

THEORETICAL STUDY OF MAGNETO-NANOFLUID FLOW AND HEAT TRANSFER IN MICROCHANNELS



Mesfin Zewde Kefene

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Science

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy
in Applied Mathematics

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Adama Science and Technology University

June, 2022
Adama, Ethiopia

Declaration

I hereby declare that this dissertation entitled "**Theoretical Study of Magneto-nanofluid Flow and Heat Transfer in Microchannels**" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials used for this dissertation have been duly acknowledged through appropriate citations.

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Recommendation

We, the supervisors of this dissertation, hereby certify that we have read and revised the dissertation entitled "**Theoretical Study of Magneto-nanofluid Flow and Heat Transfer in Microchannels**" prepared under our guidance by Mesfin Zewde Kefene submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics. Therefore, we recommend the submission of the dissertation to the department for further review and defense.

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Approval page

We hereby certify that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the dissertation entitled "**Theoretical Study of Magneto-nanofluid Flow and Heat Transfer in Microchannels**" by Mesfin Zewde Kefene.

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We, the undersigned, members of the Board of Examiners of the dissertation open defense by Mesfin Zewde Kefene have read and evaluated the dissertation entitled "**Theoretical Study of Magneto-nanofluid Flow and Heat Transfer in Microchannels**" and examined the candidate during open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in Applied Mathematics.

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Finally, approval and acceptance of the dissertation is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the candidate's Department Graduate Council (DGC) and School Graduate Committee (SGC).

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Dedication

This dissertation is dedicate to all my family members.

Abstract

Microchannel liquid flow is characterized much by laminar flow regime with low Reynolds number. This laminar flow gives rise to molecular diffusion as the main means of heat transfer, which is not sufficient to remove the required quantity of heat from the high-performance industrial equipments. Thus, the inherent laminar flow within the microchannel limits the heat transfer. To deal with the heat transfer challenge associated with the laminar nature of microchannel flow was the focus of this dissertation, and addressed by incorporating both passive and active heat transfer enhancement methods to improve the heat transfer rate of microchannel flow, where water is used as base fluid with mixed convection. Employing permeable microchannel walls with asymmetric wall temperature/ concentration (passive), loading nanoparticles to water (passive), and imposing uniform magnetic field (active) were utilized in all the problems discussed in the study for better heat transfer performance of microchannel flow for the case of single phase and two phase approaches. In addition to heat transfer enhancement, the applied magnetic field manipulates the flow fields and the heat transfer rate. Since the thermophysical properties of working fluids in microchannel rely largely on temperature, the problems also considered temperature dependent viscosity and thermal conductivity mutually or exclusively. Moreover, the physical phenomena like thermal radiation, chemical reaction, and convective wall heating were included in some of the problems. More precisely, a two-phase model approach was employed to study the effects of variable fluid properties on unsteady mixed convection of MHD nanofluid within a permeable microchannel with mutual absence and presence of thermal radiation, chemical reaction and convective heating. Moreover, a single-phase model approach has been utilized to investigate the effects of magnetic field and variable viscosity on the heat transfer analysis of unsteady mixed convection of microchannel flow using magneto-nanofluids $Fe_3O_4 - H_2O$ and $Cu - H_2O$, respectively, in the absence and presence of thermal radiation and convective heating. The governing nonlinear partial differential equations (PDEs) of the problems were formulated and then transformed into a set of dimensionless nonlinear PDEs by employing dimensionless variables and parameters. The resulting nonlinear PDEs in dimensionless form were solved numerically by semidiscretization via centered finite difference scheme with Runge-Kutta Fehlberg integration technique. Results obtained from transient analysis indicated that the velocity profiles achieved steady states faster than the temperature and concentration profiles. The findings from steady state analysis pointed out that the Nusselt number of non-radiating nanofluids decreased with rise in viscosity variation parameter and magnetic field parameter at the wall $y = 0$, but a reverse pattern happened with these parameters at the wall $y = 1$. Moreover, the Nusselt number of radiating nanofluid indicated an increasing behavior with increasing in thermal conductivity variation parameter; Biot number, thermal radiation parameter, Eckert number and thermal Grashof number at both walls. However, the Nusselt number of a radiating $Cu - H_2O$ nanofluid with convective wall heating decreased with increasing magnetic field parameter at both channel walls. Some of the useful results presented for the velocity, temperature and concentration profiles were validated qualitatively with the previous results reported in the literatures. In this respect, a good match was obtained. At the end, general closing remarks on the outcomes of the problems and suggestions for future investigations were made.

Keywords: MHD; Microchannels; Mixed convection; Nanofluids; Suction/injection; Variable fluid properties; Thermal radiation; Chemical reaction; Buongiorno model.

List of Published and Communicated Papers

The course of the study have come up with the following list of articles:

- M. Z. Kefene, O. D. Makinde, L. G. Enyadene (2020). MHD variable viscosity mixed convection of nanofluid in a microchannel with permeable walls. *Indian Journal of Pure & Applied Physics*. 58(12), 892-908.
- M. Z. Kefene, O. D. Makinde, L. G. Enyadene (202X). Magnetic Effects of Ferrofluid ($Fe_3O_4 - H_2O$) Flow in a Heated Micro-Channel with Permeable Walls. *Latin American Applied Research*. 52(3), 247-257. DOI: <https://doi.org/10.52292/j.laar.2022.666>
- M. Z. Kefene, O. D. Makinde, L. G. Enyadene (202X). Unsteady Hydromagnetic Mixed Convection of a Radiating and Reacting Nanofluid in a Microchannel with Variable Properties. (Submitted for consideration in *Engineering Transactions*)
- M. Z. Kefene, O. D. Makinde, L. G. Enyadene (202X). Hydromagnetic Mixed Convection of $Cu - H_2O$ Nanofluid in a Micro-Channel with Thermal Radiation and Convective Heating. (Under review in *Journal of Nanofluid*)

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List of Symbols

Symbols

a	Microchannel width	r	Dimensionless electrical conductivity
A	Dimensionless pressure gradient	R	Source term
A_1	Viscosity variation parameter	Re	Reynold suction/injection parameter
A_2	Thermal conductivity variation parameter	rx	Chemical reaction parameter
B, B_o	Magnetic field vector and component of B	Sc	Schmidt number
Bi	Biot number	$\bar{t}$	Time
C	Nanoparticles concentration	t	Dimensionless time
C_o	Nanofluid initial concentration	T	Temperature of nanofluid
C_1	Nanofluid concentration at the right wall	T_o	Temperature of the left/lower wall
Cf	Coefficient of skin friction	T_1	Temperature of the right/upper wall
C_p	Specific heat at constant pressure	T_∞	Ambient temperature
D_B	Brownian diffusion coefficient	T_f	Hot fluid temperature heating the left/lower wall
D_T	Thermal diffusion coefficient	T_w	Nanofluid initial temperature (or $T_w=T_1$)
E	Electric field vector	$\bar{u}$	Dimensional nanofluid velocity
Ec	Eckert number	u	Dimensionless nanofluid velocity
F_{em}	Electromagnetic force	V	Constant suction/injection velocity
F_{es}	Electrostatic force	$\bar{x}, \bar{y}$	Dimensional rectangular coordinate system
F_l	Lorentz force	x, y	Dimensionless rectangular coordinate system
F_m	Magnetophoresis force	Greek Alphabet	
g	Acceleration due to gravity	α	Thermal diffusion coefficient
Gc	Solutal Grashof number	β_1, β_2	Thermal and solutal expansion coefficient
Gr	Thermal Grashof number	γ_1, γ_2	Viscosity and thermal conductivity variation
h_f	Convective heat transfer coefficient	ε	Chemical reaction rate
H_m	Mass flux	ρ_e	Electric charge density
J	Electric current density	ζ	Dielectric constant
k	Thermal conductivity of the nanofluid	θ	Dimensionless temperature
k_o	Initial nanofluid thermal conductivity	λ	The second viscosity coefficient
M	Magnetic field parameter	μ	Dynamic viscosity coefficient
Nb	Brownian motion parameter	ν	Kinematic viscosity coefficient
Nr	Radiation parameter	ρ	Density
Nt	Thermophoretic parameter	σ	Electrical conductivity
P	Dimensionless nanofluid pressure	τ	The ratio between effective heat capacity of nanoparticles and fluid
$\bar{P}$	Nanofluid pressure	τ_w	Wall shear stress
Pr	Prandtl number	ϕ	Dimensionless nanoparticles concentration
Q	Heat added per unit mass	ϑ	Magnetic permeability
q_r	Radiative heat flux	Φ	Viscous dissipation term
q_w	Wall heat flux		

Abbreviations

CFD	Computational fluid dynamics
EDL	Electric double layer
KKL	Koo-Kleinstreuer method
MCHS	Microchannel heat sink
MEMS	Micro-electro-mechanical systems
MHD	Magnetohydrodynamics
MOL	Method of line
ODEs	Ordinary differential equations
PDEs	Partial differential equations
RKFIT	Runge-Kutta-Fehlberg integration technique

Subscripts

f	Base fluid
nf	Nanofluid
s	Solid nanoparticles

CHAPTER 1

INTRODUCTION AND OVERVIEW

1.1 Background of the Study

In micromachining process, heat sinks have been developed to a highly efficient heat removal devices due to the advantages of minimal coolant requirement and smaller dimensions. One of the most micromachining technology is microchannels. Microchannels have a wide range of applications in cooling micro-electro-mechanical systems (MEMS), micro-scale and large-scale thermal systems demanding for effective cooling capacity. This is due to their micron-sizes which provide them larger surface area to volume ratio leading to enhance the capability to dissipate much greater amount of heat flux than any conventional heat removing systems. On the other hand, the microchannels have become essential components that may serve as pipes, valves, mixers and pumps to the lab-on-chip systems in the area of biological and chemical analysis. Therefore, microchannels are useful in transport of fluids and mixing different fluids or separating different species. It shows that mass or momentum transport processes inside the microchannels are very important. Even though the microchannels are advantageous in many areas, they are not without curbs. Due to extreme compactness of the channel, high pressure drop in the working fluid with the requirement of pumping devices for running the flow and high frictional resistance of the coolants which produce high temperature rise within the fluid are the challenges associated with flows through microchannel.

To have a better heat transfer performance of microchannels, the present work considers passive and active techniques of heat transfer enhancement in microchannel flow where water is employed as common fluid with laminar regime and mixed convection flow. Using permeable microchannel walls with asymmetric wall temperature (passive), loading nanoparticles to water (passive), and imposing uniform magnetic field (active) are methods used in all the problems. Since the thermophysical properties of nanofluid like viscosity and thermal conductivity depend largely on temperature in microchannel flow, the variable fluid properties also partake in the problems of this study. Besides, the physical phenomena such as thermal radiation, chemical reaction and convective heating are used in some of the problems.

Most advanced physical systems in the fields of engineering, medicine, biology and economics are modeled by differential equations. Differential equations are used to describe a continuous change of physical systems mathematically. Partial differential equations (PDEs) and ordinary differential equations (ODEs) are the two forms of differential equations that

are employed to model and describe the physical systems features. An MHD nanofluid flow in a microchannel is a physical problem governed by a system of nonlinear PDEs. These system of equations are complex and impossible to find the solution analytically, therefore, numerical method are utilized instead to get an approximations of exact solution. Heath (2002) highlights numerical methods that played an important role in solving problems in engineering and industry. Also, Makinde et al. (2012) described that numerical method (semi-discretization scheme with RKFIT) have been employed to solve system of PDEs that characterize the flow of nanofluid in a channel. Various numerical methods such as explicit finite difference method, Galerkin weighted residual finite element method, finite volume method, method of line, etc are used to solve PDEs. Among which the method of line (semi-discretization scheme with RKFIT) which essentially used in tackling the system of PDEs that govern the problems of this study. A MAPLE software is used to perform the transient analysis of the system of PDEs and to obtain the numerical solutions of the resulting system of ODEs.

The aim of this study is to formulate and analyze the mathematical models of unsteady microchannel flow of magnetohydrodynamic (MHD) nanofluid with variable properties for the four set of problems under their respective physical scenarios to investigate the time development of flow quantities (velocity, temperature, concentration) and the aggregate effects of thermophysical parameters on transport phenomena utilizing single and two phase approaches.

1.1.1 Magnetohydrodynamics

Magnetohydrodynamics (magnetofluid dynamics or hydromagnetics) is the science which studies the flow of electrically conducting fluids in the presence of a magnetic field. Examples of electrical conducting fluids include mercury, plasma, nanofluid, and salt water or electrolyte (Sarojini et al., 2013). The word magnetohydrodynamic comes from magneto- meaning magnetic field, hydro- meaning liquid, and -dynamic meaning movement. The field of MHD was first explored by Hannes Alfven, who won the Nobel Prize in Physics in 1970. The basic idea in studying MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn creates forces on the fluid and changes the magnetic field itself (figure 1.1) (Sheikholeslami and Ganji, 2016). The MHD effect is a physical phenomenon encountered in the study of solar winds (Priest, 2003), geophysics (Freidberg, 1987), MHD-derived biomedical sensor (Laughlin, 1989), and actuators (Jang and Lee, 2000). The study of the MHD effect involves the coupling of electromagnetism, specifically regarding induction, to dynamics. The magnitude and direction of the fluid flow (q) and the magnetic field (B) both directly affect the magnitude and resultant direction of the MHD effect, creating a variance in induced Hall effect voltages (Blandford and Thorne, 2004). The mathematical equations of MHD phenomena described in terms of magnetic field (B), and Maxwell's equa-

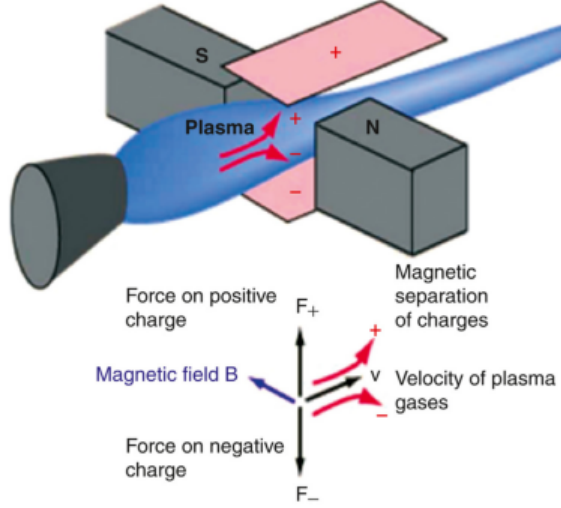


Figure 1.1: Induced current in a moving conductive fluid in the presence of a magnetic field (Sheikholeslami and Ganji, 2016).

tions (1.1)-(1.4) expressed in differential form to predict the interactions between magnetic fields and electric fields, currents, and charges

$$\nabla \cdot E = \frac{\rho_e}{\zeta}, \quad (1.1)$$

$$\nabla \cdot B = 0, \quad (1.2)$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad (1.3)$$

$$\nabla \times B = \vartheta \left(J + \zeta \frac{\partial E}{\partial t} \right). \quad (1.4)$$

Considering a highly conducting fluid flow, the displacement current become negligible as a result equation (1.4) is simplified to ignore the displacement current (second parenthetical term). This is due to inhibition of electric field propagation. To define this effect in terms of applied magnetic field, Ohm's law is used (equation 1.5)

$$J = \sigma(E + q \times B). \quad (1.5)$$

In a situation where currents are induced by a motion of a conducting fluid through a magnetic field, an additional force called Lorentz force ($F_l = J \times B$) will affect and modify the motion of the fluid. The thorough description of conducting fluid flow behavior under the exposure to external magnetic field take into account the total electromagnetic force F_{em} which is the total of Lorentz F_l , magnetophoresis F_{mp} (occur in ferrous materials), and electrostatic

F_{es} forces (Bau et al., 2003; Qian and Bau, 2009; Qin and Bau, 2011; Xiang and Bau, 2003).

$$F_{em} = F_l + F_{mp} + F_{es}. \quad (1.6)$$

During the consideration of most clinical applications of the MHD effect, the applied magnetic field is uniform and static, especially during magnetic resonance imaging (MRI), and magnetophoretic and electrostatic forces become null in Equation (1.6) (Hetsroni et al., 2005). This simplification implies that Lorentz F_l is the only remaining force in the total electromagnetic force F_{em} . In MHD, the motion modifies the field, and the field in turn reacts back and modifies the motion (Chung, 2002). This makes the theory highly nonlinear.

1.1.2 Heat Transfer

Heat transfer is the transmission of thermal energy as a result of spatial temperature gradient. The driving force of heat transfer in channel flow is the temperature difference and it occurs due to high frictional resistance of the coolant, temperature differences among fluid layer or among the fluid and the channel walls. Variations of temperature in a flowing fluid may also arise from Brownian motion effect, thermophoretic effect, viscous dissipation and absorption of thermal radiation. Heat transfer process is the main course of action in many industrial and engineering systems in which the thermal power generation based. Generally, there are three forms of heat transfer: (1) conduction (2) convection, and (3) radiation. Boiling and condensation are a combination of these modes. In reality, it is rare to happen one pure mode of heat transfer alone but a mix where one of the mode may be more important that the other can be ignored.

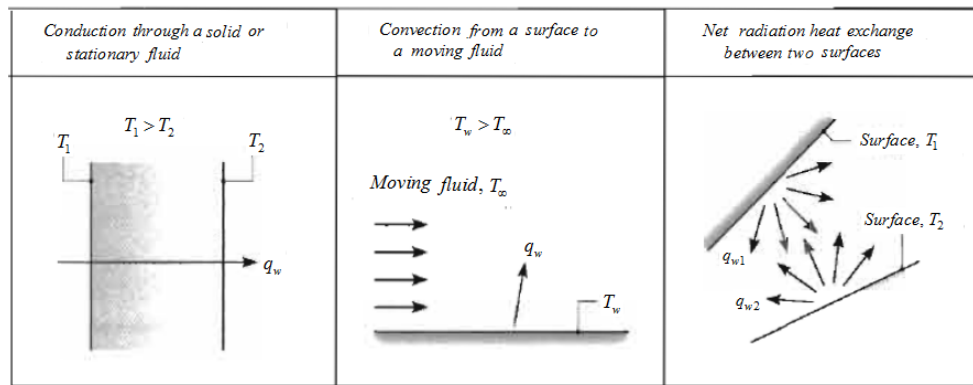


Figure 1.2: Forms of heat transfer (Moran et al., 2003).

Conduction

Conduction is the transfer of energy from one part of a substance to another part of the same substance, or from one substance to another in physical contact with it, without appreciable

displacement of molecules forming the substance. In gaseous matter, the heat conduction mechanism is by random motion of gaseous molecules. In liquids, the mechanism of heat transfer is by intermolecular interaction. In case of solids, the heat is conducted by means of lattice vibration and movement of free electrons. The comprehensive theory of heat conduction is studied and developed for the first time by Joseph Fourier. According to Fourier, the heat flux in the direction of heat flow is directly related to the temperature gradient in that direction.

Convection

Convection is the form of heat transfer in which transport of heat from a solid wall is effected by a fluid flowing adjacent to the wall. The transfer of heat can be in opposite direction (i.e., from the fluid to the wall). Convective phenomena are quite common in nature and they play a key role in many fields namely scientific areas, engineering, astrophysics and meteorology etc. Convection is of three types: natural (or free), forced and mixed convection (Figure 1.3). Natural (or free) convection is the term that refers to the heat transport by fluid flow generated by buoyancy force. The heat transfer due to free or natural convection is induced by the thermal expansion and density differences of a fluid under a temperature gradient. The heat transfer between two plates with different temperatures and concentration is characterized by thermal and solutal Grashoff number, respectively.

$$Gr = \frac{\beta_1 g a^3 (T_1 - T_o)}{\nu_o^2}, \quad \text{and} \quad Gc = \frac{\beta_2 g a^3 (C_1 - C_o)}{\nu_o^2}.$$

In micro process technology, free convection is negligible in most circumstances, however, it is the major cause for heat losses of equipment to the environment. This is because there is a thin gap in micro-system and the viscous forces reduce the convection which brings conduction as a sole heat transfer mechanism. In contrast, the term forced convection refers to heat transfer phenomena generated by external agents such as a pump, fan, compressor, blower or imposed pressure gradients which made the fluid to flow. If both natural and forced convection occurs the term mixed convection is used. Convective heat transfer depends primarily on fluid viscosity, thermal conductivity, specific heat and density of the fluid.

Radiation

Radiation is the transfer of heat through space or matter by means of electromagnetic mechanism that arise due to temperature of the body. Radiation heat is thought of as electromagnetic waves or quanta - an emanation of the same nature as light and radio waves. All bodies radiate heat; so a transfer of heat by radiation occurs because hot body emits more heat than it receives and a cold body receives more heat than it emits.

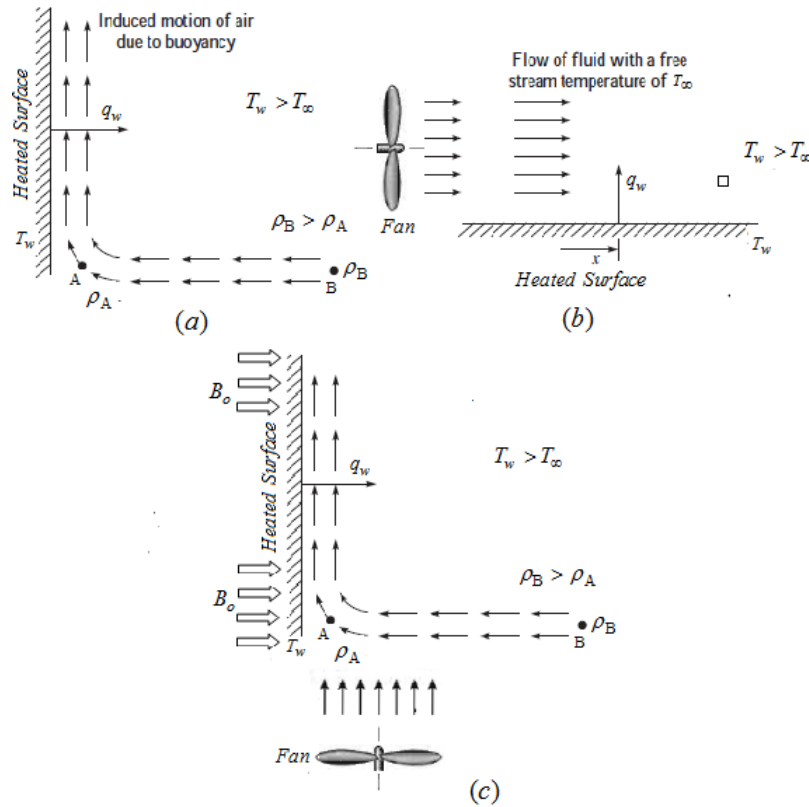


Figure 1.3: Convection heat transfer process (a) Natural convection (b) Forced convection (c) Mixed convection (Som, 2008).

Modeling Heat Transfer in Engineering Systems

An engineering system can be studied either experimentally (testing or taking measurements) or theoretically (by analysis or calculations).

- The experimental approach has the advantage that it deals with the actual physical system, and the desired quantity is determined by measurement, within the limits of experimental error. However, this approach is expensive, time-consuming, and often impractical.
- The theoretical approach (including the numerical approach) has the advantage that is fast and inexpensive, but the results obtained are subject to the accuracy of the assumptions, approximations, and idealizations made in the analysis.

1.1.3 Nanofluid

These days heat transfer technology is associated with the cooling of miniaturized high heat flux components. Conventional heat transfer fluid such as water, oil and ethylene glycol have relatively limited thermal conductivity (Xuan and Li, 2000) and do not meet the rising demands of modern technology. The usual technique to increase the heat dissipation is to augment the surface area for exchanging heat with a heat transfer fluids, but this method leads to unwanted increase in the thermal management system. So, by taking into consider

the requirements of contemporary technology, including chemical production, power stations and microelectronics, there is a need to suggest new kinds of fluids which are more effective with respect to heat transfer enhancement. For this reason, nanofluids are created to enhance the heat transfer rate and to achieve the rising demands of cooling/ heating and other needs. A Nanofluid is a colloidal suspension containing ultra-fine nanometer-sized solid particles, called nanoparticles. The nanoparticles used in nanofluids preparation are metallic elements (Al, Cu), oxides (Al_2O_3, Fe_3O_4), nitrides (AlN, SiN), or non-metallic elements (graphite, carbon nanotubes), etc. Conventional heat transfer base fluids include water, ethylene glycol, oil products, biofluids, polymer suspension, etc. The term nanofluid introduced by Choi (1995), refers to stable suspensions of nanoparticles (less than 100 nm) in the base fluid (Yu and Choi, 2003). Due to the novel characteristics of nanofluids that make them capable in many applications in hybrid-powered engines, pharmaceutical processes, fuel cells, including microelectronics and heat transfer. It has been widely used in engineering devices for ship-sand in boiler flue gas reduction of temperature and defense, in space, in machining, in grinding, nuclear reactor and heat exchanger, chiller, domestic refrigerator and engine vehicle thermal/cooling management.

Nanofluids are also consistent and have no supplementary issues like additional pressure drop, erosion and sedimentation. Buongiorno (2006) was the first researcher who dealt with convective transport in nanofluids. He noted that the nanoparticles absolute velocity can be viewed as the sum of the base fluid velocity and a relative (slip) velocity. He also discussed the effect of seven slip mechanisms: inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage and gravity setting. He concludes that in the absence of turbulent eddies, Brownian diffusion and thermophoresis dominate the other slip mechanisms. In experimental study of nanofluids, its preparation is a crucial step. Based on researches performed by (William et al., 2006; Drzazga et al., 2012; Qudoods et al., 2014; Prakash et al., 2015), nanofluid synthesis can be synthesized by one-step or two-step method. In the one-step (single-step) method, the nanoparticles production and nanofluid preparation are carried out simultaneously. The nanoparticles are directly prepared by a physical vapor deposition technique or a liquid chemical method. On the other hand, in two-step method, the nanoparticles are first produced as dry powders by physical or chemical methods e.g. grinding, laser ablation, sol-gel processing, etc. and then, the nanosized powder will be suspended in base fluid (Paul et al., 2010) in the second processing step with the help of intensive magnetic force agitation, ultrasonic agitation, high-shear mixing, homogenizing, and ball milling.

Ferrofluid

Ferrofluids (magneto-nanofluids) are colloidal suspensions prepared by dispersing magnetic nanoparticles in a base fluid. The magnetic particles are commonly magnetite (Fe_3O_4),

Cobalt (*Co*), iron (*Fe*), or Copper (*Cu*) with a mean diameter of size 10 *nm*. These particles on nanoscale are superparamagnetic; that is they are paramagnetic, meaning they are magnetic only in the presence of large magnetic moment. Their unique properties at nanoscale—such as paramagnetism—lead to create novel material and devices in nanotechnology. In ferrofluid flow, fluid features such as flow velocity, heat and mass transfer processes can be controlled by applying external magnetic fields (Odebach, 2013; Jordan et al., 1999). However, there is a possibility of particles agglomeration in such a fluids because of Van-der wall interaction and magnetic interaction between the particles. To prevent this situation (particle agglomeration), the particles are coated by surfactant layer which enables to get stable colloidal suspensions. The ferrofluids flow is applicable in various fields ranging from medical to engineering processes. Ferrofluids are widely applied in Chemotherapy such as drug targeting for cancer treatment and hyperthermia (Lubbe et al., 2001; Hiergeis et al., 1999). Other technical applications of ferrofluids found in cooling loudspeakers depicted in figure 1.4, rotary seals in computer hard drive, and contrast agent for magnetic resonance imaging (MRI). Figure 1.4 shows the use of a ferrofluid in cooling a loudspeaker. The ferrofluid is positioned by magnetic field gradients produced by permanent magnets embedded in the loudspeaker in such a way that the voice coil is suspended in a ferrofluid ring. A thermal bridge between the voice coil heated by Ohmic losses and the main structure of the loudspeaker is established, where the loudspeaker can be cooled from outside.

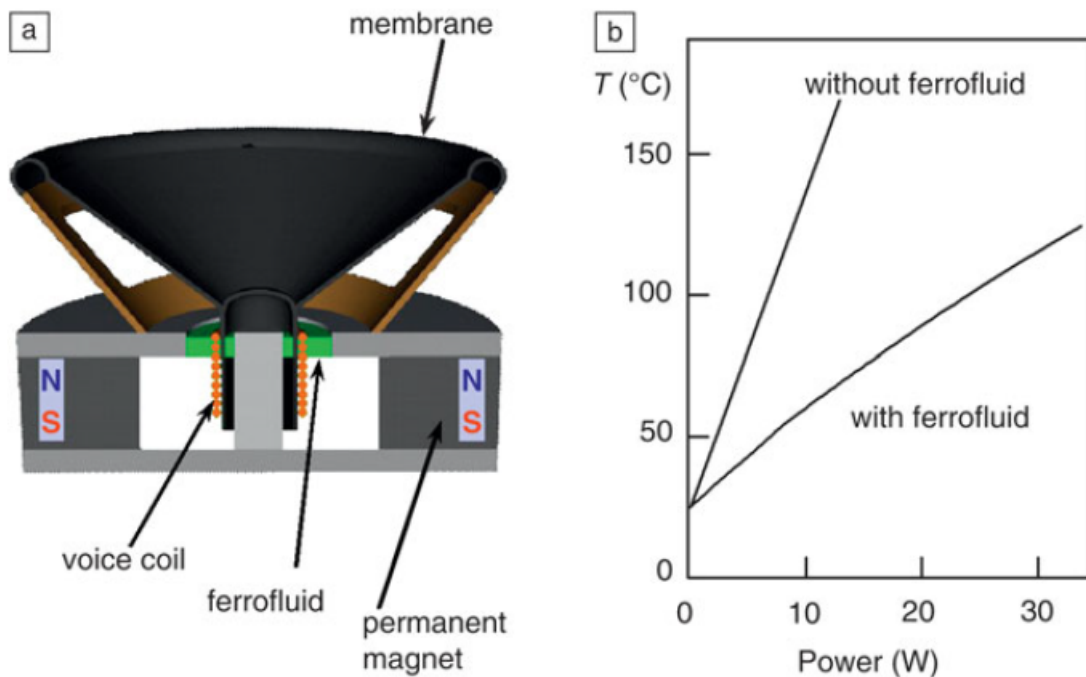


Figure 1.4: (a) Diagram of cooling loudspeaker with a ferrofluid, and (b) the power dependence of the speaker's temperature (T) with and without the ferrofluid (Odebach, 2013).

1.1.4 Statement of the Problem

The study of magneto-nanofluid with heat transfer in microchannels has a wide range of applications in high-tech industries such as microelectronics, nuclear reactor, manufacturing, metrology, vehicle thermal management, machining and space technology. These high-tech industries come against different technical challenges, among which heat transfer problem is one of their challenges. To alleviate this problem, many researchers have done experimental and theoretical researches on heat transfer problems including Aminossadati et al. (2011), Jha and Aina (2017), and Pordanjani et al. (2019). The objectives of their studies focus to moderate the flow field and augment the heat transfer in the microchannels.

The study of magneto-nanofluid flow and heat transfer in microchannels is an active area of research, where nanofluids and microchannels enhance the performance and efficiency of the heat transfer; the exerted magnetic field manipulates the suspended particles and rearranges their concentration in the fluid flow field. From engineering perspective, however, the study on the effect of variable viscosity on the heat transfer of unsteady MHD nanofluid flow in a permeable microchannels; and the analysis of thermal radiation on unsteady magneto-nanofluid flow through a permeable microchannels with convective heating and variable fluid properties in the presence/absence of first order chemical reaction have never been reported for both cases of single-phase flow model and two-phase flow model, though variable fluid properties (Makinde and Onyejekwe, 2011; Thomson and Bopp, 1970; Setayesh and Sahai, 1990), chemical reaction (Moshizi, 2015), and thermal radiation (Abbas et al., 2020) have a crucial influences on the heat transfer of nanofluid flow through microchannel. From mathematical solution technique perspective, almost all the governing nonlinear PDEs of the flow model problems have never been solved by the method of semi-discretization via finite difference along with Runge-Kutta Fehlberg integration scheme.

Therefore, the main focus of this study is on the theoretical analysis of single-phase and two-phase flow models of MHD nanofluid with variable fluid properties flow and heat transfer in permeable microchannels in the presence and absence of chemical reaction and thermal radiation mutually or unilaterally.

1.1.5 Objectives of the Study

General Objective of the Study

The general objective is to study theoretically magneto-nanofluid flow and heat transfer in a permeable microchannel.

Specific Objectives of the Study

The specific objectives of this study are:

1. to formulate the governing equations of MHD nanofluid flow through parallel-plates microchannel with permeable walls for the cases of:
 - i. two-phase flow model whose channel geometry is oriented vertically and fluid viscosity is temperature dependent (Problem I/ Chapter 4).
 - ii. single-phase flow model using ferro-fluid ($Fe_3O_4 - H_2O$) with channel geometry oriented horizontally and fluid viscosity is a function of temperature (Problem II/ Chapter 5).
 - iii. two-phase flow model using radiating-chemically reacting fluid with temperature dependent fluid viscosity and thermal conductivity where the channel flow geometry is oriented vertically (Problem III/ Chapter 6).
 - iv. single-phase flow model of radiating $Cu - H_2O$ nanofluid with variable viscosity and convective heating where the flow geometry is oriented horizontally (Problem IV/ Chapter 7).
2. to solve numerically the above problems using semi-discretization via centered finite difference scheme along with RKFIT.
3. to discuss the effects of varying thermophysical parameters on profiles of velocity, temperature, concentration, skin friction Cf , Nusselt number Nu , and Sherwood number Sh .
4. to validate qualitatively the present results by comparing them with the previous outcomes available in the open literature.

1.1.6 Significance of the Study

The results *and* numerical techniques implemented in this study may:

- i. provide useful information for application (the one who works in MHD and MCH laboratory experiment). Understanding the characteristic phenomena of microchannel flow is useful in designing efficient micro-devices.
- ii. enhance scientific knowledge in the areas of applied mathematics.
- iii. be useful in chemotherapy application. In chemotherapy, there is a possibility of missing to provide localized drug delivery which may bring side effects on the nearby organs and tissues. The precise drug targeting is accomplished through magnetic drug delivery. Medicines bound to magnetic-nanoparticles (ferrofluids) are compatible biologically and injected into the blood stream. The targeted areas are subjected to an external magnetic field that is able to reduce the flow rate of blood stream. The drug is then slowly released from the magnetic carrier (Goodwin et al., 1999).

- iv. be used as an alternative numerical method for solving governing equations.
- v. serve as a complement for the previous studies.
- vi. open gates for new researches and propose suggestions that could lead to further study.

1.2 Organization of the Dissertation

This dissertation comprises eight chapters and is organized as follows:

Chapter 1 presents a brief introduction of the topic, problems, physical features involved in the study of microchannel flow of MHD nanofluid, statement of the problem, objectives of the study, and significance of the study.

Chapter 2 provides the review of literatures on properties and applications of microchannel liquid flow, experimental and theoretical study of nanofluid flow in a microchannel in the presence and absence of magnetic field, mixed convection in microchannel, effects of wall suction/injection, and chemical reaction in microchannels that lead to the gap of the study in the thesis.

Chapter 3 provides the details of mathematical procedures used for attaining the stated objectives and the research methodology such as derivation of governing equations, computational methods and MAPLE software description for the analysis of the problem in the study.

Chapter 4 presents the combined effects of unsteadiness, buoyancy forces, pressure gradient, thermophoresis, Brownian motion, and variable viscosity of nanofluid flow in a permeable microchannel in the presence of uniform magnetic field. The physical aspects of this nanofluid flow through microchannel are formulated mathematically by two-phase flow model equations of computational fluid dynamics. The governing nonlinear PDEs with IBCs are converted to their dimensionless counterpart through non-dimensionalization process. The resulting nonlinear PDEs are investigated numerically (by semi-discretization scheme via Runge-Kutta Fehlberg integration technique) and analyzed using MAPLE software version 18. In addition, the chapter presents the results graphically for velocity, temperature, nanoparticles concentration, skin friction, Nusselt number and Sherwood number profiles of the flow system.

Chapter 5 presents the magnetic field effects of unsteady ferrofluid flow in a heated permeable microchannel with temperature dependent viscosity. Here, a single-phase flow model is used to formulate the governing PDEs of the flow system. Applying non-dimensionalization process, the governing nonlinear PDEs with associated IBCs are reduced to a set of dimensionless PDEs with IBCs. The transient analysis of the velocity and temperature distributions are displayed in 3D and 2D plot. Moreover, steady state analysis of velocity, temperature, skin friction and Nusselt number profiles are presented for varying values of governing parameters in this chapter.

Chapter 6 presents the study of unsteady hydromagnetic mixed convection of a radiating and reacting nanofluid in a vertical permeable parallel-plates microchannel with temperature dependent viscosity and thermal conductivity. A two-phase flow model equation is applied to find the governing PDEs of the physical system considered. This chapter also provide dimensionless governing PDEs, transient analysis of velocity, temperature and nanoparticles concentration, and steady state analysis of dimensionless velocity, temperature, nanoparticles concentration, skin friction, Nusselt number and Sherwood number for varying values of thermophysical parameters.

Chapter 7 presents the study of hydromagnetic mixed convection of $Cu - H_2O$ nanofluid in a microchannel with thermal radiation and convective heating

Chapter 8 recaps the results and provides concluding remarks and gives recommendation for future work.

CHAPTER 2

LITERATURE REVIEW

This Chapter will provide properties and applications of microchannel liquid flow; the presentation of important published papers on the study of nanofluid heat transfer through microchannel flow in the absence and presence of applied magnetic field (both experimental and theoretical studies), mixed convection in microchannels, variable liquid property effect in microchannel, walls suction/ injection and chemical reaction effects in microchannels. Moreover, the chapter offers the conclusions drawn from the survey of the literature.

2.1 Microchannel Liquid Flow and Its Applications

Microchannels were first proposed for electronics cooling applications by Tuckerman and Pease (1981), who employed the direct circulation of water in microchannels fabricated in silicon chips. Their pioneer work initiated many researchers to study fluid flow and heat transfer in microchannels and microchannel flows are used in a variety of electronic devices to increase heat transfer. The extensive use of microchannels has been found in medical applications and in micro-electro-mechanical systems (MEMS) such as micro heat exchangers, mixers, pumps, turbines, sensors and actuators. It is important to look at the definition of "microchannel" assigned by various intellectuals, before discussing size-related characteristic features of microchannel flow. The definition and scope of the term "microchannel" is the topics of discussion among researchers in the field. Scholars have explained microchannel with different criteria. Mehendale et al. (2000) used the following classification based on manufacturing techniques required to obtain various ranges of channel dimensions, " D ", being the smallest channel dimension:

- $1\ \mu m \leq D \leq 100\ \mu m$: Microchannels,
- $100\ \mu m \leq D \leq 1\ mm$: Minichannels,
- $1\ mm \leq D \leq 6\ mm$: Compact Passages,
- $D \geq 6\ mm$: Conventional Passages.

Kandlikar and Grande (2003) adopted a different classification based on the rarefaction effect of gases in various ranges of channel dimensions, " D " being the smallest channel dimension:

- $1\ \mu m \leq D \leq 10\ \mu m$: Transitional Microchannels,
- $10\ \mu m \leq D \leq 200\ \mu m$: Microchannels,
- $200\ \mu m \leq D \leq 3\ mm$: Minichannels,
- $3\ mm \leq D$: Conventional Passages.

A simpler classification was proposed by Obot (2003) based on the hydraulic diameter rather than the smallest channel dimension. Obot classified channels of hydraulic diameter under 1 mm ($D_h \leq 1\text{mm}$) as microchannels, which was also adopted by many other researchers such as Bahrami and Jovanovich (2006), Bahrami et al. (2006) and Bayraktar and Pidugu (2006). The definition of microchannel assigned by Obot is more appropriate for the purpose of this dissertation as it depends on the channel geometry alone and it uses the hydraulic diameter concept, which is the main representative dimension for internal flow of parallel-plates geometry. But the definition made by Kandlikar and Grande considered the behavior of gas flow in contrast to the present study which uses liquid (nanofluid), and the classification proposed by Mehendale is based on manufacturing processes which is expected to change and/or improve in the future.

The application of microchannel flow is revealed in *microfluidics*, the science and technology of systems that process/manipulate small amounts of fluidics, say, 10^{-12} to 10^{-9} liters, and sort particles in microchannel with characteristics length scale D of, say, 10 to 50 micrometer, where D represents a hydraulic diameter or channel height or width (Kleinstreuer, 2014). The rate of transport phenomena in microchannel flow depends on the surface area of the microchannel which varies with D , whereas the flow rate depends on the cross-sectional area of the microchannel, which varies with D^2 . Thus, surface-area-to-volume ratio is approximated as $\frac{1}{D}$. Consequently, $\lim_{D \rightarrow 0} \frac{1}{D} = \infty$, where $D \cong$ the microchannel hydraulic diameter, channel height, or width. This implies surface-area-related quantities such as pressure, shear forces, surface tension, electrostatic and magnetohydrodynamic forces become much dominant in microchannel flow.

In microchannel flow, the channel size is relatively so small as compared to conventional channel, and this minimal channel size gives rise to certain flow physical phenomena that are not worthy in conventional channel but become significant with microchannel flow. Below the size effects on microchannel liquid flow physics are discussed.

i. Surface Area-to-Volume Ratio

The large surface area-to-volume ratio, or specific surface area, is an important feature of microchannels and is inversely proportional to the length-scale of the channel cross-section. The large surface area-to-volume ratio produces high rates of heat and mass transfer, making the use of micro-structured devices as heat exchangers, mixers and reactors appealing.

ii. Surface Forces

Both body and surface forces that govern the flow physical phenomena in a channel directly vary with certain power(s) of the length-scale. Body forces depend up on third powers of length-scale whereas surface forces are scaled with the first (for instance viscous forces and surface tension) or second power of the length-scale. Therefore,

below a certain length scale, surface forces are more important compared with body forces (Ho and Tai, 1998). Some examples of surface forces existed in microchannel flow are surface tension, frictional (viscous) forces, wall adhesion and pressure, etc.

iii. **Wall Wettability**

Wall wettability is another important surface effect that can become significant in multiphase flow in microchannels. In continuum models, the three-phase contact angle, i.e. the angle between the solid wall and gas-liquid interface in contact with it, is used to measure wettability of a liquid on a solid wall. The static three-phase contact angle is a result of the balance among the adhesive (attraction force between liquid and solid wall, liquid and gas molecules) and cohesive (attraction force among liquid molecules) forces. When the contact line moves, the contact angle does not remain constant and the complex phenomena of contact angle dynamics need to be accounted for (Mourik et al., 2005; Ajaev and Homsy, 2006; Afkhami et al., 2009). This dissertation deals with single and two phase flow of nanofluid through microchannel with constant suction/ injection at the walls, this effect is not important.

iv. **Laminar Flow**

In microchannels the Reynolds numbers are relatively low so that laminar flow is much more common than turbulent flow. The fundamental equations of flow, heat and mass transfer can be solved directly without reference to any heuristic models, such as those required to model the effects of turbulence. This allows, in theory, the prediction of transport phenomena to be made more accurately in microchannels than in large channels, where the flow is generally turbulent. However, this laminar nature of the flow often gives rise to the need for additional means of mixing to achieve high transport or reaction rates. The use of obstacles on the channel wall (Stroock et al., 2002), varying channel cross-sections (Hardt et al., 2003), suction and injection on the channel walls, and having tortuous channel paths giving rise to secondary flow helps to enhance mixing at low Reynolds numbers.

v. **Continuum Hypothesis**

For liquid flow in a microchannel, the liquid molecules are tightly crammed together as compared to the channel hydraulic diameter, which is much greater than the molecular mean free path of the liquid. Consequently, the Knudsen number (the ratio of molecular mean free path and a characteristic dimension of the flow) is so small that the continuum assumption is valid in microchannel liquid flow. The governing equations developed for conventional channel flow, such as momentum equation and energy equation, are still applicable in microchannel liquid flow. Microscale effect on liquid flow and validity of continuum hypothesis in microchannel is also investigated by many researchers including Xu et al. (2000), Palm (2001), Sobhan and Garimella

(2001), Lelea et al. (2004). Meanwhile, Steinke and Kandlikar (2006) indicated that liquid flow in microchannel follows the continuum theory and the discrepancies were mainly attributed to the wall roughness effects, experimental uncertainties, axial conduction effects. Also, the applicability of continuum theory verified by Lee et al. (2005). Literature surveys implicated that continuum modeling theory is relevant to liquid flow in microchannel.

vi. Electric Double Layer (EDL) Formation

When a solid surfaces are brought in contact with aqueous solutions, they will get surface electric charge. This happened due to the process of ionization, ion-adsorption or ion-dissolution. If the liquid contains ions, the electrostatic charges on the solid surface will attract the counter-ions in the liquid. The cloud of ions, with the presence of high counter-ions concentration and low co-ions concentration, surrounding the charged surface is called Electrical double layer (EDL) (Hunter, 1981; Lyklema, 1995). Immediately next to the solid surface, there is a layer of ions that are strongly attracted the solid surface and are immobile. This layer is called the compact layer, normally about several Angstroms thick. From the compact layer to the uniform bulk liquid, the net charge density gradually reduces to zero. Ions in this region are affected less by the electrostatic interaction and are mobile. This region is called the diffuse layer of EDL. The interface of the compact layer and the diffuse layer is usually referred as the shear plane. The phenomenon of EDL formation has a significant importance in physical, chemical, and biological systems. For instance, it has applications in energy storage and capacitive desalination, and is favored by electrode materials of high surface area and high conductivity (Frackowiak and Beguim, 2001; Gabelich et al., 2003; Ying et al., 2002). The ion and electrical potential distributions in the EDL can be determined by solving the Poisson Boltzmann equation (Hunter, 1981; Lyklema, 1995).

Because all solid-liquid (aqueous solutions) interfaces carry electrostatic charge, there is an electrical double layer (EDL) field in the region close to the solid-liquid interface on the liquid side. Such an EDL field is responsible for at least two basic electrokinetic phenomena: electroosmosis and electrophoresis. Briefly, electroosmosis is the liquid motion in a microchannel caused by the interaction between the EDL at the liquid-channel wall interface with an electrical field applied tangentially to the wall. This electrokinetic process is first introduced by Reuss in 1809. Electrophoresis is the motion of a charged particle relative to the surrounding liquid under an applied electrical field.

vii. Wall Velocity Slip

The no-slip velocity and impermeable wall condition is one of the simplest boundary conditions, which dictates that a liquid element adjacent to the surface assumes

the velocity of the surface. This condition is acceptable for macroscale flows and is also relevant for most microchannel flows. However, in many natural and industrial microfluidics cases effective or apparent slip velocities can be observed. Examples include:

- Rarefied gases on actual solid surfaces
- Liquids on rough and hydrophobic surfaces, labeled "superhydrophobic"
- Liquid-molecule depletion-layer flow
- Electroosmotic flow of electrolytes with thin electric double layers (EDLs)
- Porous wall-layer flow.

viii. **Viscous Energy Dissipation**

Reynolds number is defined as the ratio of inertial and viscous forces. In microchannel flow, due to its minimal channel dimension the flow is characterized by small Reynolds number. This indicates that the flow through microchannel is laminar in general and viscous forces remain considerable, mainly at low velocities. Xu et al. (2003) theoretically studied the effects of viscous dissipation on the fluid flow field in microchannel. Koo and Kleinstreuer (2004) investigated the effects of viscous dissipation on the temperature field and friction factor using dimensional analysis and experimentally validated computer simulations. It is found that the channel size has an important impact on temperature of the flowing fluid. They also concluded that the viscous dissipation effect on the friction factor increases rapidly with pipe/channel size decrease.

ix. **Heat Transfer Coefficient and Pressure Drop**

It is known that, for thermally developed flow in conventional channel, the Nusselt number $Nu (= \frac{hD}{k})$ is constant where h is the convective heat transfer coefficient, k is thermal conductivity of the liquid and D is the channel width. As the channel width D reduces to micron or nano size, the heat transfer coefficient $h = Nu (\frac{k}{D})$ can increase to thousand or million times the original value in order to maintain a constant value of the Nusselt number. Consequently, the heat transfer coefficient h enhances drastically. On the other hand, the skin friction Cf for channel flow is defined as

$$Cf = \frac{\tau_w}{(\frac{1}{2})\rho V^2}, \quad (2.1)$$

where τ_w is wall shear stress and V is the mean (suction/injection) velocity. For fully developed flow through tubes, Cf can be expressed in terms of pressure drop and is referred to as the friction factor f_{fr}

$$f_{fr} = \frac{1}{2} \frac{D}{L} \frac{\Delta P}{\rho V^2}, \quad (2.2)$$

where D is diameter, L is length and ΔP is pressure drop. For fully developed laminar flow, the product of f_{fr} and Reynolds number Re is constant. That is.,

$$f_{fr}Re = constant. \quad (2.3)$$

The frictional pressure drop per unit length is given by:

$$\frac{\Delta P}{L} = \frac{2f_{fr}\rho V^2}{D}. \quad (2.4)$$

It is seen from equation (2.4) that the pressure drop increases dramatically with a reduction in the channel size (Kandlikar et al., 2006).

Since EDL is a layer formed around charged channel walls due to the movement of electrolyte solution through the microchannel and Wall wettability is a surface property which happens to exist in gas-liquid flow in microchannel, this study generally deals with nanofluid flows in microchannel by considering all the above size related effects except the formation of electric double layer and wall wettability.

Microchannels are physically well suited for high heat flux removal tasks as they provide a large heat transfer surface area per unit fluid flow volume. Its relevance on heat transfer application is found in cooling micro-electro-mechanical devices such as microchannel heat sink, microreactor, bioreactor, etc.

Microchannel heat sink contains a large number of parallel microchannels with a hydrodynamic diameter ranging from 10 to 100 μm . These parallel microchannels in MCHS are innovative cooling technology for a removal of heat through a small area. Coolant is forced to pass through these channels to carry heat away from a hot surface. Experimental studies have shown that the MCHS has several distinct features compared to conventional heat dissipation devices, i.e., the ability to produce a very high heat transfer coefficient, high surface area to volume ratio, very small size and volume per heat load, and small coolant requirement (Wu and Cheng, 2003; Qu and Mudawar, 2002). These special attributes render MCHS very suitable for cooling devices such as high performance microprocessors, laser diode arrays, radars, and high-energy-laser mirror. Microchannel heat exchanger could be easier to repair than their conventional counterparts. It offers other benefits, including increased latent capacity for microchannel evaporators. Microchannel heat exchangers improve heat transfer in two ways. First, the smaller dimensions of the refrigerant flow passage increase refrigerant-side heat transfer. Second, the flat tube orientation reduces airside flow resistance, leading to either increased airflow or reduced fan power. Though microchannel heat exchangers offer high heat transfer coefficient in compact size, its heat transfer characteristics is limited by the heat transfer fluid that is used (Mohammeda et al., 2011). Figure 2.1 below shows a schematic of microchannel heat exchanger. Its parts contain manifolds, fins, and microchannels.

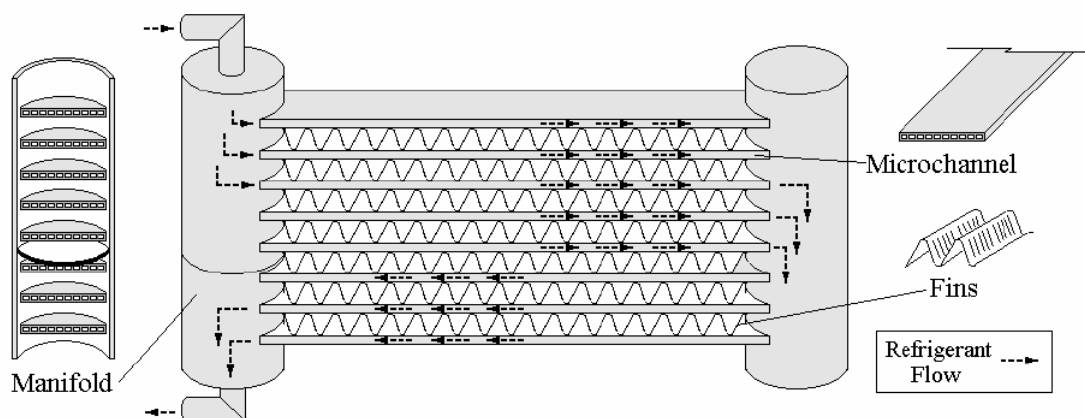


Figure 2.1: (a) Microchannel heat exchanger fluid flows through the fins, orthogonal to the microchannels (Nino et al., 2002).

A micro-reactor is a device in which chemical reactions take place in a confinement with typical lateral dimensions below 1.0 mm. When the temperature of the reactor is required to maintain, micro reactor also act as a micro channel heat exchanger. Micro reactors used in these applications are typically continuous flow reactors rather than batch reactors, with the continuous flow model providing better performance in material synthesis than is possible with batch reactors. The reactors offer many advantages over conventional scale reactors, including vast improvements in energy efficiency, reaction speed and yield, safety, reliability, scalability, on-site/on-demand production, and a much finer degree of process control. It enables miniaturization of the fuel processor because they minimize heat and mass transfer resistance. A micro reactor may also be used to perform reactions on a very small scale to determine the potential for dangerous situations such as runaway reactions or the generation of excessive levels of heat.

Bio-reactors are liquid vessels in which live populations of suspended or surface-attached cells can be maintained for extended periods of time, either for expansion and differentiation of a desired cell type, or for the mass production of a biomolecule produced by the cells. Bioreactors usually feature gas and/or liquid exchange, to replenish nutrients to the cells as they are consumed and to remove waste and other products made by the cells.

2.2 Nanofluid Heat Transfer in Microchannels Flow: Experimental Study

As revealed, nanofluids exhibit remarkable thermal and fluid properties, which in combined with microchannel system will provide enhanced heat transfer performances. This peculiar properties make nanofluid flow in microchannel useful in cooling of microelectronics and micro-electro-mechanical system (MEMS). The problem of nanofluid flow and heat trans-

fer in microchannels has been studied experimentally by a number of researchers including Singh et al. (2013), Subhedar and Ramani (2013), and Siddiqui et al. (2018) among others.

Heris et al. (2006) experimentally studied the thermal performance of laminar flow of the CuO -water and Al_2O_3 -water nanofluids through a circular tube with constant wall temperature boundary condition. The Nusselt number of both nanofluids increased with nanoparticle volume fractions as well as Peclet number. They predicted that there may exist an optimal volume fraction range of CuO and Al_2O_3 nanoparticles between 2.5% and 3% in volume.

Duangthongsuk et al. (2012) have performed experimental study on convective heat transfer performance and pressure drop of MCHS using Al_2O_3 -water nanofluids as working fluid. They examined the effect of nanoparticles concentration of 1% to 3% and the Reynolds number on heat transfer performance and pressure drop of nanofluid cooled-MCHS, and then presented the comparison of heat transfer coefficient of nanofluid cooled-MCHS and water cooled-MCHS. Their results entail that the heat transfer performance of MCHS enhanced with an increase in nanoparticles concentrations and Reynolds number. Also, it is found that the nanofluid cooled-MCHS heat transfer coefficient is higher than the water cooled-MCHS.

Bowersa et al. (2018) have done experimental investigation on flow and heat transfer of aqueous based silica and alumina nanofluids in microchannels. They observed convective heat transfer enhanced for both nanofluids, relative to their base fluid (water), at low particle concentrations. Also, heat transfer enhancement increased with increasing Reynolds number and a better heat transfer enhancement is obtained with alumina nanofluid as compared to silica nanofluid. Due to the higher viscosity of nanofluids compared to water, the pumping power required to drive nanofluid flow through the microchannels also increases. The extra pumping power required has been found to increase with nanofluid flow rate and nanoparticle volume fraction.

Lee and Mudawar (2007) have done experimental work to explore the effectiveness of Al_2O_3 nanoparticles volume fraction at enhancing single-phase and two-phase flow heat transfer in microchannel heat sinks using Al_2O_3 -water nanofluid as coolant. It was observed that the presence of nanoparticles enhances the single-phase heat transfer coefficient, especially for laminar flow. Higher heat transfer coefficients were achieved mostly in the entrance region of microchannels. However, the enhancement was weaker in the fully developed region. Higher concentrations also produced greater sensitivity to heat flux. A large axial temperature rise was associated with the decreased specific heat for the nanofluid compared to the base fluid.

Chein and Chuang (2007) have addressed microchannel heat sink (MCHS) performance using nanofluids as coolants. They have carried out a simple theoretical analysis that indicated more energy and lower microchannel wall temperature could be obtained under the assumption that heat transfer could be enhanced by the presence of nanoparticles. The theoretical predictions were followed by their own experimental results verification. It was observed that nanofluid-cooled MCHS could absorb more energy than water-cooled MCHS

when the flow rate was low. For high flow rates, the heat transfer was dominated by the volume flow rate and nanoparticles did not contribute to the extra heat absorption.

Jung et al. (2009) have studied experimentally the convective heat transfer coefficients and friction factor of nanofluids of Al_2O_3 with diameter of 170 nm in a rectangular microchannel. Appreciable enhancement of the convective heat transfer coefficient of the nanofluids with the base fluid of water and a mixture of water and ethylene glycol at the volume fraction of 1.8 volume percent was obtained without major friction loss. It has been found that the Nusselt number increases with increasing the Reynolds number in laminar flow regime.

Liu et al. (2010) presented an experimental investigation of single-phase forced convection of Al_2O_3 water nanofluid in a circular minichannel with a 1.09 mm inner diameter. The Reynolds number was varied from 600 to 4500, covering the laminar, transition, and early fully developed turbulent regions. The results revealed that in the laminar region, the nanofluids exhibit pronounced entrance region behaviors possibly due to the flattening of the velocity profile caused by the flow induced particle migration. Furthermore, the convective heat transfer of nanofluids is enhanced in the laminar flow with the penalty of increased pressure drop. The increasing in both pressure drop and heat transfer were proportional to the nanoparticle volume concentration. The results revealed that the nanofluids should be used in either the laminar flow or the fully developed turbulent flow with sufficiently high Reynolds number in order to yield enhanced heat transfer performance for engineering applications.

Rimbault et al. (2014) performed an experimental investigation to analyze flow and thermal fields of Cu -water nanofluid for three different particle volume fractions, namely 0.24%, 1.03% and 4.5% flowing through a rectangular microchannel heat sink for the case of both laminar and turbulent regime. The isothermal and heated experiments are accomplished, respectively, for Reynolds number varying up to ≈ 5000 and to ≈ 2500 . For a particular nanofluid flow rate experimental results indicated that an increase of pressure drop and friction factor as compared to water. This increase can be as high as 70%, 25%, and 0-30%, respectively, for the 4.5%, 1.03%, and 0.24% particle volume fractions. The laminar-to-turbulent transition for Cu -water nanofluid observed approximately at critical Reynolds number ≈ 1000 , which is also the critical value for laminar-turbulent transition of water. Results show a slight heat transfer enhancement with respect to water for nanofluids with low particle volume fractions, 0.24% and 1.03%, while for the 4.5% fraction a clear decrease of heat transfer was found.

The experimental study carried out by Razali et al. (2017) is to investigate the effect of heat transfer to Al_2O_3 nanofluid after used as a working fluid in a square microchannel with a cross section of $0.5 \times 0.5 \text{ mm}^2$. This experimental work was conducted in a laminar flow with Reynolds number ranging approximately 633 to 1172 at concentrations of Al_2O_3 nanoparticles of 1.0 wt. % and 2.5 wt. % dispersed in water. It is observed that higher heat transfer was achieved by using 2.5 wt. % Al_2O_3 nanofluids compared to 1.0 wt. % Al_2O_3

nanofluids and water. It is proved that the theory stated that thermal conductivity of fluid affect the heat transfer in microchannel because the thermal conductivity of Al_2O_3 nanofluid is higher than water. The experimental result for heat transfer in the microchannel has been validated with computational (CFD) simulation. The higher heat transfer was achieved in the inlet region of the microchannel with increase nanoparticles concentration to 2.5 wt. % concentration.

Markal et al. (2018) experimentally examined single-phase laminar flow of deionized water in rectangular microchannel. It is disclosed that increasing aspect ratio and Reynolds number increases Nusselt number while decreasing the skin friction.

Hatami et al. (2017) have examined experimentally the magnetic field effects on forced convection heat transfer of Fe_3O_4 -water nanofluid with laminar flow regime in a horizontal pipe under constant heat flux conditions. They considered Fe_3O_4 nanoparticles with diameters less than 100 nm dispersed in water with various volume fractions are used as a test fluid. Their results showed that by the presence of a magnetic field, increase in nanoparticle concentration caused reduction of convection heat transfer coefficient, because a reduction of Brownian motion of nanoparticles due to applied external magnetic field which dominates the other increasing factors of heat transfer; however, in the absence of an external magnetic field, adding magnetic nanoparticles increased convection heat transfer. It was observed that the Nusselt number decreased by increasing the Hartmann number at a specified concentration of magnetic nanofluids, that reduction about 25% in heat transfer rate could be found.

2.3 Nanofluid Heat Transfer in Microchannels Flow: Theoretical Study

To design and develop an effective microchannel heat sink and other micron-size devices, the basic theoretical knowledge of fluid flow and heat transfer in microchannels are important. This theoretical knowledge is acquired through mathematical analysis or calculation including numerical simulations. The potency of the numerical simulations gives an opportunity to examine small features that are impossible to observe in experiments. Therefore, many researchers give much attention to theoretical investigation of heat transfer in microchannel flow.

In the Absence of Magnetic Field

Khanafer et al. (2003) has investigated heat transfer enhancement in a two-dimensional rectangular enclosure utilizing water/copper nanofluids for various pertinent parameters. They solved the developed transport equations numerically using the finite-volume approach along with the alternating direct implicit procedure. The effect of suspended ultra-fine metallic

nanoparticles on the fluid flow and heat transfer processes within the enclosure was analyzed and effective thermal conductivity enhancement maps were developed for various controlling parameters. The heat transfer correlation of the average Nusselt number for various Grashof numbers and volume fractions was also presented.

Hung et al. (2012) numerically studied that heat transfer enhancement in 3D microchannel heat sink (MCHS) using nanofluid by finite volume scheme. The calculations in this work suggested that the best heat transfer enhancement can be obtained by using a system with an Al_2O_3 -water nanofluid cooled MCHS. Moreover, using base fluids with lower dynamic viscosity (such as water) and substrate materials with high thermal conductivity enhance the thermal performance of MCHS. The results also show that as the particle volume fraction of the nanofluid increases from 0.5% to 1%, the thermal resistance decreases.

Roy et al. (2004) has studied a steady, laminar flow and heat transfer of a nanofluid flowing inside a radial channel between two coaxial and parallel discs. The non-dimensional governing equations of mass, momentum and energy were solved by using a numerical method that is essentially based on the finite control volume approach. Results presented in this paper are for a water/aluminum oxide particle nanofluid (H_2O/Al_2O_3). Results have shown that the inclusion of nanoparticles in a traditional coolant can provide considerable improvement in heat transfer rates, even at small particle volume fractions. Increases in the resulting wall shear stresses were also noticed.

Oztop and Abu-Nada (2008) has carried out CFD study on heat transfer and fluid flow due to buoyancy forces in a partially heated rectangular enclosure filled with nanofluids. The temperature of the right vertical wall kept lower than that right wall while other two walls are insulated. The finite volume technique is used to solve the governing equations. Different types of nanoparticles were tested. An increase in mean Nusselt number was found with the volume fraction of nanoparticles for the whole range of Rayleigh number.

Investigation of the flow and heat transfer characteristics in micro pin fin heat sink with a nanofluid done by Hasan (2014). The author simulated working fluid as alumina or diamond nanoparticles dispersed in the water with the laminar flow regime. They used four rectangular microchannel geometries: square fins, triangular fins, circular fins, and unfinned channel. The governing system of equations and boundary conditions for nanofluid flow and heat transfer are numerically solved using finite volume method. Hasan found that using of nanofluid instead of pure fluid as a coolant leads to an enhanced heat transfer performance by increasing the amount of heat dissipated, but it also leads to increased pressure drop for all fins shapes and nanofluids studied.

Nebbati and Kadja (2015) have investigated, by numerical simulation rooted in finite volume method, the hydrodynamic and thermal characteristics of the combined laminar forced convection flow of nanofluids and conjugate convection/conduction heat transfer in a three-dimensional rectangular micro-channel heat sink with the assumption of a single-phase approach and in the case of constant and temperature-dependent properties. The results clearly

show that the inclusion of nanoparticles produces a considerable increase of the heat transfer coefficient. Also the temperature dependent models present higher values of local and average Nusselt number because the difference between wall temperature and the temperature of the fluid in the vicinity of this wall is maximized, thereby maximizing the heat transfer coefficient, which is proportional to this difference.

El Mghari et al. (2016) analyzed nanofluids condensation heat transfer in a horizontal square microchannel, numerically by Runge-Kutta method. Those nanofluids are suspension of water and five types of nanoparticles at volume concentration smaller than 5%: copper, aluminum, diamond, copper oxide and alumina. They concluded that the increase of the nanoparticles volume fraction is also accompanied by a very slight increase of the pressure drop. Heat transfer improvement obtained using nanofluids (56%) is much important than decreasing microchannel hydraulic diameter.

Abdollahi et al. (2017) studied the fluid flow and heat transfer characteristics of laminar forced convection of single-phase nanofluid flow in microchannel heat sink (MCHS) with V-Type inlet/outlet orientation numerically by finite volume method. The study emphasized mainly on the flow field and heat transfer enhancement aroused by various factors, such as different types of nanofluids including SiO_2 , Al_2O_3 , ZnO and CuO , different amounts of nanoparticles volume fractions (0, 1, 1.5 and 2%) and different sizes of nanoparticle diameters (30, 40 and 60 nm). It was found that SiO_2 , Al_2O_3 , ZnO and CuO nanofluids in water base fluid had better heat transfer augmentation compared to pure water. The results show that SiO_2 nanofluid presents the maximum Nusselt number value among tested nanofluids and pure water. The increase of nanoparticle volume fraction enhanced the average Nusselt number. Moreover, the Nusselt number had a small enhancement with decreasing the nanoparticle diameter. Besides, the flow fluid shows that SiO_2 nanofluid had the uppermost pressure drop among tested nanofluids and pure water. Following this, the larger volume fraction and smaller nanoparticle size increase the pressure drop.

Makinde (2018) conducted a study on the aggregate effects of convective cooling and temperature dependent viscosity on the heat transfer characteristics of ethylene glycol-water mixture base nanofluid containing silver nanoparticles in microchannel flow. The model equations for momentum and energy balance are obtained and transformed into a non-linear boundary value problem using lubrication approximation theory and tackled semi-analytically via perturbation method coupled with Hermite-Padé approximation techniques. EG to water volume ratios examined are 0:100%, 20:80%, 40:60%, 60:40%, 80:20% and 100:0% while the silver nanoparticles volume fraction utilised are 0%, 5% and 10%. It is found that the critical Eckert number for nanofluid thermal stability and the Nusselt number are enhanced with an increase in the nanoparticles volume fraction while an increase of ethylene glycol volume ratio in the base fluid lessens the critical Eckert number.

Nguyen et al. (2020) studied ferrofluid flow entropy generation and heat transfer in three types of microchannel heat sink including MCHS without ribs, with ribs, and with porous

ribs in both the flow model approaches. The range of Reynolds number Re and porosity $\bar{s}$ of ribbed blocks is $10 \leq Re \leq 80$ and $0 \leq \bar{s} \leq 75 \%$. They have shown that the highest value of the heat transfer coefficient occurred in the maximum value of volume concentration of nanoparticles, Reynolds number and porosity percent. The two-phase model has the highest Nusselt number than single-phase flow model in all the three cases of MCHS geometry.

Next, the survey of nanofluid flow heat transfer analysis through microchannel under the influence of magnetic field will be discussed.

In the Presence of Magnetic Field

Aminossadati et al. (2011) numerically solved the governing equations with the boundary conditions by the control volume formulation using Patankar's simple algorithm and simulated by FORTRAN to investigate forced convection of water/ Al_2O_3 in a horizontal microchannel. Middle part of the microchannel was exposed to a constant magnetic field and heated by a constant heat flux. The effect of parameters such as Reynolds number, volume fraction of solid nanoparticles, and Hartman number on the flow field and thermal performance of the microchannel was studied. The results have shown that the microchannel performs better heat transfers at higher values of the Reynolds and Hartmann numbers. They also found that the average Nusselt number on the middle section surface of the microchannel increases as the solid volume fraction increases for all values of the Reynolds and Hartmann numbers considered.

Melvandi and Ganji (2015) analyzed the combined effects of bouyance force, brownian motion, thermophoresis and magnetic field on nanofluid flow and heat transfer through microchannel. Method of separation of variables along with Runge-Kutta Fehlberg scheme is employed to solve the system of nonlinear ordinary differential equations with boundary conditions which govern two dimensional laminar flow of MHD alumina/water nanofluid inside a vertical channel. They found that the near-wall velocity gradient, heat transfer rate and pressure gradient drop increase in the presence of intensifying magnetic field. Also, increasing the mixed convective parameter enhances the heat transfer rate, which is more pronounced for smaller nanoparticles.

Malvandi and Ganji (2015) studied the effects of nanoparticle migration and asymmetric heating on the magnetohydrodynamic forced convection of alumina/water nanofluids in microchannels with Navier's wall slip condition. A two-component heterogeneous mixture model is used for nanofluids with the hypothesis that Brownian motion and thermophoretic diffusivities are the only significant slip mechanisms between solid and liquid phases. Assuming a fully developed flow and heat transfer, the basic partial differential equations including continuity, momentum, and energy equations have been reduced to two-point ordinary boundary value differential equations and solved numerically by method of separation of variables with Runge-Kutta Fehlberg scheme. They found that the non-uniform distribu-

tion of nanoparticles causes velocities to move toward the wall with a higher heat flux and enhances heat transfer rate there. In addition, inclusion of nanoparticles in a very strong magnetic field and slip velocity at the walls has a negative effect on performance.

Moshizi and Malvani (2016) studied magnetic field effects on nanoparticle migration of mixed convection of magnetohydrodynamic nanofluids flow in microchannels with temperature dependent thermo-physical properties. A modified Buongiorno's model is employed for the nanofluid which takes into account the effects of thermophoresis and Brownian motion. The Navier's slip condition is considered in the wall to assess the non-equilibrium region near the fluid-solid interface. After the fluid flow has been assumed to be fully developed, the governing equations including the continuity, momentum, energy, and nanoparticle volume fraction have been simplified to a set of ordinary differential equations and solved numerically by means of the Runge-Kutta Fehlberg scheme. The results are obtained with and without considering the dependency of thermophysical properties upon temperature. It is shown that neglecting the dependency of thermophysical properties to the temperature does not significantly affect the flow fields and heat transfer behavior of nanofluids, but it changes the relative magnitudes. Further, ignoring the temperature dependency of the thermophysical properties for quantities of engineering interest including the heat transfer rate and pressure drop resulted in an over prediction of the heat transfer rate and pressure drop.

Hosseini et al. (2018) carried out an investigation on heat transfer of Al_2O_3 -water nanofluid flow in microchannel heat sink in the presence of magnetic field. The Koo-Kleinstreuer (KKL) correlation method is used for computation of effective viscosity and thermal conductivity of the nanofluid. The governing equations of heat transfer of nanofluid through microchannel heat sink are solved by fourth-fifth order Runge-Kutta-Fehlberg Procedure and collocation method. Results indicated that Nusselt number has a direct relationship with applied magnetic field on microchannel heat sink.

Abbas et al. (2020) described an incompressible, fully developed natural convection flow of steady and electrically conducting viscous fluid in a microchannel formed by two vertical parallel permeable plates in the existence of magnetic field and thermal radiation. The closed-form solutions of the coupled ordinary differential equations describing the flow behavior are found for velocity and temperature profiles. The exact results calculated is used to validate the accuracy of two numerical methods namely, Keller-box and shooting method, employed in numerical solution findings and they found in good agreement with the closed-form solutions. The obtained results revealed that each of the flow velocity, fluid temperature and skin friction are a decreasing functions of suction and injection parameter.

Selimefendigil et al. (2019) studied forced convection of Fe_3O_4 -water nanofluid in a bifurcating channel under the effect of variable magnetic field numerically by Galerkin residual finite element method. Effects of various values of Reynolds number, Hartmann number, and solid nanoparticle volume fraction on the convective heat transfer characteristics were analyzed. It was observed that location and size of the re-circulation zones established in

the walls of the bifurcating channel strongly influenced by the variable magnetic field and Reynolds number. This in turn resulted in the formation of local Nusselt number peaks and changed their locations on the hot walls. Variation of average Nusselt number showed different average heat transfer behavior with respect to changes in Hartmann number for walls of the vertical and horizontal branching channels.

Zhong et al. (2021) investigated numerically the heat transfer and entropy generation of MHD ferro-nanofluid flow through three different sets of 3D microchannel heat sinks, such as microchannel heat sink without ribs, with solid ribs, and with porous ribs, respectively with single-phase and two-phase model approaches. The finite volume method was utilized to solve the governing equations of the problems. The values of non-dimensional parameters, namely Hartmann number Ha , Reynolds number Re , porosity $\bar{\epsilon}$ percentage and nanoparticles volume fractions ϕ are assumed to vary in the range of $0 \leq Ha \leq 10$, $10 \leq Re \leq 80$, $0 \leq \bar{\epsilon} \leq 75\%$ and $0 \leq \phi \leq 2\%$, respectively. The influences of these parameters on the three sets of the microchannels are analyzed. It is demonstrated that in all cases, variables such as porosity percentage, Reynolds and Hartmann numbers are directly related to the enhancement of the heat transfer coefficient. Also, in the single-phase model, the dimensionless heat transfer coefficient has a lower amount compared to the two-phase model, and the only exception is for $\phi = 0\%$.

2.4 Mixed Convection in Microchannels

Mixed convection in microchannel flow has potential applications in various science and engineering applications such as chemical processing equipments (Lab-on-chips), cooling electronic devices, transport of heated and cooled fluids, gas-cooled nuclear reactors, micro-electronic cooling, etc. It happens when the combined free and forced convection mechanism exist in the flow system and has a considerable contribution to heat and mass transfer.

Avci and Aydin (2007) analytically investigated fully developed mixed convection of Newtonian fluid in an open-ended vertical parallel plate microchannel with asymmetric wall heating at uniform heat fluxes. An increase in $\left(\frac{Gr}{Re}\right)$ results in flow acceleration near the walls ($y = 0, 1$) and consequently flow deceleration in the centerline of the microchannel. Further, increasing $\left(\frac{Gr}{Re}\right)$ increases both temperature and Nusselt profiles.

Shokouhmand and Jafari (2010) numerically investigated the problem of mixed convection in a vertical microchannel influenced by electric double layer effect with symmetric wall temperature. The nonlinear governing equations are solved numerically by finite volume method. It is found that by increasing Reynold number while keeping the electrical potential constant, the flow tends to Poiseuille flow and consequently the local Nusselt number decreases. It is also obtained that increasing the thermal Grashof number has insignificant on fluid flow and heat transfer. This is due to the very small length scale in the microchannels (Grashof number is proportional to a^3).

The laminar fully developed mixed convection of alumina/water nanofluid inside a vertical microchannel in the presence of heat source/sink is analyzed theoretically employing the modified two-component heterogeneous buongiorno model by Malvandi and Ganji (2016). The modified two-component heterogeneous used the Brownian motion and thermophoresis as the only significant slip mechanisms of nanoparticle in the base fluid. The parallel flat surfaces making the microchannel boundaries were kept at constant but different heat fluxes. The partial differential equations governing the flow are two point boundary value problems solved numerically by Runge-Kutta-Fehlberg method using Fortran code. From scaling analysis of the governing equations, it is obtained that the buoyancy effects due to the temperature difference is insignificant, however, the buoyancy effects due to concentration difference of nanoparticles have considerable effects on the flow and heat transfer characteristics of nanofluid. It is also revealed that the imposed thermal asymmetry would change the direction of nanoparticle migration and distort the asymmetry of the velocity, temperature and nanoparticles concentration profiles.

Jha and Aina (2017) presented a theoretical study of an MHD mixed convection flow in a vertical microchannel formed by two electrical non-conducting infinite vertical parallel plates. The governing equations for the motion are a set of simultaneous ODEs and their exact solutions in dimensionless form have been obtained for the velocity field, the induced magnetic field and the temperature field. It is observed that near the channel wall $y = 0$, increasing the mixed convection parameter $\left(\frac{Gr}{Re}\right)$ leads to a reduction in both fluid flow and skin friction while the results are just reverse at the channel wall $y = 1$. Further, at the center of the channel, the fluid velocity is not influenced by mixed convection parameter $\left(\frac{Gr}{Re}\right)$.

The combined effects of suction/injection, hydrodynamic slip, magnetic field, thermal radiation and convective boundary condition on the heat transfer and entropy generation of Casson fluid flow in a porous microchannel has been examined by Shashikumar et al. (2018). The dimensionless governing equations are solved numerically by using fourth-fifth-order Runge-Kutta integration method along with shooting technique. Based on numerical results, entropy generation rate increases with an increase in radiation and Biot number. Moreover, the temperature initially increases with Biot number up to a certain distance from the heated lower plate and then this behavior reverses in the rest part.

The study of Eyring-Powell fluid flow in an inclined micro porous channel with thermal radiation and convective heating effects is analyzed by Madhu et al. (2019). The governing differential equations are changed by using dimensionless parameters in to a set of non-dimensionalized differential equations. Finite difference method is employed to solve these set of dimensionless equations. Their results demonstrated an enhancement of velocity and temperature profiles for higher values of thermal Grashof number.

The impact of magnetic field on mixed convection flow in a vertical parallel plate microchannel filled with porous medium in the presence of heat generation/absorption parameter is studied by Manthesha and Patil (2021). The governing nonlinear coupled equations

are non-dimensionalised using dimensionless parameters and solved numerically by finite difference method. The study indicated that the mixed convection parameter $\left(\frac{Gr}{Re}\right)$ increases the fluid velocity and suppresses the fluid temperature. Moreover, it is observed that a rise in $\left(\frac{Gr}{Re}\right)$ increases the skin friction at $y = 0$ and decreases it at $y = 1$, whereas a reverse phenomenon happens with Nusselt number at $y = 0$ and $y = 1$.

2.5 Variable Liquid Property Effects in Microchannel

The study of fluid flow in microchannels has been increased significantly due to its relevance in a number of engineering applications and scientific disciplines, such as microscale heat exchangers and reactors used in microelectronics, automotives, bioengineering, aerospace technology, laser process equipments, power and process industries, microsenors, etc. The practical operation of fluid flow through microchannels, particularly for liquids as working fluids, is generally characterized by laminar flow regime with low Reynolds number and high heat flux. One consequence of such conditions is the large variation of liquid properties, at least the viscosity and thermal conductivity, due to steep temperature gradient within the liquid. Thus, it is important here to consider temperature dependent fluid properties and point out their effects on liquid flow and heat transfer in microchannel. Several researchers (Mahulikar and Herwig, 2006; Lui et al., 2008; Gulhane and Mahulikar, 2011; Pati and Kumar, 2018) investigated the influences of temperature dependent fluid properties on the transport behavior of liquid flow in microchannel.

Mahulikar and Herwig (2005) conducted theoretical study on scaling effects from macro-to-micro-scale forced convection of liquid in circular tube, incorporating temperature dependent fluid viscosity and thermal conductivity. They obtained that the role of temperature dependent viscosity and thermal conductivity become more important as the channel size varying from macro-to-micro-scale. It is also obtained that viscosity variation produced an increase in radial flow and thermal conductivity increased fluid axial conduction.

Nonino et al. (2010) carried out a parametric study to find the influences of viscous dissipation and temperature dependent viscosity on forced convection in simultaneously developing laminar flow of a liquid in a straight microchannels of constant cross-section. Numerical results indicated that the effects of temperature dependent viscosity and viscous dissipation can not be neglected in a wide range of operative conditions of the laminar forced convection.

The effect of temperature dependent viscosity on the behavior of laminar fully developed flow and heat transfer for Robertson-Stiff fluids in rectangular duct is investigated numerically by Sayed and El-Yaza (2003). It is concluded that the viscosity variation parameter has an apparent effect on the flow, friction factor and heat transfer rate.

The effects of temperature dependent viscosity on fully developed flow through a microchannel were numerically investigated by Kumar and Mahulikar (2015). Due to viscosity variation term, the momentum and energy equations were mutually coupled. Consequently,

the velocity and temperature profiles varied qualitatively along the micro-flow.

A detailed numerical simulation of forced convection heat transfer in silicon-based microchannel heat sink had been performed by Li et al. (2004). It was reported that the temperature varying thermophysical properties of the liquid can significantly affect both the flow and heat transfer in the microchannel heat sink.

Kumar and Mahulikar (2017) analyzed numerically the effects of temperature dependent density, viscosity, and thermal conductivity variations on single-phase laminar forced convection water in microchannel. It is observed that the axial velocity profile, temperature field and Nusselt number were affected by temperature dependent viscosity. Moreover, thermal conductivity variation directly affected the temperature field and Nusselt number, but it didn't produce any effect on the velocity field.

The effects of temperature varying viscosity and thermal conductivity on heat transfer and frictional flow characteristics of water flowing through a microchannel were studied numerically by Kumar and Mahulikar (2018).

Kumar and Mahulikar (2019) numerically studied the heat transfer characteristics of laminar single-phase forced convective water flow through a micro-tube heat exchanger. Two-dimensional simulations are performed to find the effects of variable fluid properties on heat transfer for hydrodynamically and thermally developed flow. The effects of variable fluid properties on convective heat transfer coefficient and Nusselt number are significant for micro-convective flow. It is noted that the variation in temperature-dependent thermal conductivity greatly enhances the convective heat transfer coefficient as compared to the variation in temperature-dependent viscosity.

2.6 Effects of wall Suction/Injection

Suction/Injection of a fluid through permeable walls of microchannel flow can significantly affect on the velocity, temperature, skin friction and heat transfer (Jha et al., 2013; Jha et al., 2015; Jha and Aina, 2016). Suction/Injection plays an important role in controlling flow through microchannel, hence its significance in practical problems including film cooling and mass transfer applications. Olofikua (2018) evaluated the effect of injection flow in diverging and converging microchannel on heat transfer and hydrodynamics of a nanofluid. He found that the velocity distribution is an increasing function of Reynolds number. Shiriny et al. (2019) investigated forced convection of water/ Al_2O_3 nanofluid flow in a microchannel with inclined cross-flow injection on the lower wall. There are a number of injecting holes on its lower wall. Other walls are insulated. The effect of different parameters including injection angle and number of injections on slip flow and heat transfer is investigated. It is concluded that increasing the injection angle up to 94° lead to an enhancement of the slip velocity and the Nusselt number on the microchannel walls. The results demonstrate that maximum heat transfer corresponds to the injection angle of 94° . It is revealed that the variation of Nusselt

number decreases with the number of injection.

2.7 Chemical Reaction in Microchannels

A chemical reaction is a process involving one or more substances called reactants, characterized by a chemical reaction change and yielding one or more products which are different from the reactants. A chemical change is defined as molecules attaching to each other form large molecules, molecules breaking apart to form two or more, smaller molecules or rearrangement of atom within the molecules. In nature, the presence of pure air or water is impossible. Some foreign mass may be present either naturally or mixed with the air or water. The presence of a foreign mass in air or water causes some kind of chemical reaction. During a chemical reaction between two species, heat is also generated. In most cases of chemical reaction, the reaction rate depends on the concentration of the species itself. Most transport phenomena occurring in nature and industrial engineering applications in which heat and mass transfer is the result of buoyancy effect caused by temperature and species concentration gradient. A few representative fields of interest where combined heat and mass transfer with chemical reaction plays an important role in design of chemical processing equipment, cooling devices, etc.

Chemical reaction can be codified as either homogeneous or heterogeneous process. This depends whether they occur at an interface or as single-phase volume reaction. Homogeneous reaction or chemical reaction in which the reactants are in the same phase, while heterogeneous reactions have reactants that are in two or more phases. Reactions that takes place on the surface of a catalyst of a different phase are also heterogeneous . A reaction between two gases, two liquids or two solids is homogeneous. A reaction between a gas and a liquid, a gas and solid or a liquid and a solid is heterogeneous. Practical application of heterogeneous reaction are in catalytic converter, fuel cells and chemical vapour decomposition among others. Recently, manufacturing engineers have surface reactions for synthesis of micro and nanoscale in biomedical devices.

Moshizi (2015) investigated the effects of chemical reactions on convective heat and mass transfer characteristics of Cu-water nanofluid inside a porous microchannel in the presence of a uniform magnetic field. The walls of the microchannel are subjected to constant asymmetric heat fluxes. Results show that the addition of Cu-nanoparticles to the fluid has a significant influence on decreasing temperature gradient at the both walls and velocity profile along the microchannel. In addition, total heat transfer in microchannel increases as nanoparticles add to the fluid. On the other hand, Damkohler number, which relates to the ratio between the reaction rate and diffusive mass transfer rate, has an influence on concentration profiles. When the absolute value of Damkohler number goes up, it means that the reaction rate is more than diffusive mass transfer rate. With increasing the reaction rate, reactants tend to consume progressively, consequently the mass concentration in whole mi-

crochannel experiences a downward trend.

Dhlamini et al. (2019) presented a theoretical study of the combined effects of activation energy and binary chemical reaction in an unsteady mixed convective nanofluid flow over convectively heated flat plate moving with a constant velocity. The governing equations are solved numerically to a high degree of accuracy using the spectral quasilinearization method. Increasing Brownian motion parameter and the chemical reaction constant causes a decrease in the chemical species within the boundary layer.

Raju and Reddy (2019) analyzed the aggregate impacts of Chemical reaction, thermo-diffusion and Hall effects on hydromagnetic convective heat and mass transfer flow of an electrically conducting, viscous, incompressible fluid through a porous medium in a vertical channel bounded by stretching and stationary walls. The nonlinear governing equations have been solved by using Runge-Kutta shooting technique. The velocity, temperature and concentration profiles reduced in the degenerating chemical reaction case and in the generating chemical reaction case the velocity components and the concentration enhances and the temperature reduces in the entire flow region

The impact of chemical reaction and thermal radiation on nanofluid flow in a rotating channel with Hall current is assessed by Lv et al. (2021). It is reported that on up-surging chemical reaction parameter, the nanoparticles concentration reduced owing to its consumption in the chemical reaction. Substantial studies on chemical reaction effect in channel flow with various physical aspects are cited in (Khan et al., 2021; Tarakaramu and Narayana, 2019).

2.8 Conclusion from Literature Review

Based on the literature survey from section (2.1)-(2.7), the following conclusions can be drawn:

- Liquid flow in microchannel is mostly laminar because of the hydraulic diameter that makes Reynold number very small.
- Regarding the definition of microchannel, the one suggested by Obot is adopted by many researchers and appropriate for this study due to the fact that it depends on channel geometry and uses the concept of hydraulic diameter.
- In most microchannel flow, surface-area-related quantities are much dominant.
- Continuum modeling assumption is valid in almost all liquid flow in microchannel.
- An absolute increase in the heat transfer coefficient and pressure drop are a much common features of microchannel flows.

- Experimentally, it is found that the nanofluid cooled-microchannel heat transfer is higher than the water cooled-microchannel with the requirement of higher pumping power to drive the nanofluid flow through microchannels.
- Experimental literature survey indicated that Nusselt number of the nanofluids rise with an increase in Reynolds number and nanoparticles volume fraction, which in turn enhance the heat transfer performance of the microchannels in the absence of magnetic field. In the presence of magnetic field, however, the rise in magnetic-nanoparticles concentration cause reduction of convection heat transfer coefficient, due to the fact that the applied magnetic field hinders the Brownian motion of the nanoparticle.
- Most of the papers fail to consider unsteady fluid flow and temperature dependent fluid properties like viscosity, thermal conductivity and electrical conductivity.
- A variety of numerical methods were used to solve the governing equations of the problems, however the finite volume method was comparatively the common technique to tackle the problems numerically.
- The Nusselt number increases with an increase in nanoparticles concentration in the absence of magnetic field.
- Generally, the heat transfer rate is directly related with the applied magnetic field.
- Mixed convection has considerable effect on flow and heat transfer of nanofluids in microchannel.
- The velocity profiles, the temperature profiles, the Nusselt number, and the convective heat transfer coefficient are apparently influenced by temperature varying liquid properties such as viscosity and thermal conductivity.

CHAPTER 3

RESEARCH METHODOLOGY

This chapter provides the details of mathematical procedures used for achieving the stated objectives and research methodology (Derivation of governing equations and computational techniques) for the analysis of the problems in the study.

3.1 Mathematical Procedures

The procedures used for attaining the objectives of the study are:

1. Defining (or describing) the problems.
2. Derivation of the governing equations.
3. Imposing assumptions and approximations like order of magnitude analysis, Boussinesq and Rosseland approximations on the governing equations of the problems to get simplified version of the problem's model in PDEs form (Formulation of the problem's model in PDE form).
4. Non-dimensionalizing the model of the problems.
5. Solving the non-dimensionalized form of the model problem using appropriate numerical scheme (Semi-discretization method with Runge-Kutta Fehlberg method) and developing MAPLE code for the scheme.
6. Validating the results by comparing with available data in open literature (qualitative validation).
7. Presenting the results using graphs.
8. Discussing and providing conclusions.

3.2 Research Methodology

3.2.1 Nanofluid Dynamics Model Equations in Microchannels

Computational fluid flow problems are founded on the fundamental governing equations of fluid dynamics- the continuity, momentum, and energy equations. These equations are the

mathematical statements of three physical principles upon which all of fluid dynamics is based:

- (a) Mass is conserved.
- (b) Newton's law, $F = ma$.
- (c) Energy is conserved.

In the literature, nanofluids flows with convective heat transfer can be modeled mainly using single phase flow model or two-phase flow model. In the single phase flow model, it is assumed that the suspended nanoparticles are in thermal equilibrium with the liquid phase and that the relative velocity between the two phases is negligible. The reason is that nanoparticles are so small that they follow the streamlines of the fluid exactly, making the mixture behave like a homogeneous mixture. Thus, a set of governing equations for pure fluids can be used with the effective thermophysical properties of nanofluids replacing the fluid properties. In the two phase flow model, the velocity between the fluid and particles might not be zero because of several factors such as gravity, friction between the fluid and solid particles, Brownian forces, Brownian diffusion, sedimentation, and dispersion. The study uses both single phase and two phase approaches to model and solve the flow problems.

The purpose of this section is to derive the governing equations of these equations by making use of MHD nanofluid flow in a microchannel with permeable walls at $y = 0$ and $y = a$ under the action of constant axial pressure gradient, exothermic chemical reaction and thermal radiation using infinitesimal fluid element approach. The single phase model equations will be an immediate consequences of the two phase model derivation.

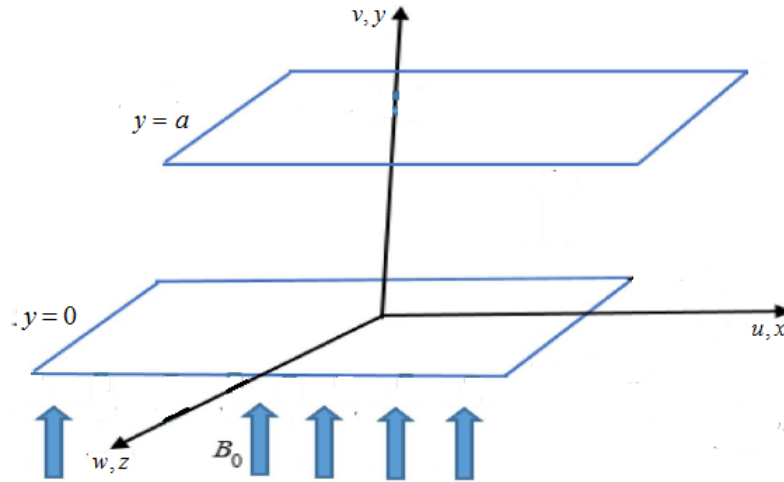


Figure 3.1: Schematic diagram of the flow.

i. Two Phase Modeling Equations

1. Continuity Equation

Consider the mass of fluid flowing with velocity $\mathbf{q} = \mathbf{q}(x, y, z, t)$ through an infinitesimal small element (rectangular-box) of sides dx , dy , and dz fixed in Cartesian coordinate space as shown in figure 3.2. Let $\rho = \rho(x, y, z)$ be the density of the fluid which is a function of

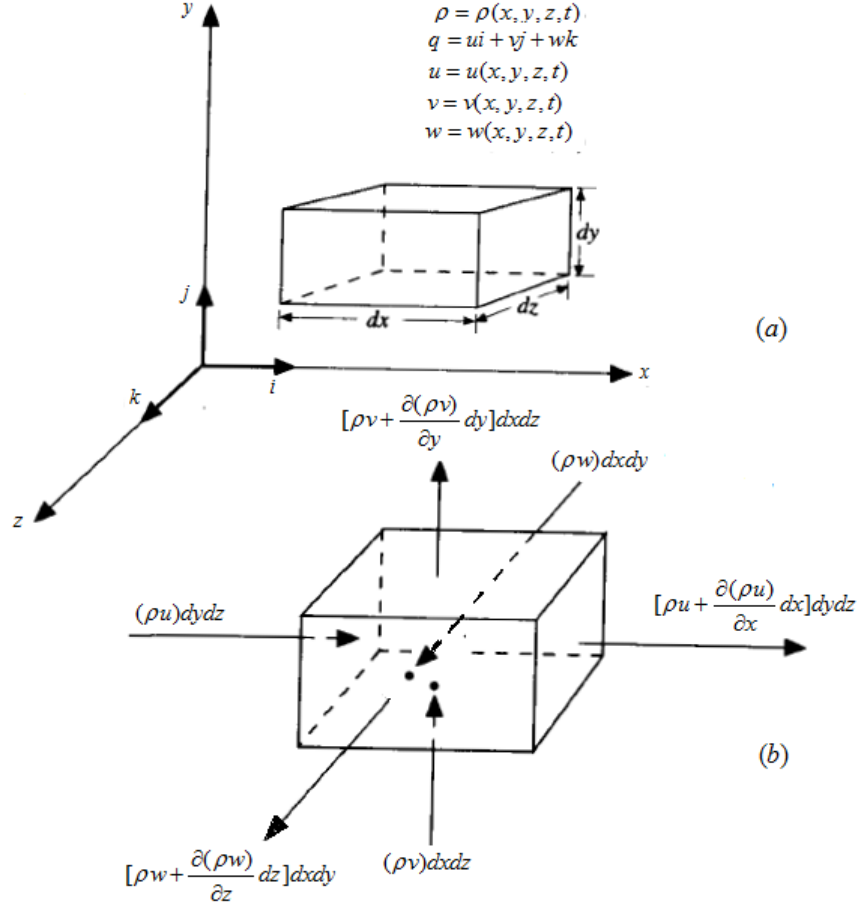


Figure 3.2: Model of infinitesimal mass fluxes.

space (x, y, z) and time t .

The left and right faces of the infinitesimal element are assumed to be orthogonal to the x -axis and their area is $dydz$. The mass flow through the left face is $(\rho u)dydz$. Since the velocity and density are functions of spatial location, the values of the mass flux across the right face will be different from that across the left face; indeed, the difference in mass flux between the two faces is simply $[\frac{\partial(\rho u)}{\partial x}]dx$. Thus, the mass flow across the right face can be expressed as $\{\rho u + [\frac{\partial(\rho u)}{\partial x}]dx\}dydz$. The mass flow across both the left and right faces is shown in figure 3.2 (b). In a similar vein, the mass flow through both the bottom and top faces, which are perpendicular to the y -axis, is $(\rho v)dx dz$ and $\{\rho v + [\frac{\partial(\rho v)}{\partial y}]dy\}dx dz$, respectively. The mass flow through both the front and back faces, which are perpendicular to the z axis, is

$(\rho w)dxdy$ and $\{\rho w + [\frac{\partial(\rho w)}{\partial z}]dz\}dxdy$, respectively. If the net outflow of mass is considered as a positive quantity, then from Fig. 3.2, it is obtained that:

$$\begin{aligned} \text{Net outflow in } x \text{ direction: } & [\rho u + \frac{\partial(\rho u)}{\partial x}dx]dydz - (\rho u)dydz = \frac{\partial(\rho u)}{\partial x}dxdydz, \\ \text{Net outflow in } y \text{ direction: } & [\rho v + \frac{\partial(\rho v)}{\partial y}dy]dxdz - (\rho v)dxdz = \frac{\partial(\rho v)}{\partial y}dxdydz, \\ \text{Net outflow in } z \text{ direction: } & [\rho w + \frac{\partial(\rho w)}{\partial z}dz]dxdy - (\rho w)dxdy = \frac{\partial(\rho w)}{\partial z}dxdydz. \end{aligned}$$

Hence, the net mass flow out of the element is given by

$$\text{Net mass flow} = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] dxdydz. \quad (3.1)$$

The total mass of fluid in the infinitesimally small element is $\rho(dxdydz)$; hence the time rate of increase of mass inside the element is given by

$$\text{Time rate of mass increase} = \frac{\partial \rho}{\partial t}(dxdydz). \quad (3.2)$$

The physical principle that mass is conserved, when applied to the fixed element in figure 3.2, can be expressed in words as follows: the net mass flow out of the element must equal the time rate of decrease of mass inside the element. Denoting the mass decrease by a negative quantity, this statement can be expressed in terms of Equations (3.1) and (3.2) as

$$\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] dxdydz = -\frac{\partial \rho}{\partial t}(dxdydz)$$

or

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] = 0. \quad (3.3)$$

In Equation (3.3), the term in brackets is simply $\nabla \cdot (\rho \mathbf{q})$. Thus, Equation (3.3) becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0. \quad (3.4)$$

Equation (3.4) is *the continuity equation* for compressible fluids in Cartesian coordinates.

For incompressible fluids, the density ρ is constant in space and time, therefore, the Equation (3.4) reduces to

$$\nabla \cdot \mathbf{q} = 0. \quad (3.5)$$

2. Momentum Equation

The momentum equation is derived by applying the fundamental physical principle to a model of a flow, namely:

Physical principle: $F = ma$ (Newton's second law)

When Newton's second law applied to the moving fluid element in figure 3.3, says that the net force on the fluid element equals its mass times the acceleration of the element. This is a vector relation, and hence can be split into three scalar relations along x, y, z axes.

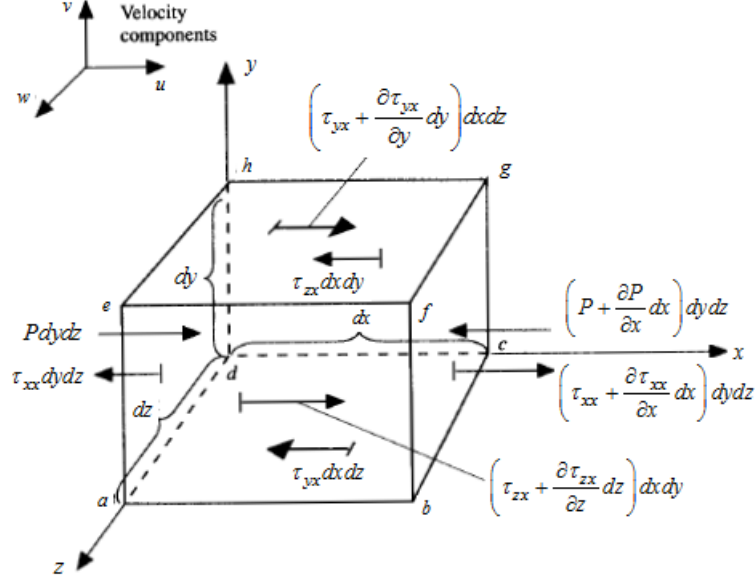


Figure 3.3: Model of infinitesimal pressure and shear stresses.

x-component Newton's second law:

$$F_x = ma_x, \quad (3.6)$$

y-component Newton's second law:

$$F_y = ma_y, \quad (3.7)$$

z-component Newton's second law:

$$F_z = ma_z. \quad (3.8)$$

The sources of the net force on fluid element that the fluid element experience are:

Body forces, which act directly on the volumetric mass of the fluid element. These forces "act at a distance"; examples are gravitational, electric, and magnetic forces. Here, gravitational and magnetic forces are the body forces incorporated in momentum equation derivation.

Surface forces, which act directly on the surface of the fluid element. They are due to only two sources: (i) the pressure distribution acting on the surface, imposed by the outside fluid surrounding the fluid element, and (ii) the shear and normal stress distributions acting on the surface, also imposed by the outside fluid "tugging" or "pushing" on the surface by means of friction. Let us denote the body force per unit mass acting on the fluid element by f , with its components f_x , f_y , and f_z along x , y , and z directions, respectively. The volume of the fluid element is $(dx dy dz)$; hence,

$$\text{Body force on fluid element acting in } x \text{ direction} = \rho f_x (dx dy dz), \quad (3.9)$$

$$\text{Body force on fluid element acting in } y \text{ direction} = \rho f_y (dx dy dz), \quad (3.10)$$

$$\text{Body force on fluid element acting in } z \text{ direction} = \rho f_z (dx dy dz). \quad (3.11)$$

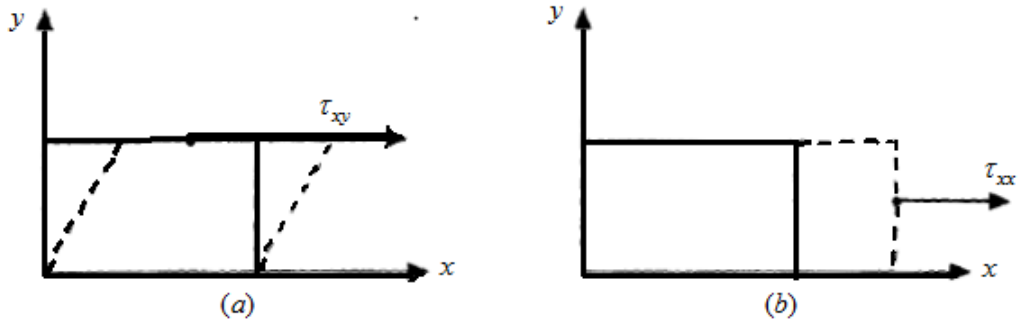


Figure 3.4: Illustration of (a) shear stress and (b) normal stress.

The shear and normal stresses in a fluid are related to the time rate of change of the deformation of the fluid element, as sketched in figure 3.4 for just the xy plane. The shear stress, denoted by τ_{xy} in Fig. 3.4 (a), is related to the time rate of change of the shearing deformation of the fluid element, whereas the normal stress, denoted by τ_{xx} in figure 3.4 (b), is related to the time rate of change of volume of the fluid element. As a result, both shear and normal stresses depend on velocity gradients in the flow, to be designated later.

Let τ_{ij} denotes a stress in the j direction exerted on a plane perpendicular to the i axis.

On face $abcd$: The stresses in x , y and z directions respectively are $\tau_{yx} dx dz$, $\tau_{yy} dx dz$ and $\tau_{yz} dx dz$.

On face $adhe$: The stresses in x , y and z -directions respectively are $\tau_{xx} dy dz$, $\tau_{xy} dy dz$ and $\tau_{xz} dy dz$.

On face $dcgh$: The stresses in x , y and z directions respectively are $\tau_{zx} dx dy$, $\tau_{zy} dx dy$ and $\tau_{zz} dx dy$.

On face $efgh$: The stresses in x , y and z directions respectively are $\left(\tau_{yx} + \frac{\tau_{yx}}{\partial y} dy \right) dx dz$, $\left(\tau_{yy} + \frac{\tau_{yy}}{\partial y} dy \right) dx dz$ and $\left(\tau_{yz} + \frac{\tau_{yz}}{\partial y} dy \right) dx dz$.

On face $bcfg$: The stresses in x , y and z directions respectively are $\left(\tau_{xx} + \frac{\tau_{xx}}{\partial x} dx \right) dy dz$, $\left(\tau_{xy} + \frac{\tau_{xy}}{\partial x} dx \right) dy dz$ and $\left(\tau_{xz} + \frac{\tau_{xz}}{\partial x} dx \right) dy dz$.

On face $abfe$: The stresses in x , y and z directions respectively are $\left(\tau_{zx} + \frac{\tau_{zx}}{\partial z} dz \right) dx dy$, $\left(\tau_{zy} + \frac{\tau_{zy}}{\partial z} dz \right) dx dy$ and $\left(\tau_{zz} + \frac{\tau_{zz}}{\partial z} dz \right) dx dy$.

On faces $adhe$, $abcd$ and $cghd$ with thier corresponding opposite faces, there is additional pressure (normal) stresses in x , y and z -directions.

On faces $adhe$ and $bcfg$: The pressure stresses are $P dy dz$ and $\left(P + \frac{\partial P}{\partial x} dx \right) dy dz$ in x -direction.

On faces $abcd$ and $efgh$: The pressure stresses are $P dx dz$ and $\left(P + \frac{\partial P}{\partial y} dy \right) dx dz$ in y -direction.

On faces $cghd$ and $bfea$: The pressure stresses are $P \, dxdy$ and $\left(P + \frac{\partial P}{\partial z} dz\right) \, dxdy$ in z -direction.

With the above in mind, for the moving fluid element it can be written that:

$$\begin{aligned} \text{Net surface force in } x \text{ direction} = & \left[P - \left(P + \frac{\partial P}{\partial x} dx \right) \right] dydz + \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx \right) - \tau_{xx} \right] dydz \\ & + \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) - \tau_{yx} \right] dxdz \\ & + \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right) - \tau_{zx} \right] dxdy, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \text{Net surface force in } y \text{ direction} = & \left[P - \left(P + \frac{\partial P}{\partial y} dy \right) \right] dxdz + \left[\left(\tau_{yy} + \frac{\partial \tau_{yy}}{\partial y} dy \right) - \tau_{yy} \right] dxdz \\ & + \left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) - \tau_{xy} \right] dydz \\ & + \left[\left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz \right) - \tau_{zy} \right] dxdy, \end{aligned} \quad (3.13)$$

$$\begin{aligned} \text{Net surface force in } z \text{ direction} = & \left[P - \left(P + \frac{\partial P}{\partial z} dz \right) \right] dxdy + \left[\left(\tau_{zz} + \frac{\partial \tau_{zz}}{\partial z} dz \right) - \tau_{zz} \right] dxdy \\ & + \left[\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy \right) - \tau_{yz} \right] dxdz \\ & + \left[\left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx \right) - \tau_{xz} \right] dydz. \end{aligned} \quad (3.14)$$

The total force in the x direction F_x , is given by the sum of Equations (3.9) and (3.12). Adding, and canceling terms,

$$F_x = \left[-\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] dxdydz + \rho f_x dxdydz. \quad (3.15)$$

The total force in the y direction F_y , is given by the sum of Equations (3.10) and (3.13). Adding, and canceling terms,

$$F_y = \left[-\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right] dxdydz + \rho f_y dxdydz. \quad (3.16)$$

The total force in the z direction F_z , is given by the sum of Equations (3.11) and (3.14). Adding, and canceling terms,

$$F_z = \left[-\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right] dxdydz + \rho f_z dxdydz. \quad (3.17)$$

Equation (3.15)-(3.17) represent the left-hand side of Equation (3.6)-(3.8), respectively.

Considering the right-hand side of Equations (3.6)-(3.8), recall that the mass of the fluid element is fixed and is equal to

$$m = \rho dx dy dz. \quad (3.18)$$

Also recall that the acceleration of the fluid element is the time rate of change of its velocity; since we are following a moving fluid element, this time rate of change is given by the substantial derivative. Thus,

$$a_x = \frac{du}{dt}, \quad (3.19)$$

$$a_y = \frac{dv}{dt}, \quad (3.20)$$

$$a_z = \frac{dw}{dt}. \quad (3.21)$$

Combining equations (3.6) and (3.15) to (3.19), we obtain

$$\rho \frac{du}{dt} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x. \quad (3.22)$$

Combining equations (3.7) and (3.16) to (3.20), we obtain

$$\rho \frac{dv}{dt} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y. \quad (3.23)$$

Combining equations (3.8) and (3.17) to (3.21), we obtain

$$\rho \frac{dw}{dt} = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z. \quad (3.24)$$

Equations (3.22) to (3.24) are the x , y , and z components, respectively, of the momentum equation.

Writing the left-hand side of Equations (3.22)-(3.24) in terms of the definition of the substantial derivative, we have

$$\rho \frac{du}{dt} = \rho \frac{\partial u}{\partial t} + \rho \mathbf{q} \cdot \nabla u, \quad (3.25)$$

$$\rho \frac{dv}{dt} = \rho \frac{\partial v}{\partial t} + \rho \mathbf{q} \cdot \nabla v, \quad (3.26)$$

$$\rho \frac{dw}{dt} = \rho \frac{\partial w}{\partial t} + \rho \mathbf{q} \cdot \nabla w. \quad (3.27)$$

Also, expanding the following derivatives,

$$\frac{\partial(\rho u)}{\partial t} = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}, \quad \frac{\partial(\rho v)}{\partial t} = \rho \frac{\partial v}{\partial t} + v \frac{\partial \rho}{\partial t}, \quad \text{and} \quad \frac{\partial(\rho w)}{\partial t} = \rho \frac{\partial w}{\partial t} + w \frac{\partial \rho}{\partial t}$$

and rearranging, we have

$$\rho \frac{\partial u}{\partial t} = \frac{\partial(\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t}, \quad \rho \frac{\partial v}{\partial t} = \frac{\partial(\rho v)}{\partial t} - v \frac{\partial \rho}{\partial t}, \quad \text{and} \quad \rho \frac{\partial w}{\partial t} = \frac{\partial(\rho w)}{\partial t} - w \frac{\partial \rho}{\partial t}. \quad (3.28)$$

Recalling the vector identity for the divergence of the product of a scalar times a vector, we have

$$\begin{aligned} \nabla \cdot (\rho u q) &= u \nabla \cdot (\rho q) + (\rho q) \cdot \nabla u, \quad \nabla \cdot (\rho v q) = v \nabla \cdot (\rho q) + (\rho q) \cdot \nabla v, \quad \text{and} \\ \nabla \cdot (\rho w q) &= w \nabla \cdot (\rho q) + (\rho q) \cdot \nabla w, \end{aligned}$$

or

$$\begin{aligned} (\rho q) \cdot \nabla u &= \nabla \cdot (\rho u q) - u \nabla \cdot (\rho q), \quad (\rho q) \cdot \nabla v = \nabla \cdot (\rho v q) - v \nabla \cdot (\rho q), \quad \text{and} \\ (\rho q) \cdot \nabla w &= \nabla \cdot (\rho w q) - w \nabla \cdot (\rho q). \end{aligned} \quad (3.29)$$

Substitute Equations (3.28) and (3.29) into (3.25)-(3.27).

$$\rho \frac{du}{dt} = \frac{\partial(\rho u)}{\partial t} - u \left[\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho q) \right] + \nabla \cdot (\rho u q), \quad (3.30)$$

$$\rho \frac{dv}{dt} = \frac{\partial(\rho v)}{\partial t} - v \left[\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho q) \right] + \nabla \cdot (\rho v q), \quad (3.31)$$

$$\rho \frac{dw}{dt} = \frac{\partial(\rho w)}{\partial t} - w \left[\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho q) \right] + \nabla \cdot (\rho w q). \quad (3.32)$$

The term in brackets in Equations (3.30)-(3.32) is simply the left-hand side of the continuity equation as Equation (3.4); hence the term in brackets is zero. Thus Equations (3.30)-(3.32) reduce to

$$\rho \frac{du}{dt} = \frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u q), \quad (3.33)$$

$$\rho \frac{dv}{dt} = \frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v q), \quad (3.34)$$

$$\rho \frac{dw}{dt} = \frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w q). \quad (3.35)$$

Substitute Equations (3.33)-(3.35) into (3.22)-(3.24), respectively:

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u q) = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x, \quad (3.36)$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v q) = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y, \quad (3.37)$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w q) = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z. \quad (3.38)$$

In the late seventeenth century, Isaac Newton stated that shear stress in a fluid is proportional

to the time rate of strain, i.e., velocity gradients. Such fluids are called *Newtonian* fluids. For such fluids, Stokes in 1845 obtained

$$\tau_{xx} = \lambda (\nabla \cdot \mathbf{q}) + 2\mu \frac{\partial u}{\partial x}, \quad (3.39)$$

$$\tau_{yy} = \lambda (\nabla \cdot \mathbf{q}) + 2\mu \frac{\partial v}{\partial y}, \quad (3.40)$$

$$\tau_{zz} = \lambda (\nabla \cdot \mathbf{q}) + 2\mu \frac{\partial w}{\partial z}, \quad (3.41)$$

$$\tau_{xy} = \tau_{yx} = \mu \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right], \quad (3.42)$$

$$\tau_{xz} = \tau_{zx} = \mu \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right], \quad (3.43)$$

$$\tau_{yz} = \tau_{zy} = \mu \left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right]. \quad (3.44)$$

where μ is the molecular viscosity coefficient and λ is the second viscosity coefficient. Stokes made the hypothesis that

$$\lambda = -\frac{2}{3}\mu.$$

Substituting Equations (3.39)-(3.44) into (3.36)-(3.38) and using vector identity (3.29), the complete Navier-Stokes equations will be obtained

$$\begin{aligned} \frac{\partial(\rho u)}{\partial t} + u \nabla \cdot (\rho \mathbf{q}) + (\rho \mathbf{q}) \cdot \nabla u = & -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left(2\mu \frac{\partial u}{\partial x} - \frac{2}{3} \mu \nabla \cdot \mathbf{q} \right) + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \\ & + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] + \rho f_x, \end{aligned} \quad (3.45)$$

$$\begin{aligned} \frac{\partial(\rho v)}{\partial t} + v \nabla \cdot (\rho \mathbf{q}) + (\rho \mathbf{q}) \cdot \nabla v = & -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left(2\mu \frac{\partial v}{\partial y} - \frac{2}{3} \mu \nabla \cdot \mathbf{q} \right) \\ & + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right] + \rho f_y, \end{aligned} \quad (3.46)$$

$$\begin{aligned} \frac{\partial(\rho w)}{\partial t} + w \nabla \cdot (\rho \mathbf{q}) + (\rho \mathbf{q}) \cdot \nabla w = & -\frac{\partial P}{\partial z} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right] \\ & + \frac{\partial}{\partial z} \left(2\mu \frac{\partial w}{\partial z} - \frac{2}{3} \mu \nabla \cdot \mathbf{q} \right) + \rho f_z. \end{aligned} \quad (3.47)$$

The vector form of the momentum equation is:

$$\frac{\partial(\rho q)}{\partial t} + q \nabla \cdot (\rho q) + (\rho q) \cdot \nabla q = -\nabla P + \frac{4}{3} \nabla(\mu \nabla \cdot q) + \nabla(q \cdot \nabla \mu) - q \nabla^2 \mu + \nabla \mu \times (\nabla \times q) - (\nabla \cdot q) \nabla \mu - \nabla \times (\nabla \times \mu q) + \rho f \quad (3.48)$$

Since the body forces are gravitational force g and magnetic force $\frac{J \times B}{\rho}$ origin, the body forces $f = g + \frac{J \times B}{\rho}$ and Equation (3.48) is written as:

$$\frac{\partial(\rho q)}{\partial t} + q \nabla \cdot (\rho q) + (\rho q) \cdot \nabla q = -\nabla P + \frac{4}{3} \nabla(\mu \nabla \cdot q) + \nabla(q \cdot \nabla \mu) - q \nabla^2 \mu + \nabla \mu \times (\nabla \times q) - (\nabla \cdot q) \nabla \mu - \nabla \times (\nabla \times \mu q) + \rho g + J \times B. \quad (3.49)$$

Equation (3.49) is called the momentum equation for compressible fluid in Cartesian coordinate system.

For incompressible fluids, ρ is constant and $\nabla \cdot q = 0$. In this case $\nabla \times (\nabla \times \mu q) = -\nabla(\nabla \cdot \nabla \mu q) + q \nabla^2 \mu$ and $\nabla \mu \times (\nabla \times q) = -\nabla \mu \cdot \nabla q$. Therefore, the momentum equation for incompressible fluid is given by:

$$\rho \left[\frac{\partial q}{\partial t} + (q \cdot \nabla) q \right] = -\nabla P - \nabla \mu \times \nabla q + \rho g + J \times B. \quad (3.50)$$

3. Energy Equation

In the derivation of energy equation, the flow model of an infinitesimally small fluid element moving with the flow is considered and is based on the first law of thermodynamics, which states that,

$$\left\{ \begin{array}{l} \text{Rate of change of} \\ \text{internal energy} \\ \text{inside fluid element} \end{array} \right\} = \left\{ \begin{array}{l} \text{Net flux of heat} \\ \text{into element} \end{array} \right\} + \left\{ \begin{array}{l} \text{Rate of work done on} \\ \text{fluid element due to} \\ \text{body and surface forces} \end{array} \right\}$$

or

$$A = B + C \quad (3.51)$$

where A, B , and C denote the respective terms above. Let us first evaluate C ; that is, let us obtain an expression for the rate of work done on the moving fluid element due to body and surface forces. It can be shown that the rate of doing work by a force exerted on a moving body is equal to the product of the force and the component of velocity in the direction of the force. Hence, the rate of work done by the body force acting on the fluid element moving at

a velocity q is

$$\rho f \cdot q(dx dy dz).$$

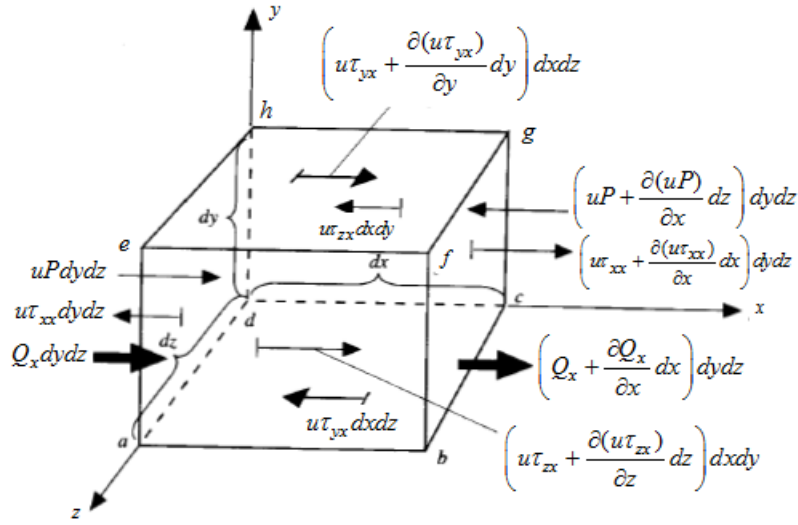


Figure 3.5: Model of infinitesimal energy fluxes.

With regard to the surface forces (pressure plus shear and normal stresses), consider just the forces on each six faces of the fluid element.

The rate of work done by surface forces (pressure plus shear and normal stresses) on the faces $abcd$ and $efgh$ are:

The rate of work done on by $p dx dz$ is $v p dx dz$.

The rate of work done by $\tau_{yx} dx dz$ is $u \tau_{yx} dx dz$.

The rate of work done by $\tau_{yy} dx dz$ is $v \tau_{yy} dx dz$.

The rate of work done on by $p dx dz$ is $[v p + \frac{\partial(v p)}{\partial y} dy] dx dz$.

The rate of work done by $\tau_{yx} dx dz$ is $[u \tau_{yx} + \frac{\partial(u \tau_{yx})}{\partial y} dy] dx dz$.

The rate of work done by $\tau_{yy} dx dz$ is $[v \tau_{yy} + \frac{\partial(v \tau_{yy})}{\partial y} dy] dx dz$.

The rate of work done by surface forces (pressure plus shear and normal stresses) on the faces $adhe$ and $bcgf$ are:

The rate of work done on by $p dy dz$ is $u p dy dz$.

The rate of work done by $\tau_{xy} dy dz$ is $u \tau_{xy} dy dz$.

The rate of work done by $\tau_{xx} dy dz$ is $v \tau_{xx} dy dz$.

The rate of work done on by $p dy dz$ is $[u p + \frac{\partial(u p)}{\partial x} dx] dy dz$.

The rate of work done by $\tau_{xy} dy dz$ is $[v \tau_{xy} + \frac{\partial(v \tau_{xy})}{\partial x} dx] dy dz$.

The rate of work done by $\tau_{xx} dy dz$ is $[u \tau_{xx} + \frac{\partial(u \tau_{xx})}{\partial x} dx] dy dz$.

The rate of work done by surface forces (pressure plus shear and normal stresses) on the faces $cghd$ and $bfea$ are:

The rate of work done on by $p dx dy$ is $w p dx dy$.

The rate of work done by $\tau_{zx} dx dy$ is $u \tau_{zx} dx dy$.

The rate of work done by $\tau_{zz}dxdy$ is $w\tau_{zz}dxdy$.

The rate of work done on by $pdxdy$ is $[wp + \frac{\partial(wp)}{\partial z}dz]dxdy$.

The rate of work done by $\tau_{zx}dxdy$ is $[u\tau_{zx} + \frac{\partial(u\tau_{zx})}{\partial z}dz]dxdy$.

The rate of work done by $\tau_{zz}dxdy$ is $[w\tau_{zz} + \frac{\partial(w\tau_{zz})}{\partial z}dz]dxdy$.

Hence, the net rate of work done by surface forces in the x -, y -, and z - directions are: $\left[\frac{\partial(up)}{\partial x} + \frac{\partial(\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(v\tau_{zx})}{\partial z} \right]dxdydz$, $\left[\frac{\partial(vp)}{\partial y} + \frac{\partial(u\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} \right]dxdydz$, and $\left[\frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xz})}{\partial x} + \frac{\partial(u\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right]dxdydz$, respectively. In total, the net rate of work done on the moving fluid element is the sum of the surface force contributions in the x , y , and z directions, as well as the body force contribution. This is denoted by C in equation (3.51) is given by

$$C = - \left[\frac{\partial(up)}{\partial x} + \frac{\partial(vp)}{\partial y} + \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(v\tau_{zx})}{\partial z} + \frac{\partial(u\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(u\tau_{xz})}{\partial x} + \frac{\partial(u\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right]dxdydz + \rho f \cdot qdxdydz,$$

or

$$C = - \left[\nabla \cdot (pq) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(v\tau_{zx})}{\partial z} + \frac{\partial(u\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(u\tau_{xz})}{\partial x} + \frac{\partial(u\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right]dxdydz + \rho f \cdot qdxdydz. \quad (3.52)$$

To find an expression B , i.e., the net flux of heat into the fluid element, consider heat flux due to (1) volumetric heating added/removed (2) heat transfer across the surface due to temperature gradient, i.e., thermal conduction (3) heat transfer by thermal radiation and (4) the effects of thermophoresis and Brownian motion of nanoparticles. Define Q as the rate of volumetric heat addition per unit mass. Since the mass of the moving fluid element in figure (3.5) is $\rho dxdydz$, it is obtained that

$$\text{Volumetric heating of element} = \rho Qdxdydz. \quad (3.53)$$

The heat transfer by thermal conduction into the moving fluid element in the x -direction on the faces $adhe$ and $bcbf$ are $Q_x dydz$ and $(Q_x + \frac{\partial Q_x}{\partial x} dx) dydz$, respectively.

The heat transfer by thermal conduction into the moving fluid element in the y -direction on the faces $abcd$ and $efgh$ are $Q_y dxdz$ and $(Q_y + \frac{\partial Q_y}{\partial y} dy) dxdz$, respectively.

The heat transfer by thermal conduction into the moving fluid element in the z -direction on the faces $dcgh$ and $abfe$ are $Q_z dxdy$ and $(Q_z + \frac{\partial Q_z}{\partial z} dz) dxdy$, respectively.

where Q_x , Q_y , and Q_z are heat transfer in the x , y , and z directions per unit time per unit area by thermal conduction.

Therefore, the net heat transfer by conduction into the fluid element is:

$$-\left[\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z}\right]dxdydz. \quad (3.54)$$

Similarly, the net heat transfer by radiation into the fluid element is:

$$-\left[\frac{\partial q_r}{\partial x} + \frac{\partial q_r}{\partial y} + \frac{\partial q_r}{\partial z}\right]dxdydz,$$

or

$$-\nabla q_r dxdydz, \quad (3.55)$$

where q_r is thermal radiation.

Applying Fourier's law of heat conduction in Equation (3.54), i. e., the heat transfer due to thermal conduction is proportional to the local temperature gradient: $Q_x = -k\frac{\partial T}{\partial x}$, $Q_y = -k\frac{\partial T}{\partial y}$, $Q_z = -k\frac{\partial T}{\partial z}$, where k is the thermal conductivity. Equation (3.54) is reduced to:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right),$$

or

$$\nabla \cdot (\nabla kT). \quad (3.56)$$

In nanofluid flow, Brownian motion and thermophoretic diffusions of nanoparticles are the main contributors of mass and energy transfer of nanofluids. By considering figure (3.2), the total nanoparticles mass diffusion flux in x , y , z -directions, respectively, are:

$$-\rho_p \left[D_B \frac{\partial C}{\partial x} + \frac{D_T}{T} \frac{\partial T}{\partial x} \right] dxdydz, -\rho_p \left[D_B \frac{\partial C}{\partial y} + \frac{D_T}{T} \frac{\partial T}{\partial y} \right] dxdydz, \text{ and } -\rho_p \left[D_B \frac{\partial C}{\partial z} + \frac{D_T}{T} \frac{\partial T}{\partial z} \right] dxdydz.$$

Thus, the heat transfer due to the effects of Brownian motion and thermophoresis in the nanofluid flow is:

$$\rho_p c_{np} \left[D_B \nabla C + \frac{D_T}{T} \nabla T \right] \cdot \nabla T dxdydz. \quad (3.57)$$

The term B in Equation (3.51) is the sum of Equations (3.53), (3.55), (3.56) and (3.57).

$$B = \left[\rho Q + \nabla \cdot (\nabla kT) - \nabla q_r + \rho_p c_{np} \left(D_B \nabla C + \frac{D_T}{T} \nabla T \right) \cdot \nabla T \right] dxdydz. \quad (3.58)$$

Finally, the term A in equation (3.51) denotes the time rate of change of internal energy of the fluid element. Let e be the internal energy of the nanofluid. Then the rate of change of the internal energy per unit mass is

$$A = \rho \frac{de}{dt} dxdydz. \quad (3.59)$$

The final form of the energy equation is obtained by substituting Eqs. (3.52), (3.58), and (3.59) into (3.51), obtaining

$$\begin{aligned}
\rho \frac{de}{dt} = & \rho Q + \nabla \cdot (\nabla k T) - \nabla q_r + \rho_p c_{np} \left(D_B \nabla C + \frac{D_T}{T} \nabla T \right) \cdot \nabla T - \nabla \cdot (pq) + \frac{\partial(u\tau_{xx})}{\partial x} \\
& + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(v\tau_{zx})}{\partial z} + \frac{\partial(u\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(u\tau_{xz})}{\partial x} + \frac{\partial(u\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \\
& + \rho f \cdot q.
\end{aligned} \tag{3.60}$$

Since $e = c_p T$, equation (3.60) can be written as

$$\begin{aligned}
\rho c_p \frac{dT}{dt} = & \rho Q + \nabla \cdot (\nabla k T) - \nabla q_r + \rho_p c_{np} \left(D_B \nabla C + \frac{D_T}{T} \nabla T \right) \cdot \nabla T - \nabla \cdot (pq) + \frac{\partial(u\tau_{xx})}{\partial x} \\
& + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \\
& + \rho f \cdot q.
\end{aligned} \tag{3.61}$$

Work done by body force $\rho f \cdot q$ is due to the gravitational force and magnetic field force. And the impact of gravitational force g on work done $\rho f \cdot q$ is meager, the only body force which has a significant effect on the work done $\rho f \cdot q$ is the magnetic field. Thus, $\rho f \cdot q = \frac{J^2}{\rho}$. Using $\tau_{xx} = -\frac{2}{3} \nabla \cdot q + 2\mu \frac{\partial u}{\partial x}$, $\tau_{yy} = -\frac{2}{3} \nabla \cdot q + 2\mu \frac{\partial v}{\partial y}$, $\tau_{zz} = -\frac{2}{3} \nabla \cdot q + 2\mu \frac{\partial w}{\partial z}$, $\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$, $\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$, $\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$, and $\rho f \cdot q = \frac{J^2}{\rho}$ in equation (3.61)

$$\rho c_p \frac{dT}{dt} = -\nabla \cdot (pq) + \nabla \cdot (\nabla k T) + \rho_p c_{np} \left[D_B \nabla C + \frac{D_T}{T} \nabla T \right] \cdot \nabla T + \rho Q - \nabla q_r + \mu \Phi + \frac{J^2}{\rho}, \tag{3.62}$$

where

$$\begin{aligned}
\Phi = & 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] \\
& - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2.
\end{aligned}$$

4. Concentration Equation

The nanoparticles concentration in the base fluid is the dependent variable and is a function of independent variables: the space coordinate (x, y, z) and time t .

The mass balance in Cartesian coordinate (x, y, z) on an incremental volume $(\Delta x) (\Delta y) (\Delta z)$

in the presence of exothermic chemical reaction is:

$$\left\{ \begin{array}{c} \text{Rate of change of} \\ \text{concentration} \\ \text{inside a fluid element} \end{array} \right\} = \left\{ \begin{array}{c} \text{Net mass flux} \\ \text{due to} \\ \text{slip mechanism} \end{array} \right\} + \left\{ \begin{array}{c} \text{mass flux due to} \\ \text{concentration} \\ \text{gradient} \end{array} \right\} \pm R$$

or

$$\begin{aligned} \rho_p \Delta x \Delta y \Delta z \frac{\partial C}{\partial t} = & \Delta y \Delta z [x H_m]_x - \Delta y \Delta z [(x + \Delta x) H_m]_{x+\Delta x} + \Delta x \Delta z [y H_m]_y - \Delta x \Delta z [(y + \Delta y) H_m]_{y+\Delta y} \\ & + \Delta x \Delta y [z H_m]_z - \Delta x \Delta y [(z + \Delta z) H_m]_{z+\Delta z} + u \Delta y \Delta z [C]_{x-\Delta x} - \Delta y \Delta z [C]_x + v \Delta x \Delta z [C]_{y-\Delta y} - \Delta x \Delta z [C]_y \\ & + w \Delta x \Delta y [C]_{z-\Delta z} - w \Delta x \Delta y [C]_z \pm R \end{aligned}$$

where H_m is the mass flux and R is a source term which represents the contribution of chemical reaction. Since the nanofluid undergo first order exothermic reaction, $\pm R = -\epsilon \rho_p \Delta C$ $\Delta x \Delta y \Delta z$.

Dividing both sides by $\Delta x \Delta y \Delta z$,

$$\begin{aligned} \rho_p \frac{\partial C}{\partial t} = & - \left[\frac{[(x + \Delta x) H_m]_{x+\Delta x} - [x H_m]_x}{\Delta x} \right] - \left[\frac{[(y + \Delta y) H_m]_{y+\Delta y} - [y H_m]_y}{\Delta y} \right] \\ & - \left[\frac{[(z + \Delta z) H_m]_{z+\Delta z} - [z H_m]_z}{\Delta z} \right] - u \rho_p \left(\frac{[C]_x - [C]_{x-\Delta x}}{\Delta x} \right) \\ & - v \rho_p \left(\frac{[C]_y - [C]_{y-\Delta y}}{\Delta y} \right) - w \rho_p \left(\frac{[C]_z - [C]_{z-\Delta z}}{\Delta z} \right) - \epsilon \rho_p \Delta C \end{aligned}$$

In the limit Δx , Δy , and Δz tend to zero,

$$\rho_p \frac{\partial C}{\partial t} = -\frac{\partial H_m}{\partial x} - \frac{\partial H_m}{\partial y} - \frac{\partial H_m}{\partial z} - u \rho_p \frac{\partial C}{\partial x} - v \rho_p \frac{\partial C}{\partial y} - w \rho_p \frac{\partial C}{\partial z} - \epsilon \rho_p \Delta C. \quad (3.63)$$

$$\rho_p \frac{\partial C}{\partial t} = -\nabla \cdot H_m - \rho_p q \cdot \nabla C - \epsilon \rho_p \Delta C. \quad (3.64)$$

Assuming the mass flux H_m is due to Brownian motion and thermophoresis effects, then

$$H_m = -\rho_p D_B \nabla C - \rho_p D_T \frac{\nabla T}{T_o} \quad (3.65)$$

Putting (3.65) in to (3.64), the concentration equation will take the form

$$\frac{\partial C}{\partial t} + q \cdot \nabla C = \nabla \cdot \left[D_B \nabla C + D_T \frac{\nabla T}{T_o} \right] - \epsilon \Delta C \quad (3.66)$$

ii. Single Phase Modeling Equations

1. Continuity Equation

The equation for conservation of mass, or continuity equation, can be written as follows:

$$\frac{\partial \rho_{nf}}{\partial t} + \nabla \cdot (\rho_{nf} q) = 0. \quad (3.67)$$

Equation (3.4) is the general form of the mass conservation equation, and is valid for compressible flows.

For incompressible fluids, the density ρ is constant in space and time, therefore, the equation reduces to

$$\nabla \cdot q = 0. \quad (3.68)$$

2. Momentum Equation

Conservation of momentum for a viscous compressible MHD nanofluid in an inertial (non-accelerating) reference frame is described in vector form by

$$\begin{aligned} \rho_{nf} \left[\frac{\partial q}{\partial t} + (q \cdot \nabla) q \right] = & -\nabla P + \frac{4}{3} \nabla (\mu_{nf} \nabla \cdot q) + \nabla (q \cdot \nabla \mu_{nf}) - q \nabla^2 \mu_{nf} + \nabla \mu_{nf} \times (\nabla \times q) \\ & - (\nabla \cdot q) \nabla \mu_{nf} - \nabla \times (\nabla \times \mu_{nf} q) + \rho_{nf} g + J \times B. \end{aligned} \quad (3.69)$$

Note that the vector equation (3.49) represents the three components of the momentum equation of motion. These three x, y, and z components are

x-momentum:

$$\begin{aligned} \rho_{nf} \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = & -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left[\mu_{nf} \left(2 \frac{\partial u}{\partial x} - \frac{2}{3} \nabla \cdot q \right) \right] + \frac{\partial}{\partial y} \left[\mu_{nf} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\ & + \frac{\partial}{\partial z} \left[\mu_{nf} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] + \rho_{nf} g_x + (J \times B)_x, \end{aligned} \quad (3.70)$$

y-momentum:

$$\begin{aligned} \rho_{nf} \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = & -\frac{\partial P}{\partial y} + \frac{\partial}{\partial y} \left[\mu_{nf} \left(2 \frac{\partial v}{\partial y} - \frac{2}{3} \nabla \cdot q \right) \right] + \frac{\partial}{\partial z} \left[\mu_{nf} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \\ & + \frac{\partial}{\partial x} \left[\mu_{nf} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \rho_{nf} g_y + (J \times B)_y, \end{aligned} \quad (3.71)$$

z-momentum:

$$\begin{aligned} \rho_{nf} \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = & -\frac{\partial P}{\partial z} + \frac{\partial}{\partial z} \left[\mu_{nf} \left(2 \frac{\partial w}{\partial z} - \frac{2}{3} \nabla \cdot q \right) \right] + \frac{\partial}{\partial x} \left[\mu_{nf} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \\ & + \frac{\partial}{\partial y} \left[\mu_{nf} \left(\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} \right) \right] + \rho_{nf} g_z + (J \times B)_z. \end{aligned} \quad (3.72)$$

Deduction of Momentum Equations for Some Particular Cases:

Case (a): For Constant Viscosity: For this case

$$\begin{aligned} \nabla \mu_{nf} &= 0 \\ \text{and} \end{aligned}$$

$$\begin{aligned} \nabla \times (\nabla \times \mu_{nf} q) &= \nabla(\nabla \cdot \mu_{nf} q) - \nabla \cdot \nabla \mu_{nf} q \\ &= \mu_{nf} \nabla(\nabla \cdot q) - \mu_{nf} \nabla^2 q. \end{aligned}$$

Equation (3.49) may be expressed as

$$\rho_{nf} \left[\frac{\partial q}{\partial t} + (q \cdot \nabla) q \right] = -\nabla P + \frac{1}{3} \mu_{nf} \nabla(\nabla \cdot q) + \mu_{nf} \nabla^2 q + \rho_{nf} g + J \times B. \quad (3.73)$$

Case (b): For Constant Density: The continuity equation for incompressible fluid is given by $\nabla \cdot q = 0$.

Since $\nabla \times (\nabla \times \mu_{nf} q) = -\nabla(q \cdot \nabla \mu_{nf}) + q \nabla^2 \mu_{nf}$ and $\nabla \mu_{nf} \times (\nabla \times q) = -\nabla \mu_{nf} \cdot \nabla q$, equation (3.49) reduces to

$$\rho_{nf} \left[\frac{\partial q}{\partial t} + (q \cdot \nabla) q \right] = -\nabla P - \nabla \mu_{nf} \cdot \nabla q + \rho_{nf} g + J \times B. \quad (3.74)$$

Case (c): Constant Viscosity and Density: The continuity equation for incompressible fluid is given by $\nabla \cdot q = 0$. Equation (3.49) reduces to

$$\rho_{nf} \left[\frac{\partial q}{\partial t} + (q \cdot \nabla) q \right] = -\nabla P + \mu_{nf} \nabla^2 q + \rho_{nf} g + J \times B. \quad (3.75)$$

3. Energy Equation

$$(\rho c_p)_{nf} \left[\frac{\partial T}{\partial t} + (q \cdot \nabla) T \right] = \nabla \cdot (k_{nf} \nabla T) + (\rho c_p)_f \left[D_B \nabla \phi \cdot \nabla T + D_T \frac{\nabla T \cdot \nabla T}{T} \right] + \mu_{nf} \Phi, \quad (3.76)$$

where

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2.$$

Effective Nanofluid Thermophysical Properties

The basic coolant mixture properties are functions of nanoparticle volume fraction ϕ and mixture temperature T . Nanoparticles are assumed to be spherical, monodisperse, and forming a homogeneous dilute suspension, although the effects of nanoparticle aggregation on k_{nf} was considered. The following set of equations represent the effective thermophysical properties of nanofluids (Brinkman, 1952; Hayat et al, 2016; Monaledi and Makinde, 2018). The subscript nf , f , s and ϕ represent nanofluid, base fluid, nanoparticles and dimensionless nanoparticles volume fraction, respectively.

$$\begin{aligned} \mu_{nf} &= \frac{\mu_f}{(1-\phi)^{2.5}} e^{-\beta(T-T_o)}, \\ \beta_{nf} &= (1-\phi)\beta_f + \phi\beta_s, \\ \rho_{nf} &= (1-\phi)\rho_f + \phi\rho_s, \\ (\rho c_p)_{nf} &= (1-\phi)(\rho c_p)_f + \phi(\rho c_p)_s, \\ \frac{k_{nf}}{k_f} &= \frac{k_s - 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}, \\ \frac{\sigma_{nf}}{\sigma_f} &= 1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi}, \\ r &= \frac{\sigma_s}{\sigma_f}. \end{aligned} \tag{3.77}$$

3.2.2 Numerical Solutions for Partial Differential Equations (PDEs)

A wide variety of PDEs arise when modeling and describing physical phenomenon of dynamical systems in the fields of physics, biology, engineering and geology. For example, various Laplace's PDEs arise in the problems of transport phenomena (Bird et al., 2006), a Turing model in describing the combination of chemical reaction and diffusion (Turing, 1952) and governing equations of fluid motions. In particular, a system of nonlinear coupled PDEs obtained in modeling the physical problem of MHD nanofluid flow in a permeable microchannel seen in section 3.2. Several techniques have been utilized to get the analytical solution of linear and nonlinear PDEs, for instance homotopy perturbation method, variation iteration method, adomain decomposition method and homotopy method. However, it is difficult to find analytical solutions for most PDEs, thus there is a need to look for efficient numerical methods to solve these PDEs due to the necessity of PDEs solutions in science

and engineering. In this line, according to (Nofal, 2011) the system of PDEs resulting from flows in tubes, channels and parallel plates may have an exact solutions for some simple cases, but most fluid mechanics problems have no exact solutions in general. So, a number of numerical methods such as explicit and implicit finite difference method, Galerkin finite element method, and finite volume method are developed and used to solve PDEs obtained from fluid mechanics.

The successfulness of a given numerical method in solving PDEs relies on the issues of consistency, stability and convergence of the method. The method of line is one such an efficient numerical routine to deal with the solutions of transient flow problems. Since the momentum equations are parabolic PDEs in general, the method of lines are appropriate (Machura and Sweet, 1980). The application of this method is not only limited to solve the parabolic PDEs but it is also helpful to tackle elliptic PDEs (Jones, 1972; Jones et al., 1972). In the method of line, the spatial independent variables are discretized using any number of techniques such as finite difference, finite element, finite volume, or collocation methods (Berzins et al., 1989; Cutlip and Shacham, 1998; Schiesser and Griffiths, 2009; Schiesser and Silebi, 1997) to obtain a system of ordinary differential equations (ODEs) in time. The resulting ODEs are then integrated by an appropriate ODE solvers such as FORTRAN, Mathematica, Matlab, or MAPLE, etc. In solving the resulting system of ODEs in time variable, Jameson and Baker (1983) have suggested to use an explicit four-point Runge-Kutta method whereas Beam and Warming (1977) have suggested to use implicit Euler method. The current study used centered finite difference approximation to space variables combined with Runge-Kutta-Fehlberg time integration scheme.

The simplicity and versatility of the method of lines has made it to be applicable in a number of engineering applications including heat transfer problems (Labuzov and Potapov, 1985), solving Navier-Stoke (Ersahin et al., 2004) and electromagnetic equations (Sadiku and Obiozor, 2000). Further, Makinde et al. (2012) used the method of line (semi-discretization via RKFIT) to solve a system of PDEs that characterize the flow of nanofluids in channel. The basic concept in the method of line to solve PDE is to replace the spatial derivatives of the PDE with algebraic approximations obtained from finite difference or finite element. By doing that, the original PDE is approximated by a system of ODEs in time variable. This approximation leaves the time variable continuous and it is called semi-discretization. The solutions of system ODEs can be computed using the existing well-established numerical methods for ODEs. This is the significant feature of MOL. As a result, the MOL computation follows two steps: first, spatial derivatives are approximated using, for instance, finite difference or finite element techniques. Second, the resulting system of semi-discrete ODEs in the initial value variable is integrated in time t .

To describe the basic elements of MOL, consider the following PDE bellow:

$$\frac{\partial u}{\partial t} = f\left(x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right), \text{ for } t \geq 0, \quad 0 \leq x \leq 1, \quad (3.78)$$

under boundary conditions,

$$u(t, 0) = \lambda_1, \quad u(t, 1) = \lambda_2, \quad (3.79)$$

and initial condition

$$u(0, x) = \omega(x). \quad (3.80)$$

Spatial Discretization

In the MOL approach, first the spatial derivatives in equation (3.78) are discretized. Let the space interval $[0, 1]$ be subdivided into $N + 1$ equal sub-intervals with mesh points

$$x_i = i\Delta x, \quad i = 0, 1, 2, \dots, N + 1, \quad \Delta x = \frac{1}{N + 1}, \quad (3.81)$$

and the spatial derivatives be replaced by centered difference approximations, and let $u(x_i, t)$ be approximated by u_i . Then the PDE (3.78) is reduced to a system of ODEs

$$\frac{du_i}{dt} = f\left(x_i, t, u_i, \frac{u_{i+1} - u_{i-1}}{2\Delta x}, \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}\right),$$

or

$$\frac{du_i}{dt} = f(t, u_{i+1}, u_i, u_{i-1}), \quad i = 1, 2, \dots, N. \quad (3.82)$$

Here the terms $u_0(t)$ and $u_{N+1}(t)$ may be obtained from the boundary conditions, Equation (3.79) (i.e., $u_0(t) = \lambda_1$ and $u_{N+1}(t) = \lambda_2$). There are many alternatives like higher order or upwind differencing in the process of spatial discretization.

Time Integration

Numerical integration of the system of ODEs (3.82) subjected to the initial conditions,

$$u_i(0) = \omega(x_i), \quad i = 1, 2, \dots, N, \quad (3.83)$$

is a simple task using an appropriate ODE integrator. In other words, combining equations (3.82) and (3.83) it is found a system of N coupled ODEs in N variables $u_i(t)$ with associated initial condition in (3.83) and the solution can be obtained by any well-established numerical methods for solving the system of ODEs.

CHAPTER 4

MHD VARIABLE VISCOSITY MIXED CONVECTION OF NANOFLUID IN A MICROCHANNEL WITH PERMEABLE WALLS

The present work investigates unsteady mixed convection of MHD nanofluid flow through a parallel-plate's microchannel with wall's suction and injection. This work assumed that the dynamic viscosity of the nanofluid is exponentially decaying function of fluid's temperature and that the parallel-plate's wall temperature and their nearby species concentration are held asymmetrical. The Buongiorno flow model, MHD model equations, Boussinesq approximation and scaling analysis are utilized to formulate the flow model of the problem. The governing model equations in partial differential equations form are solved numerically by semi-discretization finite difference method with Runge-Kutta-Fehlberg integration scheme. Graphical results on the effects of parameter variation on velocity, temperature, concentration, skin friction, Nusselt number and Sherwood number are presented and discussed.

4.1 Mathematical Formulations and Analysis

4.1.1 The Physical Problem

Consider unsteady flow of an incompressible, temperature dependent variable viscosity MHD nanofluid in a vertical parallel-plate microchannel of width a units with permeable walls, where the left wall is kept at temperature of T_o and the right wall at T_1 such that $T_o < T_1$. A uniform transverse magnetic field B_o is applied in the direction parallel to $\bar{y}$ -axis. The schematic diagram is depicted in figure 4.1. It is assumed that a 2D-coordinate system is used, wherein the left wall is fixed at $\bar{y} = 0$, the right wall is fixed at $\bar{y} = a$ (the $\bar{x}$ -axis is aligned vertically) and the $\bar{y}$ -axis is perpendicular to the wall.

The nanofluid flow within the microchannel is induced due to the combined effect of pressure gradient and the effect of buoyancy force. It is also considered that the magnetic Reynolds number is so meager that the induced magnetic field is ignored as compared to the imposed magnetic field.

Besides the above considerations and assumptions, the mathematical equations describing the flow geometry are also based on the following assumptions:

- i. Thermophysical properties of fluid except density in buoyancy force (Boussinesq hypothesis) and dynamic viscosity are constant,

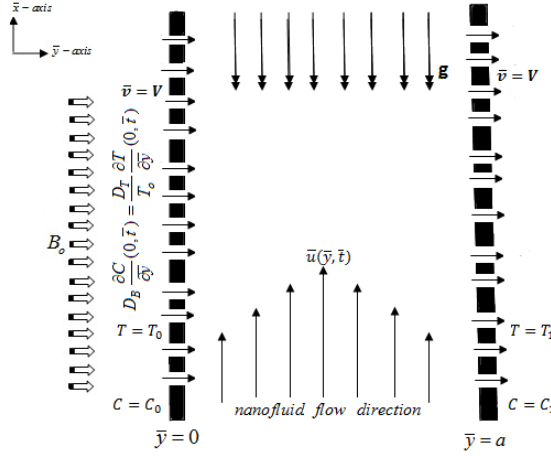


Figure 4.1: Flow geometry with coordinate system

- ii. The fluid phase and the nanoparticles are in thermal equilibrium state and thus, the heat flow has been described using one equation model,
- iii. Nanoparticles are spherical in shape,
- iv. Nanofluid is electrically conducting, Newtonian and laminar flow,
- v. The vertical parallel-plates microchannel walls are perfectly thermal conducting,
- vi. Mass transfer due to Brownian motion and thermophoretic effect at the left wall are equal,
- vii. Negligible radiation energy transfer, nuclear, and chemical reaction energy, and
- viii. Parallel stream lines.

Governing Equations of the Physical Problem

The basic MHD nanofluid flow equations that govern the above described flow configuration are as follows:

1. **Equation of Continuity:** This equation represents the conservation of mass so the electromagnetic field have no effect on it. Hence, the equation of continuity for incompressible fluid is given by

$$\nabla \cdot q = 0. \quad (4.1)$$

Since the fluid velocity $q = (\bar{u}, V, 0)$, where V is constant, equation (4.1) becomes

$$\frac{\partial \bar{u}}{\partial \bar{x}} = 0. \quad (4.2)$$

2. **Momentum Equation:** The momentum equation is derived from the principle of conservation of linear momentum. Thus, the equation of momentum for unsteady flow of temperature dependent viscosity, incompressible MHD nanofluid fluid in vector form is:

$$\rho \left[\frac{\partial q}{\partial t} + (q \cdot \nabla)q \right] = -\nabla P - \nabla \mu \cdot \nabla q + \rho g + J \times B \quad (4.3)$$

where

P is the pressure,

ρg is body force due to gravity

$J \times B$ is body force due to magnetic field.

Since $J = \sigma(E + q \times B)$ by Ohm's law, $B = (0, B_o, 0)$, $q = (\bar{u}, V, 0)$, and $E = 0$, it is obtained that $q \times B = (0, 0, B_o)$, $J = (0, 0, \sigma B_o \bar{u})$, and $J \times B = (-\sigma B_o^2 \bar{u}, 0, 0)$, where E is the electric field acting on a fluid at rest, $q \times B$ is the electric field due to a fluid moving with velocity q is in the presence of magnetic field B , and σ is the electrical conductivity of the fluid.

Substituting $(-\sigma B_o^2 \bar{u}, 0, 0)$ for $J \times B$ and using equation (4.2) in equation (4.3), the momentum equation reduced in to x- and y-momentum as follows:

x-momentum:

$$\rho \left[\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} \right] = -\frac{\partial P}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} \left[\mu(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \sigma B_o^2 \bar{u} - \rho g, \quad (4.4)$$

y-momentum:

$$0 = -\frac{\partial P}{\partial \bar{y}}. \quad (4.5)$$

But, $\frac{\partial P}{\partial \bar{y}} = 0$ which implies $P=P(\bar{x})$, pressure is a function of $\bar{x}$ only. Thus, the momentum equation of the problem is:

$$\rho \left[\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} \right] = -\frac{\partial P}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} \left[\mu(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \sigma B_o^2 \bar{u} - \rho g \quad (4.6)$$

Applying *Boussinesq approximation*, which states that density variation with temperature and concentration is important only for body force term but not for other terms, in equation (4.6) we have

$$\rho \left[\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} \right] = -\frac{\partial P}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} \left[\mu(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \sigma B_o^2 \bar{u} - \rho(T, C)g. \quad (4.7)$$

For $\rho=\rho(T, C)$, using Taylor's series expansion about (T_o, C_o) gives

$$\rho(T, C) = \rho(T_o, C_o) + \frac{\partial \rho}{\partial T}(T - T_o) + \frac{\partial \rho}{\partial C}(C - C_o) + \frac{\partial^2 \rho}{\partial T^2}(T - T_o)^2 + \frac{\partial^2 \rho}{\partial C^2}(C - C_o)^2 + \dots$$

Neglecting square and higher order terms of $(T - T_o)$ and $(C - C_o)$, we have that

$$\rho - \rho_o = \frac{\partial \rho}{\partial T}(T - T_o) + \frac{\partial \rho}{\partial C}(C - C_o). \quad (4.8)$$

This is *Boussinesq approximation* with $\beta_1 = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)$ and $\beta_2 = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial C} \right)$.

At $\bar{y} = 0$, equation (4.7) becomes

$$0 = \rho_o g \quad (4.9)$$

Subtracting (4.9) from (4.6), it obtained that

$$\rho \left[\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} \right] = -\frac{\partial P}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} \left[\mu(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \sigma B_o^2 \bar{u} - (\rho - \rho_o)g. \quad (4.10)$$

Substituting equation (4.8) in to (4.10) and using β_1 and β_2 in (4.10), the momentum equation is

$$\boxed{\left[\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} \right] = -\frac{1}{\rho} \frac{\partial P}{\partial \bar{x}} + \frac{1}{\rho} \frac{\partial}{\partial \bar{y}} \left[\mu(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \frac{\sigma}{\rho} B_o^2 \bar{u} + \beta_1 g(T - T_o) + \beta_2 g(C - C_o).} \quad (4.11)$$

3. **Equation of Energy:** The equation of energy is derived from the law of conservation of angular momentum and hence equation of energy for incompressible constant conductivity of MHD nanofluid is given by

$$\rho c_p \left[\frac{\partial T}{\partial \bar{t}} + (q \cdot \nabla) T \right] = k \nabla^2 T + \rho c_p \left[D_B \nabla C \cdot \nabla T + \frac{D_T}{T_o} \nabla T \nabla T \right] + \mu \Phi + \frac{J^2}{\sigma}. \quad (4.12)$$

Equation (4.12) can also be written as:

$$\begin{aligned} \rho c_p \left[\frac{\partial T}{\partial \bar{t}} + \bar{u} \frac{\partial T}{\partial \bar{x}} + V \frac{\partial T}{\partial \bar{y}} \right] &= k \left[\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right] + \rho c_p \left[D_B \left(\frac{\partial C}{\partial \bar{x}} \frac{\partial T}{\partial \bar{x}} + \frac{\partial C}{\partial \bar{y}} \frac{\partial T}{\partial \bar{y}} \right) \right. \\ &\quad \left. + \frac{D_T}{T_o} \left[\left(\frac{\partial T}{\partial \bar{x}} \right)^2 + \left(\frac{\partial T}{\partial \bar{y}} \right)^2 \right] + \mu \Phi + \frac{J^2}{\sigma} \right]. \end{aligned} \quad (4.13)$$

Since $J = (0, 0, \sigma B_o \bar{u})$, then $J^2 = J \cdot J = \sigma^2 B_o^2 \bar{u}^2$, $\frac{J^2}{\sigma} = \sigma B_o^2 \bar{u}^2$ and equation (4.13) is reduced to

$$\begin{aligned} \rho c_p \left[\frac{\partial T}{\partial \bar{t}} + \bar{u} \frac{\partial T}{\partial \bar{x}} + V \frac{\partial T}{\partial \bar{y}} \right] &= k \left[\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right] + \rho c_p \left[D_B \left(\frac{\partial C}{\partial \bar{x}} \frac{\partial T}{\partial \bar{x}} + \frac{\partial C}{\partial \bar{y}} \frac{\partial T}{\partial \bar{y}} \right) \right. \\ &\quad \left. + \frac{D_T}{T_o} \left[\left(\frac{\partial T}{\partial \bar{x}} \right)^2 + \left(\frac{\partial T}{\partial \bar{y}} \right)^2 \right] + \mu \Phi + \sigma B_o^2 \bar{u}^2 \right]. \end{aligned} \quad (4.14)$$

Scaling analysis

Let $X = Y \frac{\bar{x}}{a}$, $Y = \frac{y}{a}$, where $0 < Y = \frac{a}{L} \ll 1$, L is the length of the microchannel such that $a \ll L$. Then $\frac{\partial}{\partial \bar{x}} = \frac{Y}{a} \frac{\partial}{\partial X}$, $\frac{\partial}{\partial \bar{y}} = \frac{1}{a} \frac{\partial}{\partial Y}$, $\frac{\partial^2}{\partial \bar{x}^2} = \frac{Y^2}{a^2} \frac{\partial^2}{\partial X^2}$ and $\frac{\partial^2}{\partial \bar{y}^2} = \frac{1}{a^2} \frac{\partial^2}{\partial Y^2}$. Since $Y \ll 1$, $\frac{\partial}{\partial \bar{x}} \ll \frac{\partial}{\partial \bar{y}}$ and $\frac{\partial^2}{\partial \bar{x}^2} \ll \frac{\partial^2}{\partial \bar{y}^2}$. It is possible to neglect $\frac{\partial}{\partial \bar{x}}$ and $\frac{\partial^2}{\partial \bar{x}^2}$ terms in the energy equation. Equation (4.14) simplified to

$$\rho c_p \left[\frac{\partial T}{\partial \bar{t}} + V \frac{\partial T}{\partial \bar{y}} \right] = k \left[\frac{\partial^2 T}{\partial \bar{y}^2} \right] + \rho c_p \left[D_B \frac{\partial C}{\partial \bar{y}} \frac{\partial T}{\partial \bar{y}} + \frac{D_T}{T_o} \left(\frac{\partial T}{\partial \bar{y}} \right)^2 \right] + \mu \Phi + \sigma B_o^2 \bar{u}^2. \quad (4.15)$$

4. Equation of Nanoparticles Concentration: The balance for transport for nanoparticles concentration in constant-density fluid is described by:

$$\frac{\partial C}{\partial \bar{t}} + (q \cdot \nabla) C = D_B \nabla^2 C + \frac{D_T}{T_o} \nabla^2 T, \quad (4.16)$$

$$\frac{\partial C}{\partial \bar{t}} + \bar{u} \frac{\partial C}{\partial \bar{x}} + V \frac{\partial C}{\partial \bar{y}} = D_B \left(\frac{\partial^2 C}{\partial \bar{x}^2} + \frac{\partial^2 C}{\partial \bar{y}^2} \right) + \frac{D_T}{T_o} \left(\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right). \quad (4.17)$$

After scaling analysis

$$\frac{\partial C}{\partial \bar{t}} + V \frac{\partial C}{\partial \bar{y}} = D_B \frac{\partial^2 C}{\partial \bar{y}^2} + \frac{D_T}{T_o} \frac{\partial^2 T}{\partial \bar{y}^2}. \quad (4.18)$$

Using Buongiorno nanofluid flow model, MHD model equation and the usual Oberbeck-Boussinesq approximation, the governing equations for continuity, momentum, energy and concentration are given as:

$$\frac{\partial \bar{u}}{\partial \bar{x}} = 0, \quad (4.19)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{\rho} \frac{\partial}{\partial \bar{y}} \left[\mu(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \frac{\sigma}{\rho} B_o^2 \bar{u} + \beta_1 g(T - T_o) + \beta_2 g(C - C_o), \quad (4.20)$$

$$\frac{\partial T}{\partial \bar{t}} + V \frac{\partial T}{\partial \bar{y}} = \alpha \frac{\partial^2 T}{\partial \bar{y}^2} + \tau \left[D_B \frac{\partial C}{\partial \bar{y}} \frac{\partial T}{\partial \bar{y}} + \frac{D_T}{T_o} \left(\frac{\partial T}{\partial \bar{y}} \right)^2 \right] + \frac{\mu(T)}{\rho c_p} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \frac{\sigma B_o^2}{\rho c_p} \bar{u}, \quad (4.21)$$

$$\frac{\partial C}{\partial \bar{t}} + V \frac{\partial C}{\partial \bar{y}} = D_B \frac{\partial^2 C}{\partial \bar{y}^2} + \frac{D_T}{T_o} \frac{\partial^2 T}{\partial \bar{y}^2}, \quad (4.22)$$

with the initial and boundary conditions

$$\bar{u}(\bar{y}, 0) = 0, \quad T(\bar{y}, 0) = T_o, \quad C(\bar{y}, 0) = C_o, \quad (4.23)$$

$$\begin{aligned}\bar{u}(0, \bar{t}) = 0, \quad T(0, \bar{t}) = T_o, \quad D_B \frac{\partial C}{\partial \bar{y}}(0, \bar{t}) = -\frac{\partial T}{\partial \bar{y}}(0, \bar{t}), \\ \bar{u}(a, \bar{t}) = 0, \quad T(a, \bar{t}) = T_1, \quad C(a, \bar{t}) = C_1,\end{aligned}\quad (4.24)$$

where $\bar{u}$ is the nanofluid velocity in $\bar{x}$ -direction, T is the temperature of nanofluid, C is the nanoparticles concentration, V is the suction/injection velocity, P is the nanofluid pressure, $\bar{t}$ is the time, ρ is the density of nanofluid, μ is the viscosity of the nanofluid, g is the acceleration due to gravity, β_1 and β_2 are the thermal and solutal expansion coefficients respectively, α is the thermal diffusion coefficient, τ is the ratio between the effective heat capacity of nanoparticle material and heat capacity of the fluid, D_T is the thermophoretic diffusion coefficient, T_o is the temperature of the left wall, T_1 is the temperature of the right wall, σ is the electrical conductivity, a is the channel width, c_p is the specific heat at constant pressure. Using Makinde and Franks (2014), the dynamic viscosity of the nanofluid is considered to be related with temperature exponentially such that

$$\mu(T) = \mu_o e^{-\gamma(T-T_o)}. \quad (4.25)$$

The above equations (4.20)-(4.24) become non-dimensional form by using and incorporating the following variables and parameters:

$$\begin{aligned}x = \frac{\bar{x}}{a}, y = \frac{\bar{y}}{a}, u = \frac{\bar{u}a}{v_o}, t = \frac{v_o \bar{t}}{a^2}, \theta = \frac{T - T_o}{T_1 - T_o}, \phi = \frac{C - C_o}{C_1 - C_o}, \\ P = \frac{\mu_o^2 \bar{P}}{\rho a^2}, Re = \frac{Va}{v_o}, A = -\frac{d\bar{p}}{dx}, A_1 = \gamma_1(T_1 - T_o), M^2 = \frac{\sigma B_o^2 a_o^2}{\mu_o}, \\ Gr = \frac{\beta_1 g a^3 (T_1 - T_o)}{v_o^2}, Gc = \frac{\beta_2 g a^3 (C_1 - C_o)}{v_o^2}, Pr = \frac{v_o}{\alpha}, Nb = \frac{\tau D_B (C_1 - C_o)}{v_o}, \\ Nt = \frac{\tau D_T (T_1 - T_o)}{T_o v_o}, Ec = \frac{v_o^2}{c_p a^2 (T_1 - T_o)}, Sc = \frac{v_o}{D_B}.\end{aligned}\quad (4.26)$$

Then the non-dimensional partial differential equations are:

$$\frac{\partial u}{\partial t} + Re \frac{\partial u}{\partial y} = A - A_1 \cdot e^{-A_1 \cdot \theta} \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial y} + e^{-A_1 \cdot \theta} \frac{\partial^2 u}{\partial y^2} - M^2 u + Gr \cdot \theta + Gc \cdot \phi, \quad (4.27)$$

$$\begin{aligned}Pr \frac{\partial \theta}{\partial t} + Pr \cdot Re \frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial y^2} + Pr \cdot Nb \frac{\partial \phi}{\partial y} \frac{\partial \theta}{\partial y} + Pr \cdot Nt \left(\frac{\partial \theta}{\partial y} \right)^2 + Pr \cdot Ec \left(\frac{\partial u}{\partial y} \right)^2 e^{-A_1 \theta} \\ + Pr \cdot M^2 \cdot Ec \cdot u^2,\end{aligned}\quad (4.28)$$

$$Sc \frac{\partial \phi}{\partial t} + Re \cdot Sc \frac{\partial \phi}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} + \frac{Nt}{Nb} \frac{\partial^2 \theta}{\partial y^2} \quad (4.29)$$

with initial and boundary conditions

$$u(y, 0) = 0, \theta(y, 0) = 0, \phi(y, 0) = 0, \quad (4.30)$$

$$\begin{aligned} u(0, t) = 0, \theta(0, t) = 0, Nb \frac{\partial \phi}{\partial y}(0, t) = -Nt \frac{\partial \theta}{\partial y}(0, t), \\ u(1, t) = 0, \theta(1, t) = 1, \phi(1, t) = 1, \end{aligned} \quad (4.31)$$

where Re is the Reynold suction/injection parameter, A is the pressure gradient parameter, A_1 is the viscosity variation parameter, M is the magnetic field parameter, Gr is the thermal Grashof number, Gc is the solutal/mass Grashof number, Pr is the Prandtl number, Nb is the Brownian motion parameter, Nt is the thermophoretic parameter, Ec is the Eckert number and Sc is the Schmidt number.

The important physical quantities of interest in the study of the problem are the local skin friction Cf_x , the local Nusselt number Nu_x , and the local Sherwood number Sh_x , where

$$Cf_x = \frac{\tau_w}{\rho V^2}, \quad Nu_x = \frac{aq_w}{k(T_1 - T_o)}, \quad \text{and} \quad Sh_x = \frac{aq_m}{D_B(C_1 - C_o)} \quad (4.32)$$

But $\tau_w = \mu \frac{\partial \bar{u}}{\partial \bar{y}}$, $q_w = -k \frac{\partial T}{\partial \bar{y}}$, and $q_m = -D_B \frac{\partial C}{\partial \bar{y}}$ at $\bar{y} = 0, a$.

The non-dimensional form of equations (4.32) above are:

$$Re^2 Cf_x = e^{-A_1 \theta} \frac{\partial u}{\partial y} \Big|_{at \ y=0,1}, \quad Nu_x = - \frac{\partial \theta}{\partial y} \Big|_{at \ y=0,1}, \quad \text{and} \quad Sh_x = - \frac{\partial \phi}{\partial y} \Big|_{at \ y=0,1} \quad (4.33)$$

4.2 Numerical Method

The model nonlinear equations (4.27)-(4.31) are a system of IBVPs and their solutions are obtained by using a semi-discretization via finite difference scheme with Runge-Kutta Fehlberg integration technique. Spatial interval $[0, 1]$ is subdivided into $N+1$ equal subintervals. The nodal spacing and nodal points are defined respectively as $\Delta y = \frac{1}{N+1}$ and $y_i = i\Delta y$, for $i = 1, 2, \dots, N$ where N is the number of interior nodal points in $[0, 1]$. The first and second spatial derivatives are replaced by a finite difference approximation of order $(\Delta y)^2$ accuracy. Let u_i , θ_i , and ϕ_i represent $u(y_i, t)$, $\theta(y_i, t)$, and $\phi(y_i, t)$, respectively. Then the semi-discretization scheme with centered finite difference method for equations (4.27)-(4.31) reduced to:

$$\begin{aligned} \frac{du_i}{dt} + Re \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) = A - A_1 e^{-A_1 \theta_i} \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) \\ + e^{-A_1 \theta_i} \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} \right) - M^2 u_i + Gr \theta_i + Gc \phi_i, \end{aligned} \quad (4.34)$$

$$Pr \frac{d\theta_i}{dt} + PrRe \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) = \left(\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta y)^2} \right) + PrNb \left(\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta y} \right) \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) + PrNt \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right)^2 + PrEc \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right)^2 e^{-A_1\theta_i} + PrM^2 Ec u_i^2, \quad (4.35)$$

$$Sc \frac{d\phi_i}{dt} + ReSc \left(\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta y} \right) = \left(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta y)^2} \right) + \frac{Nt}{Nb} \left(\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta y)^2} \right), \quad (4.36)$$

with initial conditions

$$u_i(0) = 0, \quad \theta_i(0) = 0, \quad \phi_i(0) = 0, \quad 1 \leq i \leq N. \quad (4.37)$$

The boundary conditions at $y = 0, 1$ are transformed to incorporate as follows:

$$u_o(t) = 0, \quad \theta_o(t) = 0, \quad Nb \frac{d\phi_o(t)}{dy} = 0 \quad \left(\frac{\phi_1(t) - \phi_o(t)}{\Delta y} = 0 \right), \quad (4.38)$$

$$u_{N+1}(t) = 0, \quad \theta_{N+1}(t) = 1, \quad \phi_{N+1}(t) = 1.$$

Equations (4.34)-(4.38) is a system of first order nonlinear ordinary differential equations with known initial conditions and its solution can be obtained successively by Runge-Kutta Fehlberg integration techniques (Na, 1979).

4.3 Results and Discussion

All the effects of magnetic field, temperature dependent viscosity, Brownian motion, thermophoresis, and mixed convective heat transfer on unsteady nanofluid flow in a microchannel with permeable walls are analyzed. The semi-discretization method via centered finite difference scheme with Runge-Kutta Fehlberg integration technique is used to solve numerically the governing nonlinear initial boundary value problems (IBVPs). Numerical solutions for the velocity profiles, temperature profiles, Nanoparticles concentration profiles, skin friction, Nusselt number and Sherwood number are carried out by assigning some specific values to various governing parameters of the nanofluid flow system. The results are shown in figures (4.2) - (4.24).

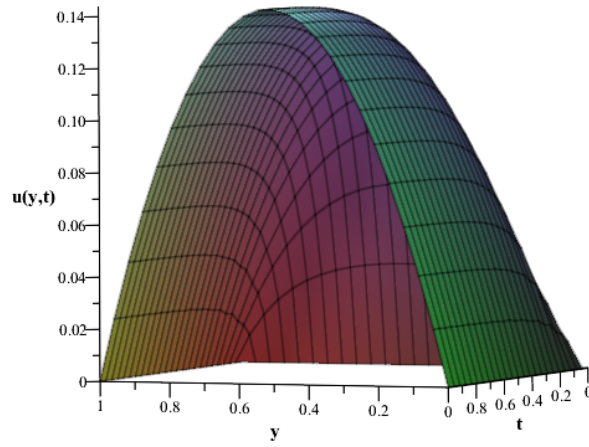
4.3.1 Transient Velocity, Temperature and Concentration Profiles

Figures (4.2)-(4.5) depict the transient effects on the velocity, temperature and concentration profiles for a fixed values of governing parameters. The fluid flow velocity increases from

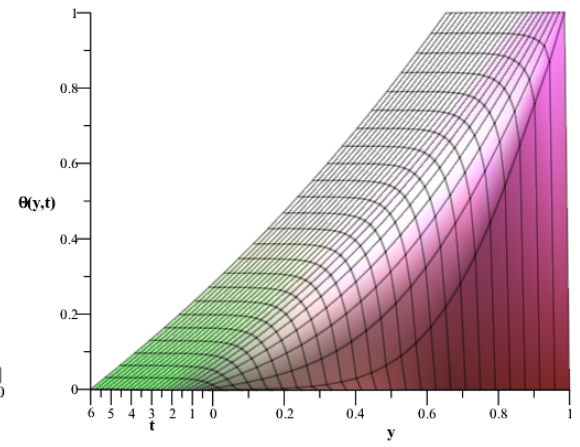
zero value in time and space at both left and right fixed plates to its maximum value within the channel as seen in figure 4.2 (a). Figure 4.2 (b) shows the fluid temperature rises from its zero value at the left plate to its maximum value at the right plate. The nanoparticles concentration of the fluid reduced from its maximum value in time and space near the left plate ($y = 0$) to its constant value at the right plate ($y = 1$) in figure 4.2 (c). Meanwhile, an increase time, rises the fluid velocity to its maximum value around the hub of the channel until a steady state is attained as illustrated in figures 4.3 (a) and (b). Moreover, the velocity reaches steady state quickly when the fluid is injected in to the channel ($Re < 0$) compared with suction ($Re > 0$). From figures 4.4 (a) and (b), it is observed that fluid temperature has noticeable value at the right plate ($y = 1$) within short period of time. As time increases, temperature of the fluid rises until the steady state is attained. It can be noticed from figures 4.5 (a) and (b) that nanoparticles concentration increases in the whole channel region for $t \leq 0.5$ and decreases for $t > 0.5$ until a steady state constant concentration value is attained at the right plate ($y = 1$). Moreover, there is a more fluctuation of concentration value near $y = 1$ for $0.1 \leq t \leq 0.5$. Again, from figures (4.3) - (4.5) it is clear that both velocity and concentration profiles reach their steady state faster than the temperature profiles. For instance, steady state temperature is attained at $t > 2$ while steady state velocity and concentration are attained at $t \geq 1.5$ and $t \geq 1.6$, respectively. This is expected, because the velocity and concentration operate as source of the temperature. The thermal energy is transported by the flow and the nanoparticles thermophoretic effect that result in enhanced thermal profile. Moreover, it can be clearly seen the model equation (4.21) that the fluid velocity, velocity gradient and concentration gradient with joule heating and viscous dissipation terms contribute additional heating source for the flow system.

4.3.2 The Steady State Velocity Profiles

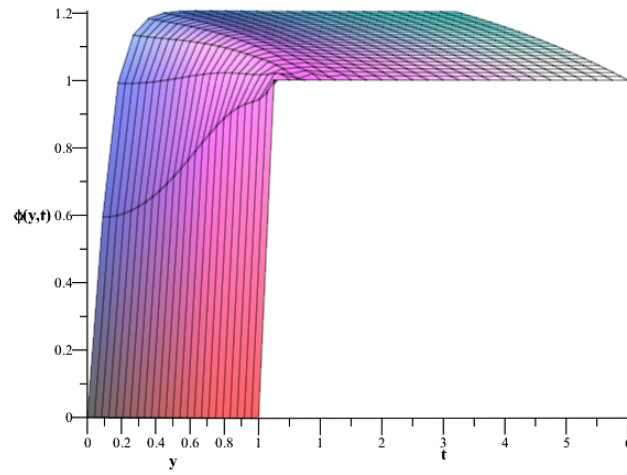
The effects of governing parameter variation on steady state velocity profile is illustrated in figures (4.6) - (4.9). Figures 4.6 (a) and (b) display the influence of suction/injection parameter Re on nanofluid velocity. It is observed in figure 4.6 (a) that as suction parameter $Re > 0$ increases, the velocity of the nanofluid decreased in the channel at the left wall and centerline before a reversal flow phenomenon near the right permeable wall happens. This is expound as increasing suction $Re > 0$ at the right hot wall imply that more nanofluid which are heated get kinetic energy at the right wall. This leads to increase the velocity of the nanofluid at right wall. Again, as injection parameter $Re < 0$ increases, the velocity of the fluid decreased at the left wall and centerline of the channel followed by increase fluid velocity near the right wall of the channel. This is due to the injection of cold nanofluid to the flow channel through the left permeable wall which allow to decrease the velocity of the nanofluid near left permeable wall and the centerline. Figure 4.6 (b) shows nanofluid velocity for different values of variable viscosity parameter A_1 in the presence of suction and injection. It can be seen that



(a) Velocity across the channel with increasing t



(b) Temperature across the channel with increasing t



(c) Concentration across the channel with increasing t

Figure 4.2: Fluid velocity, temperature and concentration across the channel with increasing time ($Re = M = Gc = Gr = Nt = Ec = A_1 = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$).

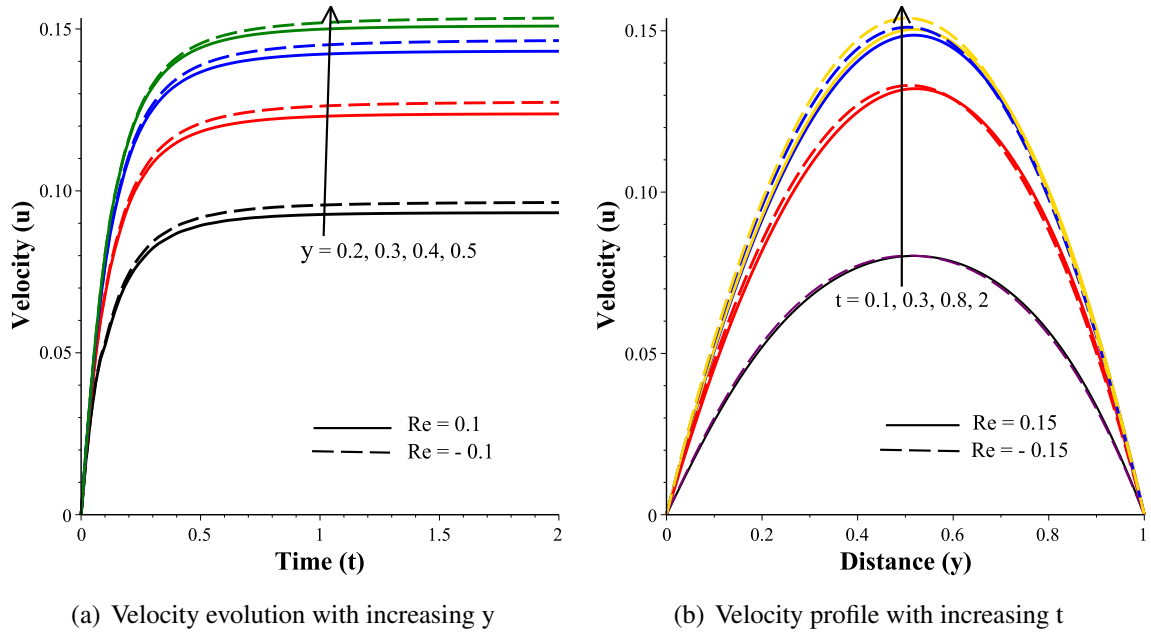


Figure 4.3: Fluid velocity across the channel with increasing time ($M = Gc = Gr = Nt = Ec = A_1 = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)

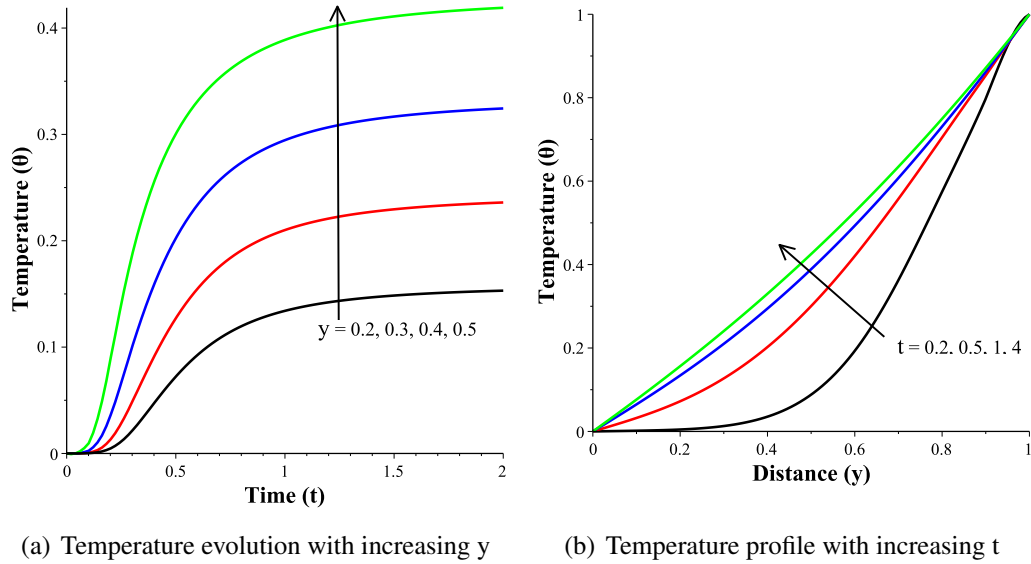


Figure 4.4: Fluid temperature across the channel with increasing time ($Re = M = Gc = Gr = Nt = Ec = A_1 = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)

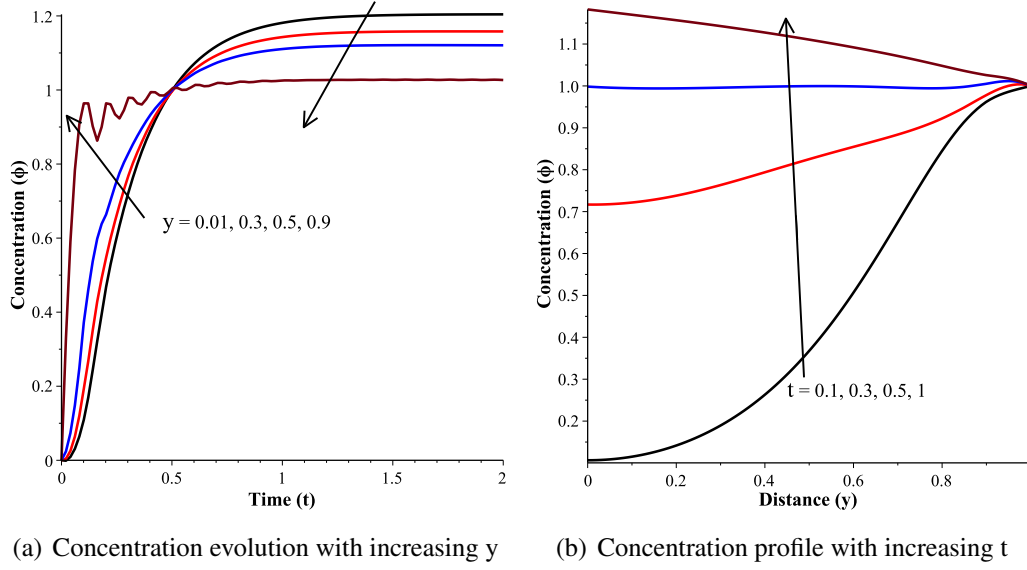
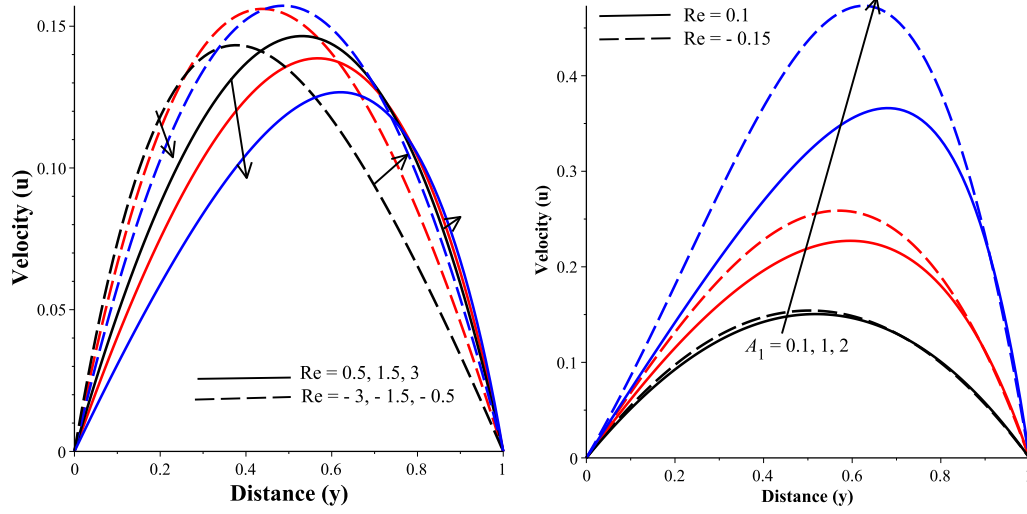


Figure 4.5: Fluid concentration across the channel with increasing time ($Re = M = Gc = Gr = Nb = 0.1$, $Nt = Ec = A_1 = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)

as A_1 increases, the nanofluid flow velocity increases in the microchannel in simultaneous presence of suction and injection. This is because increasing A_1 will make the viscosity of the fluid gets decreased. Further, the effect of A_1 on the flow velocity of nanofluid in the microchannel becomes more magnified in the presence of injection $Re < 0$ than suction $Re > 0$ except near the right wall.

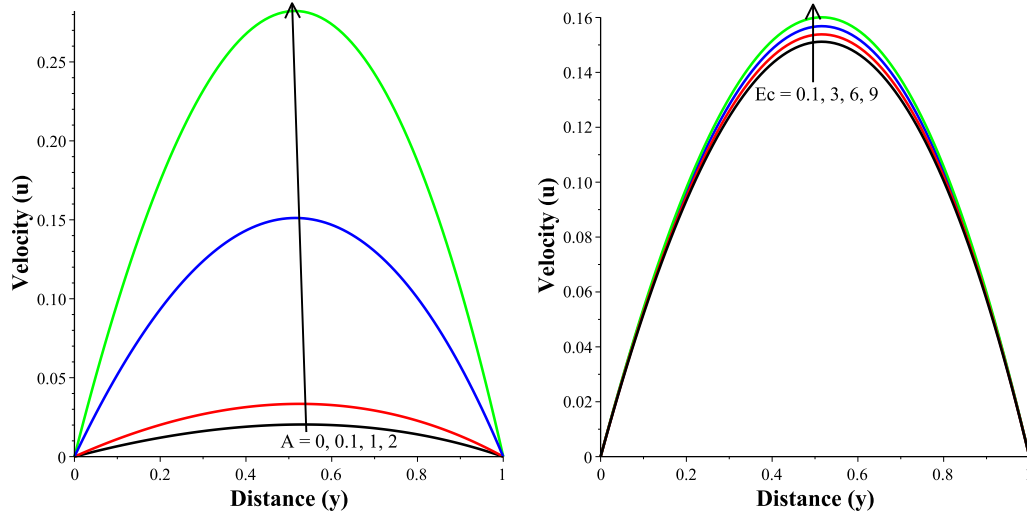
In figures 4.7 and 4.8, it is observed that the rise in nanofluid velocity with an increase in pressure gradient, Eckert number, solutal Grashof number and thermal Grashof number parameter. As Gc and Gr increase, the buoyancy effect increases due to concentration and temperature gradient, which lead to an increase in velocity profile. Similar effect of increasing velocity results from increasing viscous dissipation within the flow system as Ec increases.

In figure 4.9 (a) and (b), the effect of magnetic field and magnetic field with variable viscosity parameters are highlighted. From figure 4.9 (a) and (b), it is demonstrated that an increase in magnetic field parameter M has a retarding effect on the velocity of nanofluid across the microchannel, but strong retarding effect of velocity is observed on the centerline of the channel. It is obviously seen that the transverse magnetic field reduces the transport phenomena. This is due to the fact that change of magnetic field M gives rise to the change of Lorentz force, because of the presence of magnetic field, and the Lorentz force produces resistance to the transport phenomena. This is in agreement with the classical result. In addition, for a fixed magnetic field parameter M , as viscosity variation parameter A_1 increases there is an increase in the nanofluid velocity as shown in figure 4.9 (b).



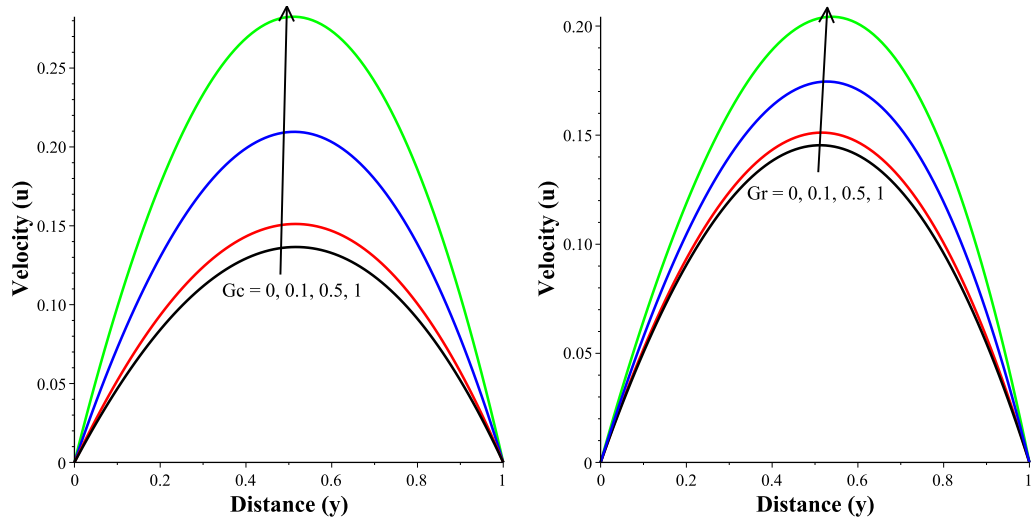
(a) Velocity profile with increasing Re , $A_1 = 0.1$ (b) Velocity profile with increasing A_1 & Re

Figure 4.6: Steady state velocity profiles with increasing Re, A_1 ($M = Gc = Gr = Nt = Ec = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)



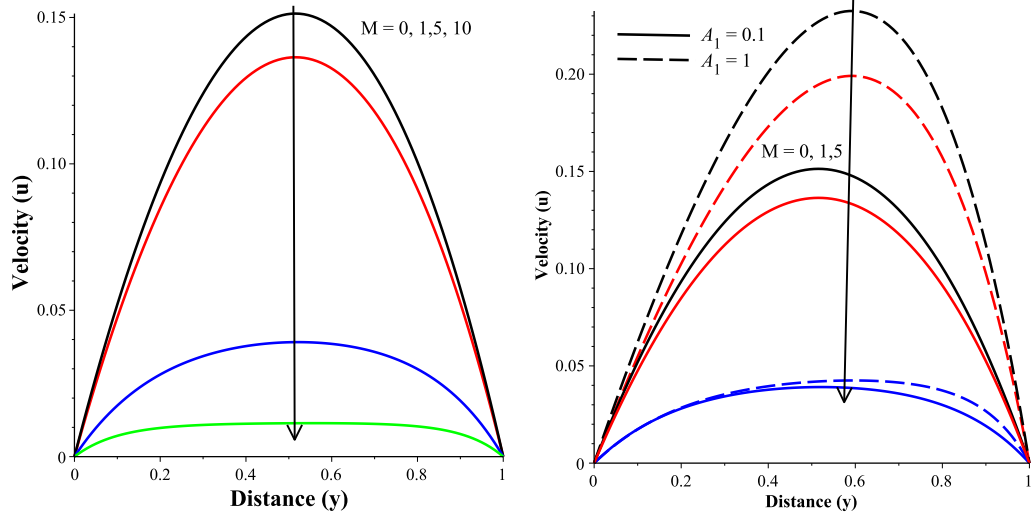
(a) Velocity profile with increasing A , $Ec = 0.1$ (b) Velocity profile with increasing Ec , $A = 1$

Figure 4.7: Steady state velocity profiles with increasing A, Ec ($Re = M = Gc = Gr = Nt = A_1 = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$)



(a) Velocity profile with increasing Gc , $Gr = 0.1$ (b) Velocity profile with increasing Gr , $Gc = 0.1$

Figure 4.8: Steady state velocity profiles with increasing Gc, Gr ($Re = M = Nt = Ec = A_1 = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)



(a) Velocity profile with increasing M , $A_1 = 0.1$ (b) Velocity profile with increasing M , A_1

Figure 4.9: Steady state velocity profiles with increasing M, A_1 ($Re = Gc = Gr = Nt = Ec = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)

4.3.3 The Steady State Temperature Profiles

Figures (4.10) - (4.12) explicate the temperature profile variation of nanofluid for different values of physical parameters. Figure 4.10 (a) depicts the temperature profile with increasing suction/injection parameter Re . It is observed that the nanofluid temperature dwindled with increasing both suction and injection parameter. The dependence of temperature profile on Prandtl number is presented in figure 4.10 (b). Prandtl number Pr is inversely related to thermal conductivity of the fluid. From the figure it is easily seen that the nanofluid temperature profile decreased with an augmentation in Prandtl number. The impact of Brownian motion parameter Nb and thermophoretic parameter Nt on nanoparticles temperature profile is shown in figures 4.11 (a) and (b), respectively. Increasing Nb enhances the Brownian motion force. There is a random movement of nanoparticles along the concentration gradient due to Brownian motion force. This random movement of nanoparticles increase the kinetic energy of the particles in the fluid which rises the heat transfer rate. Therefore, the temperature of nanofluid decreased across the channel as shown in figure 4.11 (a). An increase in Nt augments the thermophoretic force. The thermophoretic force pushes the nanoparticles from the heated wall toward the core region - in the direction of heat transfer - which leads to rise the nanoparticles concentration in the flow field. The rise of nanoparticles volume fraction in the flow field will increase the heat transfer rate which in turn reduces the temperature profile of the nanofluid as seen in figure 4.11 (b). Figure 4.12 (a) is sketched to highlight the property of temperature profile with Eckert number Ec . From the definition of Eckert number Ec , a positive Ec directly related to heating (heat is being supplied across the wall in to the fluid). There is a significant rise in the temperature of the nanofluid with an increase in Ec . This rise in nanofluid temperature is a reason for generation of internal heat due to viscous dissipation in the flow field. It is noticed that nanofluid temperature profile is increasing across the channel with an augment in variable viscosity parameter A_1 as seen in figure 4.12 (b).

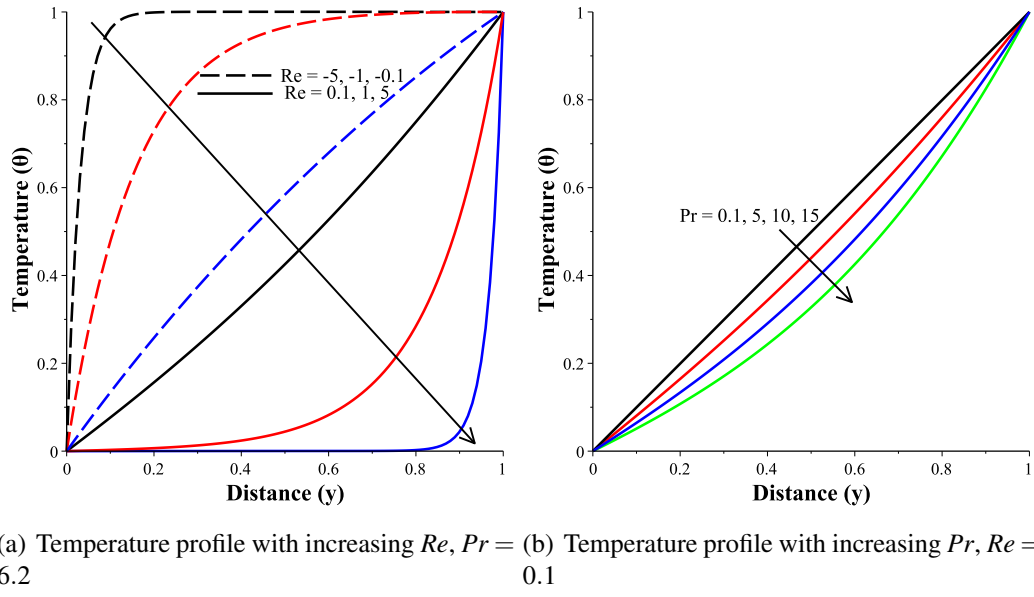


Figure 4.10: Steady state temperature profiles with increasing Re, Pr ($M = Gc = Gr = Nt = Ec = A_1 = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $A = 1$)

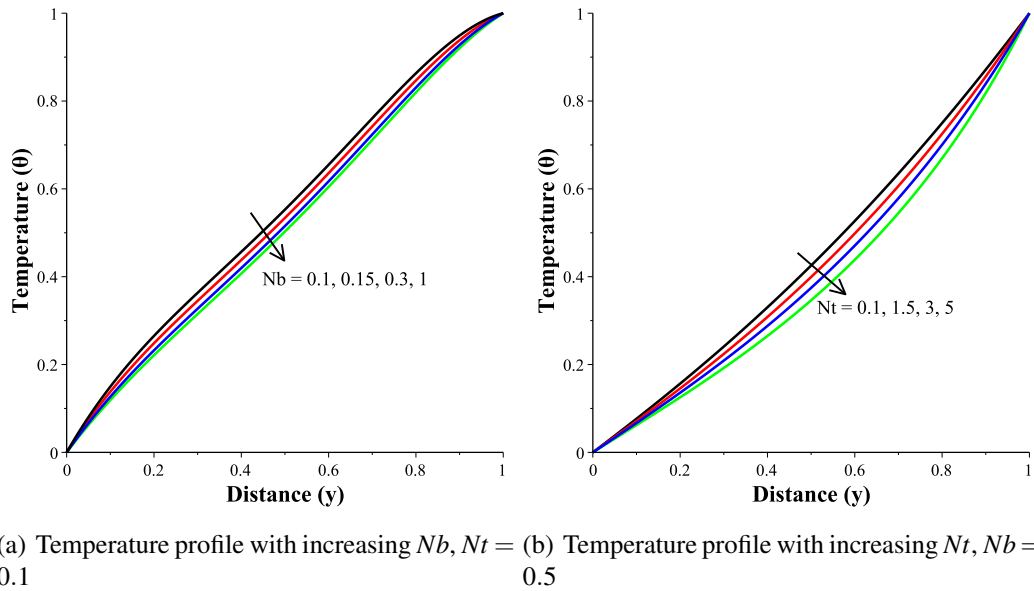


Figure 4.11: Steady state temperature profiles with increasing Nb, Nt ($Re = M = Gc = Gr = Ec = A_1 = 0.1$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)

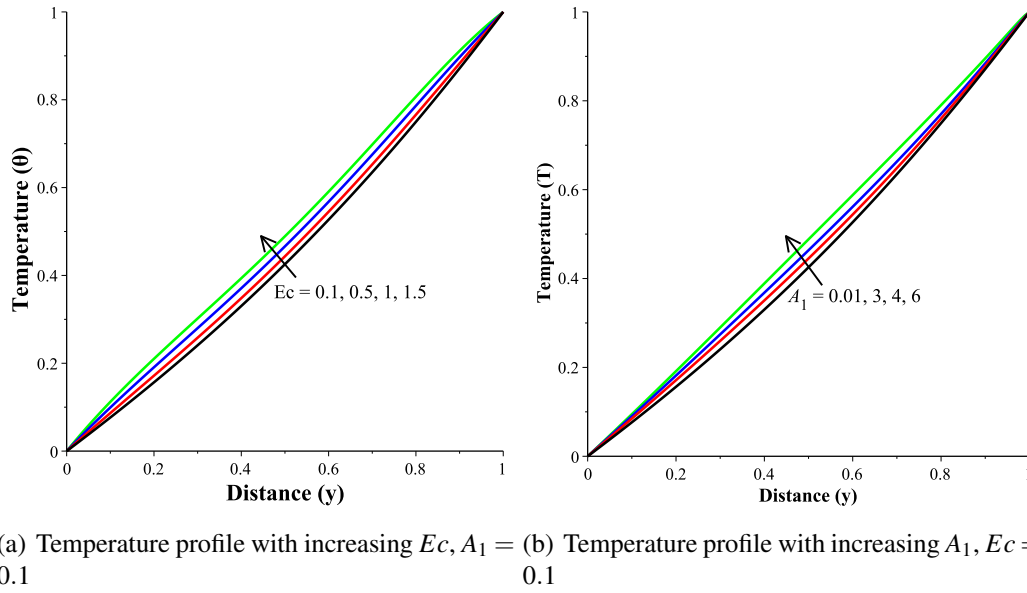


Figure 4.12: Steady state temperature profiles with increasing Ec, A_1 ($Re = M = Gc = Gr = Nt = 0.1$, $Nb = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$)

4.3.4 The Steady State Concentration Profiles

The nanoparticles concentration profiles under the influence of nanofluids parameters (Nt, Nb), pressure gradient and Schmidt number (A, Sc), buoyancy parameter (Gc, Gr), and Reynold and Eckert numbers (Re, Ec) are shown in figures 4.13 (a), 4.13 (b), 4.14 (a) and 4.14 (b), respectively. An increase in thermophoretic parameter Nt amplifies the nanoparticle concentration, whereas the Brownian motion parameter Nb decreases the nanoparticle concentration across the microchannel as shown in figure 4.13 (a). Moreover, the highest and smallest nanoparticle concentration is attained at $y = 0$ and $y = 1$ respectively for arbitrary variation of nanofluids parameters. The Schmidt number Sc describes the nanofluid flows where both momentum and mass diffusion convection processes occur simultaneously. When $Sc = 1$, the momentum and mass diffusivities are equal and the nanoparticles concentration is obtained to be higher across the microchannel in the absence of pressure gradient as shown in figure 4.13 (b). Also, an increase in pressure gradient reduces the nanoparticle concentration and a reverse trend happens with an increase in Schmidt number. Increasing both solutal and thermal Grashof number Gc, Gr reduce the nanoparticles concentration across the microchannel as shown in figure 4.14 (a). It is also seen that the highest nanoparticles concentration is found in the absence of both solutal and thermal Grashof numbers. As Eckert number Ec increases, the nanoparticles concentration decreases in the microchannel in the existence of both suction and injection as shown in figure 4.14 (b). Further, nanofluids particle concentration is found to be much concentrated across the microchannel in the presence of suction than injection.

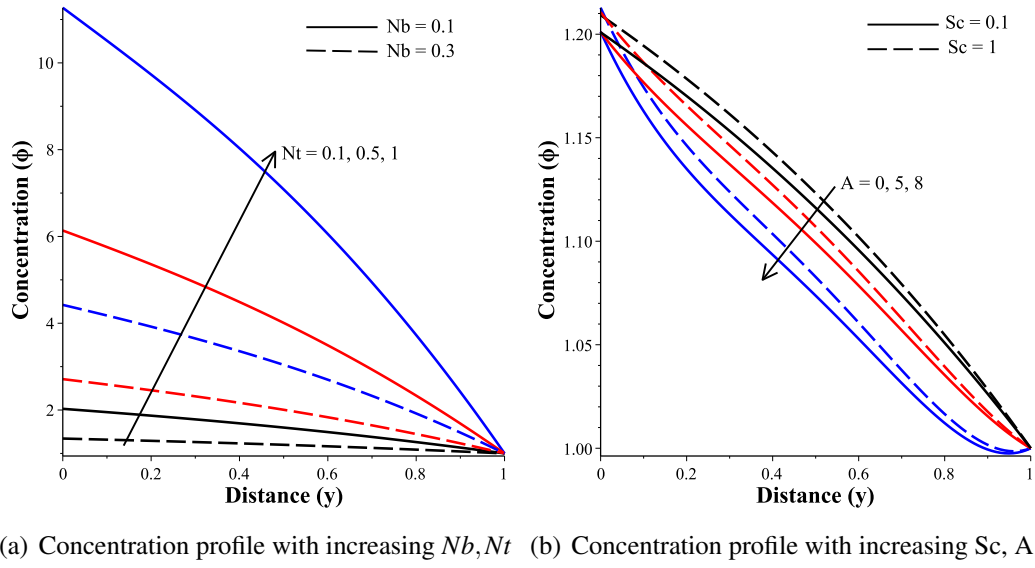


Figure 4.13: Steady state concentration profiles with increasing Nb, Nt, Sc, A , ($Re = M = Gc = Gr = Ec = A_1 = 0.1, Pr = 6.2$)

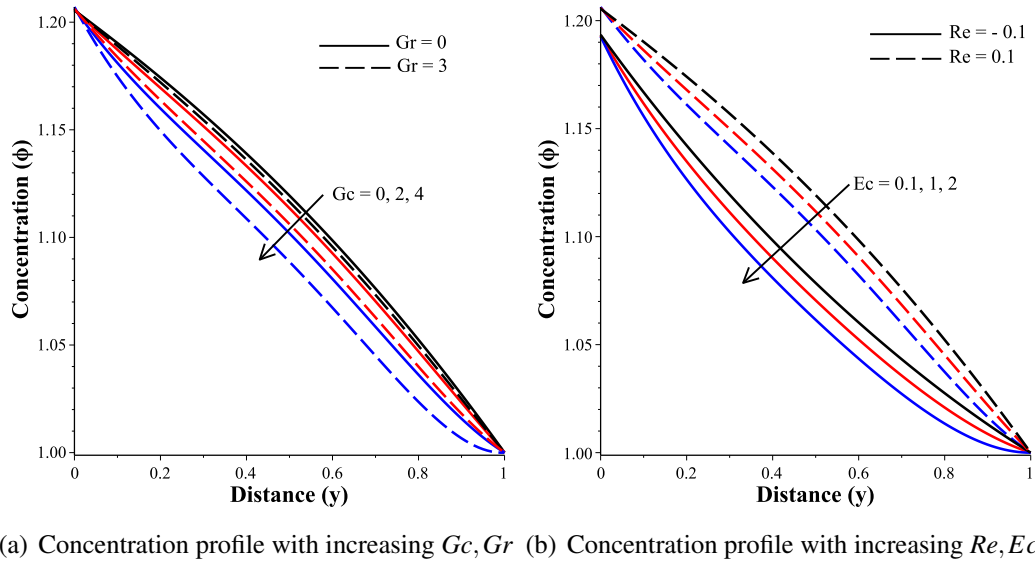


Figure 4.14: Steady state concentration profiles with increasing Gc, Gr, Re, Ec ($M = Nt = A_1 = 0.1, Nb = 0.5, Sc = 0.6, Pr = 6.2, A = 1$)

4.3.5 Skin Friction, Nusselt Number and Sherwood Number

In section 4.3.1 above, it has been shown that velocity, temperature, and concentration attained their steady state within short interval of time (respectively when $t = 1.5$, $t = 2$, and $t = 1$). Since skin friction Cf , Nusselt number Nu , and Sherwood number Sh respectively depend on change in velocity, temperature, and concentration that are steady for large time t , it is reasonable to consider these quantities of engineering interest under steady state condition for large time t .

Figures (4.15) - (4.24) show the local skin friction, the local Nusselt number, and the local Sherwood number with variation in the numerical values of thermophysical parameters embedded in the problem. The local skin friction numerical values are displayed graphically in figures (4.15) - (4.18). Figures (4.15) and (4.16) depict the effects of both suction ($Re > 0$) and injection ($Re < 0$) parameters along with viscosity variation parameter A_1 on skin friction at the microchannel walls. It can be seen from these figures that the skin friction lighten with increase of suction and escalates with an increase in magnitude of injection parameter at both microchannel plates $y = 0$ and $y = 1$. The simultaneous effects of magnetic field and thermophoretic parameters on skin friction at the microchannel walls $y = 0$ and $y = 1$ are revealed in figure 4.17 (a) and (b), respectively. It can be concluded from figure 4.17 (a) that an increase in magnetic field parameter diminishes the skin friction and it escalated with an increase in thermophoretic parameter at $y = 0$. Figure 4.17 (b) shows the skin friction increases with magnetic field parameter and decreases with thermophoretic parameter at $y = 1$. Figure 4.18 (a) and (b) illustrate the combined effect of Prandtl and thermal Grashof number on skin friction at the microchannel walls $y = 0$ and $y = 1$, respectively. From figure 4.18 (a), it is evident in general that skin friction decreases with an increase in Prandtl number and increases with thermal Grashof number at $y = 0$, while a reverse trend is observed at $y = 1$ with an increase in Prandtl and thermal Grashof number as seen from figure 4.18 (b). It is significant to note that when Grashof number $Gr = 0.1$ the skin friction at both plates surfaces is not affected with increase the value of Prandtl number. At the right wall $y = 1$ in figures 4.15 (b) - 4.18 (b), it is obtained that a negative values of skin friction which means that the wall $y = 1$ exerts a drag force on the nanofluid and the fluid also exerts a force on the wall $y = 1$ conversely.

The change of local Nusselt number versus Eckert number for different values of viscosity variation parameter A_1 at $y = 0$ and $y = 1$ is illustrated in figure 4.19 (a) and (b) respectively. It is observed that the local nusselt number at $y = 0$ decreases as both Eckert number and A_1 increase, while a reverse trend is observed at $y = 1$ with an increase in Eckert number and viscosity parameter simultaneously. Figure 4.20 (a) and (b) show the change of local Nusselt number versus Schmidt number for different values of magnetic field parameter at $y = 0$ and $y = 1$, respectively. From these figures it is observed that as Schmidt number increases, the local Nusselt number augments at $y = 0$. But, it decreases at $y = 1$ as Schmidt

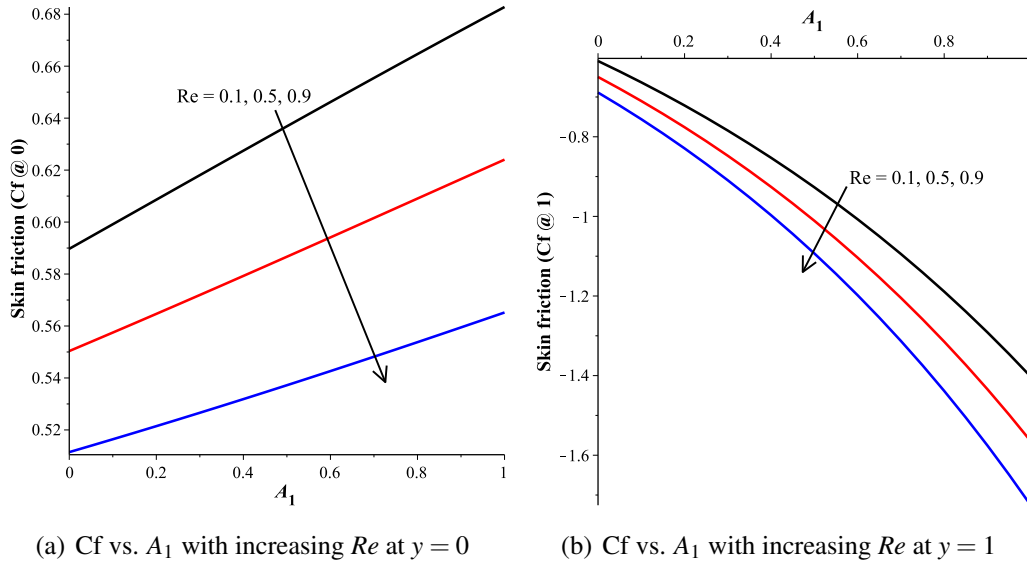


Figure 4.15: Skin friction versus A_1 with increasing Re ($M = Gc = Gr = Nt = Nb = Sc = Ec = 0.1$, $Pr = 6.2$, $A = 1$)

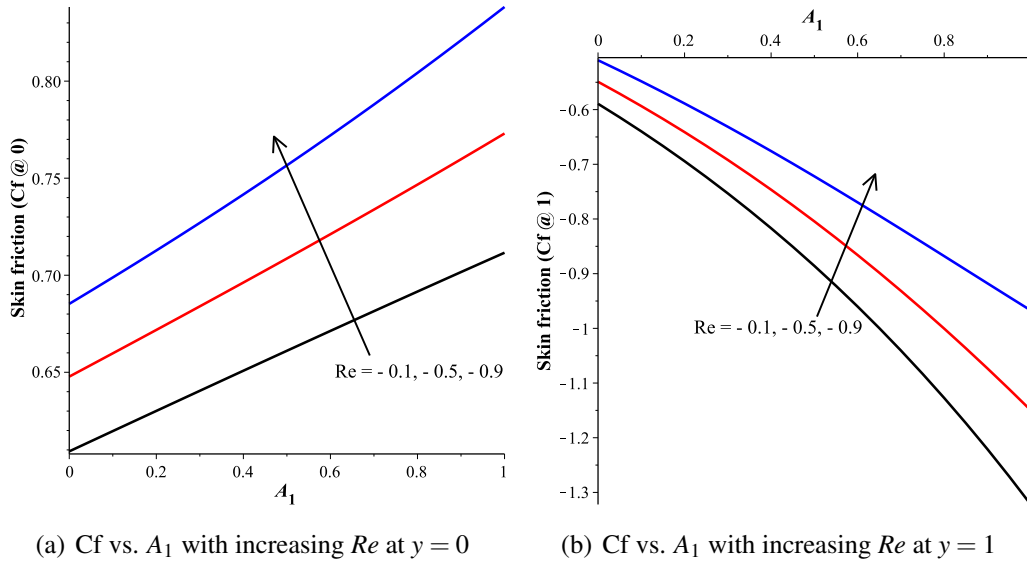


Figure 4.16: Skin friction versus A_1 with increasing Re ($M = Gc = Gr = Nt = Nb = Sc = Ec = 0.1$, $Pr = 6.2$, $A = 1$)

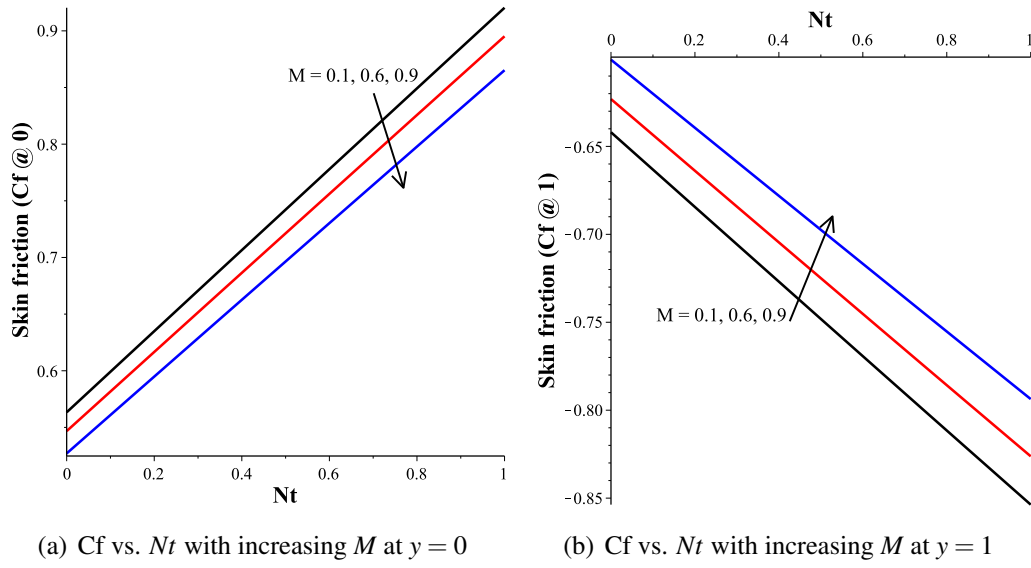


Figure 4.17: Skin friction versus Nt with increasing M ($Re = Gc = Gr = Nb = Sc = Ec = A_1 = 0.1$, $Pr = 6.2$, $A = 1$)

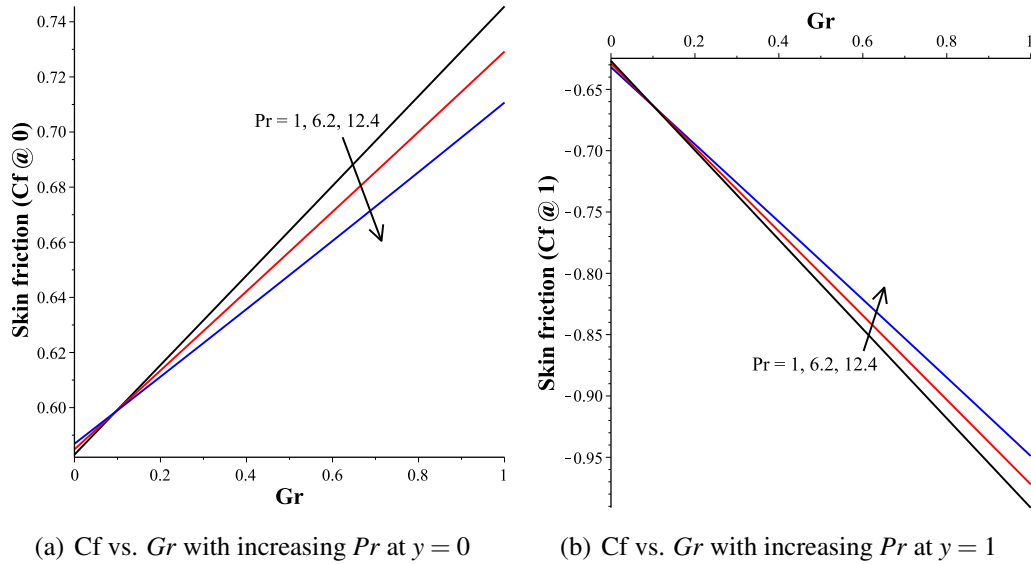


Figure 4.18: Skin friction versus Gr with increasing Pr ($Re = M = Gc = Nt = Nb = Sc = Ec = A_1 = 0.1$, $A = 1$)

number increases. However, increasing magnetic field parameter resulted in enhancement in the local Nusselt number at $y = 0$ whereas an opposite pattern of local Nusselt number is observed at $y = 1$. The variation of local Nusselt number with thermophoretic parameter Nt for various increasing values of Brownian motion parameter Nb at plates $y = 0$ and $y = 1$ is shown in figures 4.21 (a) and (b), respectively. In figure 4.21 (a), as the thermophoretic parameter Nt magnified, the local Nusselt number at the plate $y = 0$ escalated at high Brownian motion parameter $Nb = 0.9$ and diminished at low Brownian motion parameter $Nb = 0.1$. Similarly, the local Nusselt number at the plate $y = 1$ decreased for large Brownian motion parameter $Nb = 0.9$ and increased for low Brownian motion parameter $Nb = 0.1$ as the thermophoretic parameter Nt increases. The negative numerical values of the local Nusselt number at both channel walls $y = 0$ and $y = 1$ connotes that the heat transfers from fluid to the channel walls $y = 0$ and $y = 1$. This is happened due to the existence of heat generation within the flowing fluid in the vicinity of the channel walls $y = 0$ and $y = 1$ which augments the fluid temperature near the walls and may exceed the temperature of the walls. This is the reason for the transfer of heat from fluid to the channel walls.

Figure 4.22 (a) entails that increasing local Sherwood number at $y = 0$ as both variable viscosity parameter A_1 and Eckert number Ec augment. An analogous trend of dwindled local Sherwood number is observed in figure 4.22 (b) with increasing variable viscosity parameter A_1 and Eckert number Ec at $y = 1$. As Schmidt number Sc and magnetic field parameter M magnify, the local Sherwood number diminishes at $y = 0$ as illustrated in figure 4.23 (a). An overturn result on local Sherwood number at $y = 1$ is seen in figure 23 (b) with increment in Schmidt number Sc and magnetic field parameter M . Figures 4.24 (a) and (b) show the local Sherwood number versus thermophoretic parameter Nt for varying values of Brownian motion parameter Nb at $y = 0$ and $y = 1$, respectively. It is observed from these figures that local Sherwood number decreases with an increase in Brownian motion parameter Nb and increases with an increase in thermophoretic parameter Nt at both plates $y = 0$ and $y = 1$.

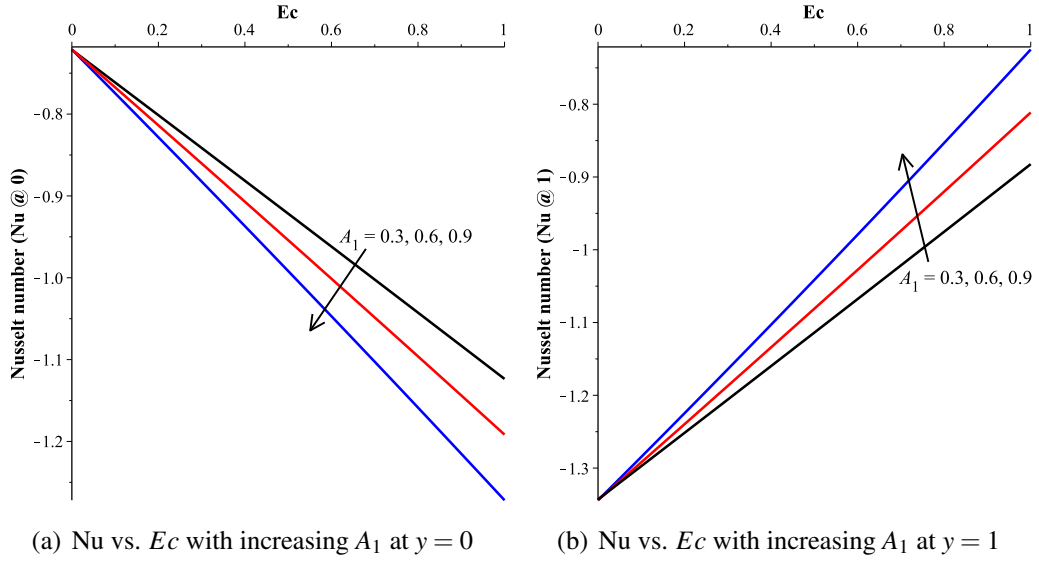


Figure 4.19: Nusselt number versus Ec with increasing A_1 ($Re = M = Gc = Gr = Nt = Nb = Sc = 0.1$, $Pr = 6.2$, $A = 1$)

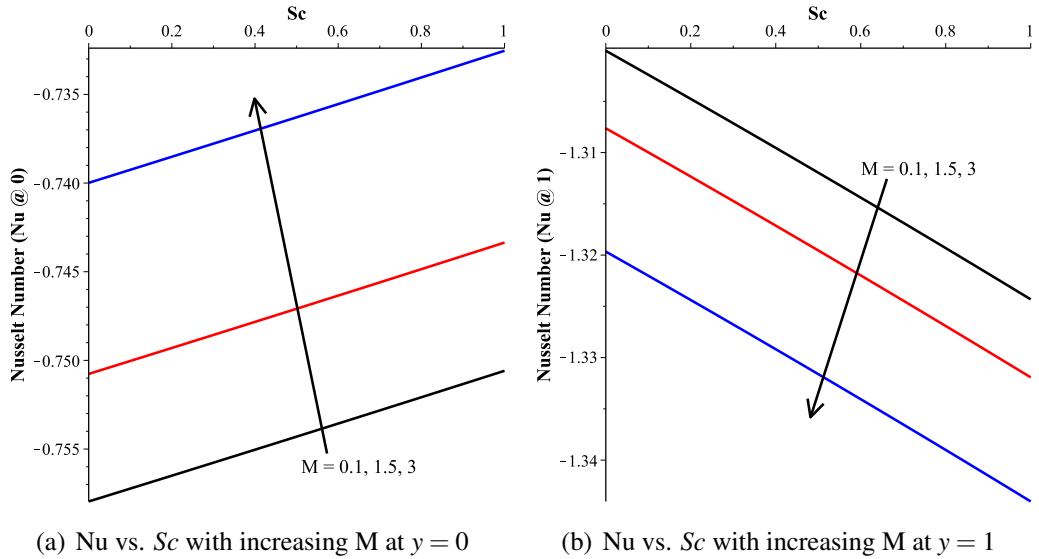


Figure 4.20: Nusselt number versus Sc with increasing M ($Re = A_1 = Gc = Gr = Nt = Nb = Ec = 0.1$, $Pr = 6.2$, $A = 1$)

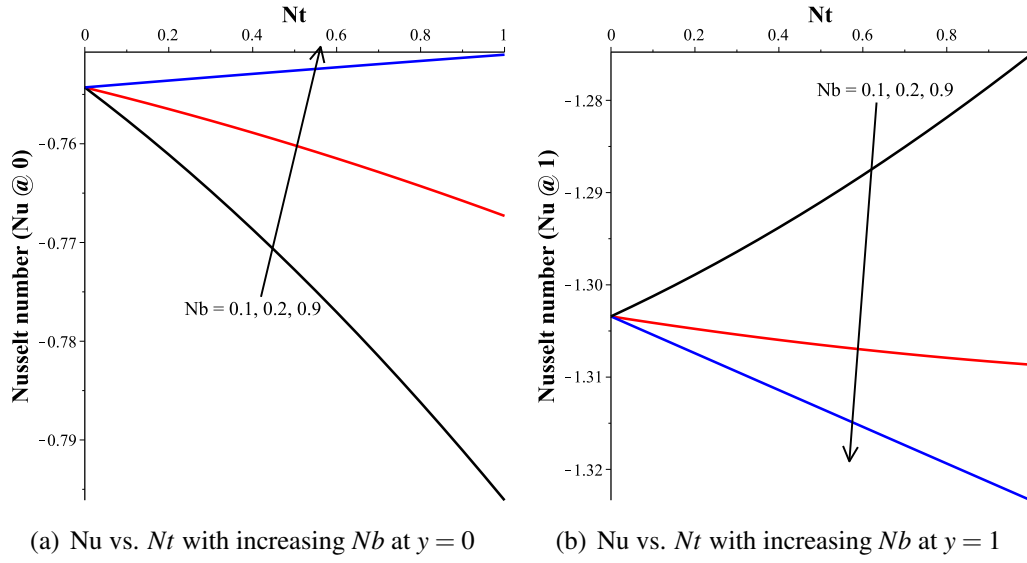


Figure 4.21: Nusselt number versus Nt with increasing Nb ($Re = M = Gc = Gr = Sc = Ec = A_1 = 0.1$, $Pr = 6.2$, $A = 1$)

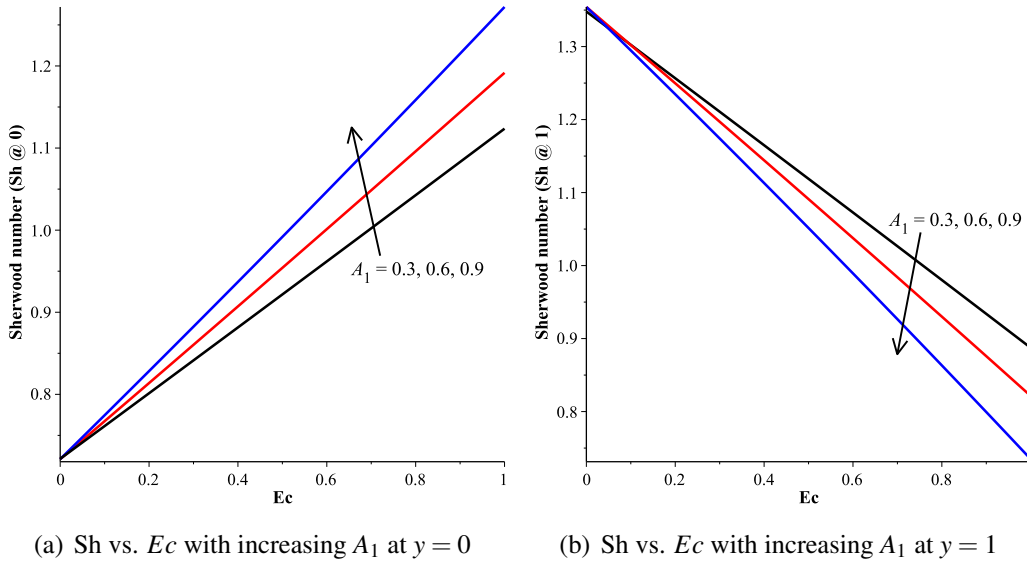


Figure 4.22: Sherwood number versus Ec with increasing A_1 ($Re = M = Gc = Gr = Nt = Nb = Sc = 0.1$, $Pr = 6.2$, $A = 1$)

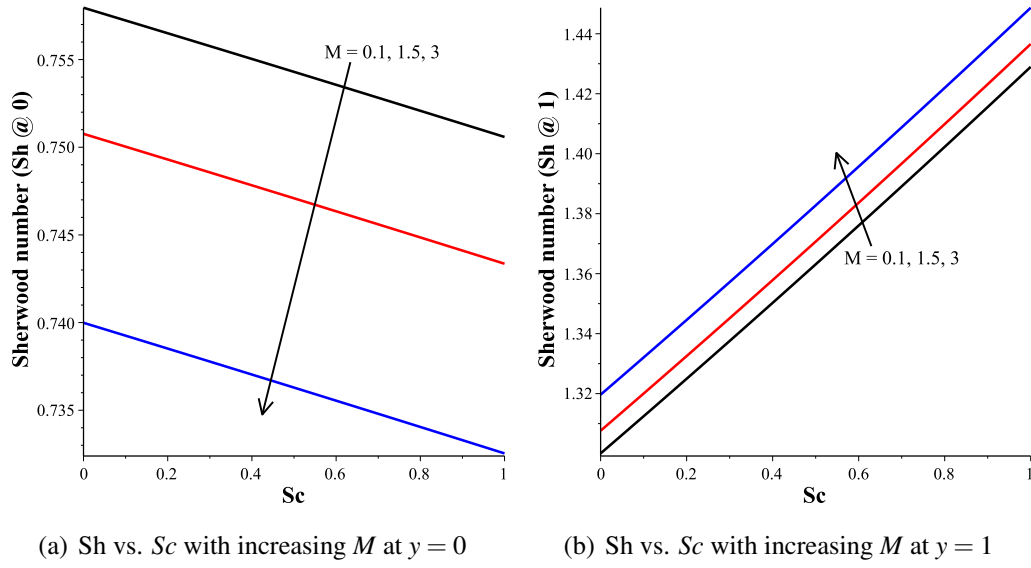


Figure 4.23: Sherwood number versus Sc with increasing M ($Re = Gc = Gr = Nt = Nb = Ec = A_1 = 0.1$, $Pr = 6.2$, $A = 1$)

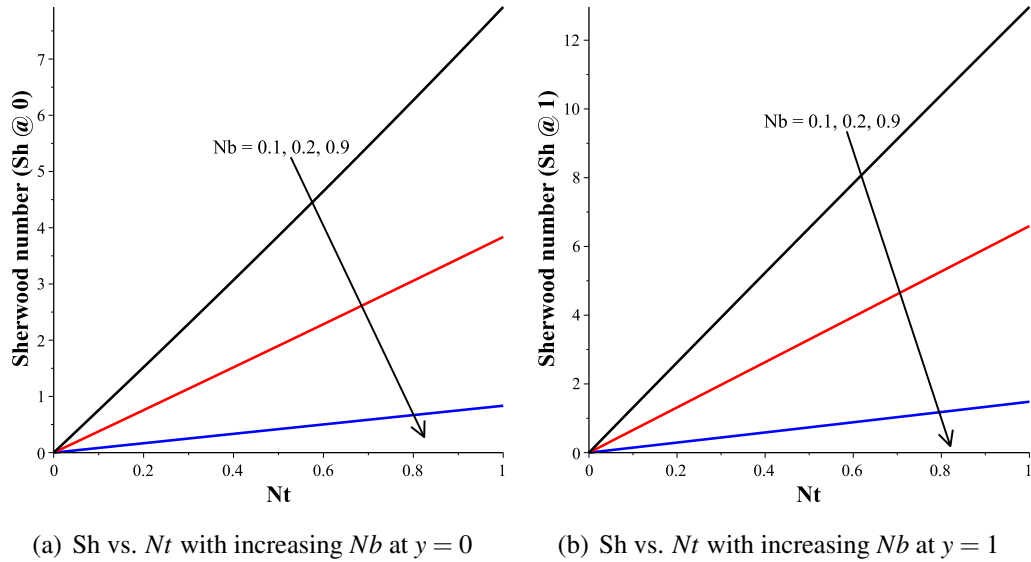


Figure 4.24: Sherwood number versus Nt with increasing Nb ($Re = M = Gc = Gr = Sc = Ec = A_1 = 0.1$, $Pr = 6.2$, $A = 1$)

CHAPTER 5

MAGNETIC EFFECTS OF FERROFLUID FLOW IN A HEATED MICROCHANNEL WITH PERMEABLE WALLS

This chapter is focused to investigate the combined effects of magnetic field, suction/injection, pressure gradient and nanoparticles volume fraction on unsteady ferrofluid flow through asymmetrically heated parallel-plate's microchannel. Such flow study has practical significance in application areas of drug delivery for medicine, cooling for microelectronics and pumping technology for medical engineering. The flow model of this problem is established using single-phase nanofluid flow model, Maxwell equations, Boussinesq approximation and scaling analysis. The resulting nonlinear partial differential equations for momentum and energy balance are tackled numerically using a semi-discretization finite difference method together with Runge-Kutta-Fehlberg integration scheme. Relevant results are displayed graphically and discussed.

5.1 Mathematical Formulation

Consider unsteady flow of an incompressible, temperature dependent variable viscosity Magneto-hydrodynamic(MHD) ferrofluid in a horizontal parallel-plates microchannel of width a units with permeable walls, where the bottom plate is kept at temperature of T_0 and the top plate at T_1 such that $T_0 < T_1$. A uniform transverse magnetic field B_0 is applied in the direction orthogonal to the $\bar{x}$ -axis. The schematic diagram is depicted in figure 5.1. It is assumed that a 2D-coordinate system is used, wherein the bottom plate is fixed at $\bar{y} = 0$, the top plate is fixed at $\bar{y} = a$ (the $\bar{y}$ -axis is aligned vertically) and the $\bar{x}$ -axis is parallel to the plates.

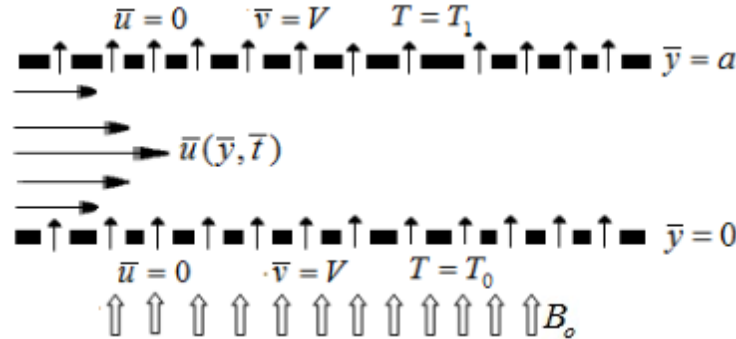


Figure 5.1: Flow Geometry and the Coordinate System

The ferrofluid flow through the microchannel is set in motion due to the effect of pressure gradient only. It is also assumed that the magnetic Reynolds number is so small that the

induced magnetic field becomes diminutive to be ignored as compared to the imposed magnetic field. Using the result (Brinkman, 1952), temperature dependent dynamic viscosity of nanofluid is given by

$$\mu(T) = \frac{\mu_f}{(1 - \phi)^{2.5}} e^{-\gamma_1(T - T_o)},$$

where μ_f is the base fluid dynamic viscosity at the ambient temperature, ϕ is the nanoparticles volume fraction, γ_1 is the viscosity variation parameter and T is the nanofluid temperature. The values of thermophysical properties of water and ferro-nanoparticles (Fe_3O_4) used in the model are tabulated in Table 1 (Sheikholeslami and Ganji, 2016).

Table 5.1: Base fluid and Ferro-nanoparticles thermophysical properties

Contents	ρ (Kg/m ³)	C_p (J/Kg.K)	k (W/m.K)	σ (S/m)
H_2O	997.1	4179	0.613	5×10^{-2}
Fe_3O_4	5180	670	9.7	2.5×10^6

Under the above assumptions and applying MHD model equation, the governing equations of the problem for continuity, momentum and energy are given as:

$$\frac{\partial \bar{u}}{\partial \bar{x}} = 0, \quad (5.1)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho_{nf}} \frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{\rho_{nf}} \frac{\partial}{\partial \bar{y}} \left[\mu_{nf}(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \frac{\sigma_{nf} B_o^2}{\rho_{nf}} \bar{u}, \quad (5.2)$$

$$\frac{\partial T}{\partial \bar{t}} + V \frac{\partial T}{\partial \bar{y}} = \frac{k_{nf}}{(\rho c_p)_{nf}} \frac{\partial^2 T}{\partial \bar{y}^2} + \frac{\mu_{nf}(T)}{(\rho c_p)_{nf}} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \frac{\sigma_{nf} B_o^2}{(\rho c_p)_{nf}} \bar{u}^2, \quad (5.3)$$

with the initial and boundary conditions

$$\bar{u}(\bar{y}, 0) = 0, \quad T(\bar{y}, 0) = T_o, \quad (5.4)$$

$$\begin{aligned} \bar{u}(0, \bar{t}) &= 0, & T(0, \bar{t}) &= T_o, \\ \bar{u}(a, \bar{t}) &= 0, & T(a, \bar{t}) &= T_1, \end{aligned} \quad (5.5)$$

where $\bar{u}$ is the nanofluid velocity in $\bar{x}$ -direction, T is the temperature of nanofluid, V is the suction/injection velocity, $\bar{P}$ is the nanofluid pressure, $\bar{t}$ is the time, T_o is the temperature of the bottom plate, T_1 is the temperature of the top plate, a is the microchannel width, ρ_{nf} is the effective density of the nanofluid, μ_{nf} is the effective dynamic viscosity of the nanofluid, σ_{nf} is the electrical conductivity of the nanofluid, k_{nf} is the thermal conductivity of the nanofluid, $(\rho c_p)_{nf}$ is the the heat capacitance of the nanofluid. Following the results (Brinkman, 1952), thermophysical properties of nanofluid in terms of its base fluid and particles volume fractions are given as

$$\rho_{nf} = (1 - \phi)\rho_f + \phi\rho_s$$

$$(\rho c_p)_{nf} = (1 - \phi)(\rho c_p)_f + \phi(\rho c_p)_s$$

$$\frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}$$

$$\frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi}$$

$$r = \frac{\sigma_s}{\sigma_f}$$

where ρ_f is the density of the base fluid, ρ_s is the nanoparticles density, $(\rho c_p)_f$ is the heat capacitance of the base fluid, $(\rho c_p)_s$ is the heat capacitance of the nanoparticles, k_f is the thermal conductivity of the base fluid, k_s is the thermal conductivity of the nanoparticle, σ_f is the electrical conductivity of the base fluid, σ_s is the electrical conductivity of the nanoparticles, r is dimensionless electrical conductivity.

Equations (5.2)-(5.5) turn in to non-dimensional equations by using and incorporating the following variables and parameters:

$$\begin{aligned} x = \frac{\bar{x}}{a}, y = \frac{\bar{y}}{a}, u = \frac{\bar{u}}{V}, t = \frac{v_f \bar{t}}{a^2}, \theta = \frac{T - T_o}{T_1 - T_o}, \\ P = \frac{a\bar{P}}{\mu_f V}, Re = \frac{Va}{v_f}, A = -\frac{dp}{dx}, A_1 = \gamma_1(T_1 - T_o), M^2 = \frac{\sigma B_o^2 a^2}{\mu_f}, \\ Pr = \frac{v_f}{\alpha_f}, \alpha_f = \frac{k_f}{(\rho c_p)_f}, Ec = \frac{V^2}{(c_p)_f(T_1 - T_o)}. \end{aligned} \quad (5.6)$$

Then the non-dimensional forms of partial differential equations (5.2)-(5.5) are:

$$\begin{aligned} \frac{\partial u}{\partial t} + Re \cdot \frac{\partial u}{\partial y} = \frac{A}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f}\right)} - \frac{e^{-A_1 \theta}}{(1 - \phi)^{2.5} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f}\right)} \cdot \left[-A_1 \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} \right] \\ - M^2 \cdot \frac{1}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f}\right)} \cdot \left[1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi} \right] \cdot u, \end{aligned} \quad (5.7)$$

$$\begin{aligned} Pr \cdot \frac{\partial \theta}{\partial t} + Pr \cdot Re \frac{\partial \theta}{\partial y} = \frac{1}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f}\right)} \cdot \left[\frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)} \right] \cdot \frac{\partial^2 \theta}{\partial y^2} \\ + Pr \cdot Ec \cdot \frac{e^{-A_1 \theta}}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f}\right)} \cdot \left(\frac{\partial u}{\partial y} \right)^2 + Pr \cdot M^2 \cdot Ec \cdot u^2, \end{aligned} \quad (5.8)$$

with initial and boundary conditions

$$u(y, 0) = 0, \theta(y, 0) = 0, \quad (5.9)$$

$$\begin{aligned} u(0, t) &= 0, \theta(0, t) = 0, \\ u(1, t) &= 0, \theta(1, t) = 1, \end{aligned} \quad (5.10)$$

where Re is the Reynolds suction/injection parameter, A is the pressure gradient parameter, A_1 is the viscosity variation parameter, M is the Hartmann number, Pr is the Prandtl number, Ec is the Eckert number.

The important physical quantities of interest in the study of the problem are the local skin friction Cf_x , and the local Nusselt number Nu_x , where

$$Cf_x = \frac{\tau_w}{\rho_f V^2}, \quad \text{and} \quad Nu_x = \frac{aq_w}{k_f(T_1 - T_o)}, \quad (5.11)$$

where τ_w is the shear stress, q_w is the heat flux; and $\tau_w = \mu_{nf} \frac{\partial \bar{u}}{\partial \bar{y}} \Big|_{\bar{y}=0,a}$ and $q_w = -k_{nf} \frac{\partial T}{\partial \bar{y}} \Big|_{\bar{y}=0,a}$.

Equation(5.11) in non-dimensional form :

$$Re^2 Cf_x = \frac{e^{-A_1 \theta}}{(1 - \phi)^{2.5}} \frac{\partial u}{\partial y} \Big|_{y=0,1}, \quad \text{and} \quad Nu_x = -m \frac{\partial \theta}{\partial y} \Big|_{y=0,1}, \quad (5.12)$$

$$\text{but } m = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}.$$

5.2 Numerical Method

The model equations (5.7)-(5.10) are a system of nonlinear IBVPs and their solutions can be tackled numerically using semi-discretization via finite difference scheme along with Runge-Kutta Fehlberg integration technique. The space interval $[0, 1]$ is subdivided into $N + 1$ equal subintervals. The nodal spacing and points are defined respectively as $\Delta y = \frac{1}{N+1}$ and $y_i = i\Delta y$, for $i = 1, 2, \dots, N$ where N is the number of interior nodal points in $[0, 1]$. The first and second spatial derivatives are replaced by a central finite difference approximation of order $(\Delta y)^2$ accuracy. Let $u(y_i, t)$ and $\theta(y_i, t)$ be represented by u_i and θ_i respectively. Then the semi-discretization method with centered finite difference scheme for equations (7)-(10)

become:

$$\begin{aligned} \frac{\partial u_i}{\partial t} + Re \cdot \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) &= \frac{A}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right)} - \frac{e^{-A_1 \theta_i}}{(1 - \phi)^{2.5} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right)} \\ &\cdot \left[-A_1 \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) + \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} \right] \\ &- M^2 \cdot \frac{1}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right)} \cdot \left[1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi} \right] \cdot u_i, \end{aligned} \quad (5.13)$$

$$\begin{aligned} Pr \cdot \frac{\partial \theta_i}{\partial t} + Pr \cdot Re \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) &= Pr \cdot Ec \cdot \frac{e^{-A_1 \theta_i}}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f} \right)} \cdot \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right)^2 \\ &+ \frac{1}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f} \right)} \cdot \left[\frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)} \right] \cdot \left(\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta y)^2} \right) \\ &+ Pr \cdot M^2 \cdot Ec \cdot u_i^2, \end{aligned} \quad (5.14)$$

with initial conditions

$$u_i(0) = 0, \quad \theta_i(0) = 0, \quad 1 \leq i \leq N. \quad (5.15)$$

The boundary conditions at $y = 0, 1$ are transformed to incorporate as follows:

$$\begin{aligned} u_o(t) &= 0, \quad \theta_o(t) = 0, \\ u_{N+1}(t) &= 0, \quad \theta_{N+1}(t) = 1. \end{aligned} \quad (5.16)$$

Equations (5.13)-(5.16) is a system of first order nonlinear ordinary differential equations with well-known initial conditions and can be solved iteratively by Runge-Kutta Fehlberg integration techniques (Na, 1979).

5.3 Results and Discussions

The effects of magnetic field and temperature dependent variable viscosity on unsteady ferrofluid flow in asymmetrically heated horizontal parallel permeable plates microchannel has been studied. Solutions to nonlinear initial boundary value problems (IBVPs) have been tackled by semi-discretization via central finite difference scheme along with Runge-Kutta Fehlberg integration technique. Numerical results of velocity profile, temperature profile, skin friction and Nusselt number are analyzed by using table 1 and considering governing parameter values of the flow $\phi = 0.1$, $Re = 0.1$, $M = 1$, $Ec = 0.1$, $Pr = 6.2$, $A = 0.1$, $A_1 = 0.1$ if not mentioned. The obtained results have been highlighted through figures (5.2)-(5.18).

5.3.1 Transient Analysis

Outlines of evolution of velocity and temperature profiles are depicted in figures (5.2)-(5.4). Figure 5.2 (a) shows the rise of fluid velocity from a value of zero in time and space at both top and bottom walls to its value of ceiling within the microchannel. The fluid temperature augments from its zero value at the bottom wall to its highest value at the top wall as illustrated in figure 5.2 (b). From figures 5.3 (a) and (b), it is observed that the fluid highest velocity value is reached near the center of the microchannel within short period of time and a steady state flow velocity value is obtained with increasing time. Also, a rise and fall of fluid flow velocity value is seen close to the bottom wall of the microchannel. It can be noted from figures 5.4 (a) and (b) that the fluid temperature boosts in the whole microchannel width before a steady state temperature is achieved and its highest value is observed at the top wall of the microchannel. In addition, a vacillation of fluid temperature value is obtained very near to top wall of the microchannel for $0.1 \leq t \leq 0.5$. This is due to the fact that magnetic-nanoparticles in the ferrofluid near the magnetic field move to the hot wall under the action of Kelvin force, as a result a dynamic fluctuating temperature distribution is produced for short while.

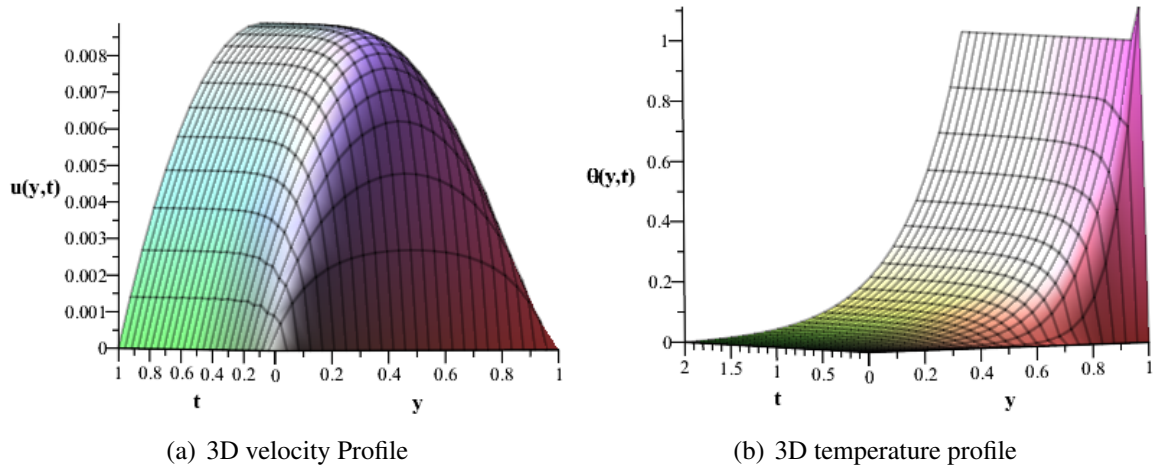
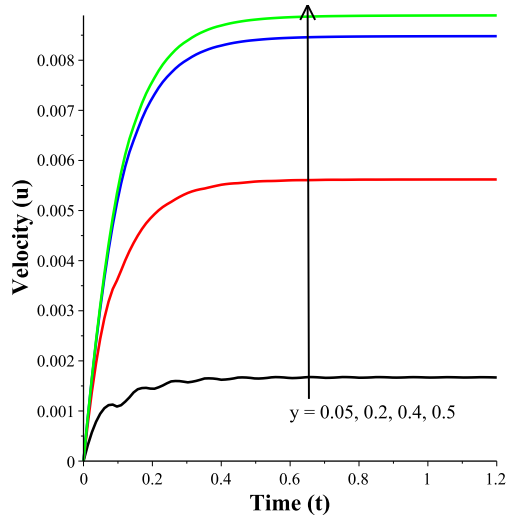


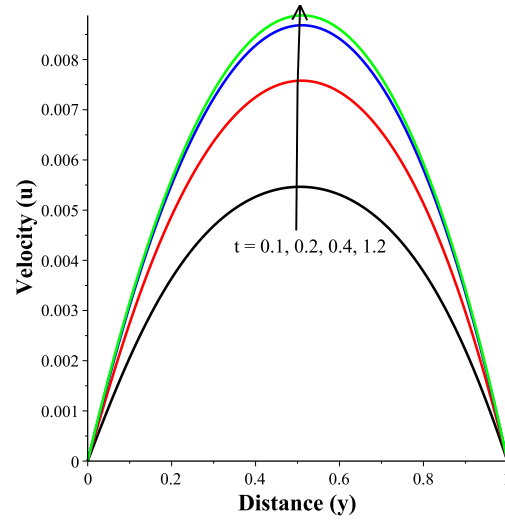
Figure 5.2: Fluid velocity and temperature profile across the channel with increasing time

5.3.2 Steady State Velocity Profiles

Figures (5.5)-(5.7) point up the effects of various model parameters on steady state nanofluid flow velocity profiles. The influence of suction/injection parameter Re on nanofluid velocity is displayed in figures 5.5 (a) and (b). In figure 5.5 (a), it is observed that increasing suction parameter $Re > 0$ decreases the nanofluid velocity near the bottom permeable wall and the center of the microchannel but there is a wavering of flow velocity before an opposite flow

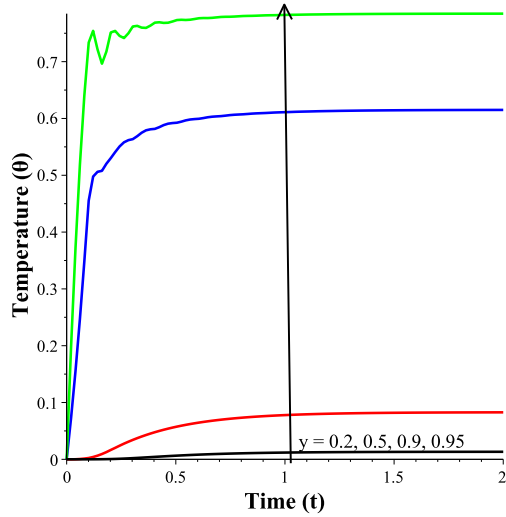


(a) Evolution of velocity profile with increasing y

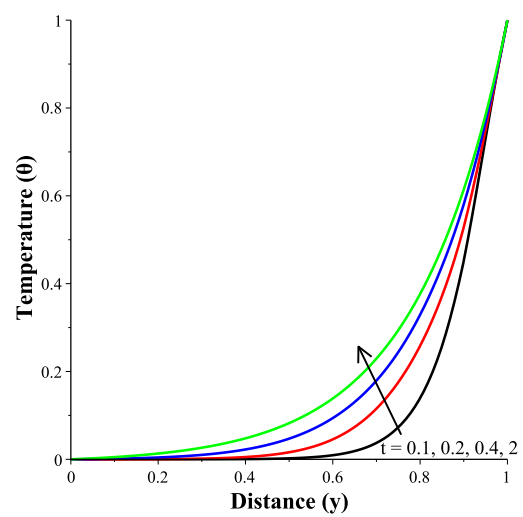


(b) Velocity profile with increasing t

Figure 5.3: Unsteady fluid velocity profile with increasing distance and time



(a) Evolution of temperature profile with increasing y



(b) Temperature profile with increasing t

Figure 5.4: Unsteady fluid temperature profile with increasing distance and time

velocity trends to happen in the vicinity of the top permeable wall. Increasing injection parameter $Re < 0$, increases the fluid velocity at the bottom permeable wall which is followed by fluctuation of fluid velocity before a reverse phenomenon happened at the centerline of the channel and top permeable wall as shown in figure 5.5 (b). Figure 5.6 (a) demonstrates fluid flow velocity for various values of variable viscosity parameter A_1 . It is observed that as A_1 increases, the flow velocity increases in the microchannel. This is because increasing A_1 will decrease the thickness of the fluid, which in turn allows increasing the flow velocity. This result coincide with the result of (Makinde, 2018; Monaledi and Makinde, 2018). The influence of magnetic field M on velocity is highlighted in figure 5.6 (b). It is seen that increasing in M reduced the fluid velocity, but a strong reducing effect is seen at the centerline and weak reducing effect is seen at both top and bottom permeable walls. In figures 5.7 (a) and (b), the effects of pressure gradient A and nanoparticles volume fraction ϕ on velocity profile are illustrated. It is seen that the rise of nanofluid velocity with an increase in pressure gradient A in figure 5.7 (a), but a decrease flow velocity with an increase in nanoparticle volume fraction ϕ in figure 5.7 (b). These results go in line with the earlier outcome reported by (Makinde, 2018; Monaledi and Makinde, 2018).

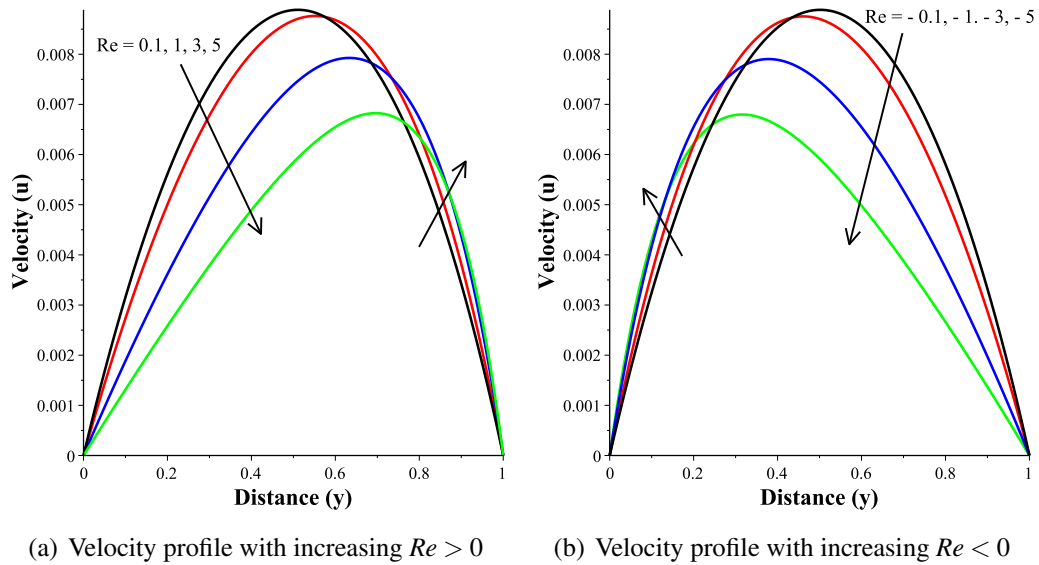


Figure 5.5: Velocity profiles with increasing $Re \geq 0$

5.3.3 Steady State Temperature Profiles

The temperature variation profiles for different values of physical parameters are graphically presented in figures 5.8 and 5.9. Figure 5.8 (a) displays the impact of Prandtl number Pr on temperature profile. From the figure, it is observed that the nanofluid temperature profile decreases with an increase in Pr . Nanofluid temperature variation profile with Eckert number

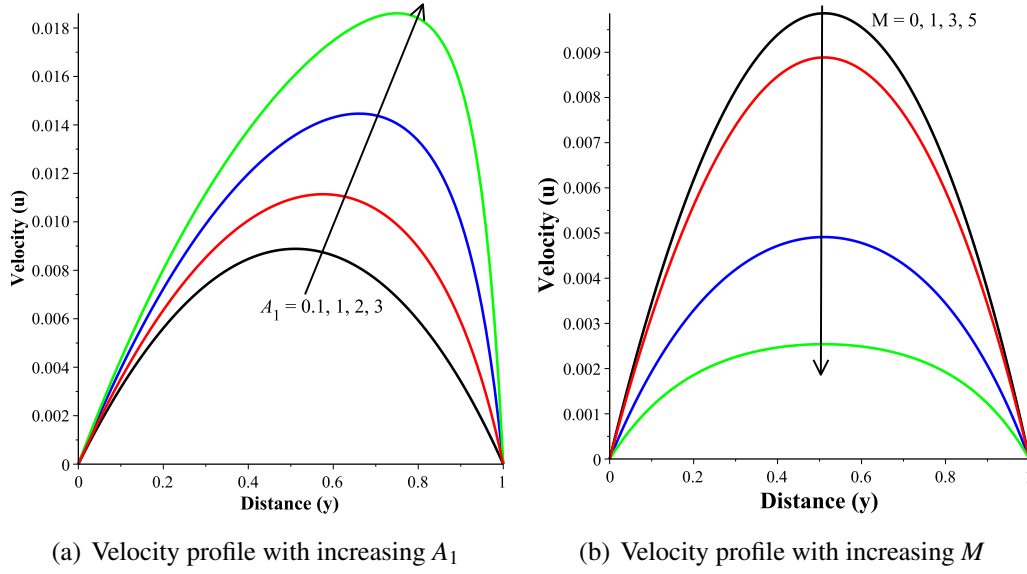


Figure 5.6: Velocity profiles with increasing A_1 , M

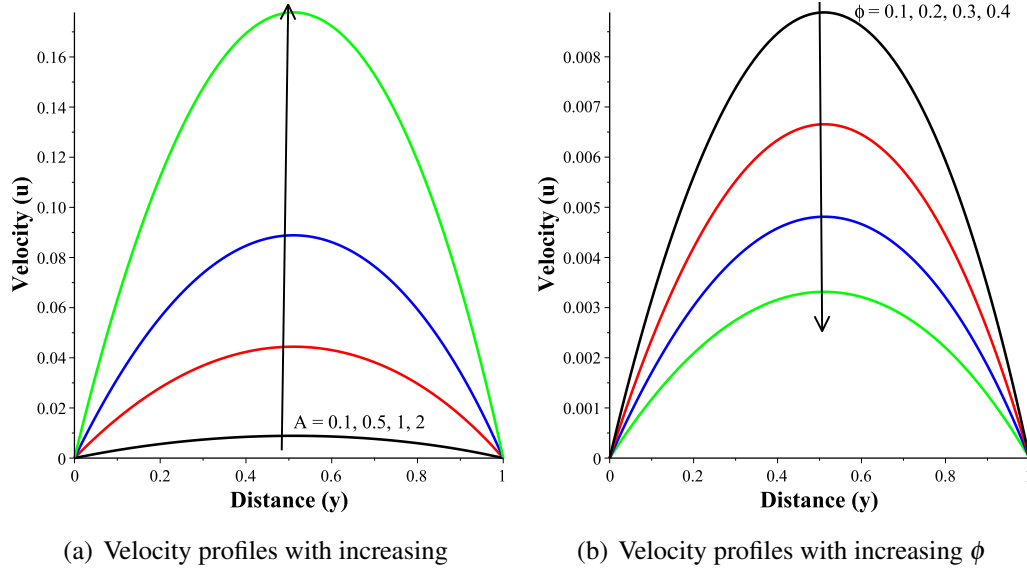


Figure 5.7: Velocity profiles with increasing A , ϕ

Ec is highlighted in figure 5.8 (b). In here, (Hindebu et al., 2021) presented a similar result. A positive Ec is directly proportional to heating (heat is being provided across the wall into the fluid). Therefore, an increase in Ec rises the temperature of the nanofluid in the microchannel. Nanofluid temperature profile enhanced with introduction of nanoparticles ϕ is displayed in figure 5.9 (a).

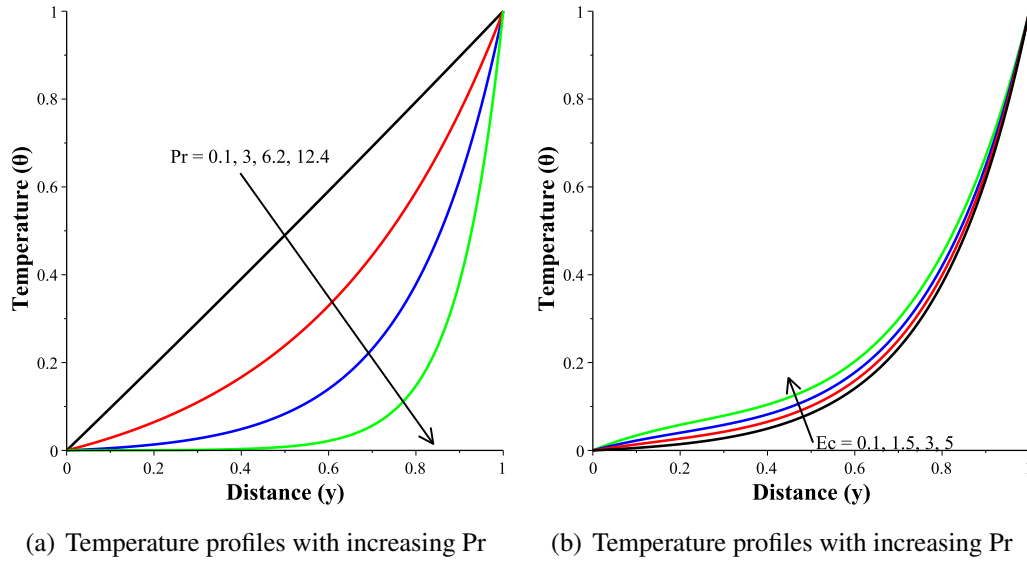


Figure 5.8: Temperature profiles with increasing Pr , Ec

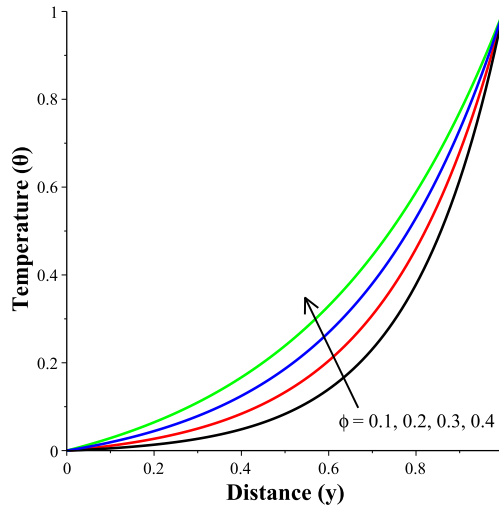


Figure 5.9: Temperature profiles with increasing ϕ

5.3.4 Skin Friction and Nusselt Number

The system profile has been attained its steady state within short period of time as shown in figures 5.3 and 5.4. Therefore, it is sensible to consider quantities of engineering interest skin friction Cf and Nusselt number Nu at steady state. Figures (5.10)-(5.18) demonstrate the skin friction and Nusselt number with variation in numerical values of various pertinent parameters. Skin frictions are plotted graphically in figures (5.10)-(5.14). The influence of suction/injection parameter Re on skin friction at the bottom and top permeable walls of the microchannel is seen in figures 5.10 and 5.11. From these figures, skin friction lessens with an increase in suction parameter $Re > 0$ and enlarges with an increase in injection parameter $Re < 0$ at both permeable walls of the microchannel. It is also found that increasing variable viscosity variation parameter A_1 escalates and lightens the skin friction at bottom and top microchannel walls, respectively. The combined effects of magnetic field parameter M and pressure gradient A on skin friction at bottom and top walls of the micro-channel are revealed in figures 5.12 (a) and (b), respectively. In figure 5.12 (a), magnetic field M has a reducing effect on skin friction Cf but pressure gradient parameter A has a magnifying effect on Cf at the bottom wall. In figure 5.12 (b), increasing M amplify Cf , but A diminishes Cf at the top wall. Figures 5.13 (a) and (b) depicts the simultaneous impact of Eckert number Ec and pressure gradient parameter A on skin friction at the bottom and top walls of the microchannel. It is observed from the figures that skin friction Cf increases with an increase in both Ec and A at bottom wall but it decreases at the top wall of the microchannel. Skin friction Cf versus pressure gradient A with increasing values of nanoparticles volume fraction ϕ at both bottom and top walls are illustrated, respectively, in figures 5.14 (a) and (b). Figure 5.14 (a) entails that increasing Cf is obtained at bottom wall as both ϕ and A augment. A similar trend of dwindled Cf is observed at top wall in figure 5.14 (b) with increasing both ϕ and A .

The Nusselt number characterizes the heat flux from a solid surface to a fluid and it is known as dimensionless heat transfer coefficient. The variation of Nusselt number Nu with viscosity variation parameter A_1 for varying values of suction/injection parameters Re at bottom and top permeable walls are shown in figures 5.15 and 5.16. From these figures, Nusselt number increase at both bottom and top walls as suction parameter $Re > 0$ increases, while a reverse trend is found at both walls as injection parameter $Re < 0$ increases. Also, increasing viscosity variation parameter A_1 lightens the Nusselt number at the bottom wall and escalates the Nusselt number at the top wall in the presence of both suction and injection parameter. Figures 5.17 (a) and (b) represent the Nusselt number Nu versus suction parameter $Re > 0$ with increasing magnetic field parameter M at bottom and top walls, respectively. In figures 5.17 (a) and (b), the Nusselt number varies directly with the magnetic field parameter M at the bottom wall, but it varies inversely at the top wall. Nusselt number dependence on simultaneous increment of magnetic field parameter M and nanoparticle particles volume fraction

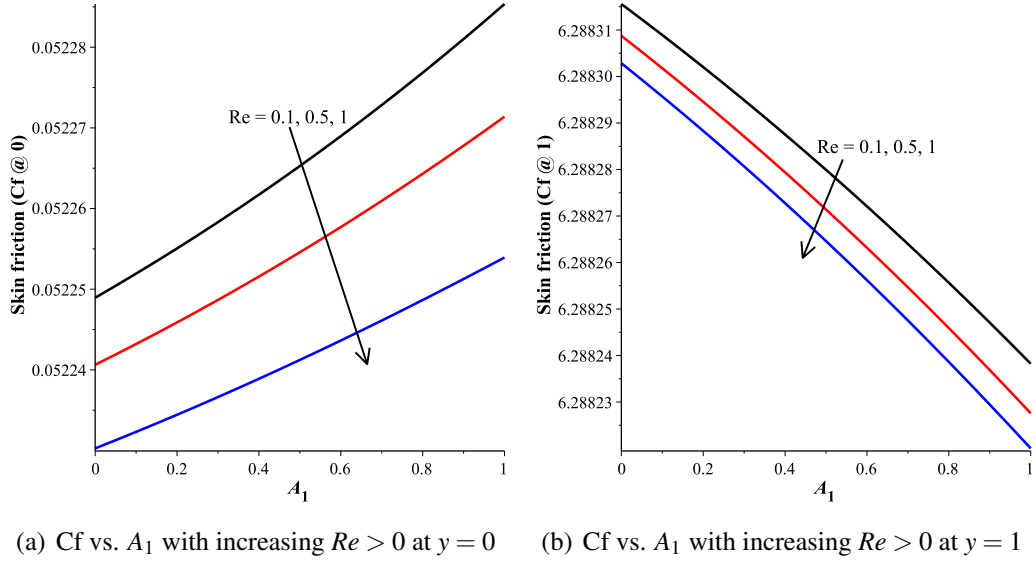


Figure 5.10: Skin friction versus A_1 with increasing $Re > 0$.

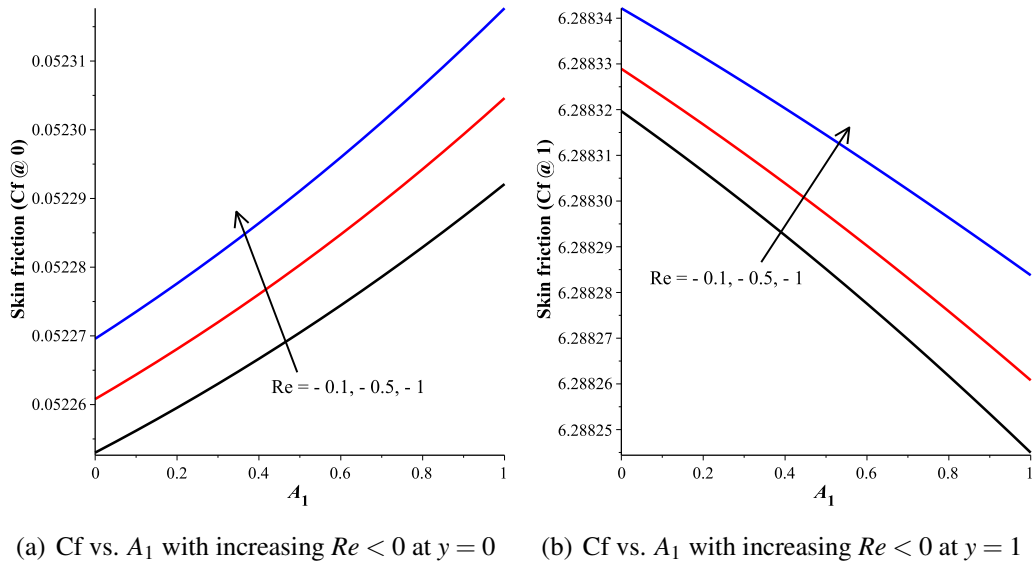


Figure 5.11: Skin friction versus A_1 with increasing $Re < 0$.

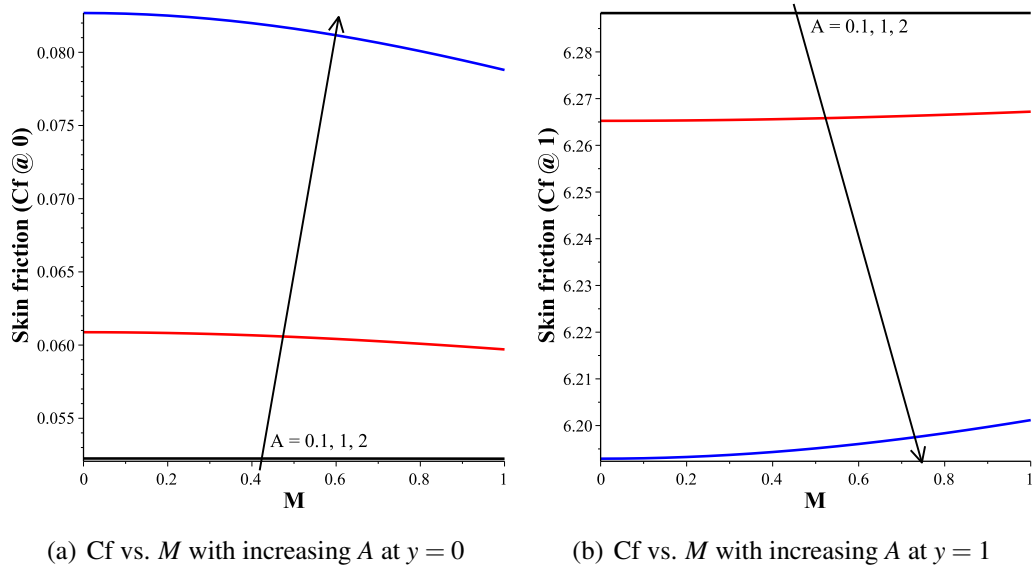


Figure 5.12: Skin friction versus M with increasing A .

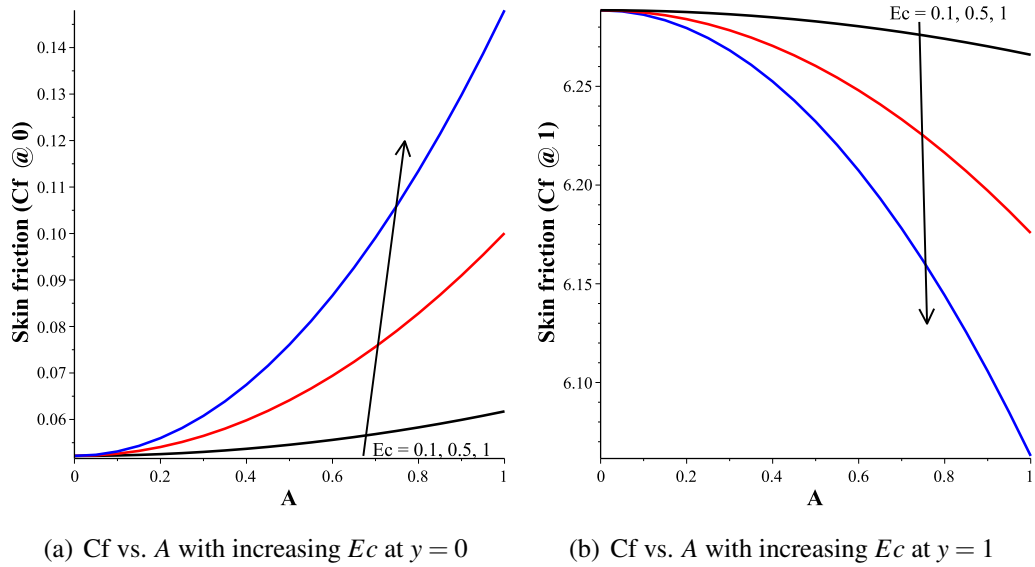


Figure 5.13: Skin friction versus A with increasing Ec .

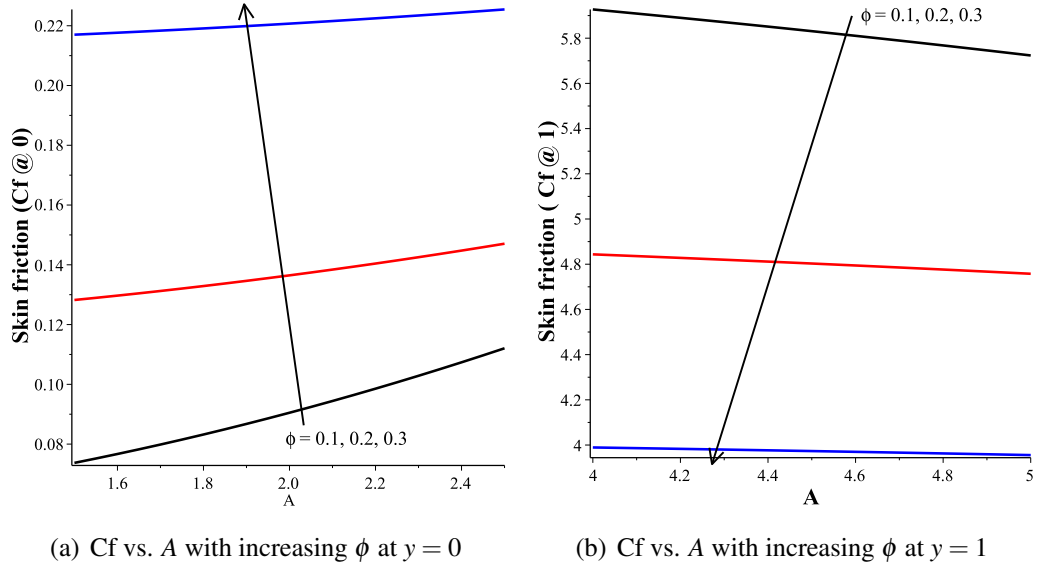


Figure 5.14: Skin friction versus A with increasing ϕ .

ϕ at bottom and top walls respectively are depicted in figures 5.18 (a) and (b). It can be seen that both magnetic field parameter M and nanoparticle particles volume fraction ϕ help in increasing Nusselt number Nu at the bottom wall but they have a decreasing effect on Nu at the top wall of the microchannel.

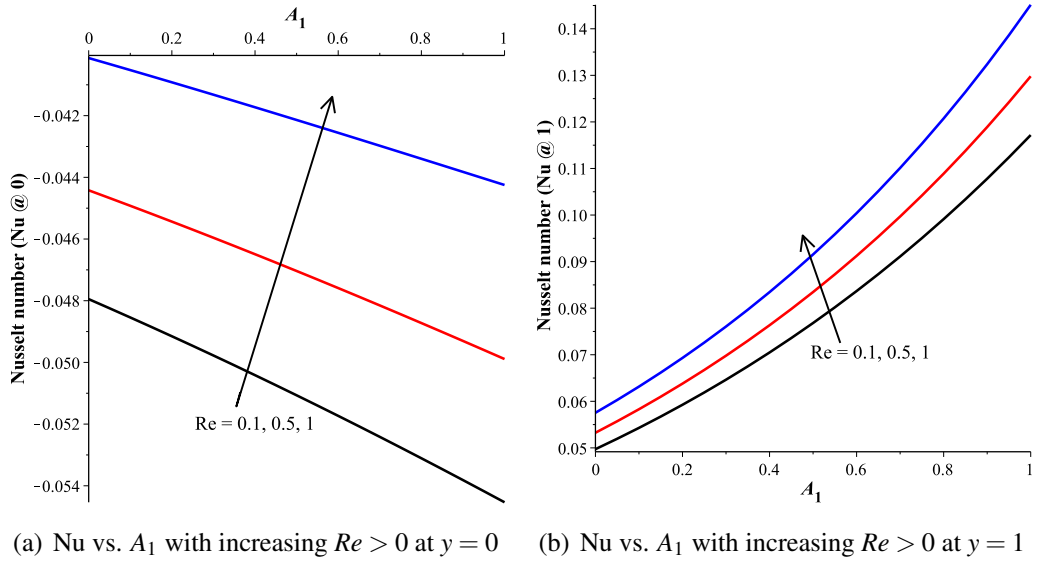


Figure 5.15: Nusselt number versus A_1 with increasing $Re > 0$.

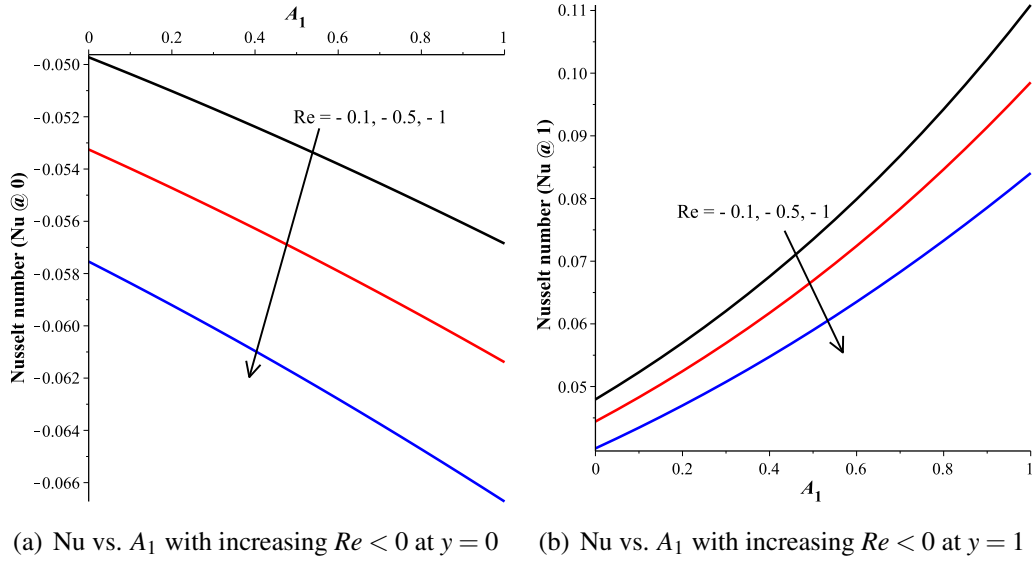


Figure 5.16: Nusselt number versus A_1 with increasing $Re < 0$.

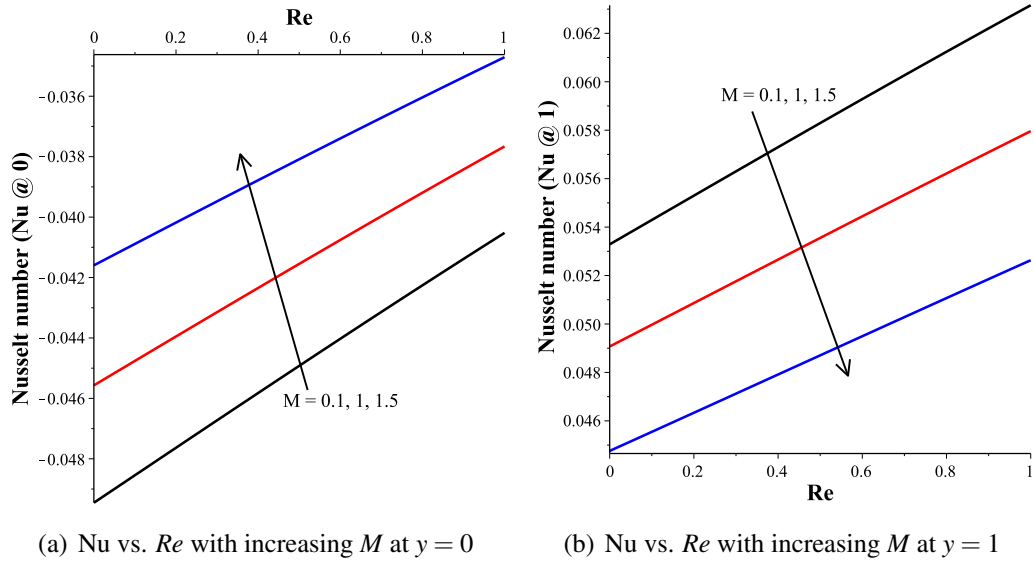


Figure 5.17: Nusselt number versus Re with increasing M .

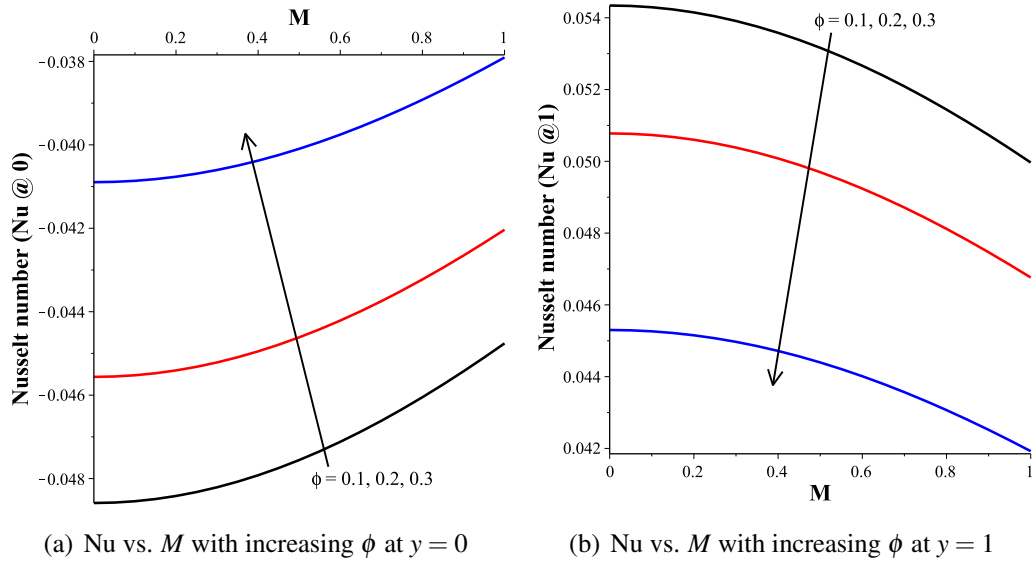


Figure 5.18: Nusselt number versus M with increasing ϕ .

CHAPTER 6

UNSTEADY HYDROMAGNETIC MIXED CONVECTION OF A RADIATING AND REACTING NANOFLUID IN A MICROCHANNEL WITH VARIABLE PROPERTIES

In this chapter, unsteady MHD mixed convection of nanofluid heat transfer in a permeable microchannel with temperature dependent fluid properties is studied under the influence of first order chemical reaction, thermal radiation and convective heating wall condition. The viscosity and thermal conductivity are assumed to be related with temperature exponentially. The governing equations of the flow problem is obtained in nonlinear PDEs form by employing Buongiorno model, Maxwell's equations, Boussinesq approximation, Rosseland's approximation and scaling analysis. The aim of the present study is to extend the work presented in chapter 4 by incorporating the combined effects of thermal radiation, chemical reaction and temperature dependent thermal conductivity on unsteady hydromagnetic nanofluid flow in a permeable microchannel with convective wall heating boundary condition.

6.1 Mathematical Model Formulation

Consider a hydromagnetic flow of an unsteady, incompressible, chemically reacting, temperature dependent thermal conductivity and variable viscosity nanofluid through a vertical parallel-plates microchannel of width a units. A uniform transverse magnetic field B_o is imposed in the direction parallel to positive $\bar{y}$ -axis and the induced magnetic field due to the motion of a conducting nanofluid is neglected. Both fluid injection and suction are assumed to take place at the left and the right walls respectively (shown in figure 6.1 below). A two-dimensional coordinate system is used where the left wall is held at $\bar{y} = 0$ and right wall is held at $\bar{y} = a$.

The nanofluid flow in the microchannel is generated due to the combined effects of pressure gradient in the $\bar{x}$ -axis direction, buoyancy forces and suction/injection velocity V at the microchannel walls. The left wall $\bar{y} = 0$ is convectively heated by a hot fluid at a temperature T_f with coefficient of heat transfer h_f and the right wall $\bar{y} = a$ is held at a temperature of T_w which is also equal to the fluid initial temperature. The fluid chemical compound could experience first order chemical reaction with Arrhenius kinetic (in the absence of reactant consumption) and without internal heat generation in the fluid. Applying the above assumptions, Buongiorno flow model, MHD model equations, and the usual Oberbeck-Boussinesq

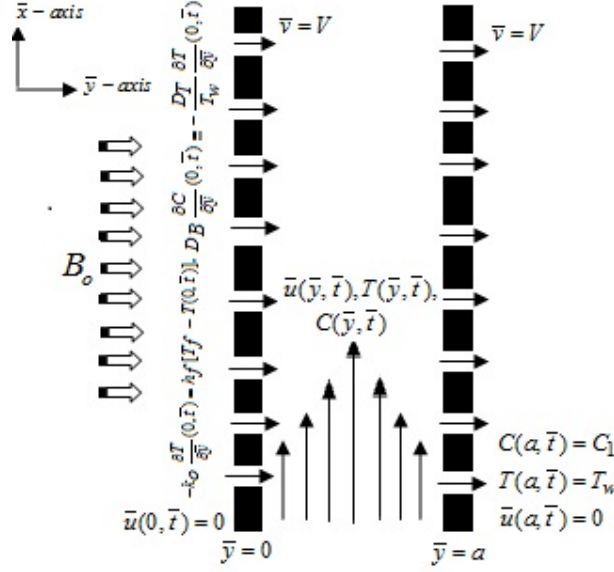


Figure 6.1: Flow set up with coordinate system

approximation, the basic unsteady conservation of mass, momentum, energy and concentration equations of nanofluid flow through microchannel in the presence of thermal radiation are written as:

$$\frac{\partial \bar{u}}{\partial \bar{x}} = 0, \quad (6.1)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{\rho} \frac{\partial}{\partial \bar{y}} \left[\mu(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \frac{\sigma}{\rho} B_o^2 \bar{u} + \beta_1 g(T - T_w) + \beta_2 g(C - C_o), \quad (6.2)$$

$$\begin{aligned} \frac{\partial T}{\partial \bar{t}} + V \frac{\partial T}{\partial \bar{y}} = & \frac{1}{\rho c_p} \frac{\partial}{\partial \bar{y}} \left[k(T) \frac{\partial T}{\partial \bar{y}} \right] + \tau \left[D_B \frac{\partial C}{\partial \bar{y}} \frac{\partial T}{\partial \bar{y}} + \frac{D_T}{T_w} \left(\frac{\partial T}{\partial \bar{y}} \right)^2 \right] + \frac{\mu(T)}{\rho c_p} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \\ & + \frac{\sigma B_o^2}{\rho c_p} \bar{u}^2 - \frac{1}{\rho c_p} \frac{\partial q_r}{\partial \bar{y}}, \end{aligned} \quad (6.3)$$

$$\frac{\partial C}{\partial \bar{t}} + V \frac{\partial C}{\partial \bar{y}} = D_B \frac{\partial^2 C}{\partial \bar{y}^2} + \frac{D_T}{T_w} \frac{\partial^2 T}{\partial \bar{y}^2} - \epsilon(C - C_o), \quad (6.4)$$

with the initial and boundary conditions

$$\bar{u}(\bar{y}, 0) = 0, \quad T(\bar{y}, 0) = T_w, \quad C(\bar{y}, 0) = C_o, \quad (6.5)$$

$$\begin{aligned} \bar{u}(0, \bar{t}) = 0, \quad -k_o \frac{\partial T}{\partial \bar{y}}(0, \bar{t}) &= h_f [T_f - T(0, \bar{t})], \\ D_B \frac{\partial C}{\partial \bar{y}}(0, \bar{t}) &= -\frac{D_T}{T_w} \frac{\partial T}{\partial \bar{y}}(0, \bar{t}), \\ \bar{u}(a, \bar{t}) = 0, \quad T(a, \bar{t}) &= T_w, \quad C(a, \bar{t}) = C_1, \end{aligned} \quad (6.6)$$

The expression for dynamic viscosity $\mu(T)$ and thermal conductivity $k(T)$ of nanofluid are varying with temperature exponentially (Makinde and Franks, 2014) and given as:

$$\mu(T) = \mu_o e^{-\gamma_1(T-T_w)} \quad (6.7)$$

$$k(T) = k_o e^{\gamma_2(T-T_w)} \quad (6.8)$$

where $\bar{u}$ is the nanofluid velocity in $\bar{x}$ -direction, T is the temperature of nanofluid, C is the nanoparticles concentration, V is the suction/injection velocity, $\bar{P}$ is the nanofluid pressure, $\bar{t}$ is the time, ρ is the density of nanofluid, μ is the dynamic viscosity of the nanofluid, g is the acceleration due to gravity, β_1 and β_2 are the thermal and solutal expansion coefficients respectively, k is the thermal conductivity of the nanofluid, τ is the ratio between the effective heat capacity of nanoparticle material and heat capacity of the base fluid, D_T is the thermophoretic diffusion coefficient, D_B is Brownian motion diffusion coefficient, q_r is the thermal radiative heat flux, ε the reaction rate, T_w is nanofluid initial temperature(nanofluid temperature at the right wall), C_o is nanofluid initial concentration, C_1 is nanofluid concentration at the right wall, σ is the nanofluid electrical conductivity, a is the channel width, c_p is the specific heat at constant pressure, μ_o is the nanofluid viscosity at temperature T_w and concentration C_o , k_o is the nanofluid thermal conductivity at temperature T_w and concentration C_o , γ_1 is viscosity variation coefficient, γ_2 is thermal conductivity variation coefficient.

The radiative heat flux by using Rosseland's approximation (Rosseland, 1931) can be written as:

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial \bar{y}}, \quad (6.9)$$

in which σ^* is the Stefan-Boltzman constant and k^* is the mean absorption coefficient. Assuming the difference of the temperature T of the flowing fluid and hot fluid temperature T_f heating the left wall is small so that the temperature T^4 can be linearized as:

$$T^4 \approx 4T_f^3 T - 3T_f^4, \quad (6.10)$$

using Taylor's series expansion and ignoring second and higher order terms in the expansion.

Substituting (6.10) into (6.9), we get

$$\frac{\partial q_r}{\partial \bar{y}} = -\frac{16\sigma^* T_f^3}{3k^*} \frac{\partial^2 T}{\partial \bar{y}^2}. \quad (6.11)$$

In transforming equations (6.2)-(6.6) into non-dimensional form, the following dimensionless variables and parameters are introduced.

$$\begin{aligned}
x = \frac{\bar{x}}{a}, y = \frac{\bar{y}}{a}, u = \frac{\bar{u}a}{v_o}, t = \frac{v_o \bar{t}}{a^2}, \theta = \frac{T - T_w}{T_f - T_w}, \phi = \frac{C - C_o}{C_1 - C_o}, \bar{P} = \frac{\rho v_o^2}{a^2} P, Re = \frac{Va}{v_o}, A = -\frac{dp}{dx}, \\
A_1 = \gamma_1(T_f - T_w), M^2 = \frac{\sigma B_o^2 a^2}{\rho v_o}, Gr = \frac{\beta_1 g a^3 (T_f - T_w)}{v_o^2}, Gc = \frac{\beta_2 g a^3 (C_1 - C_o)}{v_o^2}, Pr = \frac{v_o}{\alpha_o}, \\
Nb = \frac{\tau D_B (C_1 - C_o)}{v_o}, Nt = \frac{\tau D_T (T_f - T_w)}{T_o v_o}, A_2 = \gamma_2(T_f - T_w), Ec = \frac{v_o^2}{a^2 c_p (T_f - T_w)}, \\
Nr = \frac{4\sigma^* T_f^3}{k^* k_o}, Sc = \frac{v_o}{D_B}, rx = \frac{\varepsilon a^2}{v_o}, Bi = \frac{ah_f}{k_o}.
\end{aligned} \tag{6.12}$$

Substituting equations (6.11) and (6.12) into equations (6.2)-(6.6), a system of dimensionless partial differential equations are obtained.

$$\frac{\partial u}{\partial t} + Re \frac{\partial u}{\partial y} = A - A_1 \cdot e^{-A_1 \theta} \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial y} + e^{-A_1 \theta} \frac{\partial^2 u}{\partial y^2} - M^2 u + Gr \cdot \theta + Gc \cdot \phi, \tag{6.13}$$

$$\begin{aligned}
Pr \frac{\partial \theta}{\partial t} + Pr \cdot Re \frac{\partial \theta}{\partial y} = A_2 \cdot e^{A_2 \theta} \left(\frac{\partial \theta}{\partial y} \right)^2 + Pr \cdot Nb \frac{\partial \phi}{\partial y} \frac{\partial \theta}{\partial y} + Pr \cdot Nt \left(\frac{\partial \theta}{\partial y} \right)^2 \\
+ Pr \cdot Ec \left(\frac{\partial u}{\partial y} \right)^2 e^{-A_1 \theta} + Pr \cdot M^2 \cdot Ec \cdot u^2 + \left(e^{A_2 \theta} + \frac{3}{4} Nr \right) \cdot \frac{\partial^2 \theta}{\partial y^2},
\end{aligned} \tag{6.14}$$

$$Nb \cdot Sc \frac{\partial \phi}{\partial t} + Nb \cdot Re \cdot Sc \frac{\partial \phi}{\partial y} = Nb \cdot \frac{\partial^2 \phi}{\partial y^2} + Nt \cdot \frac{\partial^2 \theta}{\partial y^2} - Nb \cdot Sc \cdot rx \cdot \phi, \tag{6.15}$$

with initial and boundary conditions

$$u(y, 0) = 0, \theta(y, 0) = 0, \phi(y, 0) = 0, \tag{6.16}$$

$$\begin{aligned}
u(0, t) = 0, \frac{\partial \theta}{\partial y}(0, t) = Bi[\theta(0, t) - 1], \\
Nb \frac{\partial \phi}{\partial y}(0, t) = -Nt \frac{\partial \theta}{\partial y}(0, t), \\
u(1, t) = 0, \theta(1, t) = 0, \phi(1, t) = 1,
\end{aligned} \tag{6.17}$$

where Re is the Reynold suction/injection parameter, A is the pressure gradient parameter, A_1 is the variable viscosity variation parameter, M is the magnetic field parameter, Gr is the thermal Grashof number, Gc is the solutal/mass Grashof number, Pr is the Prandtl number, Nb is the Brownian motion parameter, Nt is the thermophoretic parameter, A_2 is the variable thermal conductivity variation parameter, Ec is the Eckert number, Nr is thermal radiation parameter, Sc is the Schmidt number, rx is chemical reaction parameter and Bi is the Biot number.

The associated quantities of special interest of the study are the skin friction coefficient

Cf , the Nusselt number Nu , and the Sherwood number Sh which are defined as

$$Cf = \frac{\tau_w}{\rho V^2}, \quad Nu = \frac{aq_w}{k(T)[T_f - T_w]}, \quad \text{and} \quad Sh = \frac{aq_m}{D_B(C_1 - C_o)}, \quad (6.18)$$

where τ_w is the wall shear stress, q_w is the heat flux and q_m is the mass flux at the left and right walls of the microchannel are given by

$$\tau_w = \mu \frac{\partial \bar{u}}{\partial \bar{y}} \Big|_{\bar{y}=0,a}, \quad q_w = -k \frac{\partial T}{\partial \bar{y}} \Big|_{\bar{y}=0,a}, \quad \text{and} \quad q_m = -D_B \frac{\partial C}{\partial \bar{y}} \Big|_{\bar{y}=0,a}. \quad (6.19)$$

The dimensionless form of equation(6.18) above are:

$$Re^2 Cf = e^{-A_1 \theta} \frac{\partial u}{\partial y} \Big|_{y=0,1}, \quad Nu = - \left(1 + \frac{4}{3} e^{-A_1 \theta} Nr \right) \cdot \frac{\partial \theta}{\partial y} \Big|_{y=0,1}, \quad (6.20)$$

$$\text{and} \quad Sh = - \frac{\partial \phi}{\partial y} \Big|_{y=0,1}.$$

6.2 Numerical Method

The set of dimensionless equations (6.13)-(6.15) subjected to the conditions (6.16) and (6.17) are highly nonlinear coupled system of partial differential equations whose solutions are tackled through semi-discretization finite difference scheme along with Runge-Kutta Fehlberg integration method. Spatial interval $[0, 1]$ is subdivided into $N + 1$ equal subintervals. The nodal spacing and points are defined respectively as $\Delta y = \frac{1}{N+1}$ and $y_i = i\Delta y$, for $i = 0, 1, 2, \dots, N + 1$. The first and second space derivatives are approximated by a second order accuracy $(\Delta y)^2$ of finite difference scheme. Let u_i , θ_i , and ϕ_i represent $u(y_i, t)$, $\theta(y_i, t)$, and $\phi(y_i, t)$, respectively. Then the semi-discretization scheme with centered finite difference method for equations (6.13)-(6.17) are reduced to:

$$\begin{aligned} \frac{du_i}{dt} + Re \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) &= A - A_1 e^{-A_1 \theta_i} \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) \\ &+ e^{-A_1 \theta_i} \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} \right) - M^2 u_i + Gr \theta_i + Gc \phi_i, \end{aligned} \quad (6.21)$$

$$\begin{aligned} Pr \frac{d\theta_i}{dt} + Pr Re \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) &= A_2 e^{A_2 \theta_i} \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right)^2 + Pr Nb \left(\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta y} \right) \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) \\ &+ Pr Nt \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right)^2 + Pr Ec \cdot e^{-A_1 \theta_i} \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right)^2 + \left(e^{A_2 \theta_i} + \frac{4}{3} Nr \right) \left(\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta y)^2} \right) \\ &+ Pr M^2 Ec \cdot u_i^2, \end{aligned} \quad (6.22)$$

$$Sc \frac{d\phi_i}{dt} + ReScNb \left(\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta y} \right) = Nb \left(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta y)^2} \right) + Nt \left(\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta y)^2} \right) - rx \cdot \phi_i, \quad (6.23)$$

with initial conditions

$$u_i(0) = 0, \quad \theta_i(0) = 0, \quad \phi_i(0) = 0, \quad 1 \leq i \leq N. \quad (6.24)$$

The boundary conditions at $y = 0, 1$ are transformed to incorporate as follows:

$$\begin{aligned} u_o(t) &= 0, \quad \theta_1(t) = \theta_o(t)[1 + Bi\Delta y] - Bi\Delta y, \\ \phi_1(t) &= \phi_o(t) - \frac{Nt}{Nb}[\theta_1(t) - \theta_o(t)], \\ u_{N+1}(t) &= 0, \quad \theta_{N+1}(t) = 1, \quad \phi_{N+1}(t) = 1. \end{aligned} \quad (6.25)$$

Equations (6.21)-(6.25) is a system of first order nonlinear ordinary differential equations with known initial conditions and its solution can be obtained successively by Runge-Kutta Fehlberg integration techniques (Na, 1979).

6.3 Results and Discussion

The combined effect of unsteadiness, magnetic field, temperature dependent variable viscosity and thermal conductivity, first order chemical reaction, and thermal radiation on nanofluid flow through a vertical parallel-plates microchannel in the presence of suction or injection has been investigated. Numerical solutions to nonlinear initial boundary value problems have been carried out by semi-discretization via centered finite difference method with Runge-Kutta Fehlberg integration technique. Results of representative velocity field, temperature field, nanoparticles concentration, skin friction, Nusselt number and Sherwood number are analyzed by using model parameter values $Re = M = Gc = Gr = Nt = Ec = A_1 = A_2 = rx = 0.1$, $Nb = Nr = Bi = 0.5$, $Sc = 0.6$, $Pr = 6.2$, $A = 1$, unless it is stated in figures (6.2)-(6.23).

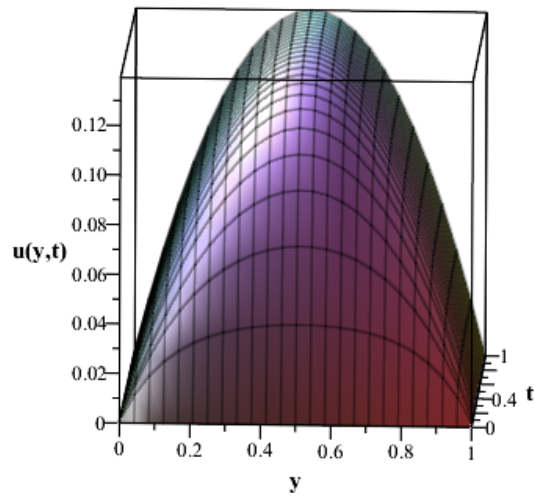
6.3.1 Time Evolution of Velocity, Temperature and Concentration Profiles

The time evolution of flow velocity, temperature and concentration profiles are highlighted in figures (6.2)-(6.5) for a fixed values of thermophysical parameters. Figure 6.2 (a) indicates that nanofluid velocity of zero value in space and time at both microchannel walls $y = 0$ and $y = 1$ start growing to attain its maximum value around the hub of the channel width. Decreasing of nanofluid temperature from its ceiling value near the left wall to its

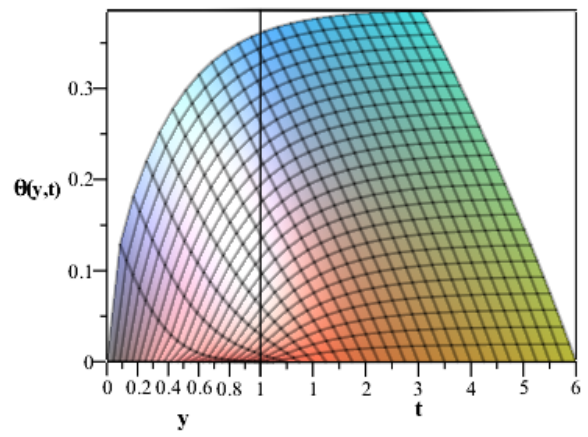
zero value at/around the right wall of the microchannel is observed, but there is an increase in temperature for a fixed point in the channel width as time increases for a certain while as depicted in figure 6.2 (b). The rise of nanoparticles concentration from its minimum value in space and time at the left wall to its highest value 1 at the right wall of the microchannel is seen in figure 6.2 (c). From Figures 6.3 (a) and (b), as the time increases, the nanofluid flow velocity attains its maximum value at the center of the microchannel before a steady state velocity value reached. Moreover, the steady state flow velocity obtained nearly $t = 2$ units. It is noted in figure 6.4 (a) that the nanofluid temperature is maximum at the left wall and decreasing toward the right wall of the microchannel until a steady state profile is attained for a given set of parameter values. The nanofluid temperature along the microchannel flow direction increases with increasing time and reaches its minimum value at the right wall of the microchannel as demonstrated in figure 6.4 (b). It can be noticed from figures 6.5 (a) and (b) that nanoparticles concentration in the fluid increases across the whole microchannel width until a steady state constant concentration value is achieved. But, there is a dynamic fluctuation of particles concentration in the fluid near the right wall $y = 1$ of the microchannel for $0 < t < 2$.

6.3.2 Effects of Model Parameters on Steady State Velocity Profiles

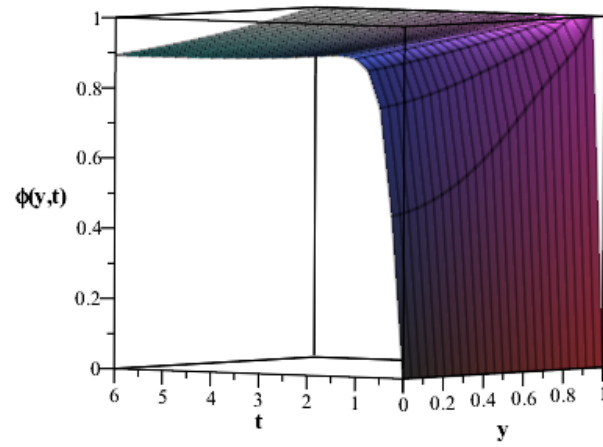
Figures (6.6)-(6.8) display the steady state velocity profiles for various values of model parameters variation. It is worth noting that the maximum nanofluid velocity is attained around the center line of the microchannel while its minimum value is at both permeable walls. The effect of suction/injection parameter Re on nanofluid velocity are illustrated in figures 6.6 (a) and (b). It is seen in figure 6.6 (a) that the nanofluid velocity get decreased around the left wall followed by a wavering of fluid velocity around the centerline and it gets increased near the right permeable wall with an increase in suction parameter $Re > 0$. From figure 6.6 (b), as injection parameter $Re < 0$ increases, the velocity of the fluid increased around the left wall followed by a fluctuation of velocity for short while before a decrease of velocity happened. The impacts of thermal and solutal Grashof numbers Gr , Gc on the nanofluid velocity are presented in figure 6.7 (a). The nanofluid velocity overshoot with an increase in both Grashof numbers. This is due to increased buoyancy force, which favors the fluid flow. This outcome agreed exactly with the result of Hindebu et al. (2021). Figure 6.7 (b) presents the effects of viscosity variation parameter A_1 on the nanofluid velocity in the presence and absence of pressure gradient A . Increment of A_1 tend to increase the nanofluid velocity in the presence and absence of A . This is in line with the observation made by (Monaledi and Makinde, 2018; Makinde, 2018; Makinde and Eegunjobe, 2013). But a significant overshoot of velocity is observed in the presence of A . Figure 6.8 (a) represents nanofluid velocity variation with chemical reaction parameter, rx and magnetic field, M . It is found that there is a considerable decline of nanofluid velocity as magnetic field M and chemical reaction



(a) 3D velocity Profile



(b) 3D temperature profile



(c) 3D concentration profile

Figure 6.2: Three dimensional velocity, temperature and concentration profile.

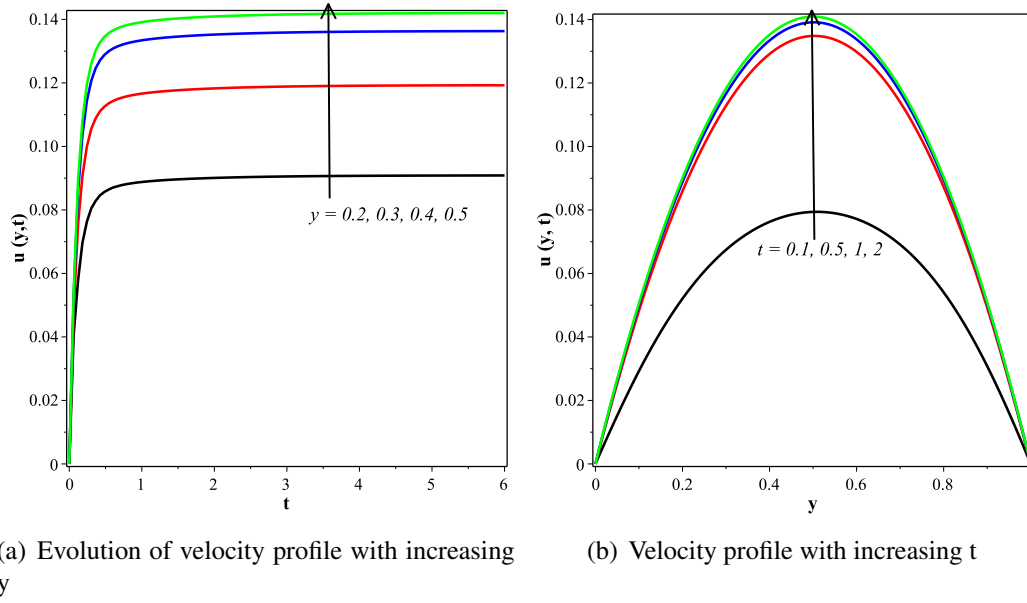


Figure 6.3: Velocity profile across the channel with increasing time t .

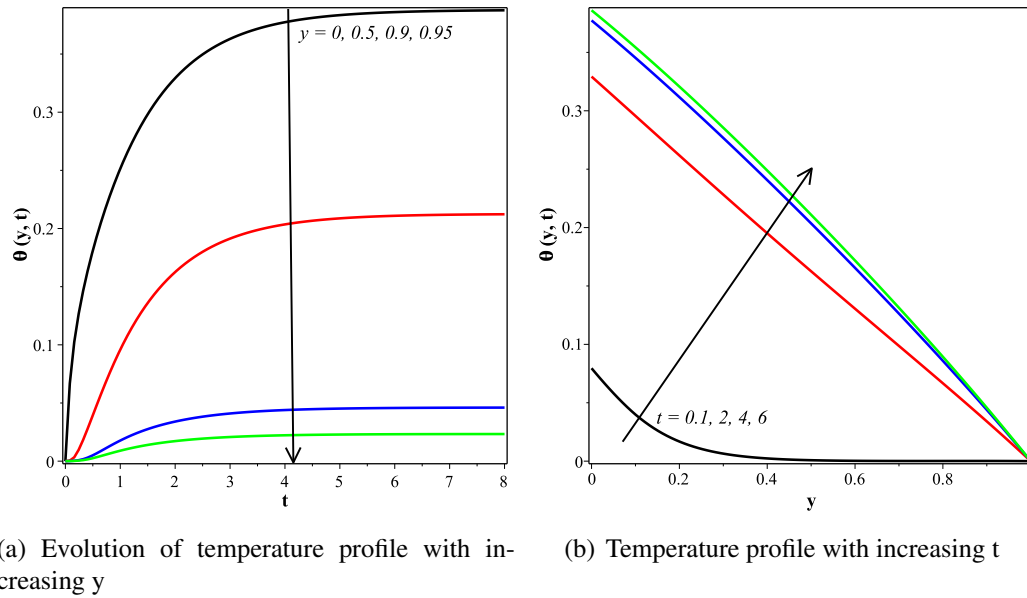


Figure 6.4: Temperature profile across the channel with increasing time t .

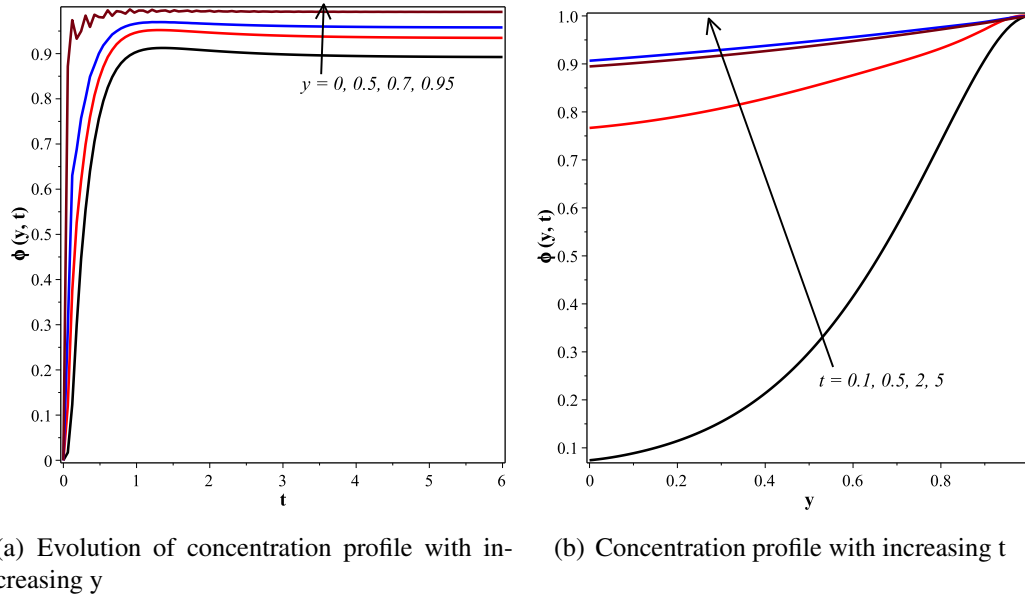


Figure 6.5: Concentration profile across the channel with increasing time t .

parameter rx increases simultaneously. The magnetic field is found to reduce the velocity across the microchannel. This is because the magnetic field provides a resisting type of force called Lorentz force. This force tends to lessen the motion of the fluid and as a result the velocity reduces. The rise of Eckert number Ec and Biot number Bi shows an increment in the nanofluid velocity as depicted in figure 6.8 (b). Here, Makinde and Eegunjobi (2013) presented a similar result. However, Biot number has no significant impact on nanofluid velocity for large values of Eckert numbers.

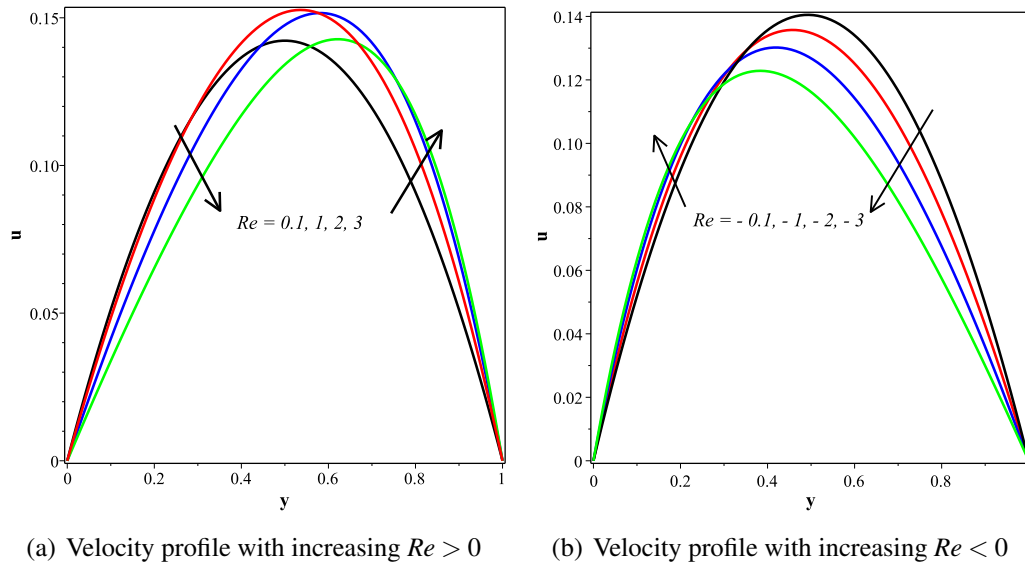


Figure 6.6: Steady state velocity profiles with increasing Re .

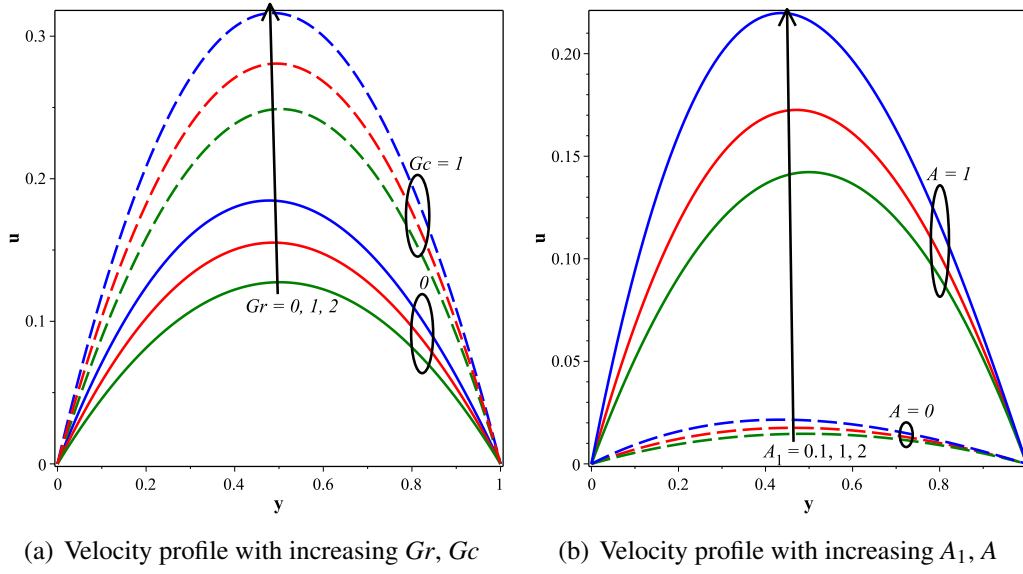


Figure 6.7: Steady state velocity profiles with increasing Gr, Gc, A_1 and A .

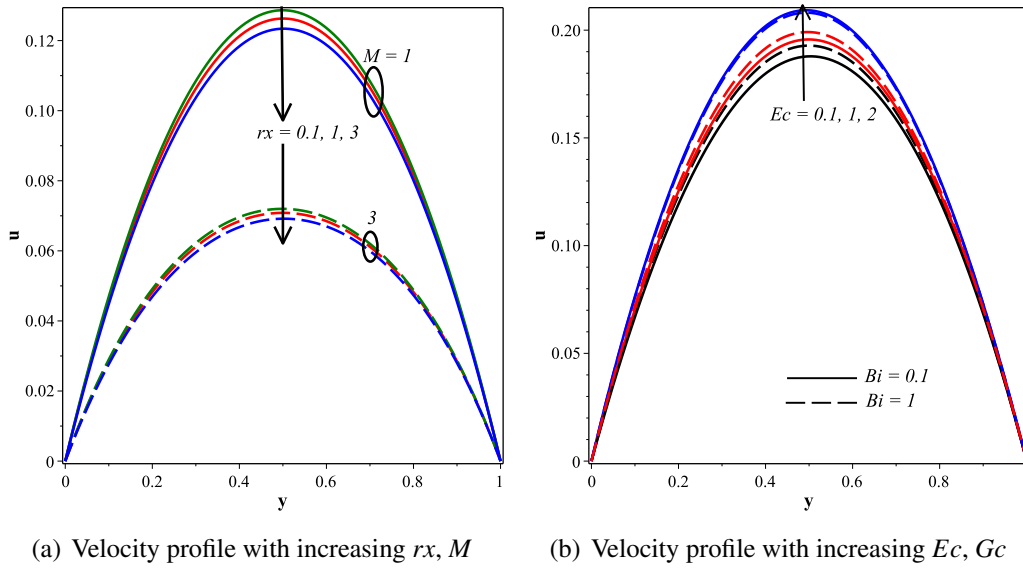


Figure 6.8: Steady state velocity profiles with increasing rx, M, Ec , and Gc .

6.3.3 Effects of Model Parameters on Steady State Temperature Profiles

The variation of nanofluid temperature is illustrated in figure 6.9 (a) with suction/injection parameters Re , in figure 6.9 (b) with viscosity variation parameter A_1 and thermal radiation parameter Nr , in figure 6.10 (a) with thermal conduction variation parameter A_2 and Biot number Bi , and in figure 6.10 (b) with Eckert number Ec and chemical reaction parameter rx . In general, the nanofluid temperature is maximum at the left wall and it continued to decrease across the channel width to attain minimal value at the right wall. An increase suction parameter $Re > 0$ tends to increase the nanofluid temperature, where as the injection parameter $Re < 0$ helps to decrease the nanofluid temperature in the microchannel. A more significant influence of suction/injection parameter Re on the nanofluid temperature is found across the microchannel width except near the right wall of the microchannel. Figure 6.9(b) demonstrates the influence of viscosity variation parameter A_1 and thermal radiation parameter Nr . It is evident that the nanofluid temperature rises as A_1 increases where as an opposite effect is seen when Nr increases. In this observation, the results are similar to the previous outcomes reported by Monaiedi and Makinde (2018). The thermal conductivity variation parameter A_2 and Biot number Bi tend to increase the nanofluid temperature observed in figure 6.10 (a). As the thermal conductivity parameter A_2 increases, the heat is more readily transferred particularly in nanofluid which leads to enhancement of nanofluid temperature in the microchannel. In figure 6.10 (b), Eckert number Ec help to increase the nanofluid temperature, while the chemical reaction parameter rx tend to decrease it.

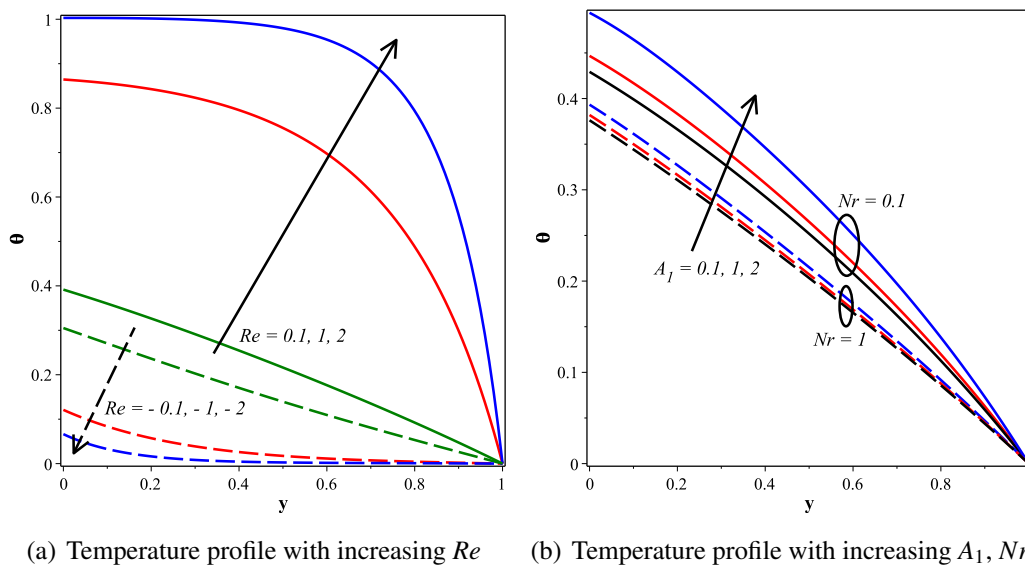
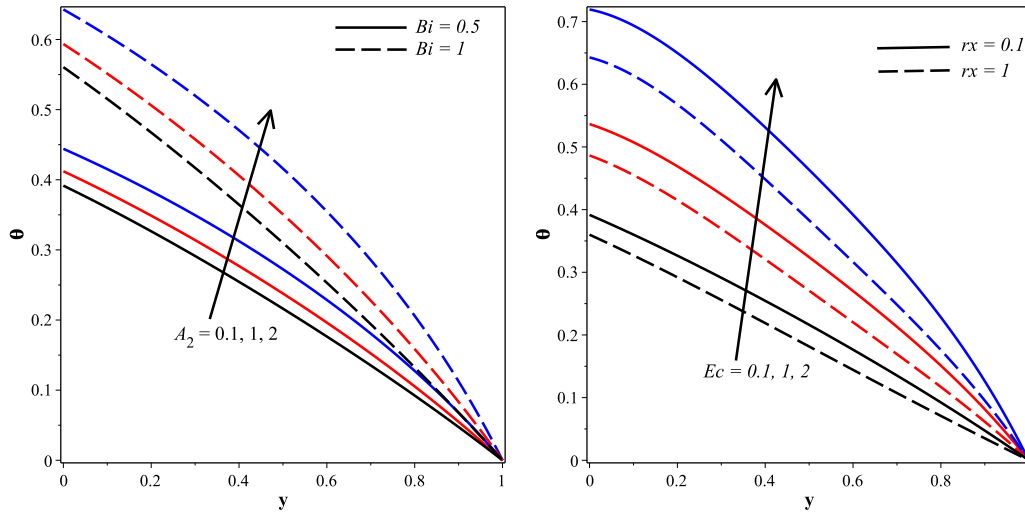


Figure 6.9: Steady state temperature profiles with increasing Re , A_1 and Nr .

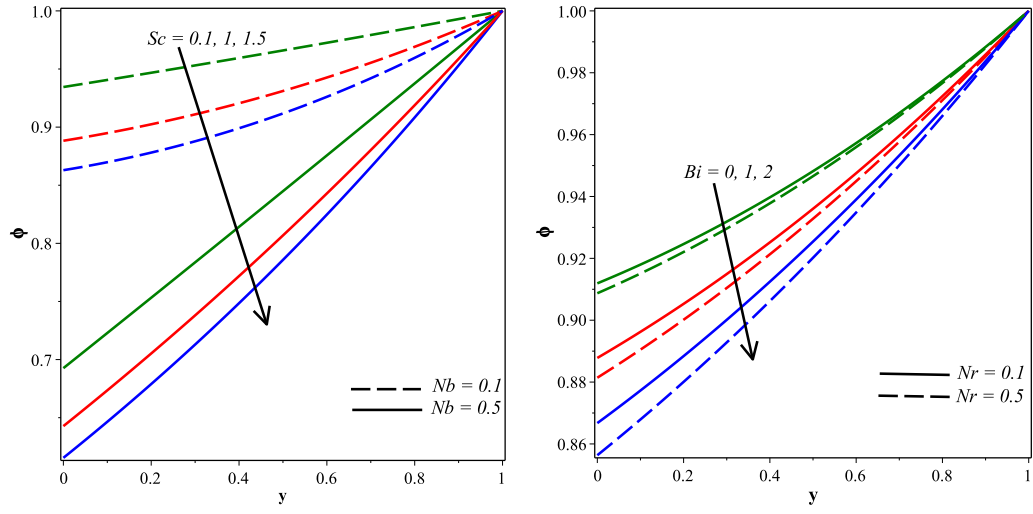


(a) Temperature profile with increasing A_2, Bi (b) Temperature profile with increasing Ec, rx

Figure 6.10: Steady state temperature profiles with increasing A_2, Bi, Ec and rx .

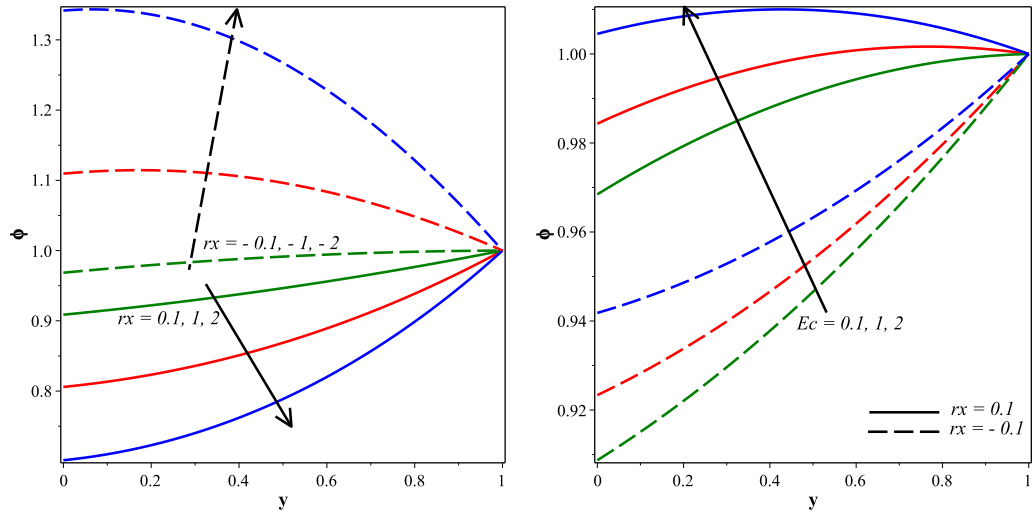
6.3.4 Effects of Model Parameters on Steady State Concentration Profiles

The nanoparticles concentration profiles under the influence of Schmidt number and Brownian motion parameter (Sc, Nb), Biot number and radiation parameter (Bi, Nr), chemical reaction parameter (generating/degenerating) ($rx < 0, rx > 0$), and Eckert number and chemical reaction parameter (Ec, rx) are shown in figures 6.11 (a), 6.11 (b), 6.12 (a) and 6.12 (b), respectively. The nanoparticles concentration in the nanofluid flow field through microchannel decreased with a rise in Schmidt number Sc , Brownian motion parameter Nb , Biot number Bi , and thermal radiation parameter Nr is illustrated in figure 6.11 (a) and (b). With respect to chemical reaction parameter rx , the nanoparticles concentration enhanced with an increase in generating chemical reaction parameter $rx < 0$ where as nanoparticles concentration reduces in degenerating chemical reaction parameter $rx > 0$ in the nanofluid flow field throughout the microchannel as depicted in figure 6.12 (a). This result is in agreement with the preceding observation reported by Ullah et al. (2017). Physically, degenerative chemical reaction leads to enhance the decomposition rate of reactant species, where as generative chemical reaction helps diffusion of nanoparticles. The nanoparticles concentration profile for different values of Eckert number Ec in the presence of degenerating/generating chemical reaction parameter rx is shown in figure 6.12 (b). It is observed that an enhancement of nanoparticles concentration in the flow field across the microchannel in the presence of both degenerating/generating chemical reaction parameter rx .



(a) Concentration profile with increasing Sc, Nb (b) Concentration profile with increasing Bi, Nr

Figure 6.11: Steady state concentration profiles with increasing Sc, Nb, Bi and Nr .



(a) Concentration profile with increasing $rx < 0$, (b) Concentration profile with increasing $Ec, rx > 0$

Figure 6.12: Steady state concentration profiles with increasing rx and Ec

6.3.5 Skin Friction, Nusselt Number and Sherwood Number

The effects of various thermophysical parameters on skin friction, Nusselt number, and Sherwood number are shown in figures (6.13)-(6.23). It is observed in figure 6.13 that increasing Bi and Pr dwindles the skin friction Cf at the left wall $y = 0$ and increases skin friction at the right wall $y = 1$. Meanwhile, a rise in Sc increases the skin friction at the left wall $y = 0$ and decreases it at the right wall $y = 1$. In figures 6.14 (a) and (b), increasing both the thermal Grashof Gr and solutal Grashof Gc diminish the skin friction Cf at the wall $y = 0$ and intensify Cf at the wall $y = 1$. However, thermal radiation parameter Nr augments Cf at $y = 0$ and reduces it at $y = 1$. The combined rise of chemical reaction parameter $rx > 0$ and magnetic field parameter M increases the skin friction Cf at the wall $y = 0$ and decreases the skin friction Cf at the wall $y = 1$ in the existence of both suction and injection as illustrated in figures 6.15 (a) and (b).

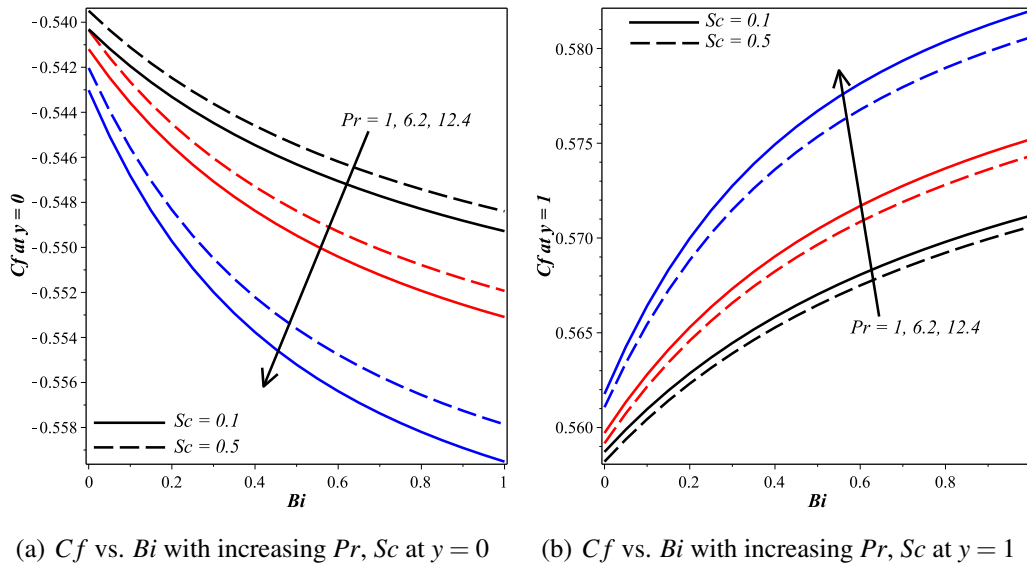
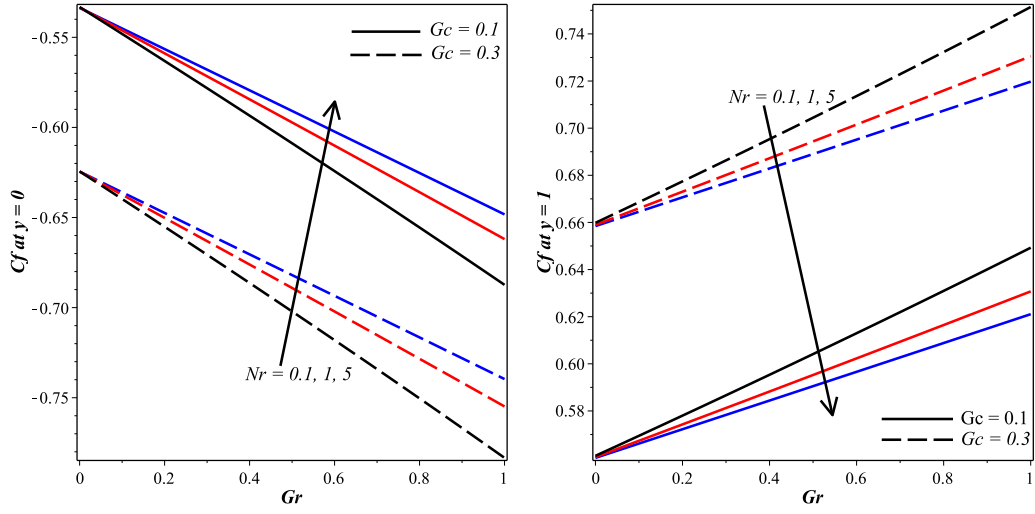


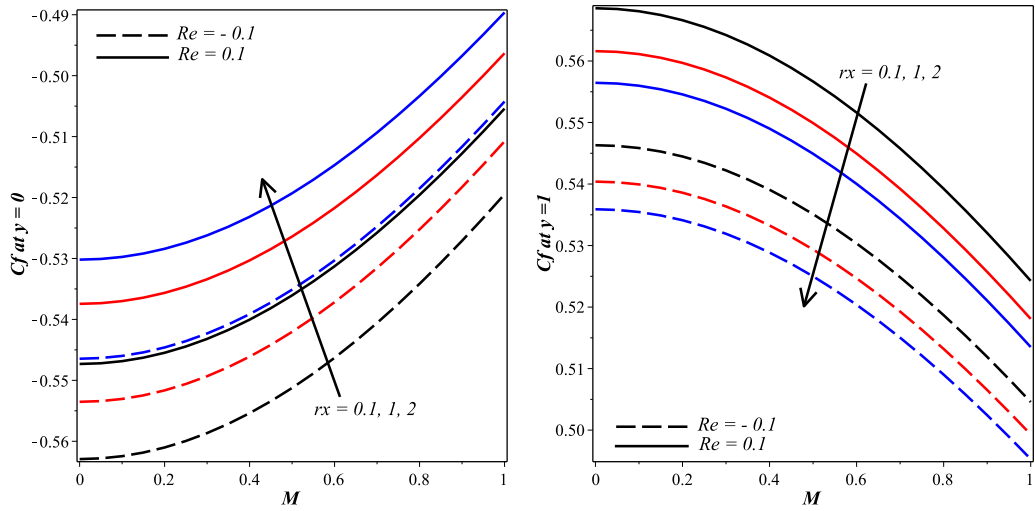
Figure 6.13: Skin friction Cf versus Bi with increasing Pr and Sc .

The effects of various model parameters on Nusselt number at both microchannel walls ($y = 0$, $y = 1$) are illustrated in figures (6.16)-(6.18). Figures 6.16 (a) and (b) represent the combined effects of pressure gradient A , thermal conductivity variation parameter A_2 and Brownian motion parameter Nb on Nusselt number at the walls $y = 0$ and $y = 1$, respectively. From these figures it is clear that Nusselt number increases with an increase in Nb and decreases with an increase in A at the wall $y = 0$, while a reverse trend is observed at the wall $y = 1$ with an increase in Nb and A . It is also seen that the thermal conductivity variation parameter A_2 has an increasing effects on Nusselt number at both left $y = 0$ and right $y = 1$ walls of the microchannel. This is because the working fluid become hotter which leading to the fluid temperature gradient at both walls augment, since thermal conductivity is dependent on temperature. In figures 6.17 (a) and (b), an increase in Eckert number Ec and magnetic



(a) Cf vs. Gr with increasing Nr, Gc at $y = 0$ (b) Cf vs. Gr with increasing Nr, Gc at $y = 1$

Figure 6.14: Skin friction Cf versus Gr with increasing Nr and Gc .



(a) Cf vs. M with increasing rx, Re at $y = 0$ (b) Cf vs. M with increasing rx, Re at $y = 1$

Figure 6.15: Skin friction Cf versus M with increasing rx and Re .

field parameter M , respectively, help to decrease and increase the Nusselt number at the left wall $y = 0$ whereas an opposite phenomenon happened at the right wall $y = 1$ of the microchannel. In addition, the Nusselt number Nu increases with increasing Biot number Bi at both left $y = 0$ and right $y = 1$ walls of the microchannel. This is due to the fact that the higher Biot number Bi mean higher level of convective heating (temperature gradient) at the left wall $y = 0$ (as it can be seen from temperature boundary condition equation (6.17)) and hence the overall temperature profile increases with increasing Biot number Bi . This attribute to a rise in heat transfer rate at both walls $y = 0$ and $y = 1$ of the microchannel which resulting an increase in Nusselt number Nu at both walls $y = 0$ and $y = 1$ of the microchannel. The change of Nusselt number Nu versus Schmidt number Sc for various values chemical reaction parameter $rx > 0$ with Eckert number $Ec = -0.1$ (heating) and $Ec = 0.1$ (cooling) at the walls $y = 0$ and $y = 1$ is portrayed in figures 6.18 (a) and (b). It is observed that Sc and $rx > 0$ help to augment the Nusselt number Nu at both walls $y = 0$ and $y = 1$ of the microchannel with joint existence of $Ec = -0.1$ (heating) and $Ec = 0.1$ (cooling). Moreover, Figures 6.19 (a) and (b) reveal the variation of Nusselt number Nu against Prandtl number Pr with changing values of thermal radiation parameter Nr in the presence of both suction ($Re > 0$) and injection ($Re < 0$) at the walls $y = 0$ and $y = 1$, respectively. Enhancement of thermal radiation parameter Nr proliferates the Nusselt number Nu at both channel walls ($y = 0$ and $y = 1$) for case of suction and injection. Further, a slight increment when $Re < 0$ and decrement when $Re > 0$ of Nusselt number Nu is seen at the wall $y = 0$ with rise in Prandtl number Pr . Nevertheless, a significant opposite trend of Nu is observed at the wall $y = 1$.

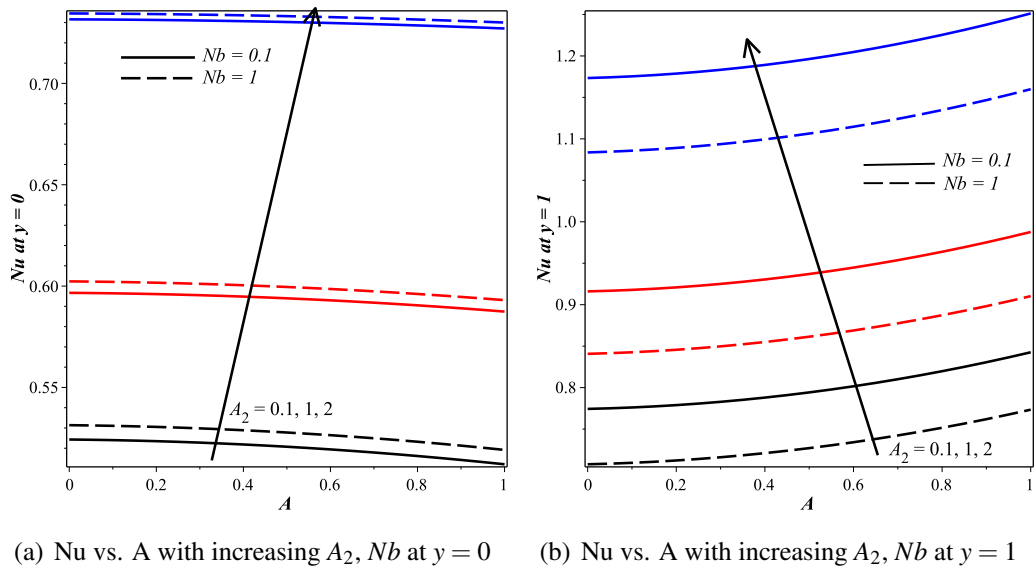


Figure 6.16: Nusselt number versus A with increasing A_2 and Nb

The effects of model parameters on Sherwood number Sh at the channel walls $y = 0$ and $y = 1$ are graphically plotted in figures (6.20)-(6.23). Figures 6.20 (a) and (b) represent

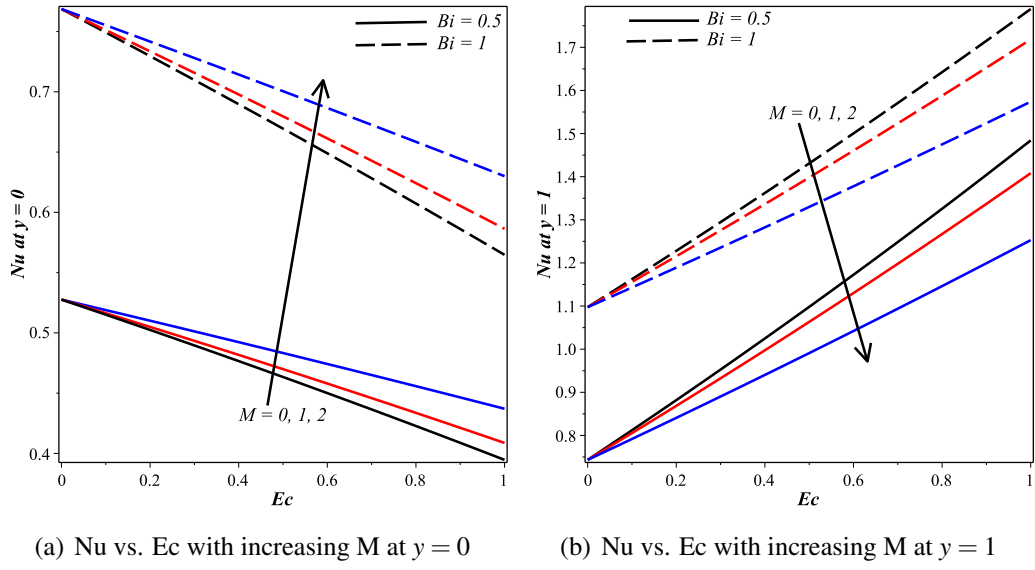


Figure 6.17: Nusselt number versus Ec with increasing M and Bi

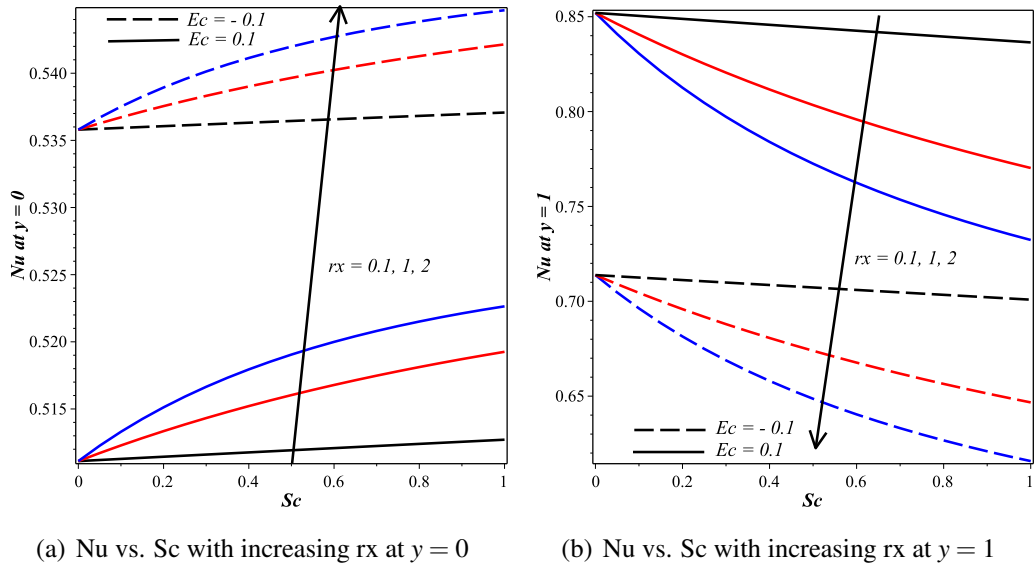


Figure 6.18: Nusselt number versus Sc with increasing rx and Ec

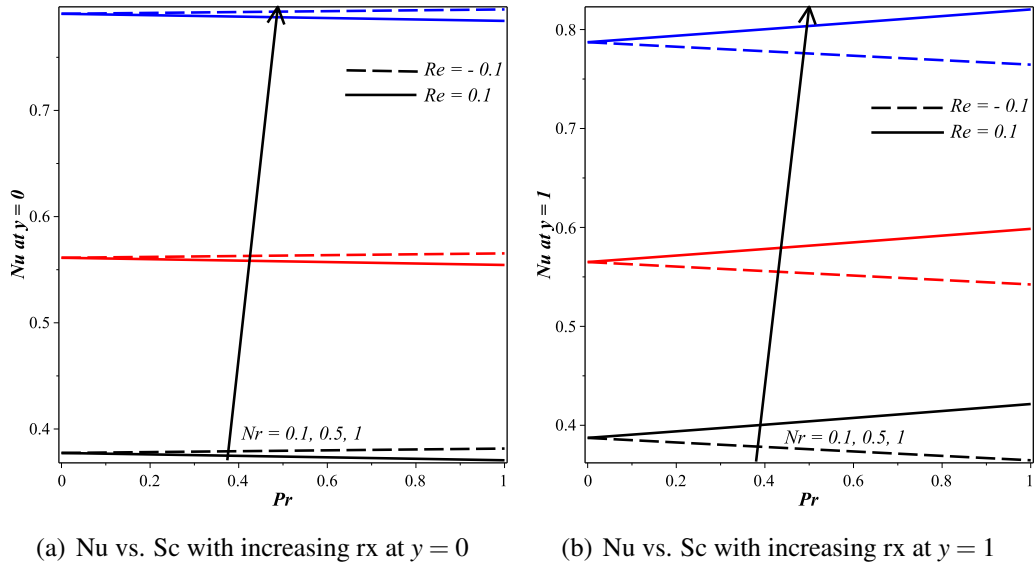


Figure 6.19: Nusselt number versus Sc with increasing rx and Ec

the simultaneous influence of pressure gradient A and chemical reaction parameter $rx > 0$ on Sherwood number Sh in the presence of suction and injection at both walls $y = 0$ and $y = 1$ of the microchannel. It is found that the rise of chemical reaction parameter $rx > 0$ increases the Sherwood number Sh at both walls of the microchannel with both presence of suction and injection. However, the Sherwood number Sh reduced with an increase in pressure gradient parameter A at the left wall $y = 0$ and is not appreciably influenced with an increase in pressure gradient parameter A at the right wall $y = 1$. Figures 6.21 (a) and (b) entail us that a decreasing Sherwood number Sh at the walls $y = 0$ and $y = 1$ as Eckert number Ec and A_1 augments concurrently. Moreover, an increased Sherwood number Sh is observed with an increase in Grashof number Gc at both walls $y = 0$ and $y = 1$. The combined effects of Prandtl number Pr , thermal conduction variation parameter A_2 and viscosity variation parameter A_1 on Sherwood number Sh at the walls $y = 0$ and $y = 1$ are depicted in figures 6.22 (a) and (b). It is found that the combined increase of Pr , A_2 and A_1 diminishes the Sherwood number Sh at both walls $y = 0$ and $y = 1$ of the microchannel. Sherwood number Sh augments at both walls of $y = 0$ and $y = 1$ of the microchannel with the rise of thermal radiation parameter Nr , Schmidt number Sc and Biot number Bi as shown in figures 6.23 (a) and (b).

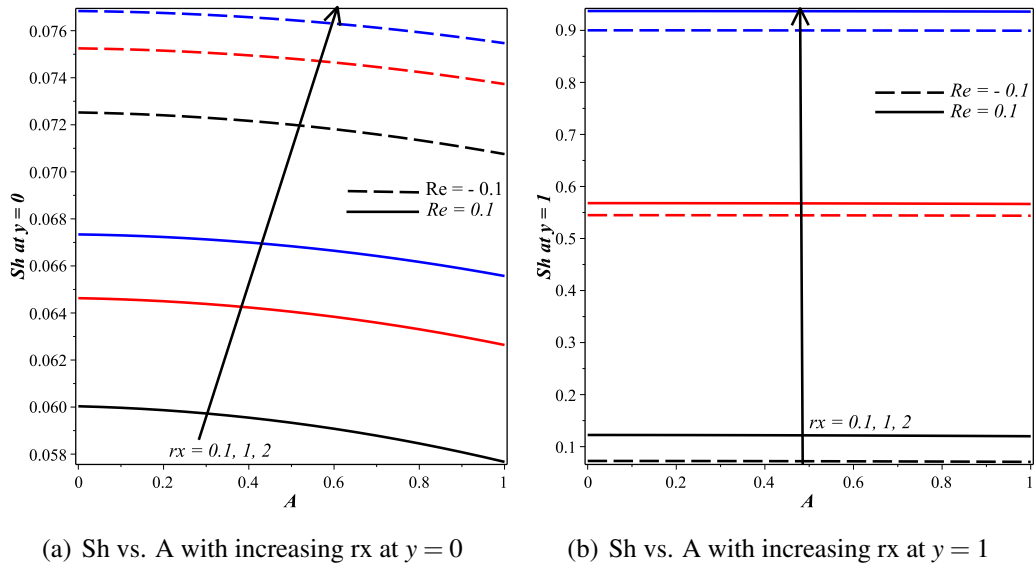


Figure 6.20: Sherwood number versus A with increasing rx and Re

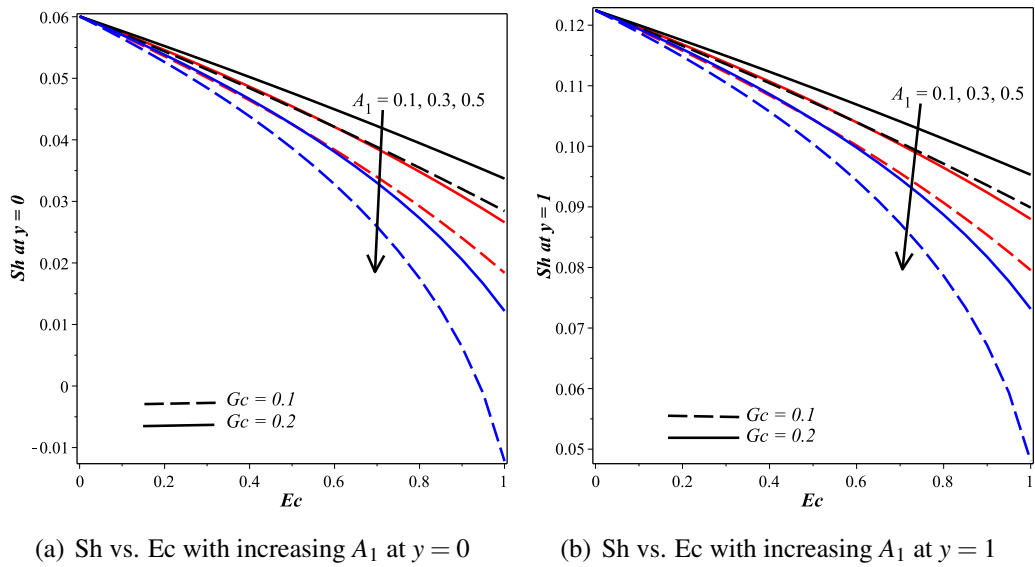


Figure 6.21: Sherwood number versus Ec with increasing A_1 and Gc

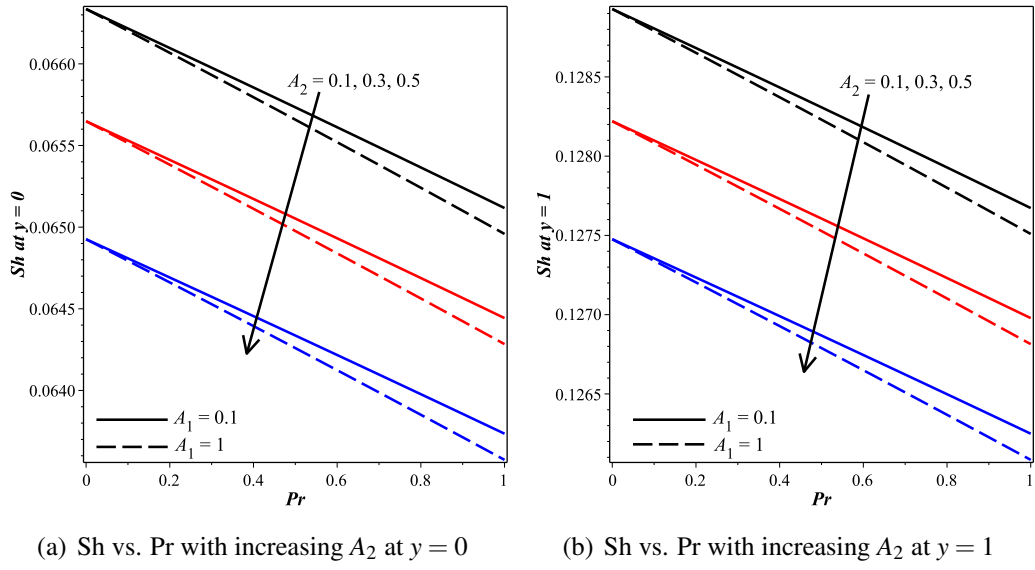


Figure 6.22: Sherwood number versus Pr with increasing A_2 and A

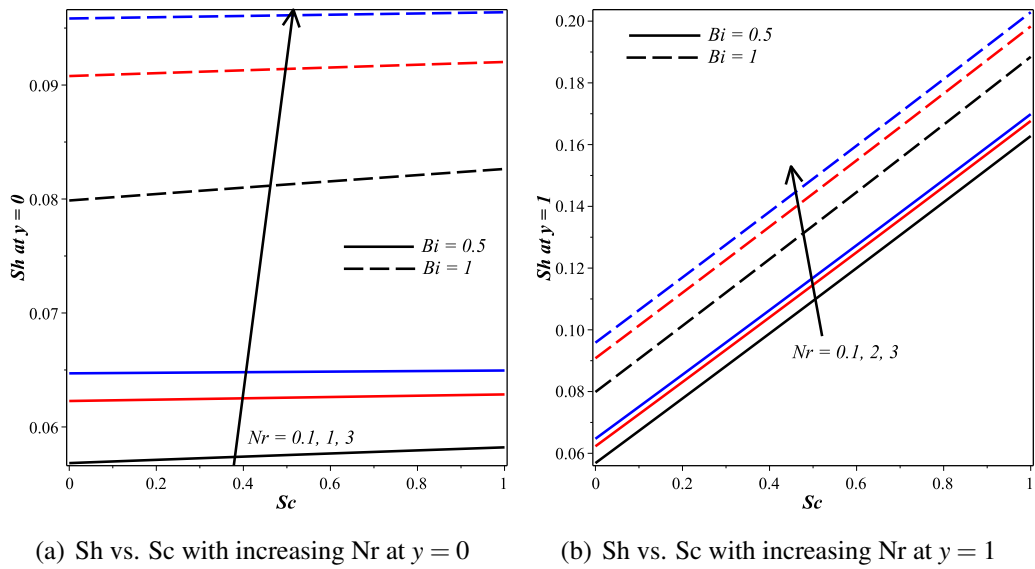


Figure 6.23: Sherwood number versus Sc with increasing Nr and Bi

CHAPTER 7

HYDROMAGNETIC MIXED CONVECTION OF COPPER/WATER NANOFLUID IN A MICROCHANNEL WITH THERMAL RADIATION AND CONVECTIVE HEATING

The present work investigates the aggregate effects of temperature dependent viscosity, magnetic field, thermal radiation, suction and injection on unsteady mixed convection of an incompressible nanofluid flow in a microchannel with convective wall heating. The nanofluid used is the mixture of copper (Cu) nanoparticles and water at various volume of fraction. The governing nonlinear partial differential equations of the flow problem are formulated using single-phase flow model, maxwell's equation, Boussinesq approximation, Rosseland's approximation and scaling analysis. The resulting PDEs are redid in to a set of dimensionless nonlinear partial differential equations using convenient non-dimensional variables, and consequently, solved numerically by semi-discretization via finite difference scheme together with Runge-Kutta Fehlberg integration technique. Transient analysis for velocity and temperature distribution have been performed across the microchannel width. Also, the effects of model parameters on velocity and temperature profiles along with shear stress and heat transfer rate are shown graphically and discussed qualitatively.

7.1 Model Formulation and Analysis

Unsteady flow of an incompressible and temperature dependent variable viscosity nanofluid in a horizontal parallel-plate microchannel is considered. The $\bar{x}$ -axis is taken along the lower wall and $\bar{y}$ -axis normal to the lower wall. A constant transverse magnetic field of B_o strength and radiative heat flux of q_r are both imposed parallel to $\bar{y}$ -axis. The fluid flow is constricted in the region $\{(\bar{x}, \bar{y}) : \bar{x}, \bar{y} \geq 0, \text{ and } \bar{y} = a\}$, where the lower wall is held at $\bar{y} = 0$ and upper wall is held at $\bar{y} = a$. The nanofluid is water based copper nanoparticles. The induced magnetic field due to the motion of a conducting fluid is assumed to be negligible when compared to applied one so that the magnetic field vector is $(0, B_o, 0)$. The flow setup of the problem is depicted in figure 7.1.

The flow is initiated by the effects of pressure gradient in the axial direction, buoyancy force, injection at the lower wall $\bar{y} = 0$ and suction at the upper wall $\bar{y} = a$ of the microchannel. The lower wall $\bar{y} = 0$ of the microchannel is convectively heated by a hot fluid at a temperature of T_f with heat transfer coefficient h_f and the upper wall $\bar{y} = a$ is held at a temperature of T_w which is also the fluid initial temperature.

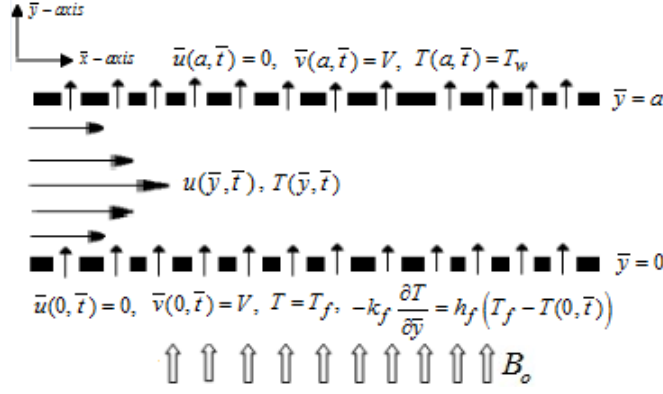


Figure 7.1: Flow setup of the problem

Under the above assumptions, using single-phase flow model and Boussinesq approximation, the system of partial differential equations that govern the microchannel flow problem are:

$$\frac{\partial \bar{u}}{\partial \bar{x}} = 0, \quad (7.1)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + V \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho_{nf}} \frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{\rho_{nf}} \frac{\partial}{\partial \bar{y}} \left[\mu_{nf}(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \frac{\sigma_{nf}}{\rho_{nf}} B_o^2 \bar{u} + \beta_{nf} g(T - T_w), \quad (7.2)$$

$$\frac{\partial T}{\partial \bar{t}} + V \frac{\partial T}{\partial \bar{y}} = \frac{k_{nf}}{(\rho c_p)_{nf}} \frac{\partial^2 T}{\partial \bar{y}^2} + \frac{\mu_{nf}(T)}{(\rho c_p)_{nf}} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \frac{\sigma_{nf} B_o^2}{(\rho c_p)_{nf}} \bar{u}^2 - \frac{1}{(\rho c_p)_{nf}} \frac{\partial q_r}{\partial \bar{y}}, \quad (7.3)$$

with the initial conditions

$$\bar{u}(\bar{y}, 0) = 0, \quad T(\bar{y}, 0) = T_w, \quad (7.4)$$

and boundary conditions given as

$$\begin{aligned} \bar{u}(0, \bar{t}) = 0, \quad -k_f \frac{\partial T}{\partial \bar{y}}(0, \bar{t}) &= h_f [T_f - T(0, \bar{t})], \\ \bar{u}(a, \bar{t}) = 0, \quad T(a, \bar{t}) &= T_w, \end{aligned} \quad (7.5)$$

Using Makinde and Franks (2014), the dynamic viscosity $\mu_{nf}(T)$ of nanofluid is:

$$\mu_{nf}(T) = \frac{\mu_f}{(1 - \phi)^{2.5}} e^{-\gamma(T - T_w)} \quad (7.6)$$

where $\bar{u}$ is the nanofluid velocity in $\bar{x}$ -direction, T is the temperature of nanofluid, V is the suction/injection velocity, $\bar{P}$ is the nanofluid pressure, $\bar{t}$ is the time, ρ_{nf} is the density of nanofluid, μ_{nf} is the dynamic viscosity of the nanofluid, g is the acceleration due to gravity, β_{nf} is the thermal expansion coefficients, k_{nf} is the thermal conductivity of the nanofluid, q_r is the thermal radiative heat flux, T_w is nanofluid initial temperature (nanofluid temperature at the upper wall), σ_{nf} is the nanofluid electrical conductivity, a is the channel width, $(\rho c_p)_{nf}$ is the specific heat at constant pressure, μ_f is the viscosity base fluid (water) and γ is viscosity variation coefficient. Table 7.1 (Hayat et al., 2016) indicates aspects of

thermophysical properties of water and copper (Cu) nanoparticles that are used in flow model analysis.

Table 7.1: Base fluid and nanoparticles thermophysical properties

Contents	ρ (Kg/m ³)	C_p (J/Kg.K)	k (W/m.K)	σ (S/m)	β (K ⁻¹)
H_2O	997.1	4179	0.613	5.5×10^{-6}	21
Cu	8933	385	400	59.6×10^6	1.67

Following (Hayat et al., 2016; Monaledi and Makinde, 2018), the thermophysical expressions of nanofluid in terms of base fluid and nanoparticles are defined as:

$$\begin{aligned} \rho_{nf} &= (1 - \phi)\rho_f + \phi\rho_s, \quad (\rho c_p)_{nf} = (1 - \phi)(\rho c_p)_f + \phi(\rho c_p)_s, \quad (\beta)_{nf} = (1 - \phi)\beta_f + \phi\beta_s, \\ \frac{k_{nf}}{k_f} &= \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}, \quad \frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi}, \quad r = \frac{\sigma_s}{\sigma_f}. \end{aligned} \quad (7.7)$$

By Rosseland's approximation (Rosseland, 1931), the radiative heat flux is given as:

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial \bar{y}}, \quad (7.8)$$

in which σ^* is the Stefan-Boltzman constant and k^* is the mean absorption coefficient. Assuming the discrepancy between the temperature T of nanofluid in a microchannel and the hot fluid temperature T_f heating the wall $y = 0$ is negligible, the temperature expression T^4 can be linearized as:

$$T^4 \approx 4T_f^3 T - 3T_f^4, \quad (7.9)$$

using Taylor's series expansion and ignoring second and higher order terms in the expansion.

Substituting (7.9) into (7.8), we get

$$\frac{\partial q_r}{\partial \bar{y}} = -\frac{16\sigma^* T_f^3}{3k^*} \frac{\partial^2 T}{\partial \bar{y}^2}. \quad (7.10)$$

To transform equations (7.2)-(7.5) into non-dimensional form, we introduce the following dimensionless variables and parameters:

$$\begin{aligned} x &= \frac{\bar{x}}{a}, y = \frac{\bar{y}}{a}, u = \frac{\bar{u}}{V}, t = \frac{v_f \bar{t}}{a^2}, \theta = \frac{T - T_w}{T_f - T_w}, \bar{P} = \frac{V v_f}{a} P, Re = \frac{Va}{v_f}, A = -\frac{dp}{dx}, \\ A_1 &= \gamma(T_f - T_w), M^2 = \frac{\sigma_f B_o^2 a^2}{\mu_f}, Gr = \frac{g \beta_f a^3 (T_f - T_w)}{v_f^2}, Pr = \frac{v_f}{\alpha_f}, \alpha_f = \frac{k_f}{(\rho c_p)_f}, \\ Ec &= \frac{V^2}{(c_p)_f (T_f - T_w)}, Nr = \frac{4\sigma^* T_f^3}{k^* k_f}, Bi = \frac{ah_f}{k_f}. \end{aligned} \quad (7.11)$$

Substituting equations (7.6), (7.7), (7.10) and (7.11) into equations (7.2)-(7.5), we obtain a system of dimensionless partial differential equations:

$$\begin{aligned} \frac{\partial u}{\partial t} + Re \frac{\partial u}{\partial y} = & \frac{A}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f}\right)} + \frac{e^{-A_1 \theta}}{(1 - \phi)^{2.5} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f}\right)} \left[-A_1 \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} \right] \\ & - M^2 \cdot \frac{1}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f}\right)} \left[1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi} \right] u + \frac{Gr}{Re} \left(1 - \phi + \phi \frac{\beta_s}{\beta_f} \right) \theta, \end{aligned} \quad (7.12)$$

$$\begin{aligned} \frac{\partial \theta}{\partial t} + Re \frac{\partial \theta}{\partial y} = & \frac{1}{Pr} \frac{1}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f}\right)} \left[\frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)} + \frac{4}{3} Nr \right] \frac{\partial^2 \theta}{\partial y^2} \\ & + M^2 Ec \left[1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi} \right] \frac{1}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f}\right)} u^2 \\ & + Ec \frac{e^{-A_1 \theta}}{(1 - \phi)^{2.5} \left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f}\right)} \left(\frac{\partial u}{\partial y} \right)^2, \end{aligned} \quad (7.13)$$

with initial and boundary conditions

$$u(y, 0) = 0, \theta(y, 0) = 0, \quad (7.14)$$

$$u(0, t) = 0, \frac{\partial \theta}{\partial y}(0, t) = Bi[\theta(0, t) - 1], u(1, t) = 0, \theta(1, t) = 0, \quad (7.15)$$

where Re is the Reynold suction/injection parameter, A is the pressure gradient parameter, A_1 is the variable viscosity variation parameter, M is the magnetic field parameter, Gr is the thermal Grashof number, Pr is the Prandtl number, Ec is the Eckert number, Nr is thermal radiation parameter and Bi is the Biot number.

The associated physical parameters of special significance for this problem are the skin friction coefficient Cf and the Nusselt number Nu which can be given in the form

$$Cf = \frac{\tau_w}{\rho V^2} \quad \text{and} \quad Nu = \frac{aq_w}{k_{nf}[T_f - T_w]} \quad (7.16)$$

where

$$\tau_w = \mu_{nf} \frac{\partial \bar{u}}{\partial \bar{y}} \Big|_{\bar{y}=0,a} \quad \text{and} \quad q_w = -k_{nf} \left(1 + \frac{16\sigma^* T_f^3}{3k^* k_{nf}} \right) \frac{\partial T}{\partial \bar{y}} \Big|_{\bar{y}=0,a} \quad (7.17)$$

are the wall shearing stress and wall heat-flux, respectively.

The non-dimensional form of Cf and Nu in equation (7.16) are:

$$Cf = \frac{e^{-A_1\theta}}{(1-\phi)^{2.5}} \frac{1}{Re} \frac{\partial u}{\partial y} \Big|_{y=0,1} \quad \text{and} \quad Nu = - \left(1 + \frac{4}{3} \frac{Nr}{m} \right) \frac{\partial \theta}{\partial y} \Big|_{y=0,1}, \quad (7.18)$$

here

$$m = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}.$$

7.2 Solution Technique

Semi-discretization scheme with Runge-Kutta Fehlberg integration method has been employed to find the numerical solution of a system of nonlinear partial differential equation (7.12)-(7.13) subject to conditions (7.14)-(7.15). The spatial domain $[0, 1]$ is discretized into $N + 1$ equal spaced interval nodes with grid size $\Delta y = \frac{1}{N+1}$ and with grid points $y_i = i\Delta y$, for $i = 1, 2, \dots, N$ where N is the number of interior nodal points in $[0, 1]$. The first and second spatial derivatives are replaced by a central finite difference approximation of order $(\Delta y)^2$ accuracy. Approximating $u(y_i, t)$ and $\theta(y_i, t)$ respectively by u_i and θ_i , the semi-discretization system for the problem is given by:

$$\begin{aligned} \frac{du_i}{dt} + Re \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) &= \frac{A}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right)} - M^2 \cdot \frac{1}{\left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right)} \left[1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi} \right] u_i \\ &+ \frac{e^{-A_1\theta_i}}{(1-\phi)^{2.5} \left(1 - \phi + \phi \frac{\rho_s}{\rho_f} \right)} \left[-A_1 \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) \left(\frac{u_{i+1} - u_{i-1}}{2\Delta y} \right) + \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} \right] \\ &+ \frac{Gr}{Re} \left(1 - \phi + \phi \frac{\beta_s}{\beta_f} \right) \theta_i, \end{aligned} \quad (7.19)$$

$$\begin{aligned} \frac{d\theta_i}{dt} + Re \left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta y} \right) &= M^2 Ec \left[1 + \frac{3(r-1)\phi}{(r+2) - (r-1)\phi} \right] \frac{1}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f} \right)} u_i^2 \\ &+ \frac{1}{Pr} \frac{1}{\left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f} \right)} \left[\frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)} + \frac{4}{3} Nr \right] \left(\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta y)^2} \right) \\ &+ Ec \frac{e^{-A_1\theta_i}}{(1-\phi)^{2.5} \left(1 - \phi + \phi \frac{(\rho c_p)_s}{(\rho c_p)_f} \right)} \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} \right)^2, \end{aligned} \quad (7.20)$$

with initial conditions

$$u_i(0) = 0, \quad \theta_i(0) = 0, \quad 1 \leq i \leq N. \quad (7.21)$$

The boundary conditions at $y = 0, 1$ are transformed to incorporate as follows:

$$\begin{aligned} u_o(t) = 0, \quad \frac{d\theta_o(t)}{dy} &= Bi[\theta_o(t) - 1], \\ u_{N+1}(t) = 0, \quad \theta_{N+1}(t) &= 0. \end{aligned} \quad (7.22)$$

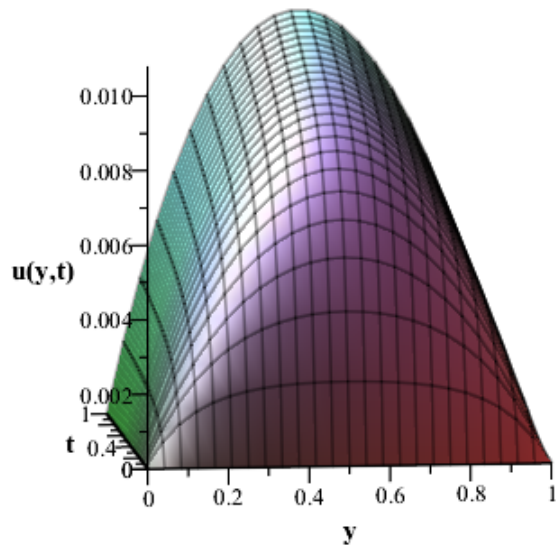
Equations (7.19)-(7.22) is a system of first order nonlinear ordinary differential equations with well-known initial conditions and can be solved iteratively by Runge-Kutta Fehlberg integration techniques (Na, 1979).

7.3 Results and Discussion

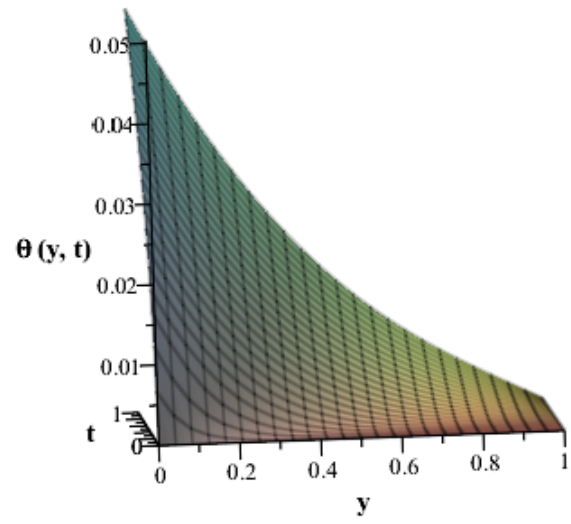
In present work, the effect of temperature dependent viscosity and thermal radiation on unsteady hydromagnetic mixed convection of an incompressible $Cu - H_2O$ nanofluid flow in a permeable microchannel with convective heating has been studied numerically. The numerical results of velocity, temperature, skin friction and Nusselt number profiles of the problem have been computed using table 1 and for fixed values of model parameters $Re=0.1$, $A=0.1$, $A_1=0.1$, $M=1$, $Gr=0.1$, $Nr=0.5$, $Ec=0.1$, $Pr=6.2$ and $\phi=0.1$ if not mentioned. The obtained results have been highlighted through figures (7.2)-(7.14).

7.3.1 Transient Effect Analysis

Figures (7.2)-(7.4) illustrate a 3D outline of time development of velocity and temperature distributions with respect to initial and boundary conditions of the problem across the microchannel width. Figure 7.2 (a) indicates that the dimensionless flow velocity starts to increase from rest both in space and time at the lower and upper walls to its highest value within the microchannel. The temperature of the fluid decreases from its highest value at the lower wall of the microchannel to zero minimum value away from the lower wall of the microchannel with increasing time as depicted in figure 7.2 (b). Figures 7.3 (a) and (b) show that the maximum flow velocity is observed around the center of the microchannel and a settled velocity profile is also observed with increasing time. It can be noted from figure 7.4 (a) and (b) that the temperature of the fluid is highest at the lower wall and decreases to zero value away from the lower wall of the microchannel meeting the boundary conditions.

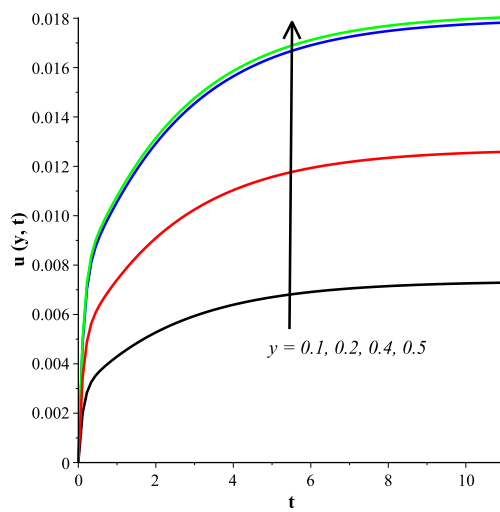


(a) 3D velocity profiles

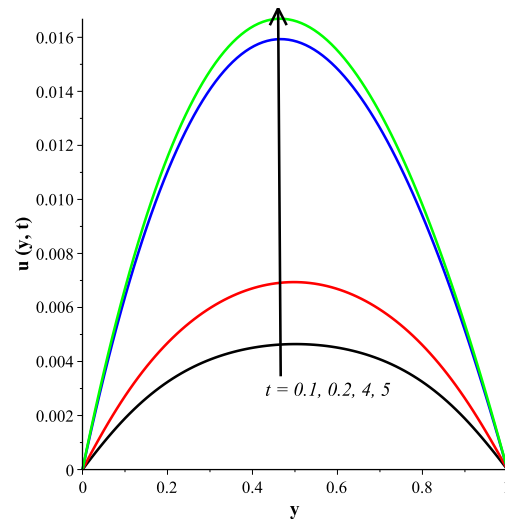


(b) 3D temperature profiles

Figure 7.2: 3D velocity and temperature profiles



(a) Developing velocity profile with increasing y



(b) Velocity profile with increasing t

Figure 7.3: Developing velocity profiles

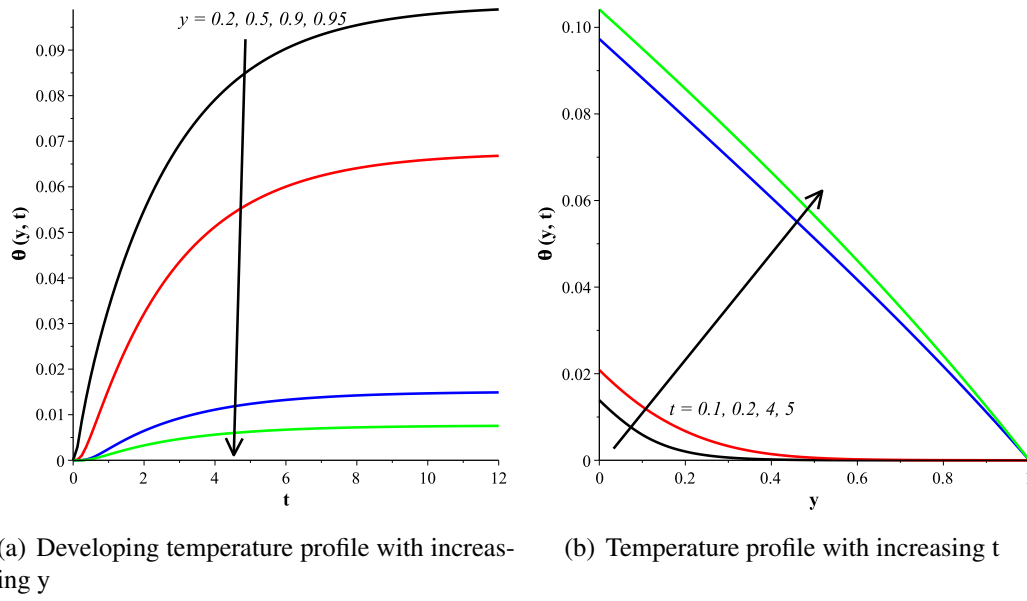


Figure 7.4: Developing temperature profiles

7.3.2 Parameter Effect on Steady-State Velocity Profiles

The effects of various model parameters on the $Cu - H_2O$ nanofluid velocity are presented in figures 7.5 and 7.6. The nanofluid velocity distributions have, in general, parabolic profiles with highest velocity value around the centerline of the microchannel width and zero velocity at both microchannel walls ($y = 0$ and $y = 1$). This result consistent with the previous observations in chapter 4-6. In figures 7.5 (a) and (b), it is observed that an increase in the values of variable viscosity parameter A_1 , Grashof number Gr and Eckert number Ec augments the nanofluid flow, whereas an increase in nanoparticles concentration ϕ dwindles the nanofluid flow in the microchannel. This is due to the fact that an increase in A_1 , Gr and Ec intensifies the internal heat generation of $Cu - H_2O$ nanofluid flow field, as a result the nanofluid becomes lighter and moves faster. On the other hand, a rise in ϕ increases the nanofluid viscosity which leads to diminish the flow velocity. These observations coincide with the previous results obtained by (Makinde, 2018; Monaledi and Makinde, 2018) and thus validate the present outcomes. In the meantime, the nanofluid velocity increases with increasing Biot number Bi and pressure gradient A , but decreased with an increase in thermal radiation parameter Nr , nanoparticles concentration ϕ and magnetic field parameter M .

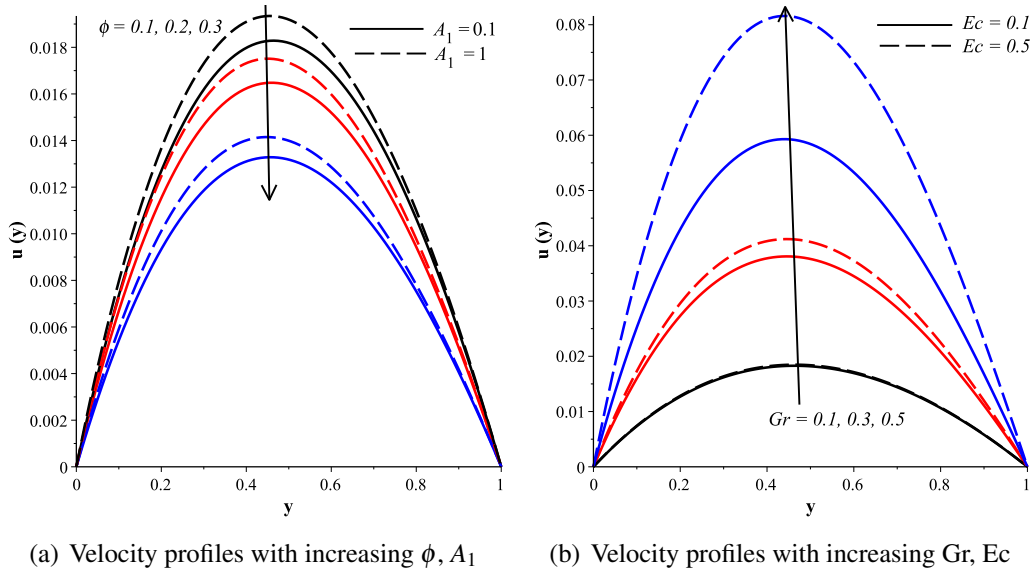


Figure 7.5: Velocity profiles with increasing ϕ, A_1, Gr and Ec

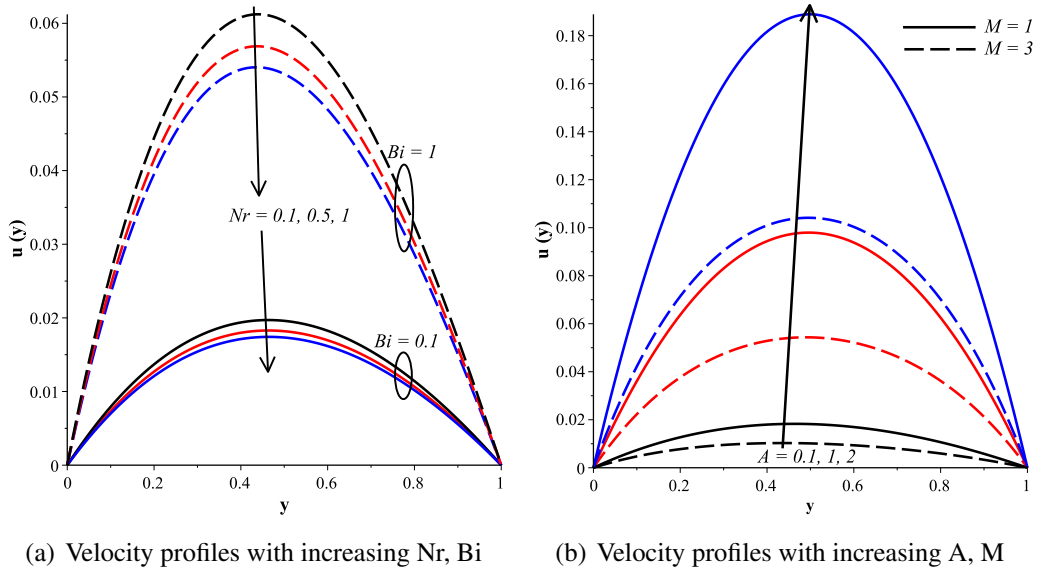


Figure 7.6: Velocity profiles with increasing Nr, Bi, A and M

7.3.3 Parameter Effect on Steady-State Temperature Profiles

The temperature profiles of $Cu - H_2O$ nanofluid for varying values of thermophysical parameters are shown in figures 7.7 and 7.8. From these figures, it is imperative to note that the maximum nanofluid temperature is attained at the lower wall $y = 0$ due to the presence of convective heating at this wall and zero temperature values at the upper wall $y = 1$ of the microchannel. Figure 7.7 (a) displays the nanofluid temperature profiles with variations in suction/injection parameter Re and Grashof number Gr . The results indicate that as Re and Gr rise in value, the $Cu - H_2O$ nanofluid temperature amplifies within the microchannel. Further, a large impact of Gr on nanofluid temperature is not seen for small suction/injection parameter Re . Increasing Re will raise the injection of more hot nanofluid in to the microchannel through the wall $y = 0$, as a result the mount of nanofluid temperature is obtained. Similarly, an increase in values of Gr intensifies the thermophoresis effect (, i.e., the movement of nanoparticles due to temperature gradient) which in turn augments the nanofluid temperature. The effect of Biot number Bi and Pressure gradient A on nanofluid temperature is illustrated in figure 7.7 (b), the results show that an increase in Bi and A enhance the nanofluid temperature in the microchannel. In figures 7.8 (a) and (b) it is observed that a rise in nanofluid temperature with higher in values of Cu nanoparticles ϕ and Eckert number Ec , but a lower nanofluid temperature has been obtained with higher in thermal radiation parameter Nr . Increase in ϕ enhance the thermal conductivity of $Cu - H_2O$ nanofluid which leads to boost the fluid temperature, similarly when values of Ec rise, the viscous dissipation effect intensified, and thus the nanofluid temperature increases. In this observation, the results go in line with the previous outcomes reported by (Kumar et al., 2018; Monaledi and Makinde, 2019).

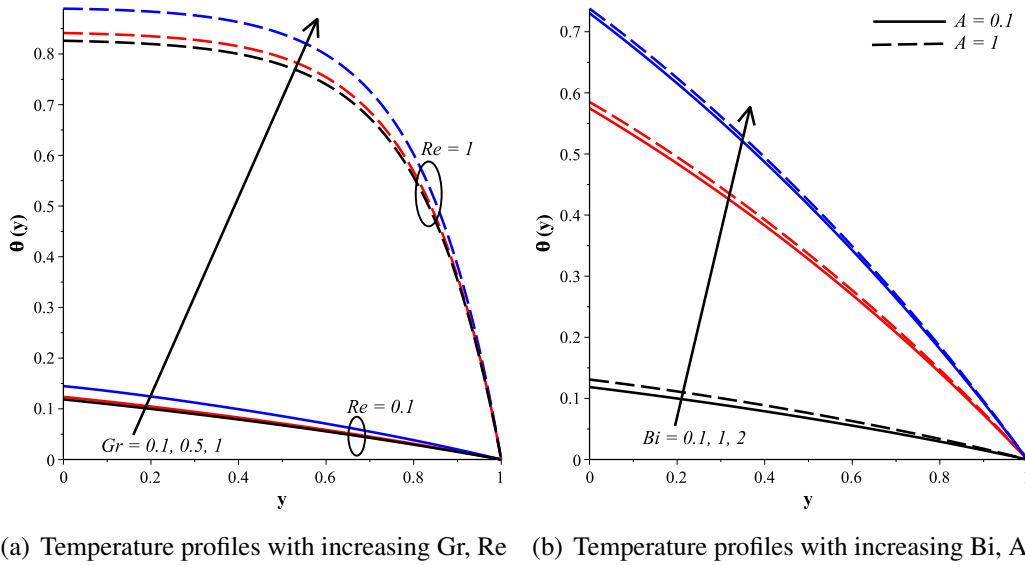


Figure 7.7: Temperature profiles with increasing Gr , Re , Bi and A

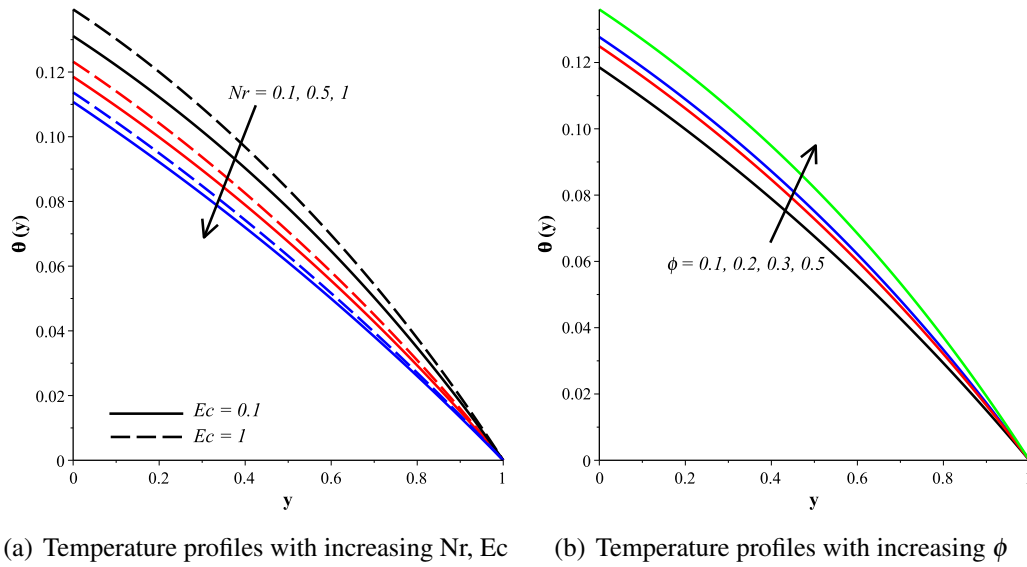


Figure 7.8: Temperature profiles with increasing Nr, Ec and ϕ

7.3.4 Skin Friction and Nusselt Number

The local skin friction Cf represents the dimensionless shearing stress due to the bulk of fluid drag force exerted on the surface of a body. The graphical description of key model parameters effect on skin friction at lower $y = 0$ and upper $y = 1$ walls of the microchannel are shown in figures (7.9)-(7.11). The effects of thermal radiation Nr on shear stress at both microchannel walls (i.e., $y = 0$ and $y = 1$) for different Eckert number Ec values demonstrated in figures 7.9 (a) and (b), respectively. A rise in thermal radiation Nr increases the shear stress and a rise Ec reduces the shear stress at $y = 0$. But an opposite phenomena is observed at $y = 1$ with rise Nr and Ec . Figures 7.10 (a) and (b) show that the wall shear stress of $Cu - H_2O$ nanofluid at the lower wall $y = 0$ and upper wall $y = 1$ in terms of Grashof number Gr for varying values of magnetic field parameter M . It is clear that the wall shear stress directly related with M and inversely related with Grashof number Gr at the lower wall $y = 0$ of the microchannel. At the upper wall, a reverse trend is observed with increase Gr and M . The impact of Biot number Bi on wall shear stress at microchannel walls $y = 0$ and $y = 1$ with varying values of Cu nanoparticles concentration are depicted in figure 7.11 (a) and (b). These figures revealed that increasing the amount of Cu nanoparticles diminish the shear stress at the lower wall $y = 0$ but an opposite event happened at the upper wall $y = 1$ of the microchannel. Further, increasing Bi leads to amplify the shear stress at both microchannel walls ($y = 0$ and $y = 1$).

The Nusselt number Nu describes the heat transfer rate from the surface of a body to the fluid and is also known as dimensionless heat transfer coefficient. In figures (7.12)-(7.14), the effects of model parameters variation on the local Nusselt number at lower $y = 0$ and upper $y = 1$ microchannel walls are presented. From these figures, it can be noted that the

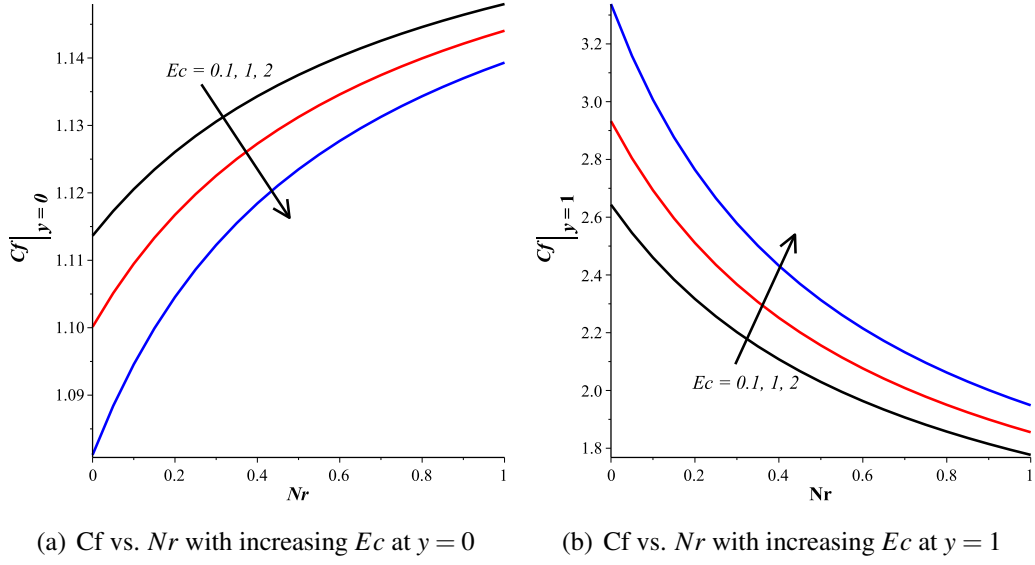


Figure 7.9: Shear stress versus Nr with increasing Ec

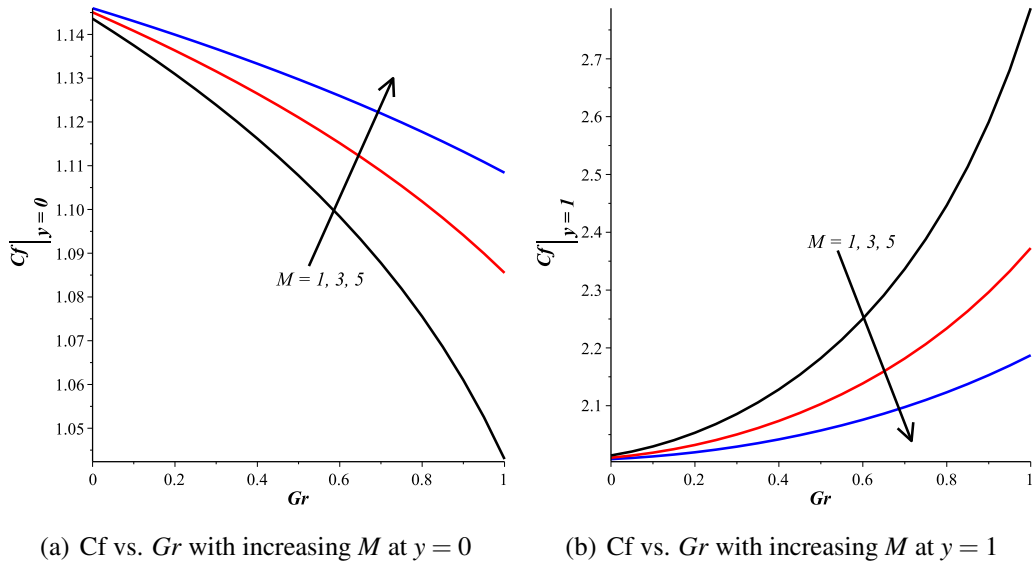


Figure 7.10: Shear stress versus Gr with increasing M

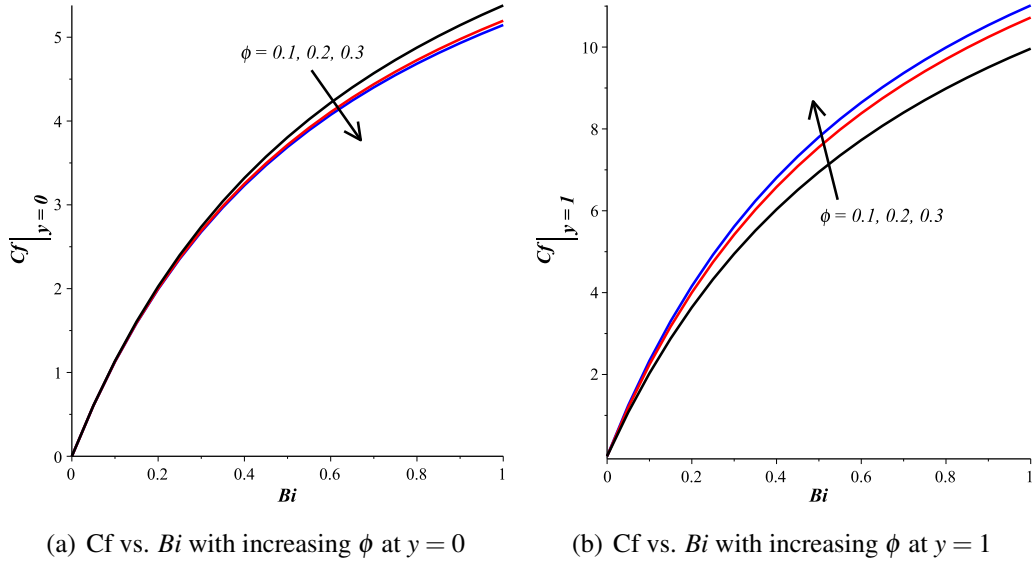


Figure 7.11: Shear stress versus Bi with increasing ϕ .

Nu at the lower wall $y = 0$ and upper wall $y = 1$ increased due to Ec , Nr , Gr , Bi and Bi , while it decreased due to M and ϕ .

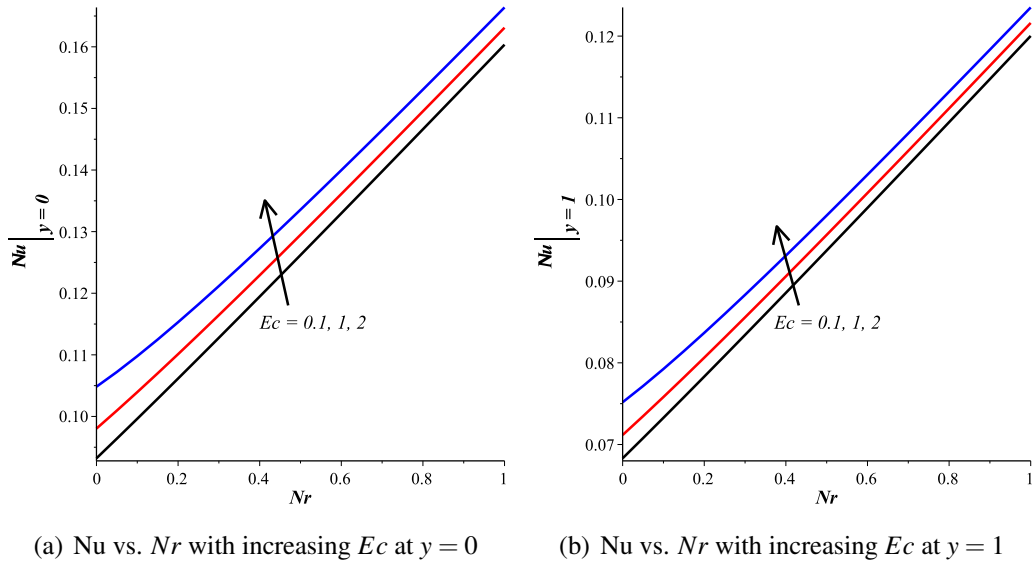
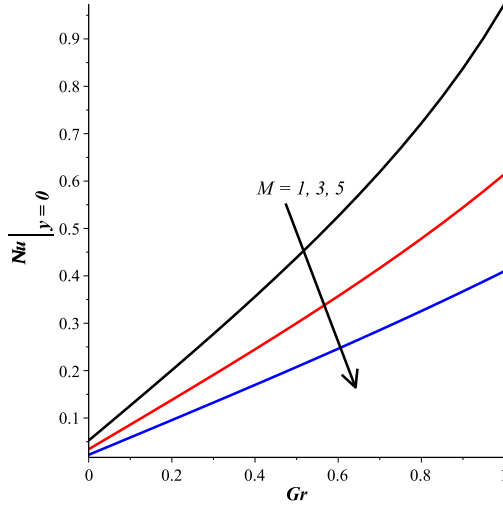
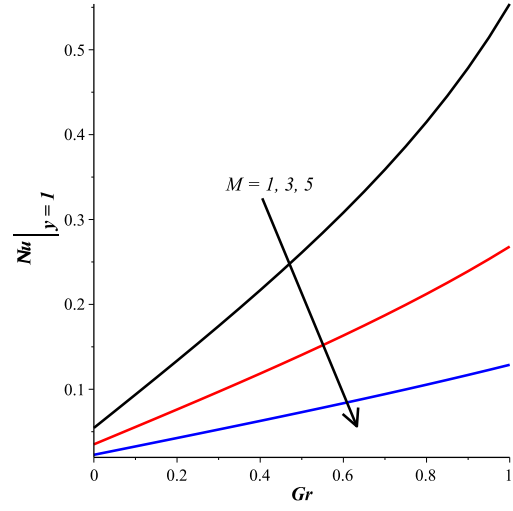


Figure 7.12: Heat transfer versus Nr with increasing Ec .

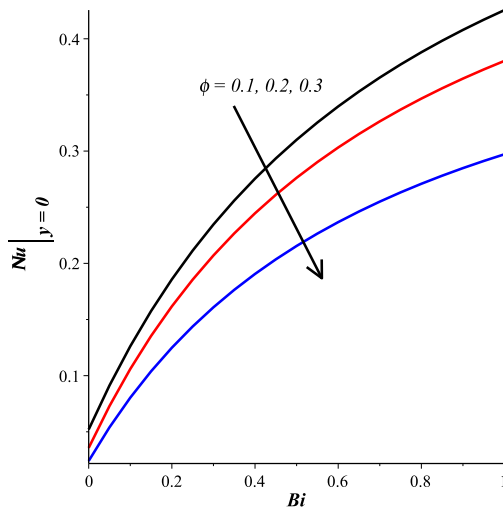


(a) Nu vs. Gr with increasing M at $y = 0$

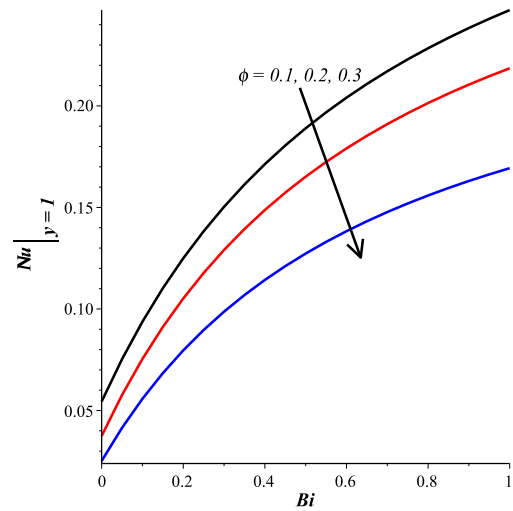


(b) Nu vs. Gr with increasing M at $y = 1$

Figure 7.13: Heat transfer versus Gr with increasing M .



(a) Nu vs. Bi with increasing ϕ at $y = 0$



(b) Nu vs. Bi with increasing ϕ at $y = 1$

Figure 7.14: Heat transfer versus Bi with increasing ϕ .

CHAPTER 8

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

8.1 Summary

In this study, the mathematical models of unsteady magneto-nanofluid flows in permeable microchannels for the case of single and two phase approaches have been formulated and analyzed to study the transient analysis of flow quantities across the channel and the steady state analysis on flow quantities across the channel, and the heat and mass transfers at the channel walls using combined effects of thermophysical parameters. Besides, the flow problems were assumed to be laminar, incompressible nanofluids with some temperature dependent fluid properties. Each of the problems were also carried out both with and without the physical situations such as derivative boundary condition, convective wall heating, asymmetric walls temperature, thermally radiating and chemically reacting nanofluids. The main body of the study followed the systematic order of the problems of: (1) two phase flow model without thermal radiation and chemical reaction, (2) single phase flow model with only asymmetric walls temperature, (3) two phase model with all above mentioned conditions and (4) single phase model without derivative boundary condition and chemical reaction. The models were commonly based on the continuity, Navier-Stokes, energy as well as Maxwell equations, and Boussinesq approximation. Some particular models were used Rosseland approximation and first order chemical reaction equation with Arrhenius kinetic. The resulting models which govern the flow problems were highly nonlinear system of PDEs and termed as initial boundary value problems (IBVPs). Subsequently, the governing IBVPs have been nondimensionalized using dimensionless parameters and solved numerically using semi-discretization centered finite difference scheme with Runge-Kutta Fehlberg integration technique. The numerical results for the velocity, temperature, concentration, skin friction, Nusselt number and Sherwood number profiles and the effect of model parameters are studied and plotted graphically. Further, qualitative validations of some selected numerical results with previously published results have been made and it is obtained a good agreement.

8.2 Conclusions

This study investigated the transport phenomena of unsteady MHD nanofluid flow through a permeable parallel plate's microchannel under various physical situations such as the effects of temperature dependent viscosity and thermal conductivity, asymmetric wall temperature

and concentration conditions, derivative boundary condition and convective wall heating. Thus, important conclusions made from the results presented in the dissertation are outlined below.

- From transient analysis, the nanofluid velocity, temperature, and concentration profiles have shown variation with time before attainment of their steady states. It also shown that the velocity profiles attained their steady state sooner than the temperature and concentration profiles.
- Generally, a significant retarding and speeding up of nanofluid velocity profiles have been observed around the centerline of the microchannel as compared to near the channel walls with an increase in influential parameters.
- For none reacting and radiating MHD nanofluid flow in microchannel in the absence of wall convective heating, the temperature and concentration profiles were observed to have a considerable variation across the channel except in the vicinity of both walls with rise in prominent parameters.
- For reacting and radiating MHD nanofluid flow in microchannel in the presence of wall convective heating, the temperature and concentration profiles were observed to have a considerable variation across the channel except in the vicinity of both right and upper walls with rise in prominent parameters.
- The skin friction Cf for none reacting and radiating MHD nanofluid has reduced with increasing Reynolds suction/injection parameter $Re > 0$ at both microchannel walls. Moreover, the model parameters viscosity variation parameter, thermophoretic parameter, pressure gradient, nanoparticles volume fraction were observed to have an increasing effects on skin friction at the wall $y = 0$, where as a decreasing effects were observed at $y = 1$.
- For none reacting and radiating MHD nanofluid, the heat transfer rate Nu has shown an increasing trend at both microchannel walls with increasing Reynold's suction/injection parameter $Re > 0$. The heat transfer rate Nu at the wall $y = 0$ decreased with viscosity variation parameter and Eckert number while increased with magnetic field parameter, Schmidt number and Brownian motion parameter. (Nu at $y = 1$ had a reverse trend.)
- For none reacting and radiating MHD nanofluid, the Brownian motion parameter influenced negatively the mass transfer rate Sh at both microchannel walls, where as an opposite trend was observed with thermophoretic parameter. In addition, Sh at $y = 0$ increased with viscosity variation parameter and Eckert number, but it decreased with magnetic field parameter and Schmidt number. (Sh at $y = 1$ had a reverse trend.)

- The skin friction Cf at the channel wall $y = 0$ for radiating but not reacting nanofluid reduced with increasing species concentration, thermal Grashof number and Eckert number but it boosted with magnetic field and thermal radiation parameter. With these influential parameters, however the skin friction had an opposite phenomena at the wall $y = 1$. Moreover, the Biot number augmented the skin friction on both channel walls $y = 0, 1$.
- An increase in solutal Grashof number, thermal radiation, magnetic field and chemical reaction parameter enhanced the wall shear stress Cf of radiating and reacting nanofluid at the wall $y = 0$ but Biot number, Schmidt number, Prandtl number and thermal Grashof number reduced it at the wall $y = 0$. On the other hand, a reverse trend of Cf was observed at the wall $y = 1$ with the prominent parameters. In addition, the Reynold's suction/injection parameter augmented Cf at both channel walls $y = 0, 1$.
- The Nusselt number Nu for radiating but not reacting MHD $Cu - H_2O$ nanofluid with convective heating at $y = 0$ observed to be increased with increasing in Eckert number, thermal radiation parameter, thermal Grashof number, and Biot number at both channel walls $y = 0, 1$ while decreased with magnetic field parameter and Cu nanoparticles concentration.
- For reacting and radiating MHD nanofluid, the Nusselt number Nu augmented with thermal conduction parameter, Brownian motion parameter, Biot number and thermal radiation parameter at both channel walls $y = 0, 1$. However, pressure gradient and Eckert number decreased Nu at $y = 0$ and increased Nu at $y = 1$. In addition, magnetic field parameter, Schmidt number and chemical reaction parameter increased Nu at $y = 0$ and decreased it at $y = 1$.
- The Sherwood number Sh for reacting and radiating MHD nanofluid intensified at both channel walls $y = 0, 1$ with chemical reaction parameter, solutal Grashof number, thermal conduction parameter, Prandtl number, thermal radiation parameter, Schmidt number, and Biot number but diminished with viscosity variation parameter and Eckert number.

8.3 Recommendations

The results of this dissertation provide an important fundamental knowledge on heat transfer enhancement and controlling techniques of MHD-nanofluids flow in permeable microchannels. As a result, the research has significant contributions on the development and design of micron-sized heat removal systems and other micron-sized devices such as microchannel heat sinks, microchannel heat exchangers, MHD micro-pumps, micromixers, and lab-on-chips by considering physical situations variable nanofluid properties, reacting and radiating

nanofluid, $Fe_3O_4 - H_2O$ and $Cu - H_2O$ nanofluids in permeable microchannel flow. Alternatively, acquiring a clear insight into better understanding of single-phase and two-phase flow in a permeable microchannel under different physical scenario has been another important contribution of the study.

Suggestions for Future Investigation

Future investigation is required to improve some aspect(s) of the problems studied in the dissertation. Thus, the problems can be improved/extended for future investigation in the following ways.

- One can consider an alternative solution technique, VIZ., explicit/implicit finite difference method with stability and convergence analysis to solve the governing equations of all problems presented here.
- The 2D flow problems herein could be extended to the 3D cases.
- For all flow diagram of microchannels flow, it is possible to take into account non-Newtonian fluids with porous medium.
- One can also incorporate the property of microchannel liquid flow that is EDL formation which is responsible for electrokinetic phenomena such as electroosmosis and electrophoresis to the presented problems.
- All problems discussed here can be considered by simply including the effects of non-linear thermal radiation, variable suction/injection velocity, heat generation/absorption, and wall velocity slip.
- Single-phase flow problems could also be investigated by using hybrid nanofluids.
- The problems discussed here also extended to include the concept of entropy generation minimization.

REFERENCES

- Abbas, Z., Naveed, M., Hussain, M., and Salamat, N. (2020). Analysis of Entropy Generation for MHD Flow of Viscous Fluid Embedded in a Vertical Porous Channel with Thermal Radiation. *Alexandria Engineering Journal*, 59(5): 3395-3405.
- Abdollahi, A., Mohammed, H.A., Vanaki, Sh. M., Osia, A., and Haghighi, G. M. R. (2017). Fluid Flow and Heat Transfer of Nanofluids in Microchannel Heat Sink With V-type Inlet/Outlet Arrangement. *Alexandria Engineering Journal*, 56(1): 161-170.
- Afkhami, S., Zaleski, S., and Bussmann, M. (2009). A Mesh-Dependent Model for Applying Dynamic Contact Angles to VOF Simulations. *Journal of Computational Physics*, 288(15): 5370-5389.
- Ajaev, V. S., and Homsy, G. M. (2006). Modeling Shapes and Dynamics of Confined Bubbles. *Annual Review of Fluid Mechanics*, 38: 277-308.
- Aminossadati, S. M., Raisi, A., and Ghasemi, B. (2011). Effect of Magnetic Field on Nanofluid Forced Convection in Partially Heted Microchannel. *International Journal of Non-Linear Mechanics*, 46(10): 1373-1382.
- Avic, M., and Aydin, O. (2007). Mixed Convection in a Vertical Parallel Plate Microchannel With Asymmetric Wall Heat Fluxes. *Journal of Heat Transfer-Transactions of the ASME*, 129(8): 1091-1095.
- Bahrami, M., Jovanovich, M. M., and Culham, J. R. (2006). Pressure Drop of Fully Developed Laminar Flow in Microchannels of Arbitrary Cross-Section. *Journal of Fluids Engineering*, 128(5): 1036-1044.
- Bahrami, M., Jovanovich, M.M., and Culham, J. R. (2006). Pressure Drop of Fully Developed, Laminar Flow in Rough Microtubes. *Journal of Fluids Engineering*, 128(3): 632-637.
- Bau, H. H., Zhu, J., and Qian, S., and Xiang, Y. (2003). A Magneto-Hydrodynamically Controlled Fluidic Network. *Sensors and Actuators B: Chemical*, 88(2): 205-216.
- Bayraktar, T., and Pidugu, S. B. (2006). Characterization of Liquid Flows in Microfluidic Systems. *International Journal of Heat and Mass Transfer*, 49(5-6): 815-824.
- Beam, R., and Warming, R. F. (1978). An Implicit Factored Scheme for the Compressible Navier-Stokes Equations. *AIAA Journal*, 16(4): 393-402.
- Berzins, M., Dew, P. M., and Furzeland, R. M. (1989). Developing Software for Time Dependent Problems Using the Method of Lines and Differential-Algebraic Integrators. *Appl. Numer. Math.*, 5(5): 375-397.
- Bird, R. B., and Stewart, W. E., and Lightfoot, E. N. (2006). *Transport Phenomena*. Second Edition. John Wiley & Sons, Inc.

- Blandford, R., and Thorne, K. (2004). Applications of Classical Physics. Airy Functions and Applications to Physics.
- Bowersa, J., Cao, H., Qiao, G., Li, Q., Zhang, G., Mura, E., and Ding, Y. (2018). Flow and Heat Transfer Behavior of Nanofluids in Microchannels. *Progress in Natural Science: Materials International*, 28(2): 225-234.
- Brinkman, H. C. (1952). The Viscosity of Concentrated Suspensions and Solutions. *J. Chem. Phys.*, 20(4): 571-571.
- Buongiorno, J. (2006). Convective Transport in Nanofluids. *Transactions of the ASME*, 128: 240-250.
- Chein, R., and Chuang, J. (2007). Experimental Microchannel Heat Sink Performance Studies Using Nanofluids. *International Journal of Thermal Sciences*, 46(1): 57-66.
- Choi, S.U.S. (1995). Enhancing Thermal Conductivity of Fluids with Nanoparticles. In: *Development and Applications of Non-Newtonian Flows*. ASME, 99-106.
- Chung, T. J. (2002). *Computational fluid dynamics*. Cambridge University Press.
- Cutli, M.B., and Shacham, M. (1998). The Numerical Method of Lines for Partial Differential Equations. *CACHE News*, 47: 18-21.
- Dhlamini, M., Kameswaran, P.K., Sibanda, P., Motsa, S., and Mondal, H. (2019). Activation Energy and Binary Chemical Reaction Effects in Mixed Convective Nanofluid Flow With Convective Boundary Conditions. *Journal of Computational Design and Engineering*, 6(2): 149-158.
- Drzazga, M., Lemanowicz, M., Dzido, G., and Gierczycki, A. (2012). Preparation of Metal Oxide-Water Nanofluids by Two-Step Method. *Ap. Chem.*, 51(5): 213-215.
- Duangthongsuk, W., Dalkilic, A. S., and Wongwises, S. (2012). Convective Heat Transfer of Al_2O_3 -water Nanofluids in a Microchannel Heat Sink. *Current Nanoscience*, 8(3): 317-322.
- El Mghari, H., Louahlia-Gualous, H., and Lepinasse, E. (2016). Numerical Study of Nanofluids Condensation Heat Transfer in a Square Microchannel. *Numer. Heat Transfer. Part A*, 69(9): 957-976.
- Ersahin, C., Tarhan, T., Tuncer, I. H., and Selcuk, N. (2004). Parallelization of a Transient Method of Lines Navier-Stokes Code. *Int. J. Comput. Fluid Dynamic*, 18(5): 81-92.
- Frackowiak, E., and F. Beguin, F. (2001). Carbon Materials for the Electrochemical Storage of Energy in Capacitors. *Carbon*, 39(6): 937-950.
- Freidberg, J. P. (1987). *Ideal Magnetohydrodynamics*. Plenum Press.
- Gabelich, C. J., Tran, T. D., and Suffet, I. H. (2002). Electrosorption of inorganic salts from aqueous solution using carbon aerogels. *Environmental Science & Technology*, 36(13): 240-250.

3010-3019.

Goodwin, S., Peterson C., Hoh, C., and Bittner, C. (1999). Targeting and Retention of Magnetic Targeted Carriers (MTCs) Enhancing Intra-arterial Chemotherapy. *Journal of Magnetism and Magnetic Materials*, 194(1): 132-139.

Gulhane, N.P., and Mahulikar, S. P. (2011). Numerical Study of Microconvective Water-Flow Characteristics with Variations in Properties. *Nano. Micro. Therm.*, 15(1): 28-47.

Hardt, S., Ehrfeld, W., Hessel, V., and Vanden Bussche K. M. (2003). Strategies for Size Reduction of Microreactors by Heat Transfer Enhancement Effects. *Chemical Engineering Communications*, 190(4): 540-559.

Hasan, M. I. (2014). Investigation of Flow and Heat Transfer Characteristics in Micro Pin Fin Heat Sink With Nanofluid. *Appl. Therm. Eng.*, 63 (2): 598-607.

Hatami, N., Banari, A.K., Malekzadeh, A., and Pournafard, A. R. (2017). The effect of magnetic field on nanofluids heat transfer through a uniformly heated horizontal tube. *Physics Letters A*, 381(5): 510-515.

Hayat, T., Kiran, A., Imtiaz, M., and Alsaedi, A. (2016). Hydromagnetic Mixed Convection Flow of Copper and Silver Water Nanofluids Due to a Curved Stretching Sheet. *Results in Physics*, 6: 904-910.

Heath, M. T. (2002). *Scientific Computing: An Introductory Survey*, 2nd ed. McGraw-Hill.

Heris, S. Z., Etemad, S. Gh., and Esfahany, N. M. (2006). Experimental Investigation of Oxide Nanofluids Laminar Flow Convective Heat Transfer. 33(4): 529-535.

Hetsroni, G., Mosyak, A., Pogrebnyak, E., and Yarin, L. (2005). Heat Transfer in Micro-Channels: Comparison of Experiments with Theory and Numerical Results. *Int. J. Heat Mass Transfer*, 48(25-26): 5580-5601.

Hiergeist, R., Andria, W., Buske, N., Hergt, R., Hilger, I., Richter, U., and Kaiser, W. (1999). Application of Magnetite ferrofluids for Hyperthermia. *Journal of Magnetism and Magnetic Materials*, 201(1): 420-422.

Hindebu, B., Makinde, O. D., and Guta, L. (2021). Unsteady mixed convection flow of variable viscosity nanofluid in a micro-channel filled with a porous medium. *Indian Journal of Physics*. <https://doi.org/10.1007/s12648-021-02116-y>.

Ho, C. M., and Tai, Y. C. (1998). Micro-Electro-Mechanical-Systems (MEMS) and fluid flows. *Annual Review of Fluid Mechanics*, 30(1): 579-612.

Hosseini, S. R., Sheikholeslami, M., Ghasemian, M., and Ganji, D. D. (2018). Nanofluid Heat Transfer Analysis in a Microchannel Heat Sink (MCHS) under the Effect of Magnetic Field by Means of KKL Model. *Powder Technology*, 324: 36-47.

- Hung, T. C., Yan, W. M., Wang, X. D., and Chang, C. Y. (2012). Heat Transfer Enhancement in Microchannel Heat Sinks Using Nanofluids. *International Journal of Heat and Mass Transfer*, 55(9-10): 2559-2570.
- Hunter, R. J. (1981). *Zeta potential in colloid science: principle and application*. Academic Press.
- Jameson, A., and Baker, T. J. (1983). Solution of the Euler Equations for Complex Configurations. *American Institute of Aeronautics and Astronauts*, 83-1929: 293-302.
- Jang, J., and Lee, S.S. (2000). Theoretical and Experimental Study of MHD (Magnetohydrodynamic) Micropump. *Sensors and Actuators A Physical*, 80(1): 84-89.
- Jha, B. K., and Aina, B. (2017). Effect of Induced Magnetic Field on MHD Mixed Convection Flow in Vertical Microchannel. *Int. J. of Applied Mechanics and Engineering*, 22(3): 567-582.
- Jha, B. K., Aina, B., and Joseph, S. B. (2013). Natural Convection Flow in a Vertical Microchannel With Suction/Injection. *Proc IMechE Part E: J Process Mechanical Engineering*, 228(3): 171-180.
- Jha, B. K., Aina, B., and Ajiya, A. T. (2015). Role of Suction/Injection on MHD Natural Convection Flow in a Vertical Microchannel. *International Journal of Energy & Technology*, 7(2): 30-39.
- Jha, B. K., and Aina, B. (2016). Magnetohydrodynamic Natural Convection Flow in a Vertical Micro-porous-annulus in the Presence of Radial Magnetic Field. *Journal of Nanofluids*, 5(2): 292-301.
- Jones, D. J. (1972). The Numerical Solution of Elliptic Equations by Method of Lines. *Computer Physics Communication*, 4(2): 165-172.
- Jone, D. J., South, J. C., and Klunker, E. B. (1972). On the Numerical Solution of Elliptic Partial Differential Equations by the Method of Lines. *J. Comp. Phys.*, 9(3): 496-527.
- Jordan, A., Scholz, R., Wust, P., Fahling, H., and Felix, R. (1999). Magnetic fluid hyperthermia (MFH): Cancer Treatment with AC Magnetic Field Induced Excitation of Biocompatible Superparamagnetic Nanoparticles. *Journal of Magnetism and Magnetic Materials*, 201(1): 413-419.
- Jung, J., Oh, H., and Kwak, H. (2009). Forced Convective Heat Transfer of Nanofluids in Microchannels. *International Journal of Heat and Mass Transfer*. 52(1-2): 466-472.
- Khan, A., Kumam, W., Khan, I., and Saeed, A., Gul, T., and Kumam, P. (2021). Chemically Reactive Nanofluid Flow Past a Thin Moving Needle With Viscous Dissipation, Magnetic Effects and Hall Current. *PLoS ONE*, 16(4): e0249264.
- Khanafer, K. Vafai, K., and Lightstone, M. (2003). Buoyancy-Driven Heat Transfer En-

hancement in a Two-Dimensional Enclosure Utilizing Nanofluids. *International Journal of Heat and Mass Transfer*, 46(19): 3639-3653.

Kandlikar, S. G., and Grande, W. J. (2003). Evolution of Microchannel Flow Passages Thermo Hydraulic Performance and Fabrication Technology. *Heat Transfer Engineering*, 24(1): 3-17.

Kandlikar, S., Garimella, S., Li, D., Colin, S., and King, M. R. (2006). *Heat Transfer and Fluid Flow in Minichannels and Microchannels*. Elsevier Ltd. ISBN:0-0804-4527-6.

Kleinstreuer, C. (2014). *Microfluidics and Nanofluidics: Theory and Selected Applications*. 211. John Wiley & Sons.

Koo, J., and Kleinstreuer, C. (2004). Viscous dissipation effects in microtubes and microchannels. *International Journal of Heat and Mass Transfer*, 47(14-16): 3159-3169.

Kumar, M. S., Sandeep, N., Kumar, B. R., and Saleem, S. (2018). Effect of Aligned Magnetic Field on MHD Squeezing Flow of Casson Fluid between Parallel Plates. *Defect and Diffusion Forum*, 384: 1-11.

Kumar, R., and Mahulikar, S. P. (2015). Effect of Temperature-Dependent Viscosity Variation on Fully Developed Laminar Microconvective Flow. *International Journal of Thermal Sciences*, 98: 179-191.

Kumar, R., and Mahulikar, S. P. (2018). Physical Effects of Variable Thermophysical Fluid Properties on Flow and Thermal Development in Micro-Channel. *Heat Transfer Engineering*, 39(4): 374-390.

Kumar, R., and Mahulikar, S. P. (2018). Variable Fluid Property Effect on Heat Transfer and Frictional Flow Characteristics of Water Flowing through Microchannel. *Journal of Engineering Thermophysics*, 27(4): 456-473.

Kumar, R., and Mahulikar, S. P. (2019). Heat transfer characteristics of water flowing through micro-tube heat exchanger with variable fluid properties. *Journal of Thermal Analysis and Calorimetry*, 140(4): 1919-1934.

Labuzov, S.G., and Potapov, M. M. (1985). Estimate of the Rate of Convergence of the Method of Lines in the Problem of Optimal Heating of a Rod. *Moscow Univ. Comput. Math. Cybern.*, 27-32.

Laughlin, D. R. (1989). A magnetohydrodynamic angular motion sensor for anthropomorphic test device instrumentation. *SAE Technical Paper 892428*: 1-35.

Lee, P.-S., and Garimella, V., and Dong, L. (2005). Investigation of Heat Transfer in Rectangular Microchannels. *Int. J. Heat Mass Transfer*, 48(9): 1688-1704.

Lee, J., and Mudawar, I. (2007). Assessment of the Effectiveness of Nanofluids for Single Phase and Two-Phase Heat Transfer in Micro-channels. *International Journal of Heat and*

Mass Transfer, 50(3-4): 452-463.

Lelea, D., Nishio, S., and Takano, K. (2004). The Experimental Research on Microtube Heat Transfer and Fluid Flow of Distilled Water. *Int. J. Heat Mass Transfer*, 47(1213): 2817-2830.

Li, J., Peterson, G. P., and Cheng, P. (2004). Three-Dimensional Analysis of Heat Transfer in a Micro-Heat Sink with Single Phase Flow. *International Journal of Heat and Mass Transfer*, 47(19-20): 4215-4231.

Liu, D., and Yu, L. (2010). Single-Phase Thermal Transport of Nanofluids in a Minichannel. *Journal of Heat Transfer*, 133(3): 1-11.

Lubbe, A. S., Alexiou, C., and Bergemann, C. (2001). Clinical Applications of Magnetic Drug Targeting. *Journal of Surgical Research*, 95(2): 200-206. Lui, J. T., and Peng, X. F., and Wang, B. X. (2008). Variable-Property Effect on Liquid Flow and Heat Transfer in Microchannels. *Chem. Eng. J.*, 141(1-3): 346-353.

Lv, Y.-P., Shaheen, N., Ramzan, M., Mursaleen, M., Nisar, S. K., and Malik, M. Y. (2021). Chemical Reaction and Thermal Radiation Impact on a Nanofluid Flow in a Rotating Channel with Hall Current. *Scientific Reports*, 11(1): 1-17.

Lyklema, J. (1995). *Fundamentals of interface and colloid science. Vol. II, Solid-Liquid Interfaces.* Academic Press.

Machura, M., and Sweet, R. A. (1980). A Survey of Software for Partial Differential Equations. *ACM Transactions on Mathematical Software (TOMS)*, 6(4): 461-488.

Madhu, M., Shashikumar, N. S., Gireesha, B. J., and KIshan, N. (2019). Second Law Analysis of Powell-Eyring Fluid Flow Through an Inclined Microchannel With Thermal Radiation. *Phys. Scr.*, 94(12): 125205.

Mahulikar, S. P., and Herwig, H. (2006). Physical Effects in Laminar Micro-Convection Due to Variations in Incompressible Fluid Properties. *Phys. Fluids*, 18(7): 073601.

Mahulikar, S. P., and Herwig, H. (2005). Theoretical Investigation of Scaling Effects from Macro-to-microscale Convection due to Variations in Incompressible Fluid Properties. *Applied Physics Letters*, 86(014105):1-3.

Makinde, O. D. (2012). Chemically Reacting Hydromagnetic Unsteady Flow of a Radiating Fluid Past a Vertical Plate with Constant Heat Flux. *Zeitschrift fñr Naturforschung A*, 67(5): 239-247.

Makinde, O. D., and Onyejekwe, O. O. (2011). A Numerical Study of MHD Generalized Couette Flow and Heat Transfer with Variable Viscosity and Electrical Conductivity. *Journal of Magnetism and Magnetic Materials*, 323(22): 2757-2763.

Makinde, O. D. (2018). Heat Transfer in Variable Viscosity Micro-Channel Flow of EG-

- Water/Ag Nanofluids With Convective Cooling. Defect and Diffusion Forum, 387: 182-193.
- Makinde, O. D., and Eegunjobi, A. S. (2013). Effects of Convective Heating on Entropy Generation Rate in a Channel with Permeable Walls. Entropy, 15(1): 220-233.
- Makinde, O. D., and Franks, O. (2014). On MHD Unsteady Reactive Couette Flow with Heat Transfer and Variable Properties. Cent. Eur. J. Eng., 4(1): 54-63.
- Malvandi, A., and Ganji, D. D. (2015). Effects of Nanoparticle Migration on Hydromagnetic Mixed Convection of Alumina/Water Nanofluid in Vertical Channels with Asymmetric Heating. Physica E, 66: 181-196.
- Malvandi, A., and Ganji, D. D. (2015). Effects of Nanoparticle Migration and Asymmetric Heating on Magnetohydrodynamic Forced Convection of Alumina/Water Nanofluid in Microchannels. Eur. J. Mech. B. Fluids, 52: 169-184.
- Malvandi, A., and Ganji, D. D. (2016). Mixed Convection of Alumina/Water Nanofluid in Microchannels Using Modified Buongiorno's Model in Presence of Heat Source/Sink. Journal of Applied Fluid Mechanics, 9(5): 2277-2289.
- Manthesha, and Mallikarjun, B. P. (2021). Magnetohydrodynamic Mixed Convection Flow in a Vertical Porous Microchannel With Heat Generation/Absorption. Palestine Journal of Mathematics, 10: 6-14.
- Markal, B., Aydin, O., and Avci, M. (2018). Experimental Study of Single-Phase Fluid Flow and Heat Transfer Characteristics in Rectangular Microchannels. Journal of Thermal Science and Technology. Journal of Thermal Science and Technology, 38(1): 65-72.
- Mehendale, S. S., Jacob, A. M., and Shah, R. K. (2000). Fluid Flow and Heat Transfer at Micro- and Meso-scales with Application to Heat Exchanger Design. Applied Mechanics Reviews, 53: 175-193.
- Mohammeda, H.A., Bhaskarana, G., Shuaiba, N.H., and Saidurb, R. (2011). Heat transfer and fluid flow characteristics in microchannels heat exchanger using nanofluids: A review. Renewable and Sustainable Energy Reviews, 15(3): 1502-1512.
- Monaledi, R. L., and Makinde, O. D. (2018). Entropy Analysis of a Radiating Variable Viscosity EG/Ag Nanofluid Flow in Microchannels with Buoyancy Force and Convective Cooling. Defect and Diffusion Forum, 387: 273-285.
- Monaledi, R. L., and Makinde, O. D. (2019). Inherent Irreversibility in $Cu - H_2O$ Nanofluid Couette Flow with Variable Viscosity and Nonlinear Radiative Heat Transfer. International Journal of Fluid Mechanics Research, 46(6): 525-543.
- Moran, M. J., Shapiro, H. N., and Munson, B. R. (2003). Introduction to Thermal Systems Engineering: Thermodynamics, Fluid Mechanics, and Heat Transfer. John Wiley & Sons, Inc. U.S.A. ISBN 0-471-20490-0.

- Moshizi, S. A. (2015). Forced Convection Heat and Mass Transfer of MHD Manofluid Flow Inside a Porous Microchannel with Chemical Reaction on the Walls. *Engineering Computations: International Journal for Computer-Aided Engineering and Software*, 32(8): 2419-2442.
- Moshizi, S.A., and Malvandi, A. (2016). Magnetic Field Effects on Nanoparticle Migration at Mixed Convection of MHD Nanofluids Flow in Microchannels with Temperature Dependent Thermophysical Properties. *J. Taiwan Inst. Chem. Eng*, 66: 269-282.
- Mourik, S.V., Veldman, A. E. P., and Dreyer, M. E. (2005). Simulation of capillary flow with a dynamic contact angle. *Microgravity Science and Technology*, 17(3): 87-93.
- Na, T. Y.(1979). *Computational Methods in Engineering Boundary Value Problems*. Volume 145, Academic Press, Inc. (London) Ltd.
- Nebbat, R., and Kadja, M. (2015). Study of Forced Convection of a Nanofluid Used as a Heat Carrier in a Microchannel Heat Sink. *Energy Procedia*, 74: 633-642.
- Nguyen, Q., Sedeh, Sh. N., Toghraie, D., Kalbasi, R., and Karimipour, A. (2020). Numerical Simulation of the Ferronanofluid Flow in a Porous Ribbed Microchannel Heat Sink: Investigation of the First and Second Laws of Thermodynamics with Single Phase and Two Phase Approaches. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 42(9): 1-14.
- Nino, V. G., Harnjak, P. S., and Newell, T. A. (2002). Characterization of Two-Phase Flow in Microchannels. ACRC TR-202, Air Conditioning and Refrigeration Center, University of Illinois at Urbana-Champaign.
- Nofal, T. (2011). An Approximation of the Analytical Solution of the Jeffery-Hamel Flow by Homotopy Analysis Method. *Applied Mathematical Sciences*, 5(53): 2603-2615.
- Nonino, C., Del Giudice, S., and Savino, S. (2010). Temperature-Dependent Viscosity and Viscous Dissipation Effects in Microchannel Flows With Uniform Wall Heat Flux. *Heat Transfer Engineering*, 31(8): 682-691.
- Obot, N.T. (2003). Toward a Better Understanding of Friction and Heat/Mass Transfer in Microchannels A Literature Review. *Microscale Thermophysical Engineering*, 6(3): 155-173.
- Odeban, S. (2013). Ferrofluids and Their Applications. *MRS Bulletin*, 38(11): 921-924.
- Olofin, J. (2018). On the Effect of Nanofluid Flow and Heat Transfer With Injection Through an Expanding or Contracting Porous Channel. *J. Comput. Appl. Mech.*, 49(1):1-8.
- Oztop, H. F., and Abu-Nada, E. (2008). Numerical Study of Natural Convection in Partially Heated Rectangular Enclosures Filled With Nanofluids. *International Journal of Heat and Fluid Flow*, 9(5): 1326-1336.

- Palm, B. (2001). Heat Transfer in Microchannels. *Microscale Thermophys. Eng.*, 5(3): 155-175.
- Pati, S., and Kumar, V. (2018). Effects of Temperature-Dependent Thermo-physical Properties on Hydrodynamic Swirl Decay in Microtubes. *Proceedings of the Institution of Mechanical Engineers, Part: Journal of Process Mechanical Engineering*, 233(3): 427-435.
- Paul, G., Chopkar, M., Manna, I., and Das, P. K. (2010). Techniques for Measuring the Thermal Conductivity of Nanofluids. A Review. *Renewable & Sustainable Energy Reviews*, 14(7): 1913-1920.
- Pordanjani, A. H., Raisi, A., and Ghasemi, A. (2019). Numerical Simulation of the Magnetic Field and Joule Heating Effects on Force Convection Flow Through Parallel-Plate Microchannel in the Presence of Viscous Dissipation Effect. *Numerical Heat Transfer, Part A: Applications. An International Journal of Computation and Methodology*, 76(6): 499-516.
- Prakash, S. B., Kotin, K. N., and Kumar, P. (2015). Preparation and Characterization of Nanofluid (*CuO*-Water, *TiO₂*-Water). *International Journal of Science and Engineering*, 1(3): 14-20.
- Priest, E. A. (2003). *Solar Magnetohydrodynamics*. Vol. 1: Springer Dordrecht.
- Qian, S., and Bau, H. H. (2009). Magneto-Hydrodynamic Based Microfluidics. *Mech. Res. Commun.*, 36(1): 10-21.
- Qin, M., and Bau, H. H. (2011). When MHD-Based Microfluidics is Equivalent to Pressure-Driven Flow. *Microfluid. Nanofluid.*, 10(2): 287-300.
- Qu, W. and Mudawar, I. (2002). Experimental and Numerical Study of Pressure Drop and Heat Transfer in a Single-Phase Micro-channel Heat Sink. *Int. J. Heat Mass Transfer*, 45(12): 2549-2565.
- Quddoos, A., Anand, A., and Mishra, G. K., and Nag, P. (2014). Nanofluids: Introduction, Preparation, Stability Analysis and Stability Enhancement Techniques. *International Journal in Physical & Applied Sciences*, 1(3): 31-36.
- Raju, K. C., and Reddy, B. S. (2019). Effect of Chemical Reaction, Thermo-Diffusion, Hall Effects on MHD Convective Heat and Mass Transfer Flow of a Viscous Fluid in a Vertical Channel Bounded by Stretching and Stationary Walls. *International Journal of Scientific and Innovative Mathematical Research (IJSIMR)*, 7(4): 12-24.
- Razali, A. A., Sadikin, A., and Ibrahim, S. A. (2017). Heat Transfer of *Al₂O₃* Nanofluids in Microchannel Heat Sink. *American Institute of Physics Conference Proceeding*, 1831(1): 020050.
- Rimbault, B., and Nguyen, C. T., and Galanis, N. (2014). Experimental Investigation of *CuO*-Water Nanofluid Flow and Heat Transfer Inside a Microchannel Heat Sink. *Interna-*

tional Journal of Thermal Sciences, 84: 275-292.

Rosseland, S. (1931). Astrophysics and Atom-theoretische Grundlagen. Springer, Berlin, Germany.

Roy, G., and Nguyen, C. T., and Lajoie, P. (2004). Numerical Investigation of Laminar Flow and Heat Transfer in a Radial Flow Cooling System With the Use of Nanofluids. Super lattices and Microstructures, 35(3-6): 497-511.

Sadiku, M. N. O., and Obiozor, C. N. (2000). A Simple Introduction to the Method of Lines. Int. J. Elec. Eng. Educ., 37(3): 282-296.

Sarojini, K. G. K., Manoj, S. V., Singh, P. K., Pradeep, T., and Das, S. K. (2013). Electrical Conductivity of Ceramic and Metallic Nanofluids. Colloids and Surfaces A: Physicochemical and Engineering Aspects, 417: 39-46.

Sayed-Ahmed, M. E., and El-Yazal, A. S. (2003). Laminar Fully Developed Flow and Heat Transfer for Robertson-Stiff Fluids in Rectangular Ducts with Temperature Dependent Viscosity. Int. Comm. Heat Mass Transfer, 30(6): 851-860.

Selimefendigil, F., Ztop, H. F., Sheremet, M. A., and Abu-Hamdeh, N. (2019). Forced Convection of Fe_3O_4 -Water Nanofluid in a Bifurcating Channel under the Effect of Variable Magnetic Field. Energies, 12(4): 666.

Setayesh, A., and Sahai, V. (1990). Heat Transfer in Developing Magnetohydrodynamic Poiseuille Flow and Variable Transport Properties. In. J. Heat mass transfer, 33(88): 1711-1720.

Shashikumar, N.S., Prasannakumara, B. C., Gireesha, B. J., and Makinde, O. D. (2018). Thermodynamics Analysis of MHD Casson Fluid Slip Flow in a Porous Microchannel with Thermal Radiation. Diffusion Foundations, 16: 120-139.

Sheikholeslami, M., and Ganji, D. D. (2016). External Magnetic Field Effects on Hydrothermal Treatment of Nanofluid: Numerical and Analytical Studies. William Andrew.

Sheikholeslami, M., and Ganji, D. D. (2016). Ferrofluid Convective Heat Transfer Under the Influence of External Magnetic Source. Alexandria Eng. J., 57(1): 49-60.

Schiesser, W. E., and Griffiths, G. W. (2009). A Compendium of Partial Differential Equation Models: Method of Lines Analysis with Matlab. Cambridge University Press.

Schiesser, W. E., and Silebi, C. A. (1997). Computational Transport Phenomena: Numerical Methods for the Solution of Transport Problems. Cambridge University Press.

Shiriny, A., Bayareh, M., and Nadooshan, A. A. (2019). Nanofluid Flow in a Microchannel With Inclined Cross-Flow Injection. SN Applied Sciences. 1(9). 1-8.

Shokouhmand, H., and A. Jafari, A. (2010). Investigation of Mixed Convection in a Vertical

- Microchannel Affected by EDL. Vol. 2. Proceedings of the World Congress on Engineering.
- Siddiqui, M. G. A., Saxena, S., and Singh, A. P. (2018). Study of Fluid Flow and Heat Transfer in Different Micro-channels- Computational Fluid Dynamics. *IOSR Journal of Engineering*, 08: 80-91.
- Singh, P. K., Harikrishna, P.V., Sundararajan, T., and Das, S. K. (2012). Experimental and Numerical Investigation into the Hydrodynamics of Nanofluids in Microchannels. *Experimental Thermal and Fluid Science*, 42: 174-186.
- Sobhan, C. B., and Garimella, S. V. (2001). Comparative Analysis of Studies on Heat Transfer and Fluid Flow in Microchannels. *Microscale Thermophys. Eng.*, 5(4): 293-311.
- Som, S. K. (2008). Introduction to Heat Transfer. PHI Learning Limited, M-97, Connaught Circus, New Delhi-110001, India. ISBN-978-81-203-3060-3.
- Steinke, M. E., and Kandlikar, S. G. (2006). Single-Phase Liquid Friction Factors in Microchannels. *Int. J. Therm. Sci.*, 45(11): 1073-1083.
- Stroock, A. D., Dertinger, S. K. W., Ajdari, A., Mezic, I., Stone, H. A., and Whitesides, G. M. (2002). Chaotic Mixers for Microchannels. *Science*, 295(5555): 647-651.
- Subhedar, D. G., and Ramani, B. (2013). Recent Studies on Single Phase Fluid Flow and Heat Transfer in Microchannel - a Review. *International Journal of Mechanical and Production*, 3(2): 169-180.
- Tarakaramu, N., and Narayana, P. V. S. (2019). Chemical Reaction Effects on Bio-Convection Nanofluid Flow Between Two Parallel Plates in Rotating System With Variable Viscosity: A Numerical Study. *J. Appl. Comput. Mech.*, 5(4): 791-803.
- Taylor, R. (1999). Engineering Computing with Maple: Solution of PDEs via Method of Lines. *CACHE News*, 49: 5-8.
- Thomson, W. J., and Bopp, G. R. (1970). The Effects of Variable Properties on MHD Flowing Infinite Ducts. *Transactions of the ASME: Journal of Applied Mechanics*. pp. 954-958.
- Tuckerman, D. B., and Pease, R. F. W. (1981). High-performance Heat Sinking for VLSI. *IEEE Electron Device Letters*, 2(5): 126-129.
- Turing, A. M. (1952). The Chemical basis of Morphogenesis. *Philosophical Transactions of the Royal Society of London Series B. Biological Science*, 27(641): 37-72.
- Ullah, I., Shafie, S., Makinde, O. D., and Khan, I. (2017). Unsteady MHD Falkner-Skan flow of Casson nanofluid with generative/destructive chemical reaction. *Chem. Eng. Sci.*, 172: 694-706.
- Williams, W. C., Bang, I. C., Forrest, E., Hu, L.W., and Buongiorno, J. (2006). Preparation and Characterization of Various Nanofluids. *NSTI-Nanotech*. 2: 408-411.

- Wu, H. Y., and Cheng, P. (2003). An experimental Study of Convective Heat Transfer in Silicon Microchannels with Different Surface Conditions. *Int. J. Mass Heat Transfer*. 45(14). 2547-2556.
- Xiang, Y., and Bau, H. H. (2003). Complex Magnetohydrodynamic Low-Reynolds-Number Flows. *Physical Review E*, 68(1): 1-11.
- Xu, B., Ooi, K. T., Wong, N. T., and Choi, W. K. (2000). Experimental Investigation of Flow Friction for Liquid Flow in Microchannels. *Int. Commun. Heat Mass Transfer*, 27(6): 1165-1176.
- Xu, B., and Ooi, K. T., and Mavriplis, C., and Zaghloul, M. E. (2003). Evaluation of viscous dissipation in liquid flow in microchannels. *Journal of Micromechanics Microengineering*, 13(1): 53.
- Xuan, Y., and Li, Q. (2000). Heat Transfer Enhancement of Nanofluids. *International Journal of Heat and Fluid Flow*. 21(1): 58-64.
- Ying, T. Y., Yang, K. L., Yiacoumi, S., and Tsouris, C. (2002). Electrosorption of Ions from Aqueous Solutions by Nanostructured Carbon Aerogel. *Journal of Colloid and Interface Science*, 250(1): 18-27.
- Yu, W., and Choi, S. (2003). The Role of Interfacial Layers in the Enhanced Thermal Conductivity of Nanofluids: a Renovated Maxwell Model. *Journal of Nanoparticle Research*, 5(1): 167-171.
- Zhong, J.-F., and Sedeh, Sh. N., and Lv, Y.-P., and Arzani, B., and Toghraie, D. (2021). Investigation of Ferro-nanofluid Flow within a Porous Ribbed Microchannel Heat Sink Using Single-Phase and Two-Phase Approaches in the Presence of Constant Magnetic Field. *Powder Technology*, 387: 251-260.

APPENDIX A

DIFFERENTIAL OPERATIONS AND DEFINITIONS OF BASIC CONCEPTS

Here, differential operators with their properties as well as the general idea of fluid and fluid flow needed in the dissertation were presented.

A.1 Differential Operations

A.1.1 Del Operator, ∇

(i) Gradient: Del applied to a scalar.

* In rectangular coordinates, $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$

* $\text{grad } P = \nabla P = \frac{\partial P}{\partial x} \hat{i} + \frac{\partial P}{\partial y} \hat{j} + \frac{\partial P}{\partial z} \hat{k}$, where P is pressure.

(ii) Divergence: Scalar product of Del operator (or Nabla) and a vector.

* $\nabla \cdot q = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$, where $q = (u, v, w)$.

(iii) Curl: Vector product of Del operator and a vector.

* $\text{curl } q = \nabla \times q = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$

(iv) Laplacian of a scalar.

* $\nabla \cdot \nabla P = \nabla^2 P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}$

(v) Rules

For scalar functions a, b of position and A, B vector functions of positions,

* Gradient: $\nabla ab = a \nabla b + b \nabla a$

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \nabla)B + (B \nabla)A$$

* Divergence: $\nabla \cdot (aA) = a(\nabla \cdot A) + A \cdot (\nabla a)$

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

* Curl: $\nabla \times aA = a(\nabla \times A) + A \times (\nabla a)$

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A)$$

A.1.2 Material Derivative of a Vector

* $\frac{dq}{dt} = \frac{\partial q}{\partial t} + (q \cdot \nabla)q$

A.2 Definitions of Key Terms

A.2.1 Classification of Fluids

Fluid is any substance which deforms continuously under the action of applied shear forces, however small. Forces which act tangentially to the surfaces to which they are applied are referred to as **shear forces**.

Microchannels are channels whose hydraulic diameter less than 1 mm ($D_h < 1\text{mm}$).

Compressible and Incompressible Fluids: A fluid is said to be compressible if its density changes with pressure but if the density of the fluid is constant or has no significant change with pressure, it is called incompressible fluid.

Viscous and Inviscid Fluids: A fluid is said to be viscous when both the normal stress (pressure) and the shearing stress exist but it is said to be inviscid fluid when only the normal stress (pressure) acts between the adjoining layers.

Newtonian and Non-Newtonian Fluids: Newton's law of viscosity is given by $\tau_{ij} = \mu \frac{du}{dy}$. It states that the shear stress is linearly related to the rate of strain or velocity gradient (rate of deformation) where dynamic viscosity (μ) is a constant of proportionality. Thus, fluids that obey the Newton's law of viscosity are called Newtonian fluids but that do not are called non-Newtonian fluids.

A.2.2 Types of Flows

Internal and External flow: The flow of an unbounded fluid over a surface is called external flow and if the fluid is completely bounded by the surface, then it is called internal flow. For example, flow over a flat plate is considered as external flow and flow through a channel or pipe/duct is internal flow.

Steady and Unsteady Flows: If a fluid flows in such a way that all its properties at every point in the flow depend upon time then it is called unsteady flow; otherwise, it is called steady flow.

Rotational and Irrotational Flows: A flow is said to be irrotational if the fluid particles along their path translate without rotation about the particles center. If along their path the fluid particles translate and rotate about the particles center, the flow is rotational.

Laminar and Turbulent Flows: A flow is said to be a laminar flow if the fluid particles move in a definite path or streamlines. In contrary, turbulent flow is characterized by the irregular movement of particles of the fluid. That is, the particles travel in irregular paths with no observable pattern and no definite layers.

APPENDIX B

THE FOLLOWING IS MAPLE CODE FOR THE MODEL IN CHAPTER 4

This code is 2D transient analysis for velocity, temperature and concentration with increasing y .

```

with(plots);

R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .1; Nb := .5; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := .1:

fcns := [u(y, t), T(y, t), C(y, t)]:

IBC := [u(y, 0) = 0, T(y, 0) = 0, C(y, 0) = 0, u(0, t) = 0, T(0, t) = 0, Nb*(D[1](C))(0, t) =
-Nt*(D[1](T))(0, t), C(1, t) = 1, u(1, t) = 0, T(1, t) = 1]:

sys:=[diff(u(y,t),t)+R1*diff(u(y,t),y)=A-A1*exp(-A1*T(y,t))*diff(u(y,t),y)*diff(T(y,t),y)+exp(-
A1*T(y,t))*diff(u(y,t),y2)-M*M*u(y,t)+Gr*T(y,t)+Gc*C(y,t),Pr*(diff(T(y,t),t))+
Pr*R1*(diff(T(y,t),y))=diff(T(y,t),“(y, 2)))+(Ec*Pr*(diff(u(y, t), y))*(diff(u(y, t),
y)))*exp(-A1*T(y, t))+(Ec*Pr*M*M)*u(y, t)*u(y, t)+Pr*Nb*(diff(T(y, t), y))*(diff(C(y, t),
y))+Pr*Nt*(diff(T(y, t), y))*(diff(T(y, t), y)), Sc*(diff(C(y, t), t))+Sc*R1*(diff(C(y, t), y))=
diff(C(y, t), “(y,2)))+Nt*(diff(T(y,t),“(y,2)))/Nb:

pds:=pdsolve(sys, IBC, fcns, numeric);

p1u:=pds:-plot(u(y, t), t = 0 .. 2, y = .2, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Velocity"], style = line, color = black);

p2u:=pds:-plot(u(y, t), t = 0 .. 2, y = .3, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Velocity"], style = line, color = red);

p3u:=pds:-plot(u(y, t), t = 0 .. 2, y = .4, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Velocity"], style = line, color = blue);

p4u:=pds:-plot(u(y, t), t = 0 .. 2, y = .5, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Velocity"], style = line, color = green);

p1t:=pds:-plot(T(y, t), t = 0 .. 2, y = .2, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Temperature"], style = line, color = black);

p2t:=pds:-plot(T(y, t), t = 0 .. 2, y = .3, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Temperature"], style = line, color = red);

p3t:=pds:-plot(T(y, t), t = 0 .. 2, y = .4, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Temperature"], style = line, color = blue);

p4t:=pds:-plot(T(y, t), t = 0 .. 2, y = .5, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Temperature"], style = line, color = green);

p1c:=pds:-plot(C(y, t), t = 0 .. 2, y = .2, numpoints = 100, labeldirections = ["horizontal",

```

```
"vertical"], labels = ["t", "Concentration"], style = line, color = black); p2c := pds:-plot(C(y,
t), t = 0 .. 2, y = .4, numpoints = 100, labeldirections = ["horizontal", "vertical"], labels =
["t", "Concentration"], style = line, color = red);
p3c := pds:-plot(C(y, t), t = 0 .. 2, y = .6, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Concentration"], style = line, color = blue);
p4c := pds:-plot(C(y, t), t = 0 .. 2, y = .9, numpoints = 100, labeldirections = ["horizontal",
"vertical"], labels = ["t", "Concentration"], style = line, color = green);
%% %% %% %% %% %% %% %% %% %% %% %% %% %% %% %% %% %% %% %% %% %%
# This code is for steady state velocity analysis with parameter A1 change.
```

```
with(plots):
R1:=0.1:M:=0.1:Gc:=0.1:Gr:=0.1:Nt:=0.1:Nb:=0.5:Sc:=0.6:Ec:=0.1:Pr:=6.2:A:=1:A1:=0.1:
fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y), "(y,2)) - (M *
M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), "(y,2))+Ec*Pr*(diff(u(y),
y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y), y))+Pr*Nt*(diff(T(y),
y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), "(y,2)) + Nt*(diff(T(y), "(y,2)))/Nb;
p1:= dsolve (sys, u(0)= 0,T(0)= 0, Nb*D(C)(0)= -Nt*D(T)(0), C(1)=1, u(1)=0, T(1)= 1, fcns,
type = numeric, method= bvp [midrich], abserr=1e-10):
p1u:=odeplot(p1,[y,u(y)],0..1,numpoints=100,labels= ["y","u"],style =line,color=black):
with(plots):
R1:=0.1:M:=0.1:Gc:=0.1:Gr:=0.1:Nt:=0.1:Nb:=0.5:Sc:=0.6:Ec:=0.1:Pr:=6.2:A:=1:A1:=1:
fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y), "(y,2)) - (M *
M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), "(y,2))+Ec*Pr*(diff(u(y),
y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y), y))+Pr*Nt*(diff(T(y),
y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), "(y,2)) + Nt*(diff(T(y), "(y,2)))/Nb;
p2:= dsolve (sys, u(0)=0, T(0)= 0, Nb*D(C)(0)= -Nt*D(T)(0), C(1)=1, u(1)=0, T(1)= 1, fcns,
type= numeric, method= bvp[midrich], abserr= 1e-10):
p2u:=odeplot(p2,[y,u(y)],0..1,numpoints=100,labels=["y","u"],style=line,color=red):
with(plots):
R1:=0.1:M:=0.1:Gc:=0.1:Gr:=0.1:Nt:=0.1:Nb:=0.5:Sc:=0.6:Ec:=0.1:Pr:=6.2:A:=1:A1:=2:
fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y), "(y,2)) - (M *
M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), "(y,2))+Ec*Pr*(diff(u(y),
y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y), y))+Pr*Nt*(diff(T(y),
y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), "(y,2)) + Nt*(diff(T(y), "(y,2)))/Nb;
p3:= dsolve (sys, u(0)=0, T(0)=0, Nb*D(C)(0)= -Nt*D(T)(0), C(1)=1, u(1)=0,T(1)=1, fcns,
type= numeric, method= bvp[midrich], abserr= 1e-10):
```



```

p2t := odeplot(p1, [y, T(y)], 0 .. 1, numpoints = 100, laledirections = ["horizontal", "vertical"], labels = ["y", "Temperature"], style = line, color = red);
with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .1; Nb := .5; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := 4;
fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*exp(-A1*T(y))*(diff(u(y), y))*(diff(T(y), y))+exp(-A1*T(y))*(diff(u(y),
“(y, 2))) - (M*M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), “(y,
2))+Ec*Pr*(diff(u(y), y))*(diff(u(y), y))*exp(-A1*T(y))+Ec*Pr*M*M*u(y)*u(y)+Pr*Nb*(diff(T(y),
y))*(diff(C(y), y))+Pr*Nt*(diff(T(y), y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y),
“(y, 2)) + Nt*(diff(T(y), “(y, 2)))/Nb;
p3 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
p3t := odeplot(p1, [y, T(y)], 0 .. 1, numpoints = 100, laledirections = ["horizontal", "vertical"], labels = ["y", "Temperature"], style = line, color = blue);
with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .1; Nb := .5; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := 6;
fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*exp(-A1*T(y))*(diff(u(y), y))*(diff(T(y), y))+exp(-A1*T(y))*(diff(u(y),
“(y, 2))) - (M*M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), “(y,
2))+Ec*Pr*(diff(u(y), y))*(diff(u(y), y))*exp(-A1*T(y))+Ec*Pr*M*M*u(y)*u(y)+Pr*Nb*(diff(T(y),
y))*(diff(C(y), y))+Pr*Nt*(diff(T(y), y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y),
“(y, 2)) + Nt*(diff(T(y), “(y, 2)))/Nb;
p4 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
p4t := odeplot (p1, [y, T(y)], 0 .. 1, numpoints = 100, laledirections = ["horizontal", "vertical"], labels = ["y", "Temperature"], style = line, color = green);
plots[display](p1t, p2t, p3t, p4t) % % %%% % % %%% % %%% % %%% % %%% %
% %%% %

```

This code is for steady state concentration analysis with parameters Nb and Nt change.

```

with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .1; Nb := .1; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := .1;
fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y), “(y, 2)) - (M *
M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), “(y, 2))+Ec*Pr*(diff(u(y),
y))*(diff(u(y), y))+Ec*Pr*M*M*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y), y))+Pr*Nt*(diff(T(y),

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```

y))*diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), ‘‘(y, 2)) + Nt*(diff(T(y), ‘‘(y, 2)))/Nb;
p1 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
p1c := odeplot(p1, [y, C(y)], 0 .. 1, numpoints = 100, laeldirections = ["horizontal", "verti-
cal"], labels = ["y", "Concentration"], style = line, color = black);
    with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .2; Nb := .11; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := .1;
fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y), ‘‘(y, 2)) - (M *
M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), ‘‘(y, 2))+Ec*Pr*(diff(u(y),
y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y), y))+Pr*Nt*(diff(T(y),
y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), ‘‘(y, 2)) + Nt*(diff(T(y), ‘‘(y, 2)))/Nb;
p2 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
p2c := odeplot(p2, [y, C(y)], 0 .. 1, numpoints = 100, laeldirections = ["horizontal", "verti-
cal"], labels = ["y", "Concentration"], style = line, color = red);
    with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .3; Nb := .1; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := .1;
fcns := C(y), T(y), u(y); sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y),
‘‘(y, 2)) - (M*M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), ‘‘(y,
2))+Ec*Pr*(diff(u(y), y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y),
y))+Pr*Nt*(diff(T(y), y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), ‘‘(y, 2)) + Nt *
(diff(T(y), ‘‘(y, 2)))/Nb;
p3 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
p3c := odeplot(p3, [y, C(y)], 0 .. 1, numpoints = 100, laeldirections = ["horizontal", "verti-
cal"], labels = ["y", "Concentration"], style = line, color = blue);
    with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .1; Nb := 1; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := .1;
fcns := C(y), T(y), u(y); sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y),
‘‘(y, 2)) - (M*M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), ‘‘(y,
2))+Ec*Pr*(diff(u(y), y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y),
y))+Pr*Nt*(diff(T(y), y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), ‘‘(y, 2)) + Nt *
(diff(T(y), ‘‘(y, 2)))/Nb;
p4 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);

```

```

p4c := odeplot(p4, [y, C(y)], 0 .. 1, numpoints = 100, laeldirections = ["horizontal", "vertical"], labels = ["y", "Concentration"], style = line, color = green);
with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .2; Nb := 1; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := .1; fcns := C(y), T(y), u(y);
sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y), "(y,2)) - (M *
M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), "(y,2))+Ec*Pr*(diff(u(y),
y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y), y))+Pr*Nt*(diff(T(y),
y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), "(y,2)) + Nt*(diff(T(y), "(y,2)))/Nb;
p5 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
p5c := odeplot(p5, [y, C(y)], 0 .. 1, numpoints = 100, laeldirections = ["horizontal", "vertical"], labels = ["y", "Concentration"], style = line, color = purple);
with(plots);
R1 := .1; M := .1; Gc := .1; Gr := .1; Nt := .3; Nb := 1; Sc := .6; Ec := .1; Pr := 6.2; A := 1;
A1 := .1;
fcns := C(y), T(y), u(y); sys := R1*(diff(u(y), y)) = A-A1*(diff(u(y), y))*(diff(T(y), y))+diff(u(y),
"(y,2)) - (M*M)*u(y) + Gr*T(y) + Gc*C(y), Pr*R1*(diff(T(y), y)) = diff(T(y), "(y,
2))+Ec*Pr*(diff(u(y), y))*(diff(u(y), y))+(Ec*Pr*M*M)*u(y)*u(y)+Pr*Nb*(diff(T(y), y))*(diff(C(y),
y))+Pr*Nt*(diff(T(y), y))*(diff(T(y), y)), Sc*R1*(diff(C(y), y)) = diff(C(y), "(y,2)) + Nt *
(diff(T(y), "(y,2)))/Nb;
p6 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
p6c := odeplot(p6, [y, C(y)], 0 .. 1, numpoints = 100, laeldirections = ["horizontal", "vertical"], labels = ["y", "Concentration"], style = line, color = yellow);
plots[display](p1c, p2c, p3c, p4c, p5c, p6c);
% % %%% % % %%% % % %%% % % %%% % % %%%
# This code is for Skin friction, Nusselt number and Sherwood number.

```

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with(plots):
R:=array(1..352):T11:=array(1..352):T12:=array(1..352):T13:=array(1..352):
i:=1:
R1:=0.1:M:=0.1:Gc:=0.1:Gr:=0.1:Nt:=0.1:Nb:=0.1:Sc:=0.1:Ec:=0.1:Pr:=6.2:A:=1:
for A1 from 0 by 0.05 to 1 do
fcns:=u(y),T(y),C(y):
sys := R1*(diff(u(y), y)) = A - A1* exp (-A1*T(y)) * (diff(u(y), y)) * (diff(T(y), y)) +
exp (-A1*T(y)) * (diff(u(y), "(y,2))) - (M * M) * u(y) + Gr * T(y) + Gc * C(y), Pr * R1 *
(diff(T(y), y)) = diff(T(y), "(y, 2)) + (Ec * Pr * (diff(u(y), y)) * (diff(u(y), y))) * exp
(-A1*T(y)) + (Ec * Pr * M * M) * u(y) * u(y) + Pr * Nb * (diff(T(y), y)) * (diff(C(y),

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y)) + Pr * Nt * (diff(T(y), y)) * (diff(T(y), y)), Sc * R1 * (diff(C(y), y)) = diff(C(y),
“(y, 2)) + Nt * (diff(T(y), “(y, 2)))/Nb;
p1 := dsolve(sys, Nb*(D(C))(0) = -Nt*(D(T))(0), C(1) = 1, T(0) = 0, T(1) = 1, u(0) = 0, u(1)
= 0, fcns, type = numeric, method = bvp[midrich], abserr = 0.1e-9);
R[i] := A1;
#‘Skin friction‘ :
T11[i]:=rhs(p1(1)[7]):
#‘ Nusselt number‘
T12[i]:=-rhs(p1(1)[5]):
#‘ Sherwood number‘
T13[i]:=-rhs(p1(1)[3]):
i:=i+1:
od:
R1:=’R1’:M:=’M’:Gc:=’Gc’:Gr:=’Gr’:Nt:=’Nt’:Nb:=’Nb’:Sc:=’Sc’:Ec:=’Ec’:Pr:=’Pr’:A:=’A’:A1:=’A1’:
points1:=seq([R[j],T11[j]],j=1..21):
points2:=seq([R[j],T12[j]],j=1..21):
points3:=seq([R[j],T13[j]],j=1..21):
pCf1:=pointplot(points1,labels=["A1","Cf"],style=line,color=black):
pNu1:=pointplot(points2,labels=["A1","Nu"],style=line,color=black):
pSh1:=pointplot(points3,labels=["A1","Sh"],style=line,color=black):
with(plots):
R:=array(1..352):T11:=array(1..352):T12:=array(1..352):T13:=array(1..352):
i:=1:
R1:=0.5:M:=0.1:Gc:=0.1:Gr:=0.1:Nt:=0.1:Nb:=0.1:Sc:=0.1:Ec:=0.1:Pr:=6.2:A:=1:
for A1 from 0 by 0.05 to 1
do
fcns:=u(y),T(y),C(y):
sys := R1*(diff(u(y), y)) = A - A1* exp (-A1*T(y)) * (diff(u(y), y)) * (diff(T(y), y)) +
exp (-A1*T(y)) * (diff(u(y), “(y, 2))) - (M * M) * u(y) + Gr * T(y) + Gc * C(y), Pr * R1 *
(diff(T(y), y)) = diff(T(y), “(y, 2)) + (Ec* Pr * (diff(u(y), y)) * (diff(u(y), y))) * exp
(-A1*T(y)) + (Ec * Pr * M * M) * u(y) * u(y) + Pr * Nb * (diff(T(y), y)) * (diff(C(y),
y)) + Pr * Nt * (diff(T(y), y)) * (diff(T(y), y)), Sc * R1 * (diff(C(y), y)) = diff(C(y),
“(y, 2)) + Nt * (diff(T(y), “(y, 2)))/Nb;
p1:= dsolve (sys,u(0)=0, T(0)=0, Nb*D(C)(0)= -Nt*D(T)(0), C(1)=1, u(1)=0, T(1)=1, fcns,
type=numeric, method= bvp[midrich], abserr=1e-10):
R[i]:=A1:
#‘ Skin friction‘
T11[i]:=rhs(p1(1)[7]):
#‘ Nusselt number‘

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T12[i]:=-rhs(p1(1)[5]):
#‘ Sherwood number‘
T13[i]:=-rhs(p1(1)[3]):
i:=i+1:
od:
R1:='R1':M:='M':Gc:='Gc':Gr:='Gr':Nt:='Nt':Nb:='Nb':Sc:='Sc':Ec:='Ec':Pr:='Pr':A:='A':A1:='A1':
points11:=seq([R[j],T11[j]],j=1..21):
points21:=seq([R[j],T12[j]],j=1..21):
points31:=seq([R[j],T13[j]],j=1..21):
pCf2:=pointplot(points11,labels=["A1","Cf"],style=line,color=red):
pNu2:=pointplot(points21,labels=["A1","Nu"],style=line,color=red):
pSh2:=pointplot(points31,labels=["A1","Sh"],style=line,color=red):
with(plots):
R:=array(1..352):T11:=array(1..352):T12:=array(1..352):T13:=array(1..352):
i:=1:
R1:=0.9:M:=0.1:Gc:=0.1:Gr:=0.1:Nt:=0.1:Nb:=0.1:Sc:=0.1:Ec:=0.1:Pr:=6.2:A:=1:
for A1 from 0 by 0.05 to 1
do
fcns:=u(y),T(y),C(y):
sys := R1*(diff(u(y), y)) = A - A1* exp (-A1*T(y)) * (diff(u(y), y)) * (diff(T(y), y)) +
exp (-A1*T(y)) * (diff(u(y), ‘(y,2)’) - (M * M) * u(y) + Gr * T(y) + Gc * C(y), Pr * R1 *
(diff(T(y),y)) = diff(T(y),‘(y, 2)’) + (Ec* Pr * (diff(u(y), y)) * (diff(u(y), y))) * exp
(-A1*T(y)) + (Ec * Pr * M *M) * u(y) * u(y) + Pr * Nb * (diff(T(y), y)) * (diff(C(y),
y)) + Pr * Nt * (diff(T(y), y)) * (diff(T(y), y)), Sc * R1 * (diff(C(y), y)) = diff(C(y),
‘(y,2)’) + Nt * (diff(T(y), ‘(y, 2)’))/Nb;
p1:= dsolve (sys, u(0)=0, T(0)=0, Nb*D(C)(0)= -Nt*D(T)(0), C(1)=1, u(1)=0, T(1)=1, fcns,
type= numeric, method= bvp[midrich], abserr=1e-10):
R[i]:=A1:
‘ Skin friction‘
T11[i]:=rhs(p1(1)[7]):
#‘ Nusselt number‘
T12[i]:=-rhs(p1(1)[5]):
#‘ Sherwood number‘
T13[i]:=-rhs(p1(1)[3]):
i:=i+1:
od:
R1:='R1':M:='m':Gc:='Gc':Gr:='Gr':Nt:='Nt':Nb:='Nb':Sc:='Sc':Ec:='Ec':Pr:='Pr':A:='A':A1:='A1':
points12:=seq([R[j],T11[j]],j=1..21):
points22:=seq([R[j],T12[j]],j=1..21):

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points32:=seq([R[j],T13[j]],j=1..21):
pCf3:=pointplot(points12,labels=["A1","Cf"],style=line,color=blue):
pNu3:=pointplot(points22,labels=["A1","Nu"],style=line,color=blue):
pSh3:=pointplot(points32,labels=["A1","Sh"],style=line,color=blue):
plots[display](pCf1,pCf2,pCf3);
plots[display](pNu1,pNu2,pNu3);
plots[display](pSh1,pSh2,pSh3);

```