Three-Sector Keynesian Model of Business Cycles

Tihtina Atle

A Thesis submitted to the Department of Applied Mathematics
School of Applied Natural Sciences

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Mathematics (Differential Equations)

Office of Graduate Studies
Adama Science and Technology University

July, 2021
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Advisor:

Tamirat Temesgen (PhD)

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Declaration

I, hereby declare that this Master Thesis entitled "**Three-Sector Keynesian Model of Business Cycles**" is my original work. That is, it has not been submitted for the award of any academic degree, deploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citations.

Name of the Student	Signature	Date

Recommendation

I, Dr. Tamirat Temesgen hearby certify that I have read the revised version of the thesis entitled "**Three-Sector Keynesian Model of Business Cycles**" prepared under my guidance by Tihtina Atle Zelelew submitted in partial fulfillment of the requirements for the degree of Master's of science in Mathematics (Differential Equations). Therefore, I recomend the submission of revised version of the thesis to the department following the applicable procedures.

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Approval Sheet

I, the advisor of the thesis entitled "**Three-Sector Keynesian Model of Business Cycles**" and developed by Tihtina Atle, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

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We, the undersigned, members of the Board of Examiners of the thesis by Tihtina Atle have read and evaluated the thesis entitled "**Three-Sector Keynesian Model of Business Cycles**" and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master's of science in applied Mathematics (Differential Equations).

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Abstract

This Thesis is concerned with three-sector Keynesian model of business cycles, describing the fluctuations of durable, fast and investment good sectors. In the Thesis, the Hiroki's two-sector Keynesian model of business cycles is modified to three-sector Keynesian model of business cycles. The stability analysis is based on the concept of Routh-Hurwitz stability criterion. To justify the accuracy of the results obtained a specific numerical example is demonstrated through simulation by use of MATLAB 2021a.

Keywords: Keynesian theory of business cycle, Three sector model, Routh-Hurwitz stability crtieria, Periodic orbit.

Chapter 1

Introduction

1.1 Background of the Study

Economic growth continues to be one of the most relevant and exciting sub-areas of economics. Its relevance stems from the questions it focuses on. The problem of economic development remains a major one for humanity at large and for economics as a science. At the time Adam Smith laid many of the foundations of modern economics, there were likely small differences between the richest and the poorest nations in the world (Acemoglu, Johnson, & Robinson, 2002; Maddison, 2007). Since then, the gaps between the rich and poor have increased to a level that would have been incomprehensible to most 18^{th} and 19^{th} century economists.

Mathematical economics is a theoretical and applied science, whose purpose is a Mathematically formalized description of economic objects, processes, and phenomena (Tarasov, 2020; Medio, 2009). Most of the economic theories are presented in terms of economic models. In mathematical economics, the properties of these models are studied based on formalizations of economic concepts and notions. In mathematical economics, theorems on the existence of extreme values of certain parameters are proved, properties of equilibrium states and equilibrium growth trajectories are studied, etc (Tarasov, 2020; Medio, 2009; Breslau & Yonay, 1999). This creates the impression that the proof of the existence of a solution (optimal or equilibrium) and its calculation is the main aim of mathematical economics (Tarasov, 2020; Clark, 1979).

The introduction of mathematical methods into economics has been a long process. Pieces of essentially mathematical reasoning applied to economic problems have been detected as far back in history as in Aristotle's work and in the 18^{th} and early 19^{th} centuries, outstanding mathematicians such as Bernoulli, Gauss, Laplace and Poisson developed truly mathematical models to discuss economic problems (Medio, 2009; Ardor et al., 2002) .

In reality, the most important purpose is to formulate economic notions and concepts in mathematical form, which will be mathematically adequate and self-consistent, and then, on their basis to construct mathematical models of economic processes and phenomena. Moreover, it is not enough to prove the existence of a solution and find it in an analytic or numerical form, but it is necessary to give an economic interpretation of these obtained mathematical results (Tarasov, 2020; Medio, 2009).

Nonlinear economic dynamics may be considered just a collection of models with essentially nonlinear ingredients that require the use of a particular set of (relatively new) mathematical tools and has recentlty become more prominent in main stream economics (Shone, 2002; Lorenz, 1993).

A mathematical model of the economy is a formal description of certain relationships between quantities, such as prices, production, employment, saving, investment, etc., with the purpose to analyze their logical implications. Some of those relationships derive from empirical observation; others are deduced from theoretical axioms concerning the assumed behavior of a "rational" economic agent, the so-called homooeconomicus (Medio, 2009).

Specifically, nonlinear models played an important role in modelling economic dynamics during the first part of this century By the 1960^s, however, the profession had largely switched to the linear approach making use of observation that stable low order linear stochastic difference equations could generate cyclic processes that mimicked actual business cycles (Slutzky, 1937).

The hypothesis of trade cycles has been one of the important fields of macroeconomics since the establishments of macroeconomics were laid by Keynes's Common General Theory. In the period between the late 1930^s and the early 1950^s, classic models based on Keynes were proposed to explain the mechanism of business cycles (Pigou, 1936; Kalecki, 2013; P. A. Samuelson, 1939; Kaldor, 1940; Hicks, 1950; Goodwin, 1951). These models of business cycles are even today useful for understanding a lot of aspects of business cycles as observed in reality.

In economic theory, persistent cyclical fluctuations such as business cycles and growth cycles are usually described by periodic orbits including limit cycles, and the main theme of theory of business cycles is, in a lot of cases, to establish the presence of a periodic orbit in dynamic models (Murakami, 2020).

According to economics, a durable good or a hard good or consumer durable good is a good that does not quickly wear out or, more specifically, one that yields utility over time rather than being completely consumed in one use. Items like bricks could be considered perfectly durable goods because they should theoretically never wear out. Highly durable goods such as refrigerators or cars usually continue to be useful for several years of use, so durable goods are typically characterized by long periods between successive purchases (Arthur & Sheffrin, 2003; Waldman, 2003). Automobiles, books, household goods (home appliances, consumer electronics, furniture, tools, etc.), sports equipment, jewelry, medical equipment, firearms, and toys are some examples of consumer durable goods.

Fast goods or soft goods (consumables) are the opposite of durable goods. They may be defined either as goods that are immediately consumed in one use or ones that have a lifespan of less than three years. Some of the goods which are included under this category are fast-moving consumer goods such as cosmetics and cleaning products, food, condiments, fuel, beer, cigarettes and tobacco, medication, office supplies, packaging and containers, paper and paper products, personal products, rubber, plastics, textiles, clothing, and footwear. while durable goods can usually be rented as well as bought, fast goods generally are not rented (Arthur & Sheffrin, 2003; Waldman, 2003; Rush, 1997).

The economic concept of a capital good (also called complex product systems and means of production) is as a series of heterogeneous commodities, each having specific technical characteristics in the form of a durable good that is used in the production of goods or services.

Capital goods are a particular form of economic good and are tangible property (Waldman, 2003).

A society acquires capital goods by saving wealth that can be invested in the means of production. People use them to produce other goods or services within a certain period. Machinery, tools, buildings, computers, or other kinds of equipment that are involved in the production of other things for sale are capital goods. The owners of the capital good can be individuals, households, corporations, or governments. Any material used to produce capital goods is also considered a capital good (Arthur & Sheffrin, 2003; Rush, 1997).

Capital goods are one of the three types of producer goods, the other two being land and labour. The three are also known collectively as "primary factors of production". This classification originated during the classical economics period and has remained the dominant method for classification. Many definitions and descriptions of capital goods production have been proposed in the literature. Capital goods are generally considered one of a kind, capital intensive products that consist of many components. They are often used as manufacturing systems or services themselves. Hand tools, machine tools, data centers, oil rigs, semiconductor fabrication plants, and wind turbines are some of the examples included under capital goods. Their production is often organized in projects, with several parties cooperating in networks (Arthur & Sheffrin, 2003; P. Samuelson & Nordhouse, 2009).

A capital good lifecycle typically consists of tendering, engineering and procurement, manufacturing, commissioning, maintenance, and (sometimes) decommissioning. Capital goods are a major factor in the process of technical innovation (Blanchard, 1997; Veldman & Alblas, 2012).

Hiroki Murakami in (Murakami, 2018) , obtained different economic equilibria and discussed different stability conditions for the two sector Keynesian model of business cycles. Therefore, the purpose of this thesis are to modify his two-sector Keynesian model of business cycles into a three- sector Keynesian model of business cycles and to investigate different equilibria conditions and stability conditions.

1.2 Statement of the Problem

Hiroki Murakami in (Murakami, 2018) , considered two sectors; consumption good and investment good. He assume that there are only two kinds of goods that differ in property. He set up a two-sector model as follows:

$$\begin{cases} \dot{y}_c = \alpha_c[C(y_c, y_i) - y_c] \ , \\ \dot{y}_i = \alpha_i[I_c(y_c, k_c) + I_i(y_i, k_i) - y_i] \ , \\ \dot{k}_c = I_c(y_c, k_c) - \delta k_c \ , \\ \dot{k}_i = I_i(y_i, k_i) - \delta k_i \ , \end{cases} \tag{1.1}$$

where,

- y_c stands for the level of income or output of the consumption good sectors.
- y_i stand for the level of income or output of the investment-good sectors,
- k_c stand for the existing stocks of the consumption-good

- k_i stand for the existing stocks of the investment good (capital);
- C is the consumption function;
- I_c and I_i are the (gross) investment functions for the consumption-good and investment-good sectors, respectively;
- α_c , α_i and δ are positive constants, where δ stands for the rate of capital depreciation and α_c and α_i are the parameters that measure the speed of quantity adjustment processes for the consumption-good and investment-good sectors, respectively.

The first two equations in (1.1) describe the quantity (or income) adjustment processes in the consumption-good and investment-good sectors, respectively.

The last two equations in (1.1) represent the capital formation processes for the consumption-good and investment-good sectors, respectively.

The classification of kinds of goods in to only two kinds does not explain well the interaction of different sectors in business cycles, on the other hand modeling the interaction of every possible sectors may leads to higher dimensional differential equations.

Therefore, in this thesis, our aim is to extending the work of Hiroki Murakami in (Murakami, 2018) to three-sector Keynesian model of business cycles . We further split the consumption good sector in to two: fast good and durable good. Thus we investigate the effect of the three sectors interactions on the business cycles.
In general, this thesis was intended in answering the following questions:

- What is the mathematical model which describes the interaction of three sectors Keynesian model of business cycle?
- What are the conditions for the existence of equilibria of the three- sector Keynesian model of business cycle ?
- What are the different stability conditions for the dynamic system?

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this thesis is to investigate a three-sector Keynesian model of business cycle.

1.3.2 Specific Objectives

The specific objectives of this thesis are to:

- develop a three-sector Keynesian model of business cycle .
- assess the existence of economic equilibria of the three-sector Keynesian model of business cycles.
- set stability conditions for the three- sector Keynesian model of business cycles.
- demonstrate the applications of the results using specific numerical examples.

1.4 Significance of the Study

The outcomes of this thesis have the following importance

- It will provide different stability conditions for a three- sector Keynesian model of business cycles .
- It will deliver the impacts of sectoral interactions.
- It will serve for other analysts on the investigation of a three- sector Keynesian model of business cycles .

Chapter 2

Literature Review

There are two main frameworks in modeling economic growth with capital accumulation in continuous time. These are Solow's one-sector growth model and Uzawa's two-sector growth model (Zhang, 2019). The Solow model is the starting point for almost all analyses of economic growth. As consumer behavior is not described by utility optimization in the Solow model, it does not have a rational mechanism to deal with issues related to optimal consumption over time. Ramsey's 1928 paper on optimal savings has influenced modeling of consumers' behavior since the mid 1960^s (Ramsey, 1928; Barro, Sala-i Martin, et al., 1995) . This approach assumes that utility is addable over time. It has become evident from extensive publications in the economic literature based on this approach over the last fifty years that even a simple model tends to lead to a complicated dynamic system. For instance, Barro and Sala-I-Martin in 1995 propose a one-sector growth model with endogenous time within the Ramsey framework. As demonstrated by Barro and Sala-I-Martin, the model even with simple utility and production becomes too complicated to get explicit conclusions.

Solow's one-sector growth model and Uzawa's two-sector growth model have played the role of the key models in the neoclassical growth theory (Uzawa, 1963; Solow, 1971; Burmeister, Dobell, et al., 1970). These two models and their various extensions and generalizations are fundamental for the development of new economic growth theories as well (Barro et al., 1995; Aghion et al., 1998). Since Uzawa proposed the model in (Uzawa, 1963) , many works have been published to extend and generalize the model in from the 1960^ss till today (Drandakis, 1963; Diamond, 1965; Von Weizsäcker, 1966; Corden, 1966; Stiglitz, 1967; Gram, 1976; Mino, 1996; Drugeon & Venditti, 2001) .

The Uzawa model extends the Solow model by a break down of the productive system into two sectors using capital and labor, one of which produces capital goods, the other consumption goods (Solow, 1971) .

The main goals of macroeconomic studies is to explain the mechanism of business cycles. Soon after the basis of macroeconomics was established by Keynes'General Theory, a lot of theories of business cycles were proposed from the late 1930^s to the 1950^s . For example, Kalecki (1935, 1937) and Kaldor (1940) put forward models of business cycles by synthesizing the Keynesian multiplier theory and the profit principle of investment, while Harrod (1936) and Samuelson (1939) initiated the so-called multiplier-accelerator model of business cycles by combining the multiplier theory and the acceleration principle of investment (Murakami, 2018; Murakami & Zimka, 2020) .

These classic models of business cycles can be characterized by the following Keynesian features:

i. quantity or income adjustment governed by the principle of effective demand prevails

ii. variations in investment are the main source of business cycles.

In their models, the mechanism of business cycles is explained as follows: investment is linked to aggregate income and capital stock and aggregate income and capital stock are varied through the multiplier process and capital formation induced by investment, respectively.

It is true that a lot of models of business cycles, including the aforementioned ones, can describe some aspects of actual business cycles, but a certain important viewpoint is missing in them: the role of sectoral interactions in business cycles. This aspect is lacking in one-sector or one-commodity models of business cycles, but it should not be ignored in discussing actual business cycles. It goes without saying that propagation's of shocks from one industry to another do enhance economic fluctuations in reality.

Hiroki Murakami in (Murakami, 2018) put forward a two-sector model of business cycles by dis-aggregating the economy into two sectors: the consumption good sector and the investment good sector. He examined the stability of equilibrium and the possibility of existence of a periodic orbit in two-sector model. As a result, he revealed that the counterpart of the Keynesian stability condition plays a key role in the stability of the two-sector model and that a periodic orbit may arise by way of a Hopf bifurcation if the stability condition is not satisfied . He also observed that the consumption good sector lags behind the investment good sector along the periodic orbit and that interactions between these two sectors do play a significant role in business cycles. Furthermore, he numerically investigated the characteristics of a periodic orbit generated by a Hopf bifurcation in the two-sector model. By performing numerical simulations, verified that a periodic orbit, representing persistent business cycles, is actually generated by a Hopf bifurcationin in the two-sector model.

Both consumption and investment are components of aggregate expenditure. As Keynes (1936) argued, however, they differ on the demand side in that the former responds passively to aggregate income (through the'consumption function') while the latter has a decisive role in determination of aggregate income (through the'multiplier process') and is the main cause of economic fluctuations. As Harrod (1939) noticed, they also differ on the supply side in that the former is not utilized as a factor of production while the latter is durable and functions as a factor of production (the duality of investment). These differences cannot properly be described in one-sector models, but they believe that their two-sector model may shed a new light on them (Murakami, 2018) .

Hiroki Murakami and Rudolf Zimka in (Murakami & Zimka, 2020) have examined mathematical details on the two-sector Keynesian model proposed by Murakami in (Murakami, 2018) . In their analysis they provided the criterion on the stability (or instability) of limit cycles which are generated by Hopf bifurcations. The performed numerical simulations show that they are consistent with the achieved theoretical results. Two-sector formalizations are more appropriate descriptions of macroeconomic systems than conventional and traditional one-sector ones, and they believe that their analysis contributes to deeper understanding of real economies.

Chapter 3

Preliminaries

3.1 System of Nonlinear Differential Equations

Definition 1. *A system of first-order equations in general may be written in the form*

$$\begin{cases} \dot{x_1} = f_1(t, x_1, x_2, ..., x_n), \\ \dot{x_2} = f_2(t, x_1, x_2, ..., x_n), \\ \qquad . \\ \qquad . \\ \qquad . \\ \dot{x_n} = f_n(t, x_1, x_2, ..., x_n), \end{cases} \tag{3.1}$$

where x_i are unknown functions and t is the independent variable. f_j are defined in some region $\mathbb{R}$ of $(n+1)$-dimensional space $(t, x_1, x_2, ..., x_n)$.

If we introduce the vector notation

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{bmatrix}, \mathbf{F} = \begin{bmatrix} f_1(t, x_1, x_2, ..., x_n) \\ f_2(t, x_1, x_2, ..., x_n) \\ . \\ . \\ . \\ f_n(t, x_1, x_2, ..., x_n), \end{bmatrix}$$

then the system (3.1) simply takes the form

$$\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}). \tag{3.2}$$

For further details on these, we kindly refer the reader to (Tyn, 1978).

By a solution to the differential equation, $\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x})$ we mean a vector-valued function $\mathbf{x}(t)$ that is defined and continuously differentiable on an interval $a < t < b$, and, moreover, satisfies the differential equation on its interval of definition. Each solution $\mathbf{x}(t)$ serves to parametrize a curve $C \in \mathbb{R}^n$, also known as a trajectory or orbit of the system (Perko, 2013).

Definition 2. *The function $\boldsymbol{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable at $\boldsymbol{x}_0 \in \mathbb{R}^n$ if there is a linear transformation $D\boldsymbol{F}(\boldsymbol{x}_0) \in L(\mathbb{R}^n)$ that satisfies*

$$\lim_{|h| \longrightarrow 0} \frac{|\boldsymbol{F}(\boldsymbol{x}_0 + h) - \boldsymbol{F}(\boldsymbol{x}_0) - D\boldsymbol{F}(\boldsymbol{x}_0)|}{|h|} = 0.$$

The linear transformation $D\boldsymbol{F}(\boldsymbol{x}_0)$ is called the derivative of $\boldsymbol{F}$ at $\boldsymbol{x}_0$.

Theorem 1. *If $\boldsymbol{F}:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable at $\boldsymbol{x}_0$, then the parial derivatives $\frac{\partial \boldsymbol{F}_i}{\partial x_j}$, $i,j = 1,2,...,n$, all exist at $\boldsymbol{x}_0$ and for all $\boldsymbol{x} \in \mathbb{R}^n$,*

$$D\boldsymbol{F}(\boldsymbol{x}_0)\boldsymbol{x} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{F}}{\partial x_j}\boldsymbol{x}_j.$$

Thus, if $\mathbf{F}$ is differentiable function, the derivative $D\mathbf{F}$ is given by the $n \times n$ Jacobian matrix

$$D\mathbf{F} = [\frac{\partial \mathbf{F}_i}{\partial x_j}]$$

3.2 Stability and Linearization

Definition 3. *(Tyn, 1978) A point C in $\mathbb{R}^n$ at which $\boldsymbol{F}(C) = 0$ is called an equilibrium point or critical point of the autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$.*

For more details regarding the following basic definitions and theorems, we refer the readers to (Zhang, 2005).

Consider a general autonomous system of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}). \tag{3.3}$$

Suppose $\mathbf{x}^*$ is an equilibrium point of (3.3). Introduce

$$\mathbf{X}(t) = \mathbf{x}(t) - \mathbf{x}^*$$

From the Taylor theorem for functions, we know

$$\mathbf{F}_j(\mathbf{x}) = \mathbf{F}_j(\mathbf{x}^*) + \sum_j \frac{\partial \mathbf{F}_i}{\partial x_j}(\mathbf{x}^*)\mathbf{X}_k + g_i(\mathbf{X}),$$

where $g_i(\Phi)$ are higher order terms and

$$\frac{g_i(\mathbf{X})}{\|\mathbf{X}\|} \longrightarrow 0 \;\; as \;\; \|\mathbf{X}\| \longrightarrow 0,$$

where

$$\|\mathbf{X}\| = \sqrt{\sum_i X_i^2}.$$

Using $\dot{\mathbf{x}} = \dot{\mathbf{X}}$ and $\mathbf{F}(\mathbf{x}^*) = 0$, equation (3.3) can be expressed in vector form as

$$\dot{\mathbf{X}} = A\mathbf{X} + g(\mathbf{X}),$$

where the matrix

$$A = (\frac{\partial \mathbf{F}_i}{\partial X_j})_{n\times n},$$

is the Jacobian matrix of $\mathbf{F}$ at $\mathbf{x}^*$. The linear system

$$\dot{\mathbf{X}} = A\mathbf{X},$$

is called the linearized system of equation (3.3).

Example 1. *Consider*

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2 + x_3 \\ x_1 - x_2 \\ 2x_2^2 + x_3 - 2, \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The Jacobian at equilibrium is

$$J = \begin{bmatrix} 2x_1 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 2x_2 & 0 \end{bmatrix}.$$

The system contains two equilibrium points $[-2 \ \ -2 \ \ -6]^T$ and $[1 \ 1 \ 0]^T$. Hence

$$J([-2 \ \ -2 \ \ -6]^T) = \begin{bmatrix} -4 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix}, \quad J([1 \ 1 \ \ -6]^T) = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Definition 4. *An equilibrium point $\boldsymbol{x}^*$ of the system* (3.3) *is called*

i. a sink if all of the eigenvalues of the Jacobian matrix A at $\boldsymbol{x}^*$ have negative real part;

ii. a source if all of the eigenvalues of A have positive real part

iii. a saddle if it is a hyperbolic equilibrium point and A has at least one eigenvalue with a positive real part and at least one with a negative part.

Theorem 2. *If at a zero of an autonomous differential equation the linearization is a sink or a source, then zero is itself a sink or a source. Furthermore all solutions sufficiently close to the zero tend to it exponentially fast as $t \longrightarrow \infty$ for a sink or as $t \longrightarrow -\infty$ for a source.*

The following example shows that it is possible for a zero to be a sink without the linearization being a sink.

Example 2. *Consider*

$$\begin{cases} \dot{x_1} = -x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x_2} = x_1 - x_2(x_1^2 + x_2^2). \end{cases}$$

The linearization at $(0, 0)$ is a center. Introduce

$$d(t) = (x_1^2 + x_2^2).$$

Then $\dot{d} = -2d^2$ for the original system. The general solution to this new system is $u = \frac{1}{(2t-C)}$. As d is the distance from the origin of a solution to the original equation, the distance goes to zero as $t \longrightarrow \infty$.

Theorem 3. *Let $\boldsymbol{F}$ be a C^1 function. If all the eigenvalues of the Jacobian matrix A have negative real parts, then the equilibrium point $\boldsymbol{x}^*$ of the differential equation*

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$$

is asymptotically stable.

Theorem 4. *Let $\boldsymbol{F}$ be a C^1 function. If at least one of the eigenvalues of the Jacobian matrix A has positive real parts, then the equilibrium point $\boldsymbol{x}^*$ of the differential equation $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ is unstable.*

3.3 Routh-Hurwitz Stability Criterion

Suppose we are given an n^{th} order homogeneous system of differential equations with constant coefficients:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$$

A nonlinear autonomous system can be reduced to the linear system by performing a linearization around an equilibrium point. Then without loss of generality, we may assume that the equilibrium point is at the origin. It is always possible to reach by choosing a suitable coordinate system.

Routh Hurwitz Stability Criterion is based on ordering the coefficients of the characteristic equation into an array, also known as Routh Array (Slemrod & Infante, 1972).
Consider the characteristic equation of a system :

$$a_n r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + ... + a_0 = 0.$$

Now, from the given equation, we will form Routh Array as shown below: the table has n + 1 rows and is in the following structure:

$$\begin{array}{|c|c|c|c|}\hline a_n & a_{n-2} & a_{n-4} & \cdots \\ \hline a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ \hline b_1 & b_2 & b_3 & \cdots \\ \hline c_1 & c_2 & c_3 & \cdots \\ \hline \vdots & \vdots & \vdots & \ddots \\ \hline \end{array},$$

where

$$b_i = \frac{a_{n-1} \times a_{n-2i} - a_n \times a_{n-(2i+1)}}{a_{n-1}}$$
$$c_i = \frac{b_1 \times a_{n-(2i+1)} - a_{n-1} \times b_{i+1}}{b_1}$$
$$i = 1, 2, 3, ...$$

When completed, the number of sign changes in the first column will be the number of non-negative roots, in this case the system is unstable. For stability, all the elements in the first column of the Routh array must be positive. However, in this criteria there are some special conditions in which some assumptions are needed to be made. These special cases are mentioned as follows:

Case I: When the first term in any row of the Routh array is zero while rest of the row has at least one non-zero term.

Because of this zero term, the terms in the next row become infinite and Routh's test breaks down. To overcome this difficulty, substitute a small positive number ε for zero and proceed to evaluate the rest of Routh Array. Then check the signs of the first column of the array by substituting $\varepsilon \to 0$.

Case II: When all the elements in any one row of the Routh Array are zero.

In this case the polynomial whose coefficients are the elements of the row just above the row of zeros in the Routh array is called an auxiliary polynomial. The next step is to differentiate auxiliary polynomial,after that the coefficients of the row containing zero now became the coefficient of the equation we get after differentiating the auxiliary polynomial.The process of Routh array is proceeded using these values.

Example 3. *The characteristic equation of a system is given below. Determine the stability of the system.*

$$4r^4 + 8r^3 + 2r^2 + 10r + 3 = 0$$

Solution: Applying Routh Hurwitz Criteria and forming Routh array. We get,

a_4	a_2	a_0
a_3	a_1	0
b_1	b_2	0
c_1	*0*	*0*
d_1	0	*0*

,

where

$$\begin{aligned}
a_4 &= 4,\ a_3 = 8,\ a_2 = 2,\ a_1 = 10,\ a_0 = 3,\\
b_1 &= \frac{a_3a_2 - a_4a_1}{a_3} = \frac{(8)(2)-(4)(10)}{8} = -3,\\
b_2 &= \frac{a_3a_0 - a_4 \times 0}{a_3} = \frac{a_3a_0}{a_3} = a_0 = 3,\\
c_1 &= \frac{b_1a_1 - a_3b_2}{b_1} = \frac{(-3)(10)-(8)(3)}{-3} = 18,\\
d_1 &= \frac{c_1b_2 - b_1 \times 0}{c_1} = \frac{c_1b_2}{c_1} = b_2 = 3,
\end{aligned}$$

The resulting Routh array becomes

4	2	a_0
8	10	0
-3	3	0
18	*0*	*0*
3	0	*0*

,

In the first column of the Routh Array formed above, there is one negative element. Also, there are two sign changes in the first column. First is from 8 to -3 and second is from -3 to 18. Hence the system in is unstable.

Example 4. *The characteristic equation of a system is given below. Find the range of values of k for which the system would be stable.*

$$r^3 + 3r^2 + 7r + k = 0$$

Solution: Applying Routh Hurwitz Criteria and forming Routh array. We get,

a_3	a_1
a_2	a_0
b_1	*0*
c_1	*0*

,

where

$$a_3 = 1,\ a_2 = 3,\ a_1 = 7,\ a_0 = k,$$
$$b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2} = \frac{(3)(7) - (1)(k)}{3} = \frac{21 - k}{3},$$
$$c_1 = \frac{b_1 a_0 - a_2 \times 0}{b_1} = a_0 = k$$

The resulting Routh array becomes

1	7
3	k
$\frac{21-k}{3}$	*0*
k	*0*

,

The system is stable when all elements in the first column are positive. i.e.,

$$21 - k > 0 \text{ and } k > 0$$
$$k < 21 \text{ and } 0 < k$$

Therefore, the system is stable for $0 < k < 21$.

Chapter 4

Results and Discussions

4.1 Model Formulation

In this section, we extend the two sector Keynesian model of business cycle presented in [24, 40] to three sectors. We considered three kinds of goods that differ in property namely, durable goods, fast goods and investment goods. The formulation of the model is based on the following main assumptions:

1. Based on the Keynesian theory, we assume that aggregate consumption, denoted by C, is a function of all sectors' income.

$$C = C(y_d, y_f, y_i), \tag{4.1}$$

   where C is twice continously differentiable and all partial derivatives $\frac{\partial C}{\partial y_d}$, $\frac{\partial C}{\partial y_f}$ and $\frac{\partial C}{\partial y_i}$ lies between 0 and 1.

   In the expression (4.1), y_d stands for the income or output of durable good, y_f stand for output of the fast good and y_i stands for the income of the investment good.

2. The output (supply) of the fast good y_f is assumed to be varied in response to the existing excess demand or supply. Specifically, it is given by

$$\dot{y}_f = \alpha_f [C(y_d, y_f, y_i) - y_f] \tag{4.2}$$

   where α_f is a positive parameter which stands for the speed of adjustment. Equation (4.2) expresses the Keynesian quantity adjustment process in the fast good sector.

3. Next, we take a look at the demand side of the investment good, which is assumed to be dependent on the out put of the sector and on the existing stock of capital of the sector. In particular, we assumed that the gross investment functions of the durable good sector I_d, the fast good sector I_f and of the investment good sector I_i are designated as:

$$I_d = I_d(y_d, k_d), I_f = I_f(y_f, k_f), I_i = I_i(y_i, k_i) \tag{4.3}$$

   where k_d, k_f, k_i denote the existing stock of the investment function of durable good sector, fast good sector and investment good sector, respectively.

4. Similarly the output (supply) of the investment good y_i and y_d the durable good are assumed to satisfy the following equation

$$\dot{y}_d = \alpha_d[I_d(y_d, k_d) + I_f(y_f, k_f) - y_d]$$
$$\dot{y}_i = \alpha_i[I_f(y_f, k_f) + I_i(y_i, k_i) - y_i]$$

where α_d and α_i are positive parameters which represent the speed of adjustment. The above equations are, the Keynesian income adjustment process in the durable good sector and investment good sector, respectively. These equations mean that a change in the output (or of income) of each good is proportional to the existing excess demand or supply and imply that the level of output is adjusted to meet the demand for each good.

5. Finally, the demand for the investment good of each sector I_d, I_f, I_i are realized as the growth increment of the stock of the investment good in the sector k_d, k_f, k_i respectively in the following way:

$$\dot{k}_d = I_d(y_d,\ \ k_d) - \delta k_d,\ \ \dot{k}_f = I_f(y_f, k_f) - \delta k_f,\ \ \dot{k}_i = I_i(y_i, k_i) - \delta k_i,$$

where δ is a positive constant which stands for the rate of capital depreciation. These three equations represent the capital formation processes for the durable good, fast good and investment good sectors, respectively.

Thus, the three sector Keynesian model of business cycle is described by the following system of differential equations.

$$\begin{cases} \dot{y}_f = \alpha_f[C(y_d, y_f, y_i) - y_f]\ , \\ \dot{y}_d = \alpha_d[I_d(y_d, k_d) + I_f(y_f, k_f) - y_d], \\ \dot{y}_i = \alpha_i[I_f(y_f, k_f) + I_i(y_i, k_i) - y_i]\ , \\ \dot{k}_f = I_f(y_f, k_f) - \delta k_f\ , \\ \dot{k}_d = I_d(y_d, k_d) - \delta k_d\ , \\ \dot{k}_i = I_i(y_i, k_i) - \delta k_i\ , \end{cases} \tag{4.4}$$

In this model, note that the rate of depreciation is considered the same for the three sectors, but this assumption can be relaxed without so much difficulty.
For further analysis and numerical computations we need to specify the functions C, I_d, I_f and I_i.

The consumption function C

Keynes was interested in the level of total spending in general, he was particularly concerned about consumption, which was a major concern because it is by far the largest slice of the total spending pie. He made three basic points about consumption (Arnold, 2015):

1. Consumption depends on disposable income (income minus taxes).

2. Consumption and disposable income move in the same direction.

3. When disposable income changes, consumption changes by less.

The above three points gives us a specific statement about the relationship between consumption and disposable income: the consumption function, which can be written as:

$$C(y_d, y_f, y_i) = c_0 + c(y_d + y_f + y_i),$$

where,

- $y_d + y_f + y_i$ is disposable income,
- c is marginal propensity to consume, and
- c_0 is autonomous consumption, which changes not as disposable income changes, but rather because of other factors.

And also we have,

$$c = \frac{\Delta C}{\Delta(y_d + y_f + y_i)}$$

Based on Keyne's points 2 and 3 above, c is always a positive number between 0 and 1.

The Investment functions I_d, I_f and I_i

Assume that at each period, each firm puts its rate of (gross) capital formation to one of the two alternative levels: higher and lower ones, say, $\delta + \mu_j$ and $\delta - \lambda_j \geq 0$ with $0 < \lambda_j \leq \delta$ (The restriction of $\lambda_j \leq \delta$ is, of course, imposed for the rate of gross capital formation to be nonnegative for each sector). And $\mu_j > 0$ (It can differ from sector to sector). Since we can safely suppose that the share of firms choosing the higher level of investment (among each sector), say p_j , increases as the output- capital ratio, which is proportionate to the rate of profit under the assumption of constant capital share, rises, the ratio p_j can be related to the output-capital ratio $\frac{y_j}{k_j}$ in the following way:

$$\ln\left(\frac{p_j}{1 - p_j}\right) = \beta_j \frac{y_j}{k_j} - \beta_{0j}.$$

Moreover,

$$\frac{I_j}{k_j} = p_j(\delta + \mu_j) + (1 - p_j)(\delta - \lambda_j)$$

By substituting the first relationships into the second, we obtain the following logistic investment function.

$$I_j(y_j, k_j) = \left[\delta + \frac{\mu_j e^{(\beta_j(\frac{y_j}{k_j}) - \beta_{0j})} - \lambda_j}{1 + e^{(\beta_j(\frac{y_j}{k_j}) - \beta_{0j})}}\right] k_j$$

where $\beta_j > 0$ and $\beta_{0j} > 0$ are measures the sensitivity of capital formation to changes in the output-capital ratio, $\lambda_j > 0$ and $\mu_j > 0$ are constants that determine the minimum and maximum rates of gross capital formation, respectively for $j = d, f, i$ (Murakami & Zimka, 2020). Thus, keeping all the above descriptions we get the following functions.

$$C(y_d, y_f, y_i) = c_0 + c(y_d + y_f + y_i), \tag{4.5}$$

$$I_f(y_f, k_f) = \left[\delta + \frac{\mu_f e^{(\beta_f(\frac{y_f}{k_f}) - \beta_{0f})} - \lambda_f}{1 + e^{(\beta_f(\frac{y_f}{k_f}) - \beta_{0f})}}\right] k_f \tag{4.6}$$

$$I_d(y_d, k_d) = \left[\delta + \frac{\mu_d e^{(\beta_d(\frac{y_d}{k_d}) - \beta_{0d})} - \lambda_d}{1 + e^{(\beta_d(\frac{y_d}{k_d}) - \beta_{0d})}}\right] k_d \tag{4.7}$$

$$I_i(y_i, k_i) = \left[\delta + \frac{\mu_i e^{(\beta_i(\frac{y_i}{k_i}) - \beta_{0i})} - \lambda_i}{1 + e^{(\beta_i(\frac{y_i}{k_i}) - \beta_{0i})}}\right] k_i \tag{4.8}$$

4.2 Analysis

4.2.1 The Equilibrium Point

An equilibrium point of the system, (4.4) is defined as a point at which

$$\dot{y}_f = \dot{y}_d = \dot{y}_i = \dot{k}_d = \dot{k}_f = \dot{k}_i = 0.$$

That means, it is the solution of the following simultaneous equation .

$$\begin{aligned}
&\alpha_f[c_0 - (1-c)y_f + c(y_d + y_i)] = 0,\\
&\alpha_d\left\{\left[\delta + \frac{\mu_f e^{(\beta_d(\frac{y_d}{k_d})-\beta_{0d})} - \lambda_d}{1 + e^{(\beta_d(\frac{y_d}{k_d})-\beta_{0d})}}\right]k_d + \left[\delta + \frac{\mu_f e^{(\beta_f(\frac{y_f}{k_f})-\beta_{0f})} - \lambda_f}{1 + e^{(\beta_f(\frac{y_f}{k_f})-\beta_{0f})}}\right]k_f - y_d\right\} = 0,\\
&\alpha_i\left\{\left[\delta + \frac{\mu_f e^{(\beta_f(\frac{y_f}{k_f})-\beta_{0f})} - \lambda_f}{1 + e^{(\beta_f(\frac{y_f}{k_f})-\beta_{0f})}}\right]k_f + \left[\delta + \frac{\mu_i e^{(\beta_i(\frac{y_i}{k_i})-\beta_{0i})} - \lambda_i}{1 + e^{(\beta_i(\frac{y_i}{k_i})-\beta_{0i})}}\right]k_i - y_i\right\} = 0,\\
&\frac{\mu_f e^{(\beta_f(\frac{y_f}{k_f})-\beta_{0f})} - \lambda_f}{1 + e^{(\beta_f(\frac{y_f}{k_f})-\beta_{0f})}}k_f = 0,\\
&\frac{\mu_d e^{(\beta_d(\frac{y_d}{k_d})-\beta_{0d})} - \lambda_d}{1 + e^{(\beta_d(\frac{y_d}{k_d})-\beta_{0d})}}k_d = 0,\\
&\frac{\mu_i e^{(\beta_i(\frac{y_i}{k_i})-\beta_{0i})} - \lambda_i}{1 + e^{(\beta_i(\frac{y_i}{k_i})-\beta_{0i})}}k_i = 0.
\end{aligned}$$

Since the case of $k_d = 0$, $k_f = 0$ or $k_i = 0$ has no impact from economic point of view, we give emphasis to the cases when $k_d \neq 0$, $k_f \neq 0$ and $k_i \neq 0$. Denote an equilibrium point for the given system by $(y_f^*,\ y_d^*,\ y_i^*,\ k_f^*,\ k_d^*,\ k_i^*)$. Then, the equilibrium point of the system is given by in the following form.

$$\begin{aligned}
(y_f^*,\ y_d^*,\ y_i^*,\ k_f^*,\ k_d^*,\ k_i^*) = (&\frac{c_0}{1-c} + \frac{c}{1-c}(y_d^* + y_i^*),\ \frac{\delta\sigma_f}{(1-c)(1-\delta\sigma_d) - c\delta\sigma_f}c_0,\\
&\frac{\delta\sigma_f}{(1-c)(1-\delta\sigma_i) - c\delta\sigma_f}c_0,\ \sigma_f y_f^*,\ \sigma_d y_d^*,\ \sigma_i y_i^*)
\end{aligned}$$

where,

$$\begin{aligned}
\sigma_f &= \frac{\beta_f}{\beta_{0f} + \ln\lambda_f - \ln\mu_f} > 0,\\
\sigma_d &= \frac{\beta_d}{\beta_{0d} + \ln\lambda_d - \ln\mu_d} > 0,\\
\sigma_i &= \frac{\beta_i}{\beta_{0i} + \ln\lambda_i - \ln\mu_i} > 0,\\
&(1-c)(1-\delta\sigma_d) - c\delta\sigma_f > 0,\ \ and\\
&(1-c)(1-\delta\sigma_i) - c\delta\sigma_f > 0.
\end{aligned}$$

We assumed all the above are positive in order to assure that the unique equilibrium $(y_d^*,\ y_f^*,\ y_i^*,\ k_d^*,\ k_f^*,\ k_i^*)$ lies on the economically meaningful domain $\mathbb{R}_+^6$. These conditions to be satisfied, β_{0d}, β_{0f} and

β_{0i} has to be sufficiently large and δ has to be sufficiently small.

For further analysis, we transform our system by introducing new variables as follows.

$$\tilde{y}_f = y_f - y_f^*, \quad \tilde{y}_d = y_d - y_d^*, \quad \tilde{y}_i = y_i - y_i^*$$
$$\tilde{k}_f = k_f - k_f^*, \quad \tilde{k}_d = k_d - k_d^*, \quad \tilde{k}_i = k_i - k_i^*$$

The following system of equations is obtained after plugging the above new variables in (4.4),

$$\dot{\tilde{y}}_f = \alpha_f[c_0 - (1-c)\tilde{y}_f + c(\tilde{y}_d + \tilde{y}_i)] \tag{4.9}$$

$$\dot{\tilde{y}}_d = \alpha_d \left\{ \left[\delta + \frac{\mu_d e^{(\beta_d(\frac{y_d^*+\tilde{y}_d}{k_d^*+\tilde{k}_d}) - \beta_{0d})}}{1 + e^{(\beta_d(\frac{y_d^*+\tilde{y}_d}{k_d^*+\tilde{k}_d}) - \beta_{0d})}} - \lambda_d \right] (k_d^* + \tilde{k}_d) + \left[\delta + \frac{\mu_f e^{(\beta_f(\frac{y_f^*+\tilde{y}_f}{k_f^*+\tilde{k}_f}) - \beta_{0f})}}{1 + e^{(\beta_f(\frac{y_f^*+\tilde{y}_f}{k_f^*+\tilde{k}_f}) - \beta_{0f})}} - \lambda_f \right] (k_f^* + \tilde{k}_f) - (y_d^* + \tilde{y}_d) \right\} \tag{4.10}$$

$$\dot{\tilde{y}}_i = \alpha_i \left\{ \left[\delta + \frac{\mu_f e^{(\beta_f(\frac{y_f^*+\tilde{y}_f}{k_f^*+\tilde{k}_f}) - \beta_{0f})}}{1 + e^{(\beta_f(\frac{y_f^*+\tilde{y}_f}{k_f^*+\tilde{k}_f}) - \beta_{0f})}} - \lambda_f \right] (k_f^* + \tilde{k}_f) + \left[\delta + \frac{\mu_i e^{(\beta_i(\frac{y_i^*+\tilde{y}_i}{k_i^*+\tilde{k}_i}) - \beta_{0i})}}{1 + e^{(\beta_i(\frac{y_i^*+\tilde{y}_i}{k_i^*+\tilde{k}_i}) - \beta_{0i})}} - \lambda_i \right] (k_i^* + \tilde{k}_i) - (y_i^* + \tilde{y}_i) \right\} \tag{4.11}$$

$$\dot{\tilde{k}}_f = \left[\frac{\mu_f e^{(\beta_f(\frac{y_f^*+\tilde{y}_f}{k_f^*+\tilde{k}_f}) - \beta_{0f})}}{1 + e^{(\beta_f(\frac{y_f^*+\tilde{y}_f}{k_f^*+\tilde{k}_f}) - \beta_{0f})}} - \lambda_f \right] (k_f^* + \tilde{k}_f) \tag{4.12}$$

$$\dot{\tilde{k}}_d = \left[\frac{\mu_d e^{(\beta_d(\frac{y_d^*+\tilde{y}_d}{k_d^*+\tilde{k}_d}) - \beta_{0d})}}{1 + e^{(\beta_d(\frac{y_d^*+\tilde{y}_d}{k_d^*+\tilde{k}_d}) - \beta_{0d})}} - \lambda_d \right] (k_d^* + \tilde{k}_d) \tag{4.13}$$

$$\dot{\tilde{k}}_i = \left[\frac{\mu_i e^{(\beta_i(\frac{y_i^*+\tilde{y}_i}{k_i^*+\tilde{k}_i}) - \beta_{0i})}}{1 + e^{(\beta_c(\frac{y_i^*+\tilde{y}_i}{k_i^*+\tilde{k}_i}) - \beta_{0c})}} - \lambda_i \right] (k_i^* + \tilde{k}_i) \tag{4.14}$$

The newly obtained system has a unique equilibrium point , $(0, 0, 0, 0, 0, 0)$.

4.2.2 Stability and Instability Analysis

Before we are going to see stability and instability conditions, we have to linearize the system at the unique equilibrium point. Computation of the Jacobian matrix at this unique equilibrium point gives :

$$J(0,0,0,0,0,0) = \begin{bmatrix} \alpha_f c & -\alpha_f(1-c) & \alpha_f c & 0 & 0 & 0 \\ \alpha_d A_d & \alpha_d A_f & 0 & \alpha_d(\delta - B_d) & \alpha_d(\delta - B_f) & 0 \\ \alpha_i A_d & 0 & -\alpha_i(1 - A_i) & \alpha_i(\delta - B_d) & 0 & \alpha_i(\delta - B_i) \\ 0 & A_f & 0 & 0 & -B_f & 0 \\ A_d & 0 & 0 & -B_d & 0 & 0 \\ 0 & 0 & A_i & 0 & 0 & -B_i \end{bmatrix},$$

where,

$$\begin{aligned}
A_f &= \frac{\beta_f \lambda_f \mu_f}{\lambda_f + \mu_f} > 0 \\
A_d &= \frac{\beta_d \lambda_d \mu_d}{\lambda_d + \mu_d} > 0 \\
A_i &= \frac{\beta_i \lambda_i \mu_i}{\lambda_i + \mu_i} > 0 \\
B_f &= \frac{(\beta_{0f} + ln\lambda_f - ln\mu_f)\lambda_f \mu_f}{\lambda_f + \mu_f} > 0 \\
B_d &= \frac{(\beta_{0d} + ln\lambda_d - ln\mu_d)\lambda_d \mu_d}{\lambda_d + \mu_d} > 0 \\
B_i &= \frac{(\beta_{0i} + ln\lambda_i - ln\mu_i)\lambda_i \mu_i}{\lambda_i + \mu_i} > 0
\end{aligned}$$

The characteristic equation associated with this Jacobian matrix is given by

$$a_6 r^6 + a_5 r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0, \tag{4.15}$$

To find these coefficients let us use the following notations for the entries of the Jacobian matrix

$$\begin{aligned}
&c_1 = -\alpha_d(1-c), \qquad c_2 = \alpha_d c, \\
&c_3 = \alpha_f A_d, \qquad c_4 = \alpha_f A_f, \\
&c_5 = \alpha_f(\delta - B_d), \qquad c_6 = \alpha_f(\delta - B_f), \\
&c_7 = \alpha_i A_d, \qquad c_8 = -\alpha_i(1 - A_i), \\
&c_9 = \alpha_i(\delta - B_d), \qquad c_{10} = \alpha_i(\delta - B_i).
\end{aligned}$$

After using MATLAB R2021a with an appropriate code, we get the coefficients as follows

$$\begin{aligned}
a_6 =& 1 \\
a_5 =& B_i - c1 - c4 - c8 \\
a_4 =& c_1c_4 - A_ic_{10} - B_ic_1 - B_ic_4 - B_ic_8 - A_fc_5 - c_2c_3 - c_1c_7 + c_1c_8 + c_4c_8 - B_dB_f \\
a_3 =& A_fB_dc_6 - B_dB_fB_i - A_fB_ic_5 + B_dB_fc_1 + B_dB_fc_4 + B_dB_fc_8 - A_dc_2c_6 + A_fc_1c_5 + A_fc_5c_8 \\
&+ A_ic_1c_{10} + A_ic_4c_{10} + B_ic_1c_4 - B_ic_2c_3 - B_ic_1c_7 + B_ic_1c_8 + B_ic_4c_8 + c_1c_4c_7 - c_1c_4c_8 + c_2c_3c_8 \\
a_2 =& A_fB_dB_ic_6 + A_iB_dB_fc_{10} + B_dB_fB_ic_1 + B_dB_fB_ic_4 + B_dB_fB_ic_8 + A_fA_ic_5c_{10} + A_dB_fc_2c_5 \\
&- A_fB_dc_1c_6 + A_dB_fc_1c_9 - A_fB_dc_6c_8 - A_dB_ic_2c_6 + A_fB_ic_1c_5 + A_fB_ic_5c_8 - B_dB_fc_1c_4 \\
&+ B_dB_fc_2c_3 + B_dB_fc_1c_7 - B_dB_fc_1c_8 - B_dB_fc_4c_8 + A_dc_2c_6c_8 - A_fc_1c_3c_9 + A_fc_1c_5c_7 \\
&- A_fc_1c_5c_8 - A_ic_1c_4c_{10} + A_ic_2c_3c_{10} + B_ic_1c_4c_7 - B_ic_1c_4c_8 + B_ic_2c_3c_8 \\
a_1 =& A_dA_ic_2c_6c_{10} - A_dA_fc_1c_6c_9 - A_fA_ic_1c_5c_{10} - A_dB_fc_1c_4c_9 - A_fB_dc_1c_6c_7 - A_dB_fc_2c_5c_8 + A_fB_dc_1c_6c_8 \\
&+ A_dB_ic_2c_6c_8 - A_fB_ic_1c_3c_9 + A_fB_ic_1c_5c_7 - A_fB_ic_1c_5c_8 - B_dB_fc_1c_4c_7 + B_dB_fc_1c_4c_8 - B_dB_fc_2c_3c_8 \\
&- A_fA_iB_dc_6c_{10} + A_dB_fB_ic_2c_5 - A_fB_dB_ic_1c_6 + A_dB_fB_ic_1c_9 - A_iB_dB_fc_1c_{10} - A_fB_dB_ic_6c_8 \\
&- A_iB_dB_fc_4c_{10} - B_dB_fB_ic_1c_4 + B_dB_fB_ic_2c_3 + B_dB_fB_ic_1c_7 - B_dB_fB_ic_1c_8 - B_dB_fB_ic_4c_8 \\
a_0 =& A_fA_iB_dc_1c_6c_{10} - A_dA_iB_fc_2c_5c_{10} - A_dA_fB_ic_1c_6c_9 - A_dB_fB_ic_1c_4c_9 - A_fB_dB_ic_1c_6c_7 \\
&- A_dB_fB_ic_2c_5c_8 + A_fB_dB_ic_1c_6c_8 + A_iB_dB_fc_1c_4c_{10} - A_iB_dB_fc_2c_3c_{10} - B_dB_fB_ic_1c_4c_7 \\
&+ B_dB_fB_ic_1c_4c_8 - B_dB_fB_ic_2c_3c_8
\end{aligned}$$

By Routh tabular method, we have,

a_6	a_4	a_2	a_0
a_5	a_3	a_1	0
b_1	b_2	b_3	0
c_1	c_2	c_3	0
d_1	d_2	0	0
e_1	0	0	0
f_1	0	0	0

,

where

$$
\begin{aligned}
b_1 &= \frac{a_5a_4 - a_6a_3}{a_5}, \quad b_2 = \frac{a_5a_2 - a_6a_1}{a_5}, \quad b_3 = \frac{a_5a_0 - a_6 \times 0}{a_5} = a_0 \\
c_1 &= \frac{b_1a_3 - b_2a_5}{b_1}, \quad c_2 = \frac{b_1a_1 - b_3a_5}{b_1}, \quad c_3 = \frac{b_1 \times 0 - a_5 \times 0}{b_1} = 0 \\
d_1 &= \frac{c_1b_2 - b_1c_2}{c_1}, \quad d_2 = \frac{c_1b_3 - 0 \times b_1}{c_1} = b_3 \\
e_1 &= \frac{d_1c_2 - d_2c_1}{d_1}, \quad f_1 = \frac{e_1d_2 - d_1 \times 0}{e_1} = d_2
\end{aligned}
$$

As an example let us take the following specified parameters from (Murakami & Zimka, 2020)

$$
\begin{aligned}
&c = 0.53, \ \ c_0 = 1, \ \ , \lambda_d = \lambda_f = \lambda_i = 0.09, \ \ \mu_d = \mu_f = \mu_i = 0.21, \ \ \beta_d = \beta_f = \beta_i = 9.4 \\
&\beta_{0d} = \beta_{0f} = \beta_{0i} = 5.8, \ \ \delta = 0.09, \ \ \alpha_d = \alpha_f = \alpha_i = 2.55336
\end{aligned}
$$

After evaluating the above values and substituting in their respective place on the table, we get

1	−1.5696	−2.1324	5544.8
1.0413	−6.5034	2508.2	0
4.6761	−2410.9	5544.8	0
530.3539	1273.5	0	0
−2422.1	5544.8	0	0
2487.6	0	0	0
5544.8	0	0	0

,

As we can see it from the first column of the above Routh array, there are two sign changes, one is from 530.3539 to −2422.1 and the other is from −2422.1 to 2487.6. Hence, the system, (4.9)-(4.14) is unstable for the specified parameters. Moreover, since sign changes twice in the first column, so the characteristic polynomial has two roots with positive real parts.

In general, the system (4.9)-(4.14) is stable whenever all the first column entries of the Routh array positive. i.e.,

$$a_6, \ a_5, \ b_1, \ c_1, \ d_1, \ e_1, \ f_1 \ > 0$$

4.2.3 Numerical Simulations

To assess the validity of our analysis, let us perform numerical simulations. We took the following specified parameters from (Murakami & Zimka, 2020)

$$c = 0.53,\ \ c_0 = 1,\ \ ,\lambda_d = \lambda_f = \lambda_i = 0.09,\ \ \mu_d = \mu_f = \mu_i = 0.21,\ \ \beta_d = \beta_f = \beta_i = 9.4$$
$$\beta_{0d} = \beta_{0f} = \beta_{0i} = 5.8,\ \ \delta = 0.09,\ \ \alpha_d = \alpha_f = \alpha_i = 2.55336$$

After plugging the above specific values in places of each parameters in our system, (4.4) we have

$$\dot{y}_f = 1.3532808[y_i + y_d] - 1.2000792y_f + 2.55336$$
$$\dot{y}_d = \left[0.2298024 + \frac{0.5362056e^{9.4\frac{y_d}{k_d}-5.8} - 0.2298024}{1 + e^{9.4\frac{y_d}{k_d}-5.8}}\right]k_d + \left[0.2298024 + \frac{0.5362056e^{9.4\frac{y_f}{k_f}-5.8} - 0.2298024}{1 + e^{9.4\frac{y_f}{k_f}-5.8}}\right]$$
$$\times k_f - 2.55336y_d$$
$$\dot{y}_i = \left[0.2298024 + \frac{0.5362056e^{9.4\frac{y_f}{k_f}-5.8} - 0.2298024}{1 + e^{9.4\frac{y_f}{k_f}-5.8}}\right]k_f + \left[0.2298024 + \frac{0.5362056e^{9.4\frac{y_i}{k_i}-5.8} - 0.2298024}{1 + e^{9.4\frac{y_i}{k_i}-5.8}}\right]$$
$$\times k_i - 2.55336y_i$$
$$\dot{k}_f = \frac{0.5362056e^{9.4\frac{y_f}{k_f}-5.8} - 0.2298024}{1 + e^{9.4\frac{y_f}{k_f}-5.8}}k_f$$
$$\dot{k}_d = \frac{0.5362056e^{9.4\frac{y_d}{k_d}-5.8} - 0.2298024}{1 + e^{9.4\frac{y_d}{k_d}-5.8}}k_d$$
$$\dot{k}_i = \frac{0.5362056e^{9.4\frac{y_i}{k_i}-5.8} - 0.2298024}{1 + e^{9.4\frac{y_i}{k_i}-5.8}}k_i$$

Making all the equations in the above system equal to 0, we get the following equilibrium point

$$(y_f^*,\ y_d^*,\ y_i^*,\ k_f^*,\ k_d^*,\ k_i^*) = (3.9740,\ 0.8187,\ 0.8187,\ 7.5425,\ 1.5538,\ 1.5538).$$

We set the initial condition for our simulation as follows

$$\begin{aligned}(y_f^*(0),\ y_d^*(0),\ y_i^*(0),\ k_f^*(0),\ k_d^*(0),\ k_i^*(0)) &= (y_f^*,\ 1.3y_d^*,\ 1.3y_i^*,\ k_f^*,\ k_d^*,\ k_i^*)\\ &= (3.9740, 1.06431, 1.06431, 7.5425, 1.5538, 1.5538)\end{aligned}$$

The following figure described the solution path of system $(y_f(t), y_d(t), y_i(t), k_f(t), k_d(t), k_i(t))$. One can see from this simulation that the solution paths are periodic, describing the fluctuations of durable, fast and investment good sectors.

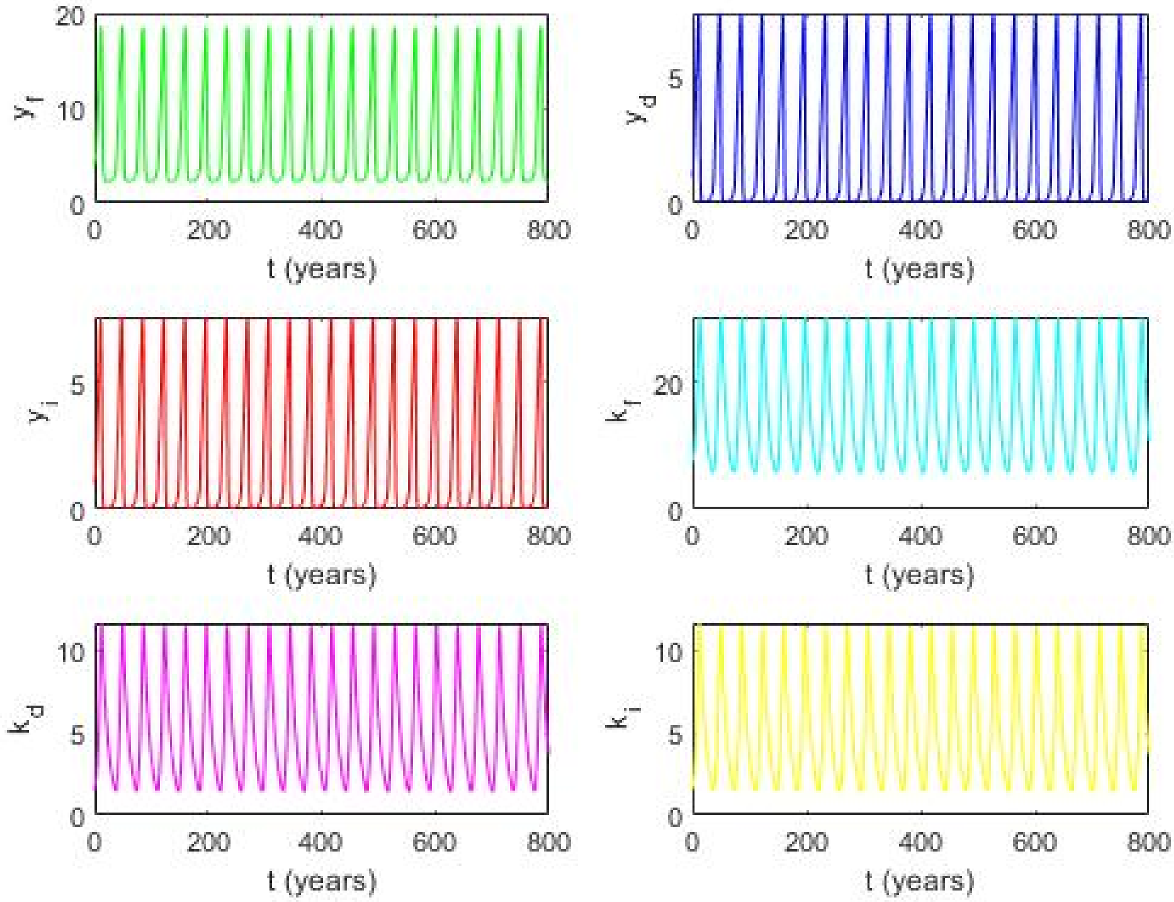

Figure 4.1: The Graphs of y_f, y_d, y_i, k_f, k_d and k_i .

Chapter 5

Conclusions and Recommendations

In this thesis, we have described mathematical details on the three sector Keynesian model of business cycle, which is a modification of the Hiroki's two-sector Keynesian model of business cycles. We considered three kinds of goods that differ in property namely, durable goods, fast goods and investment goods. We formalized the consumption and investment functions from Keynesian perspectives and then set up a dynamical system composed of differential equations. Moreover, we have investigated the periodic nature of the solution paths and the performed numerical simulations appear that they are reliable with the accomplished hypothetical comes about.

To conclude this thesis, we would like to suggest for any researcher's who are interested in this area to use this thesis as a reference and make further investigations on multi-sectoral Keynesian model of business cycles by giving attentions to the effect of changes in parameter values and investigate the existence and stability of a limit cycle. Moreover, it is possible to assess sectoral interactions in view of Ethiopian economy.

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Appendix

Listing 5.1: MATLAB R2021a Code for the coefficients of the characteristic equation (4.15)

```
syms c1 c2 c3 c4 c5 c6 c7 c8 c9 c10 Ad Bd Bf Bi Af Ai
A=[c1 c2 c1 0 0 0; c3 c4 0 c5 c6 0; c7 0 c8 c9 0 c10; 0 Af 0 0 -Bf 0; Ad 0 0 -Bd 0 0; ...
    0 0 Ai 0 0 -Bi ]
charpoly(A)
```

Listing 5.2: MATLAB R2021a Code for the Equilibrium Point Based on the Specified Parameters

```
syms ad af c0 c x1 x2 x3 x4 x5 x6 k1  ai D Md Mf Mi Bd Bf Bi B0d B0f B0i Ld Lf Li x1* ...
    x2* x3* x4* x5* x6* x11 x21 x31 x41 x51 x61
k1=[af*(c0-(1-c)*x1+c*(x2+x3));...
    (D + ((Md*exp((Bd*x1/x4)-B0d)-Ld)/(1+exp((Bd*x1/x4)-B0d))))*ad*x4+...
    (D + ((Mf*exp((Bf*x2/x5)-B0f)-Lf)/(1+exp((Bf*x2/x5)-B0f))))*ad*x5-ad*x2;...
    (D + ((Mf*exp((Bf*x2/x5)-B0f)-Lf)/(1+exp((Bf*x2/x5)-B0f))))*ai*x5+...
    (D + ((Mi*exp((Bi*x3/x6)-B0i)-Li)/(1+exp((Bi*x3/x6)-B0i))))*ai*x6-ai*x3;...
    (((Md*exp((Bd*x1/x4)-B0d))-Ld)/(1+exp((Bd*x1/x4)-B0d)))*x4;...
    (((Mf*exp((Bf*x2/x5)-B0f))-Lf)/(1+exp((Bf*x2/x5)-B0f)))*x5; ...
    (((Mi*exp((Bi*x3/x6)-B0i))-Li)/(1+exp((Bi*x3/x6)-B0i)))*x6];
subs(k1,{c,c0,Ld,Lf,Li,Md,Mf,Mi,Bd,Bf,Bi,B0d,B0f,B0i,D,ad,af,ai},...
    {0.53,1,0.09,0.09,0.09,0.21,0.21,0.21,9.4,9.4,9.4,5.8,5.8,5.8,0.09,...
    2.55336,2.55336,2.55336})
f=@(x) [];
x=fsolve(f,[5 5 5 10 10 10])
```

Listing 5.3: MATLAB 2021a Code for Figure 4.1

```
global ad af c0 c ai D Md Mf Mi Bd Bf Bi B0d B0f B0i Ld Lf Li
 ad=2.55336;af=2.55336;ai=2.55336;c0=1;c=0.53;Ld=0.09;Lf=0.09;...
 Li=0.09;Md=0.21;Mf=0.21;Mi=0.21;Bd=9.4;Bf=9.4;Bi=9.4;B0d=5.8;...
 B0f=5.8;B0i=5.8;D=0.09;
 y0=[3.9740,1.06431,1.06431,7.5425,1.5538,1.5538];
 ts=[0,1000];
 [t,y]=ode45('tihtinaatle',ts,y0);
subplot(3,2,1), plot(t,y(:,1),'g','Linewidth' ,0.1)
%xlim([9900,10000]);
 xlabel('t (years)')
 ylabel('y_f')
 subplot(3,2,2), plot(t,y(:,2),'b','Linewidth' ,0.1)
 xlabel('t (years)')
 ylabel('y_d')
 subplot(3,2,3), plot(t,y(:,3),'r','Linewidth' ,0.1)
 xlabel('t (years)')
 ylabel('y_i')
 subplot(3,2,4), plot(t,y(:,4),'c','Linewidth' ,0.1)
 xlabel('t (years)')
 ylabel('k_f')
 subplot(3,2,5), plot(t,y(:,5),'-m','Linewidth' ,0.1)
 xlabel('t (years)')
 ylabel('k_d')
 subplot(3,2,6), plot(t,y(:,6),'y','Linewidth' ,0.1)
 xlabel('t (years)')
 ylabel('k_i')
```

Listing 5.4: MATLAB 2021a Code for entries of Routh array

```
ad=2.55336;af=2.55336;ai=2.55336;
 c0=1;c=0.53;Ld=0.09;Lf=0.09;Li=0.09;Md=0.21;Mf=0.21;
 Mi=0.21;Bd=9.4;Bf=9.4;Bi=9.4;B0d=5.8;B0f=5.8;B0i=5.8;D=0.09;
Af=(Bf*Lf*Mf)/(Lf+Mf);
Ad=(Bd*Ld*Md)/(Ld+Md);
Ai=(Bi*Li*Mi)/(Li+Mi);
Cf=((B0f+log(Lf)-log(Mf))*Lf*Mf)/(Lf+Mf);
Cd=((B0d+log(Ld)-log(Md))*Ld*Md)/(Ld+Md);
Ci=((B0i+log(Li)-log(Mi))*Li*Mi)/(Li+Mi);
c1=-ad*(1-c);
c2=ad*c;
c3=af*Ad;
c4=af*Af;
c5=af*(D-Cd);
c6=af*(D-Cf);
c7=ai*Ad;
c8=-ai*(1-Ai);
c9=ai*(D-Cd);
c10=ai*(D-Ci);
a6=1;
a5=Ci - c1 - c4 - c8;
a4=c1*c4 - Ai*c10 - Ci*c1 - Ci*c4 - Ci*c8 - Af*c5 - c2*c3 ...
     - c1*c7 + c1*c8 + c4*c8 - Cd*Cf;
a3=Af*Cd*c6 - Cd*Cf*Ci - Af*Ci*c5 + Cd*Cf*c1 + Cd*Cf*c4 + Cd*Cf*c8 ...
     - Ad*c2*c6 + Af*c1*c5 + Af*c5*c8 + Ai*c1*c10 ...
     + Ai*c4*c10 + Ci*c1*c4 - Ci*c2*c3 - Ci*c1*c7 + Ci*c1*c8 + ...
     Ci*c4*c8 + c1*c4*c7 - c1*c4*c8 + c2*c3*c8;
a2= Af*Cd*Ci*c6 + Ai*Cd*Cf*c10 + Cd*Cf*Ci*c1 + Cd*Cf*Ci*c4 + Cd*Cf*Ci*c8...
     + Af*Ai*c5*c10 + Ad*Cf*c2*c5 - Af*Cd*c1*c6 + Ad*Cf*c1*c9...
     - Af*Cd*c6*c8 - Ad*Ci*c2*c6 + Af*Ci*c1*c5 + Af*Ci*c5*c8 - Cd*Cf*c1*c4 ...
     + Cd*Cf*c2*c3 + Cd*Cf*c1*c7 - Cd*Cf*c1*c8 - Cd*Cf*c4*c8 ...
     + Ad*c2*c6*c8 - Af*c1*c3*c9 + Af*c1*c5*c7 - Af*c1*c5*c8 - Ai*c1*c4*c10...
     + Ai*c2*c3*c10 + Ci*c1*c4*c7 - Ci*c1*c4*c8 + Ci*c2*c3*c8;
a1=Ad*Ai*c2*c6*c10 - Ad*Af*c1*c6*c9 - Af*Ai*c1*c5*c10 - Ad*Cf*c1*c4*c9 ...
     - Af*Cd*c1*c6*c7 - Ad*Cf*c2*c5*c8 + Af*Cd*c1*c6*c8 + Ad*Ci*c2*c6*c8...
     - Af*Ci*c1*c3*c9 + Af*Ci*c1*c5*c7 - Af*Ci*c1*c5*c8 - Cd*Cf*c1*c4*c7 ...
     + Cd*Cf*c1*c4*c8 - Cd*Cf*c2*c3*c8...
     - Af*Ai*Cd*c6*c10 + Ad*Cf*Ci*c2*c5 - Af*Cd*Ci*c1*c6 + Ad*Cf*Ci*c1*c9...
     - Ai*Cd*Cf*c1*c10 - Af*Cd*Ci*c6*c8 - Ai*Cd*Cf*c4*c10 - Cd*Cf*Ci*c1*c4 ...
     + Cd*Cf*Ci*c2*c3 + Cd*Cf*Ci*c1*c7 - Cd*Cf*Ci*c1*c8 - Cd*Cf*Ci*c4*c8;
a0=Af*Ai*Cd*c1*c6*c10 - Ad*Ai*Cf*c2*c5*c10 - Ad*Af*Ci*c1*c6*c9- ...
     Ad*Cf*Ci*c1*c4*c9 - Af*Cd*Ci*c1*c6*c7- Ad*Cf*Ci*c2*c5*c8 + Af*Cd*Ci*c1*c6*c8 ...
     + Ai*Cd*Cf*c1*c4*c10- Ai*Cd*Cf*c2*c3*c10 - Cd*Cf*Ci*c1*c4*c7...
     + Cd*Cf*Ci*c1*c4*c8 - Cd*Cf*Ci*c2*c3*c8;
x=a6
y=a5
z=a4
t=a3
r=a2
s=a1
u=a0
b1=(a5*a4-a6*a3)/a5
b2=(a5*a2-a6*a1)/a5
b3=a0
f1=(b1*a3-b2*a5)/b1
f2=(b1*a1-b3*a5)/b1
d1=(f1*b2-b1*f2)/f1
d2=b3
e1=(d1*f2-d2*f1)/d1
g1=d2
```