{R}$ and variance σ^2 if its probability density function (p.d.f.) is given by*

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{3.1}$$

for all $x \in \mathbb{R}$, or equivalently if its cumulative distribution function (c.d.f.) is given by

$$\mathbb{P}(X \leq x) = \int_{-\inf}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \tag{3.2}$$

for all $x \in \mathbb{R}$. In both cases, this is denoted by

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

Definition. *(Walck et al., 1996)*

A random variable X is said to be log-normally distributed if $Y = ln(X)$ follows a normal distribution. In other words, if Y follows a normal distribution then $X = e^Y$ is said to follow a log-normal distribution. This means that a log-normal random variable is a positive random variable.we say that X follows a log-normal distribution with parameters μ and σ^2 ,which we will denote by

$$X \sim \mathscr{L}\mathscr{N}(\mu, \sigma^2).$$

If $Y = \ln(X) \sim \mathscr{L}\mathscr{N}(\mu, \sigma^2)$ Then, its p.d.f. is given by:

$$f_X(x) = \begin{cases} \dfrac{1}{x\sqrt{2\pi\sigma^2}} e^{\left(\dfrac{-(\ln(x)-\mu)^2}{2\sigma^2}\right)}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases} \tag{3.3}$$

Mean and Variance of Log-normal distribution

If we consider the random variable $X \sim \mathscr{L}\mathscr{N}(\mu, \sigma^2)$, then $Y = ln(X) \sim \mathscr{N}(\mu, \sigma^2)$ and its mean and variance is calculated as follows.

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[e^Y] \\ &= e^{\mu + \frac{\sigma^2}{2}}. \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[X^2] &= \mathbb{E}[e^{2Y}] \\ &= e^{2\mu + 2\sigma^2}. \end{aligned}$$

Finally, its variance is given by

$$\begin{aligned} Var(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \\ &= e^{2\mu + \sigma^2}(e^{\sigma^2 - 1}). \end{aligned}$$

Truncated expectations of log-normal distribution

When the domain of a certain probability distribution is constrained, a resulting conditional distribution is known as a truncated distribution. Here we introduce the way of calculating the expectation of such distribution, which we will later use to find the solution of the Black-Scholes option pricing model.

Let us consider the truncated expectation

$$\mathbb{E}\left[x\mathbb{1}_{x\leq a}\right],$$

where,$X \sim \mathscr{L}\mathscr{N}(\mu,\sigma^2)$ and $a > 0$.

Let

$$Z = \frac{ln(x)-\mu}{\sigma} \sim \mathscr{N}(0,1).$$

Then we can write $X = e^{\mu+\sigma Z}$ and the truncated expectation of X will take the form of

$$\mathbb{E}\left[x\mathbb{1}_{x\leq a}\right] = e^{\mu}\mathbb{E}\left[e^{\sigma Z}\mathbb{1}_{Z\leq\left(\frac{ln(a)-\mu}{\sigma}\right)}\right] \tag{3.4}$$

Using the substitution

$$c = \frac{(ln(a)-\mu)}{a},$$

and write as

$$\mathbb{E}\left[e^{aZ}\mathbb{1}_{x\leq c}\right] = \int_{-\infty}^{c} e^{aZ}\frac{e^{\frac{-z^2}{2}}}{\sqrt{2x}}dz.$$

To complete the square in integral above and make evaluation easier we use the fact that we can write

$$\sigma z - \frac{z^2}{2} = \frac{z^2}{2} - \frac{(z-\sigma)^2}{2}.$$

Then we have

$$\begin{aligned}\mathbb{E}\left[e^{aZ}\mathbb{I}_{\{x\leq c\}}\right] &= e^{\frac{\sigma^2}{2}}\int_{-\infty}^{c}\frac{e^{\frac{-(z-\sigma)^2}{2}}}{\sqrt{2x}}dz\\ &= e^{\frac{\sigma^2}{2}}N(c-\sigma).\end{aligned}$$

Substituting back and comparing with (3.4) we have the following result.

$$\mathbb{E}\left[x\infty_{x\leq a}\right] = e^{\mu+\frac{\sigma^2}{2}} N\left(\frac{ln(a)-\mu}{\sigma} - \sigma\right) \tag{3.5}$$

Next for the other side truncated part: $\infty_{x>a}$ we use the fact that: $\infty_{x>a} = 1 - \infty_{x\leq a}$ Therefore, we can write its truncated expectation as

$$\begin{aligned}
\mathbb{E}\left[X\infty_{x>a}\right] &= \mathbb{E}\left[X\left(1-\infty_{x\leq a}\right)\right] \\
&= \mathbb{E}\left[X\right] - \mathbb{E}\left[\infty_{x\leq a}\right] \\
&= e^{\mu+\frac{\sigma^2}{2}} - e^{\mu+\frac{\sigma^2}{2}} N\left(\frac{ln(a)-\mu}{\sigma} - \sigma\right) \\
&= e^{\mu+\frac{\sigma^2}{2}} \left(1 - N\left(\frac{ln(a)-\mu}{\sigma} - \sigma\right)\right) \\
&= e^{\mu+\frac{\sigma^2}{2}} N\left(-\left(\frac{ln(a)-\mu}{\sigma} + \sigma\right)\right).
\end{aligned}$$

3.3.4 Stop-loss formula

Definition. *(Boudreault & Renaud, 2019) The function is defined by*

$$(x)_+ = max\{x,0\} = \begin{cases} 0 \text{ } if \, x \leqq 0 \\ x \text{ } if \, x > 0 \end{cases} \tag{3.6}$$

is known as the positive part function.

Definition. *(Boudreault & Renaud, 2019) For a fixed real number a, we consider the maximum function $x \mapsto max\{x,a\}$, defined by*

$$max\{x,a\} = \begin{cases} a \text{ } if \, x \leqq a \\ x \text{ } if \, x > a \end{cases} \tag{3.7}$$

with this notation, a is a parameter of the function while x is the variable. Here if $a = 0$ we get the positive part function hence maximum function is the generalization of the positive part function.

Definition. *For a fixed real number a, we will call the function $x \mapsto (x-a)_+$, the stop-loss function at a. It can be written using the positive part function*

$$(x-a)_+ = max\{x-a, 0\} = \begin{cases} 0 \text{ } if \, x \leqq 0 \\ x-a \text{ } if \, x > 0 \end{cases} \tag{3.8}$$

For the stop-loss function (3.8) Its stop-loss transform is given by

$$\mathbb{E}[(X-a)_+] = \mathbb{E}[X\infty_{\{x>a\}}] - a\mathbb{E}[\mathbb{1}_{\{x>a\}}] \tag{3.9}$$

$$= \mathbb{E}[X\infty_{\{x>a\}}] - a\mathscr{P}(X>a) \tag{3.10}$$

If we use equation (3.5) in equation (3.9) we get the following useful results

$$E\left[(X-a)_+\right] = e^{\mu+\frac{\sigma^2}{2}} N\left(-\frac{ln(a)-\mu}{\sigma} + \sigma\right) - aN\left(-\frac{ln(a)-\mu}{\sigma}\right) \tag{3.11}$$

Definition. *(Williams, 1991) A filtered space is, $(\Omega, \mathscr{F}, (\mathscr{F})_{n=0}^{\infty})$, where $(\Omega, \mathscr{F}, P)$ is a probability space and $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \cdots \subseteq \mathscr{F}$ are $\sigma - algebras$, jointly called filtration. It can also be defined as $\mathscr{F}_\infty := (\cup_n \mathscr{F}_n) \subseteq \mathscr{F}$.*

Definition. *(Williams, 1991) A process (sequence of random variables) X_n is said to be adapted to the filtration $(\mathscr{F}_n)n \geq 0$, if for every n, the variable X_n is $\mathscr{F}_n - measurable$.*

Definition. *(Williams, 1991) A process $(M_n)_{n\geq 0}$ in a probability space $(\Omega, \mathscr{F}, P)$ is a martingale with respect to a filtration $(\mathscr{F}_n)_{n\geq 0}$, if*

- *It is adapted to $(\mathscr{F}_n)_{n\geq 0}$,*
- $\mathbb{E}\,|M_n| < \infty \; \forall n \geq 0$ *and*
- $\mathbb{E}\,(M_{n+1}| < \infty) = M_n \quad a.s \quad \forall n \geq 0.$

If instead, in the last equality, we have $\leq$, then we call M a super martingale, while if it is $\geq$ then it is a sub martingale.

Definition. *A function $f : [0,\infty) \longrightarrow \mathbb{R}$ is of strictly sub linear growth if*

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0.$$

$u : [0,\infty) \times [0,T] \longrightarrow \mathbb{R}$ is of strictly sub linear growth if

$$|u(x,t)| \leq f(x).$$

Consider a function g:I $\longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$. We say that g is a convex function if, for any two points x and y in I and any $\alpha \in [0,1]$, we have

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y).$$

We say that g is concave if

$$g(\alpha x + (1-\alpha)y) \geq \alpha g(x) + (1-\alpha)g(y).$$

Theorem 3.1 (**Monotone Convergence Theorem**(Bartle & Sherbert, 2000))**.** *A monotone sequence of real numbers is convergent if and only if it is bounded. Furthermore*

(a) If $a = (a_n)$ is a bounded increasing sequence, then

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

This implies that for a sequence $a = (a_n)$ that is both monotone increasing and bounded, its limit exists and is equal to the supremum of the set of its terms.

(b) If $b = (b_n)$ is a bounded decreasing sequence, then

$$\lim_{n\to\infty} b_n = \inf\{b_n : n \in \mathbb{N}\}.$$

This indicates that for a sequence $b = (b_n)$ that is monotone decreasing and bounded, its limit exists and is equal to the infimum of the set of its terms.

Theorem 3.2 (**Jensen's Inequality** (Björk, 2009))**.** *If $g(x)$ is a convex function on $\mathbb{R}_X$ and $\mathbb{E}[g(X)]$ and $g(\mathbb{E}[X])$ are finite, then for any distribution P on M we have the following.*

$$\mathbb{E}_{m\sim p}[g(X)] \geq g(\mathbb{E}_{m\sim p}[X]). \tag{3.12}$$

Proof. Let $M = \{m_1, m_2, \dots, m_k\}$ assuming that M is finite, $X_i = X(m_i)$ and $p_1 = P(m_i)$. We use proof by mathematical induction.
First, we check the theorem for $M = 2$. From inequality we have

$$\begin{aligned}\mathbb{E}_{m\sim p}[g(X)] &= p_1 g(x_1) + p_2 g(x_2)\\ &\geq g(p_1x_1 + p_2x_2)\\ &= g(\mathbb{E}_{m\sim p}[X]).\end{aligned}$$

Hence it worked for the initial steps of induction. Next, we assume for $M = k-1$

$$\begin{aligned}\mathbb{E}_{m\sim p}[g(X)] &\geq g(\mathbb{E}_{m\sim p}[X]).\\ &= g(p_1x_1 + p_2x_2 + p_3x_3 + \cdots + p_{k-1}x_{k-1})\end{aligned}$$

Finally, we will check if it works for $M = k$

$$\begin{aligned}\mathbb{E}_{m\sim p}[g(X)] &= p_1g(x_1) + p_2g(x_2) + p_3(x_3) + \cdots + p_kg(x_k)\\ &= (p_1+p_2)\left(\left(\frac{p_1}{p_1+p_2}\right)g(x_1) + (\left(\frac{p_2}{p_1+p_2}\right)g(x_2)\right) + p_3g(x_3) + \cdots + p_kg(x_k)\\ &\leq (p_1+p_2)g\left(\left(\frac{p_1}{p_1+p_2}\right)x_1 + \left(\frac{p_2}{p_1+p_2}\right)x_2\right) + p_2g(x_3) + \cdots + p_kg(x_k)\end{aligned}$$

Now from induction assumptions inequality hold for $m = k-1$, Hence:

$$\begin{aligned}&\leq g\left((p_1+p_2)\left(\frac{p_1x_1}{p_1+p_2} + \frac{p_2x_2}{p_1+p_2}\right) + p_3x_3 + \dots p_kx_k\right)\\ &= g(p_1x_1 + p_2x_2 + p_3x_3 + \cdots + p_kx_k)\\ &= g(\mathbb{E}_{m\sim M}[X])\end{aligned}$$

Therefore, it is proved by mathematical induction. □

3.3.5 Stochastic Process

Definition. *(L. C. Evans, 2012) The Borel subsets of $\mathbb{R}^n$, denoted $\mathbb{B}$ comprise the smallest σ-algebra of subsets of $\mathbb{R}^n$ containing all open sets.*

Definition. *(L. C. Evans, 2012) Let $(\Omega, \mathscr{F}, P)$ be a probability space. A mapping*

$$X : \Omega \mapsto \mathbb{R}^n$$

is called an n-dimensional random variable if for each $B \in \mathbb{B}$ we have

$$X^1(B) \in \mathscr{F}$$

We equivalently say that X is $\mathscr{F}$ -measurable.

Example 1. *If $A_1, A_2, \ldots, A_m \in \mathscr{F}$, with $\Omega = \cup_{i=1}^{\infty} A_i$ and and $a_1, a_2, \ldots, a_m$ are real numbers, then*

$$X = \sum_{i=1}^{m} a_i \mathbb{1}_A$$

Where, $\mathbb{1}_A$ is indicator function of A defined as:

$$\mathbb{1}_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A, \\ 0 \text{ if } \omega \notin A \end{cases}$$

is a random variable, called a simple function.

A stochastic process is defined as a collection of random variables $X = (X_t)_{t \in T}$ defined on the same probability space (Ω, F, P) and indexed by time. Where T is associated with the set of days or years for some fixed $T \in \mathbb{N}$ and the triple (Ω, F, P) is probability space.

3.3.6 Brownian Motion

In 1827, when botanist Robert Brown examined particles found in pollen grains under a microscope, he saw that the particles moved through the water, but he was unable to identify the mechanisms causing this mobility(Nelson, 2020). The direction of the force of atomic bombardment is constantly changing. At different times particle is hit more on one side than another, leading to the seemingly random nature of the motion. Brownian motion describes the random movement of particles floating in a fluid resulting from the fluid's fast-moving atoms or molecules bombarding them (Mörters & Peres, 2010).

Definition. *(Mörters & Peres, 2010) Brownian motion is a stochastic process* $B(t)$ *characterized by the following three properties*

1. $B(0) = 0$*; It starts at time* $t = 0$.

2. *Normal Increments:* $B(t) - B(s)$ *has a normal distribution with mean* 0 *and variance* $t - s$.

3. *Independence of Increments:* $B(t) - B(s)$ *is independent of the past.*

4. *Continuity of Paths:* $B(t), t > 0$*are continuous functions of* t.

These three properties alone define Brownian motion, but they also show why Brownian motion is used to model stock prices. Property 2 shows stock price changes will be independent of past price movements.

Linear Brownian motion

Linear Brownian motion, also known as arithmetic Brownian motion is obtained by transforming a standard Brownian motion using an affine function(Boudreault & Renaud, 2019). For two constants $\mu \in \mathbb{R}$ and $\sigma > 0$, we define the corresponding linear Brownian motion, formed from $X_0 \in \mathbb{R}$ by $X = \{X_t, t \geq 0\}$ where,

$$X_t = X_0 + \mu t + \sigma W_t$$

Here,μ is called the drift coefficient and σ is the volatility coefficient or diffusion coefficient of X

Geometric Brownian motion

A geometric Brownian motion is obtained by modifying a linear Brownian motion with an exponential function(Boudreault & Renaud, 2019). For two constants $\mu \in \mathbb{R}$ and $\sigma > 0$, we define the corresponding geometric Brownian motion, formed from $S_0 \in \mathbb{R}$ by $X = \{S_t, t \geq 0\}$ where,

$$S_t = S_0 exp(\mu t + \sigma W_t) = e^{X_t} \tag{3.13}$$

where X is a linear Brownian motion obtained from $X_0 = ln(S_0)$, with drift coefficient μ and volatility coefficient σ. Therefore, a geometric Brownian motion is a continuous-time stochastic process taking only positive values.

3.3.7 Stochastic Differential Equation

Differential equations describe the evolution of systems. Stochastic Differential Equations (SDEs) come into play when random noise is introduced into these equations. Let $(\Omega, \mathscr{F}, P)$ be a probability space, and consider $X_t, t \geq 0$ as a stochastic process $X : \Omega \times [0, \infty) \rightarrow \mathbb{R}$.

Assume that $a(X_t, t) : \Omega \times \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $b(X_t, t) : \Omega \times \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ are stochastically integrable functions of $t \in \mathbb{R}_+$. Then, the equation:

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t \tag{3.14}$$

where W_t is the geometric Brownian motion, is known as a stochastic differential equation.

Equation (SDE) symbolically represents the stochastic integral equation:

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s \tag{3.15}$$

The functions $a(X_t, t)$ and $b(X_t, t)$ are referred to as the drift term and diffusion term, respectively.

Well posed problem

Definition. *(Strauss, 2007) A partial differential equation in a domain with initial and boundary conditions is said to* ***well posed*** *if it satisfies the following conditions*

- ***Existence****: There exists at least one solution u(x, t) satisfying all these conditions.*
- ***Uniqueness:*** *There is at most one solution.*

3.3.8 Ito's Formula

Ito's formula (Commonly called Ito's chain rule) is very useful for evaluating stochastic integrals.

Definition. *(Mikosch, 1998) An Ito process or stochastic integral is a stochastic process on* $(\Omega, \mathscr{F}, P)$ *adopted to* $\mathscr{F}_t$ *which can be written in the form:*

$$X_t = X_0 + \int_0^t U_s ds + \int_0^t V_s dB_s \tag{3.16}$$

Equation(3.16) can be written in a stochastic differential equation as follows.

$$dX_t = U_t dt + V_t dB_t \tag{3.17}$$

Example 2. *B_t^2 is an Ito's process.*

Theorem 3.3 (Ito's Formula)**.** *:*

Let X_t be an Ito process given by $dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t$, where W_t is a Wiener process. Consider a function $f(t,X_t)$ that is continuously differentiable in t and twice continuously differentiable in X_t. Then, $Y_t = f(t,X_t)$ satisfies:

$$dY_t = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dW_t$$

Proof. Let $Y_t = f(t,X_t)$. We want to find dY_t using Taylor's expansion.

Consider the function $f(t+\Delta t, X_{t+\Delta t})$ expanded about (t,X_t) using Taylor's theorem

$$f(t+\Delta t, X_{t+\Delta t}) = f(t,X_t) + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta X_t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta X_t)^2 + R(\Delta t, \Delta X_t)$$

where $R(\Delta t, \Delta X_t)$ is the remainder term. Dividing both sides by Δt and taking the limit as $\Delta t \to 0$ gives

$$\frac{f(t+\Delta t, X_{t+\Delta t}) - f(t,X_t)}{\Delta t} \approx \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dX_t}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\frac{dX_t}{dt}\right)^2$$

Rewriting dX_t using the Ito process $dX_t = \mu dt + \sigma dW_t$

$$\frac{dX_t}{dt} = \mu(t,X_t) + \sigma(t,X_t)\frac{dW_t}{dt} = \mu + \sigma\frac{dW_t}{dt} = \mu + \sigma \cdot 0 = \mu$$

Hence,

$$\frac{f(t+\Delta t, X_{t+\Delta t}) - f(t,X_t)}{\Delta t} \approx \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mu^2$$

Multiply both sides by Δt and rearrange terms

$$f(t+\Delta t, X_{t+\Delta t}) - f(t,X_t) \approx \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\mu\Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mu^2\Delta t$$

Divide both sides by Δt and take the limit as $\Delta t \to 0$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t^2$$

Using the property $dX_t^2 = \sigma^2 dt$ in the Ito process

$$df = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dW_t$$

This proves Ito's Lemma,where W_t is Wiener process □

3.3.9 Feynman-Kač formula

The Feynman-Kač formula provides a solution to partial differential equations as the family of parabolic PDEs. This solution is expressed as the conditional expectation of a diffusion process sampled at a fixed time. In this section, we introduce the suitable version of the formula for our case (option pricing).

Given the partial differential equation of the form

$$\frac{\partial F}{\partial t}(t,x) + a(x)\frac{\partial F}{\partial x} + \frac{1}{2}b^2(x)\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0 \tag{3.18}$$

Given a parameter $r > 0$ and given functions $a(.)$, $b(.)$, $h(.)$.For $T > 0$ is fixed and all $(t,x) \in (0,T) \times \mathbb{R}$ with final condition $F(T,x) = h(x)$, for all $x \in \mathbb{R}$. Then the solution to the PDE (3.18)is given by:

$$F(t,x) = e^{-r(T-t)}\mathbb{E}[h(X_T)|X_t = x] \tag{3.19}$$

where $\{X_s, s \geq 0\}$ is the diffusion process determined by the following SDE

$$dX_s = a(X_s)ds + b(X_s)dW_s \tag{3.20}$$

3.3.10 Portfolio

Investors naturally try to reduce their risk in investment. There may be different ways of reducing and avoiding risk. One of these is using a portfolio in investment. Portfolio investing capital on different assets that are not correlated. Hence, a Portfolio is any collection of financial assets such as stocks, bonds, and cash equivalents held by an investment institution or company.

Arbitrage Portfolio

A portfolio is said to be an arbitrage portfolio if today it has zero value, and in the future, it has a positive value. There is no doubt that arbitrage occurs in reality. One illustration is the simple difference in prices for various assets in various marketplaces. A person could be able to purchase something in one market, sell it for more money in another market, and make a profit. But situations like these are typically resolved swiftly. When arbitrage possibilities arise, others catch wind of them and seize them, driving up prices and making the opportunities disappear.

3.3.11 Self Financing strategy

A trading strategy (Θ, Δ) is said to be self-financing if, when it is rebalanced, no money is injected or withdrawn: This means the portfolio value, before re-balancing and the portfolio value after re-balancing is equal (Boudreault & Renaud, 2019). A self-financing portfolio is a trading strategy where the value of the portfolio at any time after a transaction is equal to its value before that transaction. Therefore for such a trading strategy, since the portfolio does not require any additional investments, any change in its value is caused by a change in the value of the stocks.(Ayana & Al-Swej, 2021).

It is mathematically given as

$$\Theta_k B_k + \Delta_k S_k = \Theta_{k+1} B_k + \Delta_{k+1} S_k \tag{3.21}$$

Definition (Self-Financing Portfolio)**.** *A portfolio process* $\{\theta(t); t = 1, 2, \ldots, T\}$ *is said to be self-financing if*

$$V(t) = \sum_{i=0}^{n} \theta_i(t+1)\Delta S_i(t), t = 0, 1, \ldots, T-1. \tag{3.22}$$

In Equation (3.22), the left-hand side is the portfolio value at time t that represents the value of the portfolio before any transactions are made, while the right-hand side is the time-t value of the portfolio just after any transactions take place at that time.

3.3.12 Replicating portfolio

A replicating portfolio is a collection of securities designed to resemble (replicate) the market values or cash flows of a collection of liabilities over a wide range of stochastic situations(Boekel & Verheugen, 2009).

A contingent claim is a random variable X representing a payoff at some future time T . Contingent claim can be considered as part of a contract that a buyer and a seller make at the current time $t = 0$. The seller promises to pay the buyer the amount $X(\omega)$ at time T if the state $\omega \in \Omega$ occurs in the economy. Hence, when viewed at time $t = 0$, the payoff X is a random variable.

Definition. *(Kijima, 2016) A contingent claim X is said to be attainable if there exists some self-financing trading strategy $\{\theta(t) : t = 1,2,\ldots,T\}$ called a replicating portfolio, such that $V(T) = X$. That is,*

$$X = V(0) + \sum_{i=0}^{n} \sum_{t=0}^{T-1} \theta_i(t+1)\Delta G_i(t) \tag{3.23}$$

for some self-financing portfolio process $\{\theta(t)\}$. In this case, the portfolio process is said to generate the contingent claim, X.

Chapter Four

Analysis of Black–Scholes Equation and Its Application in Finance

In this chapter, we discuss the Black-Scholes equation. Some mathematical and financial preliminaries will be used to derive the equation.

4.1 Stock Price Model

Stock prices move randomly because of the efficient market hypothesis. There are different forms of this hypothesis, but the following two things are commonly considered(Del Giudice et al., 2015).

(i). The history of the stock is fully reflected in the present price.

(ii). Markets respond immediately to new information about the stock.

With these two assumptions, changes in a stock price follow a Markov process. A Markov process is a stochastic process where only the present value of the variable is relevant for predicting the future(Ross, 2014). The stock model states that predictions for the future price of the stock should be unaffected by the price one week, one month, or one year ago.

Now consider the price of a stock S at time t. Consider a small time interval dt during which the price of the underlying asset S changes by an amount dS. The most common model separates the return on the asset,dS/S into two parts. This first part is typically risk-free and completely deterministic. This part yields a contribution of:

$$\mu dt$$

to dS/S. Here μ is a measure of the average rate of growth of the stock, also known as the drift. In this model, μ is assumed to be the risk-free interest rate on a bond, but it can also be represented as a function of S and t. The second part of the model takes into consideration the stochastic fluctuations in the stock price that arise from external variables, including unexpected news. It contributes in the following ways and is best represented by a random sample taken from a normal distribution with a mean zero:

$$\sigma dB_t$$

to dS/S. In this formula, σ is defined as the volatility of the stock, which measures the standard deviation of the returns. Like the term μ, σ can be represented as a function of S and t. The B in dB denotes Brownian motion. Putting this information together, we obtain the stochastic differential equation

$$dS/S = \mu dt + \sigma dB_t \tag{4.1}$$

4.2 Deriving Black–Scholes Equation

The basic assumptions of the Black–Scholes model are the following:

1. Markets are always open.
2. There is no cost arbitrage.
3. The risk-free interest rate is constant over time.
4. The volatility of the price of the underlying asset is constant.
5. The underlying asset's price movements follow a log-normal distribution, which implies a normal distribution of capitalized returns.
6. The underlying asset does not pay dividends.

The Black-Scholes model is a continuous-time financial market model composed of two assets(Boudreault & Renaud, 2019): risk-free asset $\{B_t, t \geq 0\}$ and a risky asset $\{S_t, t \geq 0\}$.

The following ODE can determine the dynamics of the price of the risk-free asset

$$dB_t = rB_t dt \tag{4.2}$$

Its solution is $B_t = B_0 e^{rt}$.

The following SDE can determine the dynamics of the price of the risky asset

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.3}$$

The solution for (4.3) is given by geometric Brownian motion

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

We can now, consider a simple option with payoff $V_T = h(S_T)$. We want to compute the no-arbitrage price of this derivative and find its replicating portfolio. And let us assume the value V_t of this option at any time t will be,$F(t,S_t)$, where $F(t,x)$ is a sufficiently smooth function.We assume also that there exists a replicating portfolio $\{(\Theta_t,\Delta_t), 0 \leq t \leq T\}$ for this payoff $V_T = \Pi_T$, where

$$\Pi_T = \Delta_T S_T + \Theta_T B_T,$$

By the no-arbitrage principle, we have $V_t = \Pi_t$, for all $0 \leq t \leq T$, where the value of replicating portfolio at time t is given by

$$\Pi_t = \Delta_t S_t + \Theta_t B_t,$$

Therefore by these assumptions, we have $F(t,S_t) = \Pi_t$ for all 0 $\leq$ t $\leq$T. We now assume the functions $F(t,S_t)$ and replicating portfolio {(Θ_t , Δ_t), 0 $\leq$ t $\leq$ T} exist.We then find a trading strategy that is both replicating and self-financing.
In a continuous-time model, the self-financing condition is given by the following stochastic equation

$$d\Pi_t = \Theta_t dB_t + \Delta_t dS_t \tag{4.4}$$

Since $F(t,S_t) = \Pi_t$, for all $0 \leq t \leq T$ we have:

$$d\Pi_t = dF(t,S_t)$$

This means, that at each instant, the variation in the portfolio value must match the variation in the derivative's price. It is known as the replicating condition. $dF(t,S_t)$ can be written using Ito's formula as

$$dF(t,S_t) = \frac{\partial F}{\partial t}(t.S_t)dt + \frac{\partial F}{\partial x}(t,S_t)dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)dt$$

Substituting for dS_t , as given by equation (4.3) We have:

$$dF(t,S_t) = \frac{\partial F}{\partial t}(t,S_t)dt + \frac{\partial F}{\partial x}(t,S_t)(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)dt \tag{4.5}$$

Equation (4.5) can be re-arranged as

$$dF(t,S_t) = \left(\frac{\partial F}{\partial t}(t,S_t) + \frac{\partial F}{\partial x}(t,S_t)\mu S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)\right) dt + \left(\frac{\partial F}{\partial x}(t,S_t)\sigma S_t\right) dW_t \tag{4.6}$$

On the other hand since $\Pi_t = F(t,S_t)$, the self financing condition (4.4) is given by:

$$dF(t,S_t) = \Theta_t r B_t dt + \Delta_t(\mu S_t dt + \sigma S_t dW_t) \tag{4.7}$$

In equation (4.7) We have substituted equation (4.2) and (4.3) for the terms dB_t and dS_t respectively.

Next, equation (4.7) can be written as

$$dF(t,S_t) = rF(t,S_t)dt + (\mu - r)\Delta_t dS_t dt + \sigma \Delta_t dS_t dW_t \tag{4.8}$$

We have used in equation (4.8) the fact that $\Theta_t B_t = F(t,S_t) - \Delta_d S_t$ from the definition of the portfolio's value. In comparing equations (4.6) and (4.8) We have the following pair of equation

$$\begin{cases} rF(t,S_t) + (\mu - r)\Delta_t S_t = \frac{\partial F}{\partial t}(t,S_t) + \mu S_t \frac{\partial F}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t) \\ \sigma S_t \Delta_t = \sigma S_t \frac{\partial F}{\partial x}(t,S_t) \end{cases} \tag{4.9}$$

Now, from the second equation of (4.9), we get

$$\Delta_t = \frac{\partial F}{\partial x}(t,S_t) \tag{4.10}$$

Then using the relation (4.10) in the first equation of (4.9) We will arrive at the desired Black-Scholes partial differential equation. The equation is given as:

$$\frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0 \tag{4.11}$$

With boundary (final) condition $F(T,x) = h(x)$, where h is the payoff function.

4.3 Well Posedness of the Black–Scholes PDE

In this section, we will prove the well-posedness of the Black-Scholes equation. The theory of well-posedness which is associated with the existence and uniqueness of the solution has a vital role in the study of differential equations.

4.3.1 Existence and Uniqueness of the solution

Existence

Here we will show that there is at least one solution to the Black–Scholes equation (4.11). To do so we state the following theorem and prove it.

Theorem 4.1. *Assume that the payoff function defined as* $g : [0,\infty] \longrightarrow [0,\infty]$ *is continuous and of at most linear growth. Then the option price u defined as:*

$$u(x,t) = E_{x,t}g(X(T)) \tag{4.12}$$

is the classical solution to the Black-Scholes equation. Moreover, it is of at most linear growth.

Proof of theorem 4.1 guarantees the existence of at least one classical solution to the Black–Scholes equation.

Proof:

Since g is at most linear function without the loss of generality we can assume

$$g(x) \leq C(1+x)$$

Then we have

$$\begin{aligned} u(x,t) &= E_{x,t}g(X(T)) \\ &\leq CE_{x,t}(1+(X(T))) \\ &= C(1+x) \end{aligned}$$

Because X is a super martingale. Here we have shown that u is at most linear growth.
Next, we show that u is the classical solution to the Black-Scholes equation.

First, let's check u satisfies the initial conditions.

Let $X_{x,t}(s)$ be the solution to

$$dX(t) = \alpha(X(t),t)dW$$

and also Let,

$$X^M_{x,t}(S) = X_{x,t}(S \wedge \gamma_M)$$

where,

$$\gamma_M inf\{s \geq : X_{x,t}(s) \geq M\}$$

Then X^M is a bounded local martingale and hence it is a martingale. The path of $X^M_{x,t}$ and $X^M_{y,t}$ are independent,hence we have the following result

$$E\left|X^M_{x,t}(T) - X^M_{y,t}(T)\right| = |x-y| \tag{4.13}$$

For continuity in the initial time variable , Let $t_1 \leq t_2$ with $t_2 - t_1$=δ and

$$E\left|x_{t_1}(t_2) - x\right|^2 = E\int_{t_1}^{t_2} \alpha^2(X_{x,t_1}(S),S)1_{s\leq\tau_M} dS \leq C\delta \tag{4.14}$$

where,

$$C = max\left\{\alpha^2(x,t);(x,t) \in [0,M] \times [0,T]\right\}$$

Let $\varepsilon >$ and $\eta > 0$. conditioning on $X^M_{x,t_1}(t_2)$, it follows from (4.13)that

$$P\left(\left|X^M_{x,t_2}(T) - X_{x,t_1}(T)\right| > \varepsilon; |X_{x,t_1}(t_2) - X| \leq \delta_0\right) \leq \frac{\eta}{2}$$

for some $\delta_0 > 0$, Let $\delta = \frac{\eta\delta_0^2}{2C}$.

Next, if we apply Chebychev's inequality to (4.14) we have:

$$P\left(\left|X^M_{x,t_1}(T) - X_{x,t_1(T)^M}\right| > \varepsilon\right) \leq \eta \tag{4.15}$$

provided that $t_2 - t_1 \leq \delta$

From (4.13) and (4.15) we have

$$X^M_{y,s}(T) \longrightarrow X^M_{x,t}(T)$$

As $(y,s) \longrightarrow (x,t)$ Let we again assume g be non negative concave function with $g(0) = 0$ We claim that

$$u^M x,t = Eg\left(X_{x,t}(T)\right) 1_{\{T<\gamma\}}$$

satisfies

$$u_t^M + \frac{\alpha^2}{2} u_{xx}^M = 0$$

For $(x,t) \in [o,M) \times [0,T)$ and

$$\lim_{(y,s)\to(x,T)} u^M(y,S) = g(x), \forall x \in (0,M)$$

To show our claim let $g_n : [0,M] \longrightarrow \mathbb{R}$ be an increasing sequence of continuous functions such that $g_n(M) = 0$ and $g_n(x) \uparrow g(x)$ $\forall x \in [0,M]$.

Now each g_n is bounded and hence the family

$$\left\{g_n\left(x_{y,s}^M(T)\right)\right\}_{(y,s)} \in v$$

where v is some small neighborhood of (x,t), is uniformly integrable. Consequently, the functions

$$u_n^M(x,t) \uparrow Eg_n(X_{x,t}^M(T))$$

are continuous on $[0,M] \times [0,T]$ Since $\alpha(x,t) > 0$ for a positive x a continuous stochastic solution is a classical solution, so u_n^M satisfies the Black–Scholes equation in $[0,M] \times [0,T]$.

By monotone convergence theorem

$$u_n^M(x,t) \uparrow u^M(x,t) \; as \; n \to \infty \tag{4.16}$$

Hence, the function u^M solves the Black–Scholes equation on $(0,M) \times [0,T)$ and (4.16) gives

$$\liminf_{(y,s)\to(x,t)} u^M(y,s) \geq g(x)$$

But since g is concave we have

$$u^M(x,t) \leq g(x) \forall (x,t)$$

This ends the proof of our claim.

Again for $u(x,t) = Eg(X_{x,t}(T))$;

$$u^M(x,t) \uparrow u(x,t) \;\; as \;\; M \to \infty \quad \text{by monotone convergence theorem}$$

Therefore u will be the solution of the Black–Scholes equation on $(0,\infty) \times [0,T)$, hence existence proved.

Uniqueness of solution

Here, we show that the classical solution found above is unique. To show the uniqueness of the solution it needs to restrict the class of function. to do so we state and prove the following theorem.

Theorem 4.2. *Assume that the payoff function g is strictly sub-linear growth. Then the option price u is the unique classical solution of strictly sub-linear growth to the Black–Scholes equation.*

Proof. From theorem 4.1 we have u as the classical solution to the Black–Scholes equation. If g is of strictly sublinear growth, then so is its smallest concave majorant $\bar{g}$. then we have For monotone function g

$$\bar{g}(E_{x,t}X(T)) \leq \bar{g}(x)$$

Jensen's inequality

$$\bar{g}(E_{x,t}X(T)) \geq E_{x,t}\bar{g}(X(T))$$

Therefore

$$u(x,t) \leq \; \bar{g}(x)$$

Thus u is a solution of strictly sub-linear growth.

To prove uniqueness we assume that there is another classical solution to the equation.

Suppose that v is the classical solution to

$$\begin{cases} v_t = \frac{1}{2}\alpha^2 v_{xx} \\ v(x,0) = 0 \\ v(0,t) = 0 \end{cases}$$

of strictly sub-linear growth.

If we use the standard change of variable: $t \to T-t$ the terminal condition will be replaced by the initial condition.

Next, define $h(x,t)$ as follows

$$h(x,t) = e^t(1+x)$$

and

$$v^{\varepsilon}(x,t) = v(x,t) + \varepsilon h(x,t)$$

Then,

$$\begin{aligned} v_t^{\varepsilon} - \frac{1}{2}\alpha^2 v_{xx}^{\varepsilon} = &\ \varepsilon h_t - \varepsilon\frac{1}{2}\alpha^2 h_{xx} \\ = &\ \varepsilon e^t(1+x) \\ > &\ 0 \end{aligned} \tag{4.17}$$

Again, define:

$$\Gamma = \{(x,t) \in [0,\infty)\ \times [0,T] : v^{\varepsilon}(x,t) < 0\}$$

And now, we claim that Γ is not empty. Let,

$$t_0 = \inf\left\{t \in [0,t] : \exists x \in [0,\infty), (x,t) \in \bar{\Gamma}\right\} \tag{4.18}$$

Since, v^{ε} is of linear growth the set Γ is contained in $[0,M] \times [0,T]$ for some constant M.Which shows that $\bar{\Gamma}$ is compact and (4.18) is attained for some points (x_0,t_0) with $v^{\varepsilon}(x_0,t_0) = 0$.

On other hand since

$$v^{\varepsilon}(0,t) = \varepsilon e^t > 0$$

And

$$v^{\varepsilon}(x,0) = 1+x > 0$$

We have both $x_0\,, t_0\ >\ 0$ But, from definition of t_0 the function: $x \rightarrowtail v^{\varepsilon}(x,t_0)$ has a minimum value at $x = x_0$. Therefore we have the following result

$$v_{xx}^{\varepsilon} \geq 0$$

in similar manner we get $v_t^\varepsilon \leq 0$.

Therefore

$$v_t^\varepsilon - \frac{1}{2}\alpha^2 v_{xx}^2 \leq 0 \tag{4.19}$$

But we can observe that equation (4.19) contradicts with (4.17). From this contradictions Γ is empty and $v^\varepsilon \geq 0$. But since ε is arbitrary we specify it to $v \geq 0$.

Next, we will do the same thing for $-v$ to arrive at $v \leq 0$.

From these two results, it follows that v is identically zero. hence our equation has a unique solution. □

4.4 Solving Black–Scholes Equation Using Feynman-Kac Formula

We have discussed the derivation of the Black-Scholes equation by the self-financing approach in section 4.2. Now, in this section, we will obtain an analytical solution to the equation (4.11) by using the Feynman-Kac formula introduced in the previous chapter. Considering,

$$\frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0$$

According to (3.18), we can identify the parameters as follows:

$$a(x) = rx$$
$$b(x) = \sigma x$$

Hence, the diffusion process (3.20) can be written as

$$dX_s = rX_s ds + \sigma X_s dW_s$$

This equation is the same as the geometric Brownian motion introduced in 3.3.6. Its solution is given as

$$X_s = X_0 exp\left(r - \frac{\sigma^2}{2}\right)s + \sigma W_s \tag{4.20}$$

Using Feynman Kac formula (3.19) the solution for the differential equation is given by

$$F(t,x) = e^{-(T-t)}\mathbb{E}\left[h(X_T)|X_t = x\right]$$

we next form independence between X_T and X_t in the following way

$$\begin{aligned} F(t,x) =& e^{-(T-t)}\mathbb{E}\left[h(X_T)|X_t = x\right] \\ =& e^{-(T-t)}\mathbb{E}\left[h\left(\frac{X_T}{\frac{X_t}{X_t}}\right)|X_t = x\right] \\ =& e^{-(T-t)}\mathbb{E}\left[h\left(x\frac{X_T}{X_t}\right)\right] \end{aligned}$$

Substituting for X_t from equation (4.20) It can be written as

$$F(t,x) = e^{-(T-t)}\mathbb{E}\left[h\left(x\frac{X_0 exp(r-\frac{\sigma^2}{2})T + \sigma W_T}{X_0 exp(r-\frac{\sigma^2}{2})t + \sigma W_t}\right)\right]$$

which can finally be simplified to

$$F(t,x) = e^{-(T-t)}\mathbb{E}\left[h\left(x\exp\left\{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right\}\right)\right] \tag{4.21}$$

Equation (4.21) is the pricing formula for simple derivatives with payoff $V_T = h(S_T)$ (Boudreault & Renaud, 2019).

Again in (4.21) the function $h(S_T)$ is the final value for the pricing formula. Hence we can now simply calculate the value of the call option and put option by substituting their final values.

4.4.1 Solution for European option

We will substitute the notation $F(t,x)$ with $C(t,S)$. In the case of the Call option, the final condition will be the payoff given by $C(T,S) = (S-K)_+$ as given by equation (1.1). Therefore from equation (4.21) we have

$$C(t,S) = e^{-(T-t)}\mathbb{E}\left[\left(S\exp\left\{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right\} - K\right)_+\right] \tag{4.22}$$

The expectation above is the stop-loss transform of a log-normally distributed random variable.

Hence,

$$S\exp\left\{\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right\} \sim \mathscr{L}\mathscr{N}\left(ln(S) + \left(r-\frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$$

Therefore using the stop-loss transform formula given by equation (3.11),equation(4.22) can be written as

$$C(t,S) = e^{-(T-t)}\left[e^{\mu+\frac{\sigma^2}{2}}N\left(-\frac{ln(K)-r}{\sigma}-\sigma\right)-KN\left(-\frac{ln(k)-r}{\sigma}\right)\right] \tag{4.23}$$

But the value for the stock price is log-normally distributed with mean $r-\frac{\sigma^2}{2}$ and standard deviation σ^2

$$ln\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}\left(\left(r-\frac{\sigma^2}{2}\right)T, \sigma^2 T\right) \tag{4.24}$$

From (4.23) and (4.24) we have the value for the call option as follows

$$C(t,S) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad 0 \leq t \leq T, \tag{4.25}$$

where,

$$d_1 = \frac{ln\,(S/K) + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \tag{4.26}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} \tag{4.27}$$

Put-call parity relationship for the European option is given as

$$C(t,S) + k = S + P(t,S),$$

Using this formula we obtain the put option as follows

$$P(t,S) = S\,[N(d_1-1)] + K[1-e^{-r(T-t)}N(d_2)] \tag{4.28}$$

d_1 and d_2 are the same as given in equation (4.26)

4.5 Computations and Simulations

Montecarlo Simulation

Monte Carlo simulations are a class of computational algorithms that use repeated random sampling to solve problems with a probabilistic interpretation. The main goal of Monte Carlo simulations is to generate many sample possible outcomes or scenarios-often over a set amount of time. The next step is discretization, which is the division of the horizon into a given number of time steps.

Algorithm for Monte Carlo Simulation

To compute the stock return prediction using Montecarlo simulation in Python we follow the following algorithm.

- Importing libraries.
- Defining the parameters for downloading data. This includes the data to be downloaded with starting and ending date
- Downloading the data from Yahoo Finance
- Dividing the data into the training and test sets.
- Defining the function for running Montecarlo's simulation
- Running the simulations.

Numerical

In most cases, it is very difficult to obtain the analytical solution of a partial differential equation, even if we obtain an analytical solution but in a very complex form. Therefore one needs to solve the PDE numerically. The finite difference method is one of the popular methods that have been used to solve partial differential equations. In this section, a finite difference scheme is developed in order to solve the Black-Scholes model numerically. Finite Difference Method aims to represent the differential equation in the form of a difference equation. We form a grid by considering equally space time values and stock price values. Consider an American put option on a stock paying no dividend. The differential equation that the option must satisfy is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{4.29}$$

Let the life of the option is T. Divide this into N equally spaced intervals of length $\Delta t = \frac{T}{N}$. A total of $N+1$ times are therefore considered

$$0, \Delta t, 2\Delta t, \ldots, T$$

Suppose that $S_{\max}$ is a stock price sufficiently high that, when it is reached, the put has virtually no value. We define $\Delta S = \frac{S_{\max}}{M}$ and consider a total of $M+1$ equally spaced stock prices:

$$0, \Delta S, 2\Delta S, \ldots, S_{\max}$$

The time points and stock price points define a grid consisting of a total of $(M+1)(N+1)$ points, as shown in Figure 4.1. Define $f_{i,j}$ as the value of f at time $i\Delta t$ when the stock price is $j\Delta S$.

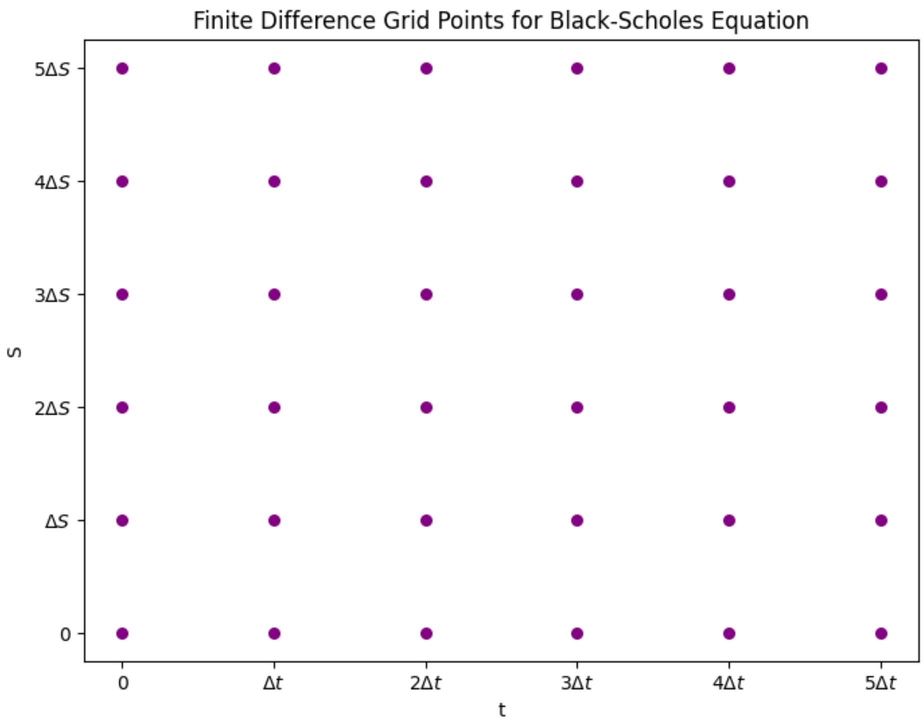


Figure 4.1: Grid for finite difference approach

Implicit Finite Difference Method

For an interior point (i,j) on the grid, $\frac{\partial f}{\partial S}$ can be approximated as

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j}}{\Delta S} \tag{4.30}$$

or as

$$\frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \tag{4.31}$$

Equation (4.30) is known as the **forward- difference** approximation; equation (4.31) is known as the **backward difference** approximation. By averaging the two

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \tag{4.32}$$

For $\frac{\partial f}{\partial t}$, we will use a forward difference approximation so that the value at time $i\Delta t$ is related to the value at time $(i+1)\Delta t$

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \tag{4.33}$$

Consider next $\frac{\partial^2 f}{\partial S^2}$. The backward difference approximation for $\frac{\partial f}{\partial S}$ at the (i,j) point is given by equation (4.31). The backward difference at the $(i,j+1)$ point is

$$\frac{f_{i,j+1} - f_{i,j}}{\Delta S}$$

Hence a finite difference approximation for $\frac{\partial^2 f}{\partial S^2}$ and the (i,j) point is

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S}\right) / \Delta S \text{ or}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2} \tag{4.34}$$

Substituting equation (4.32), (4.33), and (4.34) into the differential equation (4.29) and noting that $S = j\Delta S$ gives

$$\frac{f_{i,j+1} - f_{i,j}}{\Delta t} + rj\Delta S + \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} + \frac{1}{2}\sigma^2 j^2 \Delta S^2 \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2} = rf_{i,j} \tag{4.35}$$

for $j = 1,2,\ldots,M-1$ and $i = 0,1,\ldots,N-1$. Rearranging terms, we obtain

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \tag{4.36}$$

where,

$$\begin{aligned}
a_j &= \frac{1}{2} r j \Delta t - \frac{1}{2} \sigma^2 j^2 \Delta t \\
b_j &= 1 + \sigma^2 j^2 \Delta t + r \Delta t \\
c_j &= -\frac{1}{2} r j \Delta t - \frac{1}{2} \sigma^2 j^2 \Delta t
\end{aligned}$$

The value of the put at time T is $\max(K - S_T, 0)$. Hence,

$$f_{N,j} = \max(K - j\Delta S, 0), \quad j = 0,1,\ldots,M \tag{4.37}$$

The value of the put option when the stock price is zero is K. Hence

$$f_{i,0} = K, i = 0,1,\ldots,N \tag{4.38}$$

We assume that the put option is worth zero when $S = S_{\max}$, so that

$$f_{i,M} = 0, \quad i = 0,1,\ldots,N \tag{4.39}$$

For the points corresponding to time $T - \Delta t$, equation (4.35) with $i = N-1$ gives

$$a_j f_{N-1,j-1} + b_j f_{N-1,j} + c_j f_{N-1,j+1} = f_{N,j} \tag{4.40}$$

for $j = 1,2,\ldots,M-1$. The right sides of these equations are known from equation (4.37). Furthermore, from equations (4.38) and (4.39),

$$f_{N-1,0} = K \tag{4.41}$$

$$f_{N-1,M} = 0 \tag{4.42}$$

Equation (4.40) are therefore $M-1$ simultaneous equations that can be solved for the $M-1$ unknowns: $f_{N-1,1}, f_{N-1,2}, \ldots, f_{N-1,M-1}$. After this has been done, each value of $f_{N-1,j}$ is computed with $K - j\Delta S$. If $f_{N-1,j} < K - j\Delta S$, early exercise at time $T - \Delta t$ is optimal and $f_{N-1,j}$ is equal to $K - j\Delta S$. The nodes corresponding to time $T - 2\Delta t$ are handled in a similar way, and so on.

Example 3. *For the demonstration of the implicit difference method, consider the following parameters for the Black-Scholes equation*

$$S_0 = 20.0, \quad K = 20.0, \quad r = 0.05, \quad T = \frac{5}{12}, \quad \sigma = 0.25, \quad S_{max} = 40.0, \quad dS = 2, \quad dt = \frac{5}{120}$$

Using this set of parameters, the American put option is computed for each grid point using Python code. The results are shown in the tableTable 4.1. Each grid point in the table represents the price of corresponding asset price at a time. From the table price of American put options for given set of parameters will be 1.02 *as exercise price was 20.*

Table 4.1: American put options computation using implicit difference scheme.

$S \diagdown t$	5.0	4.5	4.0	3.5	3.0	2.5	2.0	1.5	1.0	0.5	0.0
40.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0
38.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0
36.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0
34.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0
32.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0
30.0	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.0
28.0	0.02	0.02	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.0
26.0	0.06	0.05	0.04	0.03	0.02	0.01	0.01	0.00	0.00	0.00	0.0
24.0	0.16	0.14	0.11	0.09	0.07	0.05	0.03	0.02	0.01	0.00	0.0
22.0	0.42	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.06	0.02	0.0
20.0	**1.02**	0.96	0.90	0.83	0.76	0.68	0.58	0.48	0.35	0.19	0.0
18.0	2.20	2.16	2.13	2.10	2.07	2.05	2.02	2.01	2.00	2.00	2.0
16.0	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.0
14.0	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.0
12.0	8.00	8.00	8.00	8.00	8.00	8.00	8.00	8.00	8.00	8.00	8.0
10.0	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.0
8.0	12.00	12.00	12.00	12.00	12.00	12.00	12.00	12.00	12.00	12.00	12.0
6.0	14.00	14.00	14.00	14.00	14.00	14.00	14.00	14.00	14.00	14.00	14.0
4.0	16.00	16.00	16.00	16.00	16.00	16.00	16.00	16.00	16.00	16.00	16.0
2.0	18.00	18.00	18.00	18.00	18.00	18.00	18.00	18.00	18.00	18.00	18.0
0.0	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.0

4.6 Application in Finance

4.6.1 Stock price prediction using geometric Brownian motion

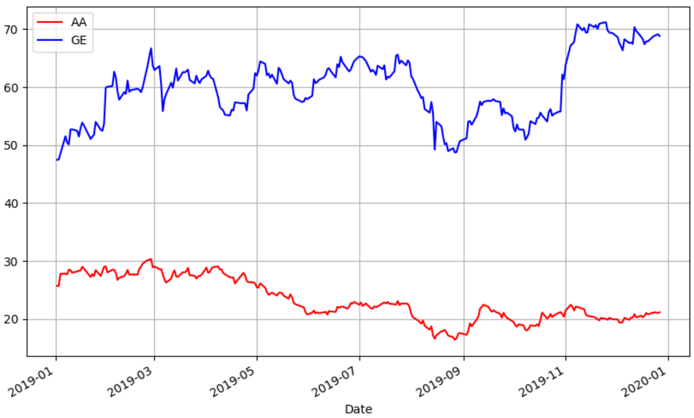
We consider two different stocks from Yahoo Finance and we asses their historical data using Python. Alcoa corporation stocks and general electric stocks are selected for this purpose. We made the investigation for the periods **(2019-01-01)** to **(2021-10-30)** by dividing into three different periods. The period is selected so that we will have different market behaviors. In this case it includes the period of COVID-19 pandemic where the market fluctuations expected to be high. For the stock prediction, we have used the Montecarlo simulation. For each simulation adjusted close price of stock is considered and and number of simulation used is 1000.The historical price movement of both stocks is given for each period so that it can easily be understood in which period the market stock price rapidly change.

For the result the historical data downloaded from yahoo finance between the each given period is divided into train data set and test data set for which simulation done for test data set using train data set historical price. Hence the result given below is the prediction of stock price for a last few days depending on their past several days historical price. The values of calculated(predicted) and adjusted close price of both stock in each period is given by tabular form. Finally, we have displayed using graph to compare the predicted price and adjusted close price.

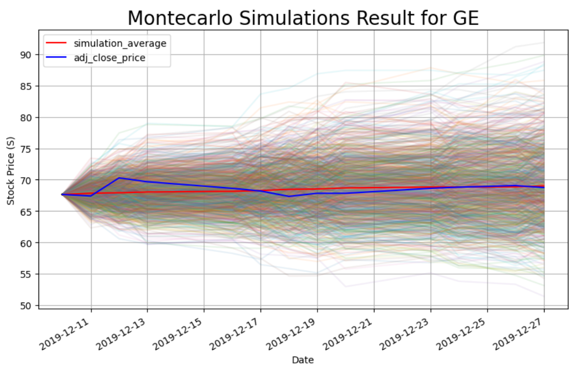
We divided the the periods into three as follows:
First period: **2019-01-01** to **2019-12-30**
Second period: **2020-01-01** to **2020-12-30**
Third Period:**2021-01-01** to **2021-11-20**

Table 4.2: Calculated stock price and actual stock price comparison for Genera Electric Company and Alcoa Corporation (2019-01-01 to 2019-12-30)

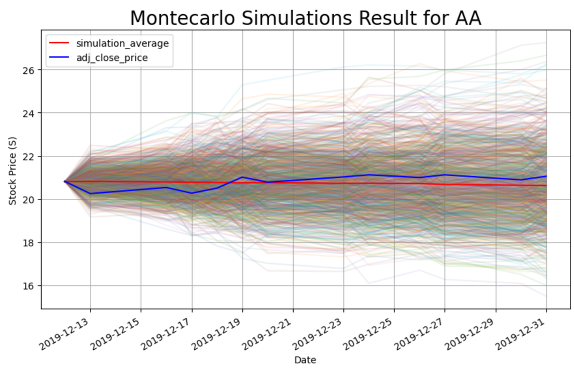
Date	GE (Actual	GE(Predictd)	AA (Actual)	AA(Predicted)
2019-12-10	67.779678	67.656570	-	-
2019-12-11	67.533440	67.661304	20.359003	20.359003
2019-12-12	70.426842	67.682368	20.960405	20.355697
2019-12-13	69.811226	67.674591	20.398441	20.364347
2019-12-16	68.764671	67.663277	20.684355	20.360689
2019-12-17	68.333740	67.673662	20.418159	20.363553
2019-12-18	67.471870	67.674833	20.654779	20.365048
2019-12-19	67.964371	67.686181	21.167450	20.365280
2019-12-20	67.964371	67.698104	20.930834	20.367151
2019-12-23	68.765396	67.688525	21.177309	20.369242
2019-12-24	68.950264	67.693529	21.275900	20.366597
2019-12-26	69.196732	67.686291	21.147732	20.370187
2019-12-27	68.888634	67.700393	21.275900	20.361269



(a) Stock price movement for two stocks (**2019-01-01** to **2019-12-31**)



(b) Simulation result display for GE



(c) Simulations result for Alcoa corporation

Figure 4.2: Montecarlo Simulation result display from (**2019-01-01**) to (**2019-12-31**)

Table 4.3: Calculated stock price and actual stock price comparison for General Electric Company and Alcoa Corporation (2020-01-01 to 2020-12-30)

Date	GE (Predicted)	GE (Actual)	AA (Predicted)	AA (Actual)
2020-12-10	69.938721	69.938721	19.471581	23.501537
2020-12-11	70.038963	68.950195	19.542753	22.356319
2020-12-14	70.154012	66.911339	19.578169	21.299189
2020-12-15	70.090043	68.641258	19.558537	21.70050
2020-12-16	70.001333	67.529160	19.546975	21.122999
2020-12-17	70.147839	67.220261	19.611284	21.710297
2020-12-18	70.225205	66.849228	19.692332	21.543894
2020-12-21	70.272157	66.601860	19.741115	21.641775
2020-12-22	70.320061	65.612404	19.768584	21.152365
2020-12-23	70.199149	67.158409	19.842129	21.749449
2020-12-24	70.280998	65.859779	19.892959	21.494951
2020-12-28	70.287552	65.797935	19.846724	21.769022
2020-12-29	70.316048	65.303215	19.920242	21.573261
2020-12-30	70.267754	66.230827	19.969405	22.463989

Table 4.4: Calculated stock price and actual stock price comparison for General Electric Company and Alcoa Corporation (2021-01-01 to 2021-10-30)

Date	GE (Predicted)	GE (Actual)	AA (Predicted)	AA (Actual)
2021-10-28	81.551529	81.551529	45.565628	45.565628
2021-10-29	81.642341	81.249374	45.736900	45.075142
2021-11-01	81.755124	82.303047	45.934958	45.840294
2021-11-02	81.908465	82.659439	46.176853	46.183632
2021-11-03	81.962357	82.101608	46.309763	46.526962
2021-11-04	82.000507	81.512787	46.426803	45.487148
2021-11-05	82.215161	84.247719	46.733783	46.948780
2021-11-08	82.348703	83.999779	46.951060	47.929737
2021-11-09	82.437685	86.223351	47.124688	46.340588
2021-11-10	82.567495	84.418159	47.346624	45.290951
2021-11-11	82.690314	82.899612	47.558962	50.234993
2021-11-12	82.787669	83.356735	47.741723	50.931480
2021-11-15	82.807921	82.643944	47.833967	49.666042
2021-11-16	82.929688	80.071732	48.046853	47.282307
2021-11-17	83.067325	79.018059	48.279015	47.007633

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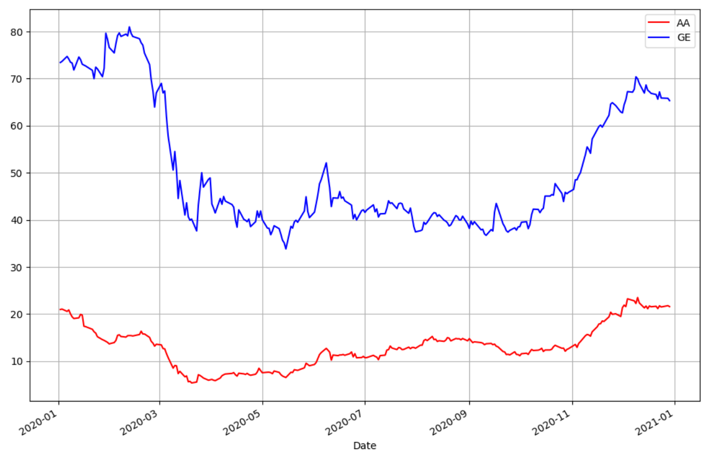
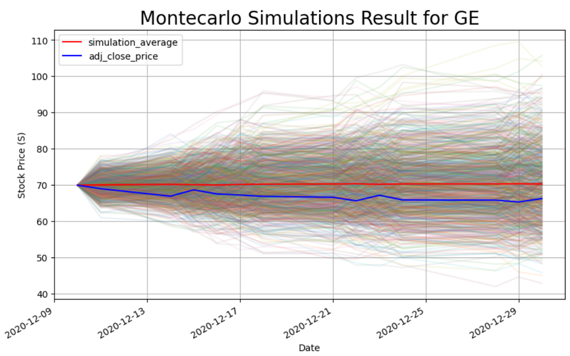
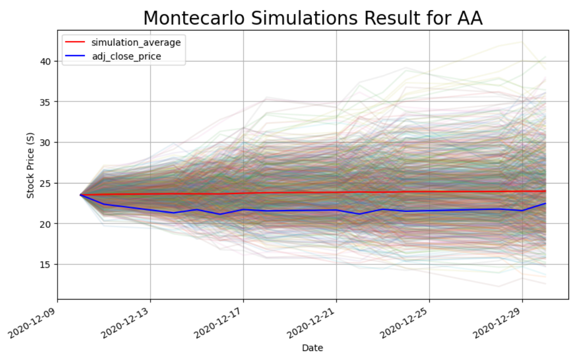


(a) Stock price movement

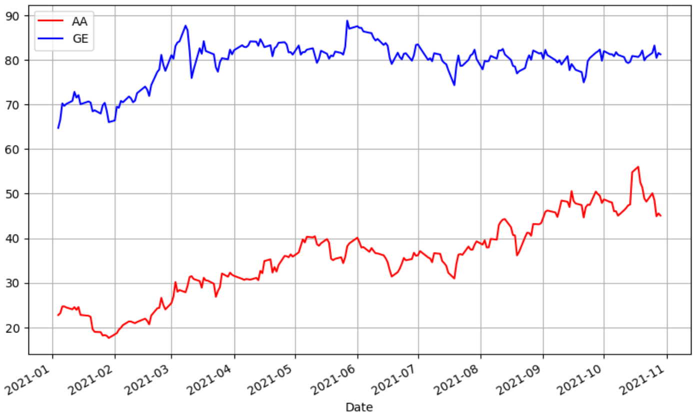


(b) Simulation result display for GE

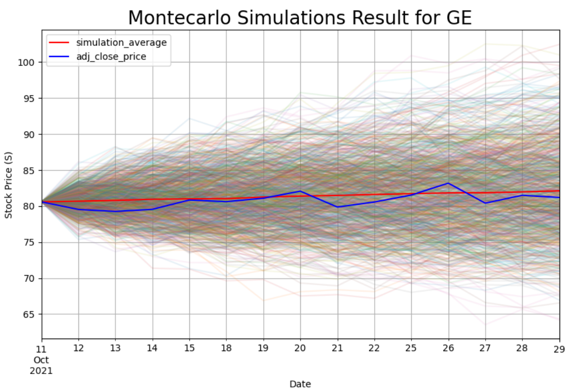


(c) Simulation result display for AA

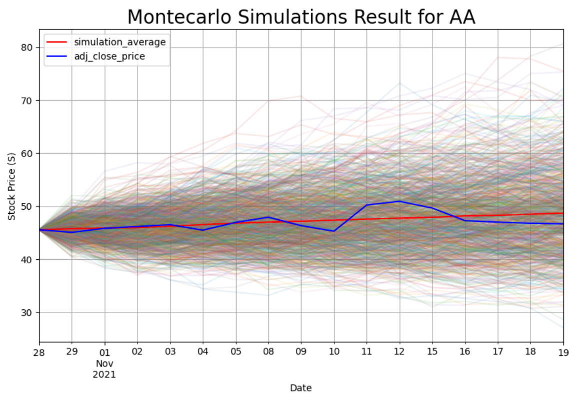
Figure 4.3: Montecarlo Simulation result display from (**2020-01-01**) to (**2020-12-31**)



(a) Stock price movement



(b) Simulation result display for GE stock



(c) Simulation result display for AA stock

Figure 4.4: Montecarlo Simulation result display from (**2021-01-01**) to (**2021-10-30**) .

4.6.2 Evaluating Black-Scholes equation for European options using American Air Lines(AAL) and Lucid Group(LCID) stocks data.

We conducted an analysis utilizing historical data sourced from Yahoo Finance for a span of one year for particular stocks. The aim was to calculate the prices of European call and put options using the Black-Scholes equation. To do this, we derived all the necessary parameters from this historical data.

The parameters used in the Black-Scholes equation were determined as follows:

- S_0 : This represents the current stock price at the time of simulation, obtained directly from Yahoo Finance.
- r: The risk-free rate was set at 1.932%, based on the 10-year US treasury yield.
- σ: The volatility of the stock was calculated using the annualized standard deviation of stock returns from the historical data. It was computed as the product of the daily returns' standard deviation, scaled by the square root of 252 (the number of trading days in a year).
- K: This value was derived from the strike prices available in the extracted option chain from Yahoo Finance,corresponding to the contract.
- T: The time to maturity (expressed in years) was calculated using the length of the option contracts available in the option chain divided by 252, assuming 252 trading days in a year.

Using the Black-Scholes model formulae for both call and put options, given as 4.25 and 4.28 respectively, we calculated the theoretical prices of these options based on the aforementioned parameters.To assess the accuracy and reliability of these calculated prices, we compared them with the last traded prices obtained directly from the option chain available on Yahoo Finance for the same maturity date. The result computed in Python is given below in tabular form and displayed by figures.

Table 4.5: Calculated Black Scholes call and put prices for American Air Lines (AAL) and Lucid group(LCID) with their respective last price for maturity December 01,2023

(a) Call Values for AAL

T	Black Scholes(Call)	Last price(Call)
0	7.0069	6.45
1	5.00965	4.8
2	4.01103	3.8
3	3.01241	2.75
4	2.5131	2.67
5	2.01379	2.32
6	1.51448	1.85
7	1.01518	1.33
8	0.519956	0.88
9	0.12527	0.44
10	0.00669811	0.17
11	5.20946e-05	0.05
12	5.63802e-08	0.02
13	9.48671e-12	0.01
14	5.46205e-16	0.01
15	0	0.02
16	0	0.05
17	0	0.16

(b) Put Values for AAL

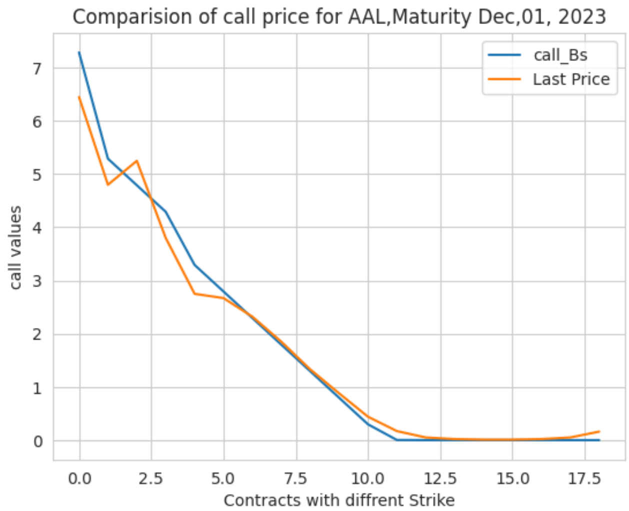
T	Black Scholes(Put)	Last price(Put)
0	0	0.04
2	0	0.05
3	0	0.02
4	0	0.03
5	0	0.01
6	0	0.02
7	1.21112e-15	0.01
8	5.17425e-10	0.01
10	0.00214543	0.02
11	0.069945	0.1
12	0.389645	0.32
13	0.870429	0.75
14	1.36932	1.28
15	1.86855	1.67
16	2.36778	2.17
17	2.86702	2.74
18	3.36625	4.35
19	5.36319	5.44

(c) Call Values for LCID

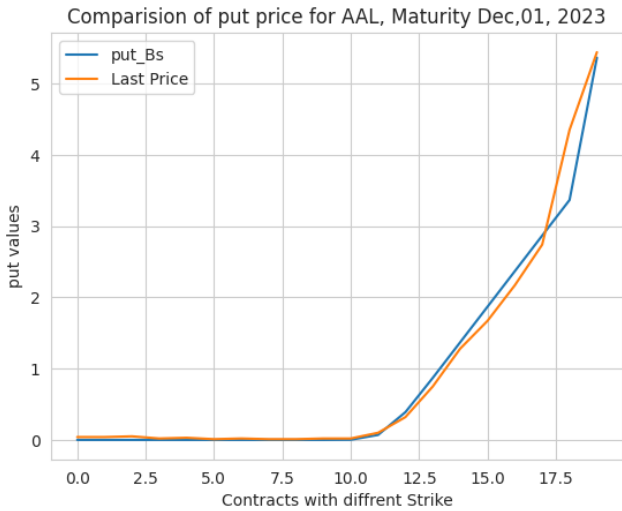
T	Black Scholes(Call)	Last price(Call)
0	3.7438	3.75
1	3.24461	3.3
2	2.74541	2.68
3	2.24622	2.2
4	1.74725	1.9
5	1.2537	1.23
6	0.795805	0.8
7	0.43235	0.3
8	0.199479	0.06
9	0.0792158	0.03
10	0.0277038	0.02
14	0.000181491	0.01
15	4.57068e-05	0.02
16	1.12224e-05	0.04
17	2.70732e-06	0.01

(d) Put Values for LCID

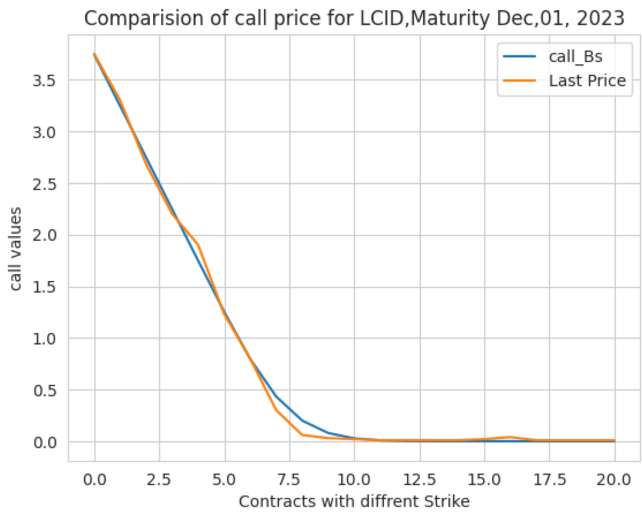
T	Black Scholes(Put)	Last price(Put)
0	2.00101e-08	0.01
1	2.43886e-05	0.01
2	0.00183175	0.01
3	0.0266324	0.02
4	0.141671	0.08
5	0.405834	0.33
6	0.799611	0.72
7	1.26312	1.2
8	1.75282	1.78
9	2.25005	2.29
10	2.74905	2.68
11	3.24839	3.31
12	3.74781	3.91
13	4.24723	4.29
14	6.24493	6.9



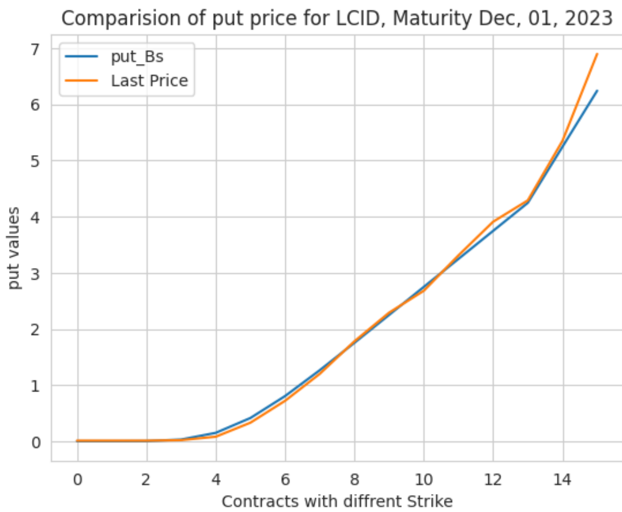
(a) Call prices display for AAL



(b) Put prices display for AAL



(c) Call prices display for LCID



(d) Put prices display for LCID

Figure 4.5: Black-Scholes call and put prices display for American Air Lines(AAL) and Lucid group(LCID) compared with their corresponding last price for maturity December 01,2023

4.7 Discussion

Result for the first period shown in Figure 4.2 and third period shown by Figure 4.4 shows the actual price of stocks followed the predicted price.During these time periods, the Monte Carlo simulation showed reasonably accurate approximations of actual stock prices. The simulated and actual prices significantly converged in both cases. Predictability is mostly due to historical stability, steady market behavior, and reasonably predictable behavioral patterns during these times.As we can see from Figure 4.2a ans Figure 4.4a historical price history for both stock specially were uniform and stable. But,if we look at Table 4.3 and Figure 4.3 for the second period result difference between the trajectory of actual price and predicted price is wider and we can say that it is not a good prediction.In this phase, the simulation's performance differed significantly from the success observed in the first and third periods. Stock prices fluctuated rapidly and randomly due to unprecedented market instability brought on by the global COVID-19 outbreak. The excessive volatility that the simulation was unable to adjust to caused a divergence between the simulated and actual prices, which in turn affected the simulation's forecast accuracy. we can observe from Figure 4.3a.

The efficiency of standard Black-Scholes formula has been tested using the last price of two stocks (AAL and LCID) extracted from Yahoo Finance. Black-Scholes model parameters were derived from historical data of stocks and the contract prices.The result of calculation for both stocks compared with their respective last traded price is given above in tabular form and also displayed graphically. The analysis shows the Black-Scholes model's efficiency as a reliable tool in low-volatility environments.This aligns with its strength in accurately estimating option prices when market fluctuations remain minimal and stock prices follow a smooth trajectory as it can be observed from Figure 4.5c and Figure 4.5d above. Despite this good approximation and price alignments for low volatile markets, Black-Scholes assumptions constant volatility may not efficient in rough markets. We can see Figure 4.5a Where there is price fluctuations of a given stock while the Black-Scholes trajectory still predicts smooth price curve.

Conclusion and Recommendation

Conclusion

In this thesis, we have investigated the Black-Scholes equation with its application in finance. We have proved that the option function the only solution to this equation which ensures the existence and uniqueness of Black-Scholes equation. We have solved Black-Scholes equation by Feyman Kac formula and we have used implicit difference method to approximate contracts for American options. Additionally, the application of the equation extended to stock price prediction, utilizing geometric Brownian motion and the Monte Carlo algorithm. Through empirical analysis using stock data from Yahoo Finance, it became evident that the Monte Carlo algorithm demonstrates efficiency in stable market conditions its accuracy diminishes in highly fluctuating market prices. We have also tested the efficiency of Black-Scholes-formula using historical data from Yahoo Finance by comparing with its last traded price. Result showed Black-Scholes formula is efficient for the market which are stable and has low volatile. But we have observed that this efficiency of Black-Scholes formula is limited to stable markets and diminishes in high volatile and rough market conditions due to its assumptions.

Recommendation

In this thesis, standard Black-Scholes equation which is limited to various assumptions was studied. Inefficiency of Black-Scholes formula observed in the result arises from these assumptions. There are other versions of Black-Scholes equation that may overcome this inefficiency but may lead to complexity of calculation. We recommend that for future researchers to develop efficient model which is understandable and suitable for use in real life. In addition we recommend the following:

- The calculated prices in this thesis are for research purposes only and may not be reliable in unstable markets. Therefore,it is not recommended for practical use.
- Although other modified and efficient Black Scholes models are developed, we can still use the standard one in stable markets.

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