UNIFORMLY CONVERGENT NUMERICAL METHOD FOR SECOND ORDER SINGULARLY PERTURBED FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

Solomon Regasa Badeye

A Thesis Submitted to

The Department of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of

Master's in Applied Mathematics (Numerical Analysis)

Office of Graduate Studies

Adama Science and Technology University

June, 2022

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Declaration

I hereby declare that this Master Thesis entitled "Uniformly convergent numerical method for second order singularly perturbed Fredholm integro-differential equations" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation

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Recommendation of Advisors

we, the advisor(s) of this thesis, hereby certify that we have read the revised version of the thesis entitled "Uniformly convergent numerical method for second order singularly perturbed Fredholm integro-differential equations" prepared under our guidance by Solomon Regasa Badeye submitted in partial fulfillment of the requirements for the degree of Mater's of Science in Applied Mathematics. Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

we, the advisors of the thesis entitled "Uniformly convergent numerical method for second order singularly perturbed Fredholm integro-differential equations" and developed by Solomon Regasa Badeye, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the thesis by Solomon Regasa Badeye have read and evaluated the thesis entitled "Uniformly convergent numerical method for second order singularly perturbed Fredholm integro-differential equations" and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics (Numerical Analysis).

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LIST OF ACRONYMS

ADM	Adomain decomposition method
FDM	Finite difference method
FIDE	Fredholm integro-differential equation
NSFDM	Nonstandard finite difference method
SPFIDEs	Singularly perturbed Fredholm integro-differential equations
SPPs	Singularly perturbed problems

Abstract

In this thesis, a parameter-uniformly convergent numerical scheme is designed for solving linear second-order singularly perturbed Fredholm integro-differential equations. A parameter-uniform numerical method was constructed via the non-standard finite difference method to approximate the differential part and the composite Simpson's 1/3 rule for the integral part. A convergence analysis has been carried out to show the parameter-uniform convergence of the proposed scheme. The maximum absolute errors and rate of convergence for different values of perturbation parameter ε *and mesh sizes are tabulated for three model examples. The proposed scheme is shown to be second-order uniformly convergent.*

Keywords: Fredholm integro-differential equation, singularly perturbed problem, nonstandard finite difference method, parameter-uniform convergence.

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Mathematical modeling of real-life problems usually results in some form of functional equations, e.g., algebraic equations, differential equations, integral equations and others. The occurrence of differential equations and integral equations is common in many areas of the sciences and engineering. However, research work in this field results in a specific topic, where both differential and integral operators appear together in the same equation. These type of equations are known as integro-differential equations. Many mathematical formulations in natural science, i.e., study of fluid, biology and chemical kinetics, contain integro-differential equations. Depending on the limits of integration and the kernel of the equation Integro-differential equations are classified as Fredholm integro-differential equations, Volterra integro-differential equations and also Volterra-fredholm integro-differential equations (Abdul-Majid, 2011).

Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in Acta-mathematica played a major role in the establishment of operator theory. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x), n \geq 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed. The equation is given in the form of,

$$u^{(n)}(x) = f(x) + \int_a^b K(x,t)u(t)dt,\ u^{(k)}(0) = b_k,\ 0 \leq k \leq n-1,$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$.
Fredholm integro-differential equations (FIDEs) have in large quantities applications in every branches of science. FIDEs arise from the mathematical modeling of many scientific phenomena, such as the study of fluid, physics, chemistry, biology, mechanics, astronomy, potential theory, electrostatics, control theory of industrial mathematics, chemical kinetics, signal processing and the theory of neural networks (Amiraliyev, Durmaz, & Kudu, 2021).

Volterra studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterra integro-differential equations, given in the form

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)u(t)dt,\ u^{(k)}(0) = b_k,\ 0 \le k \le n-1,$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$.

The Volterra integro-differential equation appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert.

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic, and from various physical and biological models.

The Volterra-Fredholm integro-differential equations appear in two types, namely:

$$u^{(n)}(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)u(t)dt\ + \lambda_2 \int_a^b K_2(x,t)u(t)dt,$$

and the mixed form

$$u^{(n)}(x) = f(x) + \lambda \int_0^x \int_a^b K(r,t)u(t)dtdr.$$

The first type contains split integrals and the second type contains mixed integrals such that the Fredholm integral is the interior one, and Volterra is the exterior integral.

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order derivative terms in the differential equations. Singularly perturbed integral equations or integro-differential equations are shown in the mathematical model of population variability, polymer rhythm and glucose tolerance (Cimen and Cakir, 2021). In particular, singularly perturbed Fredholm integral equation is given by optimal control problems Nefedov and Nikitin (2007). In general, the solutions of such equations exhibit multi-scale phenomena. Within certain thin sub-regions of the domain, the scale of some derivatives is significantly larger than other derivatives. These thin regions of rapid change are called, boundary or interior layers, as appropriate (Amiraliyev & Yapman, 2019).

A conventional finite difference operator on a uniform mesh was used to get early numerical solutions of singularly perturbed problems (Miller et al., 2012). The mesh is refined sufficiently to capture the boundary or interior layers when the singular perturbation parameter reduces in magnitude in this method. Such methods are inefficient and imprecise even when dealing with problems in one dimension. In order to solve SPPs, two approaches are often used in the context of finite difference methods. The first approach, commonly referred to as fitted mesh finite difference methods, uses an upwinding scheme to discretize a nonuniform mesh. A discrete operator involving either a fitting factor (of exponential type, for example) or a denominator function is used in the second approach, which falls under the category of fitted operator finite difference methods. The above approaches have been used extensively to solve singularly perturbed differential equations. However, very little effort in their use is observed for singularly perturbed integro-differential equations (Iragi & Munyakazi, 2018). A wide verity of articles have been published in the recent years, describing various methods for solving singular perturbation problems.

Because of the necessity to quantify the error in the extremely small domains where the boundary or interior layers occur, the maximum norm was chosen for error measurement. Other norms, such as the root mean square, use error averages to smooth out quick changes in the solutions and hence may miss the local behavior of the error in these layers. Hegarty et al. (1993) and Farrell et al. (2000) provide more information on how to choose an appropriate norm.

1.2 Statement of the Problem

It is well known that classical numerical methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is very small. The accuracy and convergence of the methods need attention because the numerical treatment of singular perturbation problems is not easy and the solution depends on the perturbation parameter and mesh size (Turuna et al., 2020). This recommends that the numerical treatment of singularly perturbed second order Fredholm integro-differential equations should be improved. The presence of the singular perturbation parameter ε leads to occurrences of oscillations or divergence in the computed solutions while using classical numerical methods (Woldaregay and Duressa, 2019). To avoid these oscillations or divergence, a large number of mesh points are required when ε is very small. It is difficult and sometimes impossible to handle such cases. Therefore, to overcome this drawback associated with classical numerical methods, this study aimed at developing a non-standard finite difference method together with a composite Simpson's $1/3$ rule on uniform mesh to treat the problem without creating an oscillation.

1.3 Objectives

1.3.1 General Objective

The general objective of this study is to develop an accurate and uniformly convergent numerical method for solving singularly perturbed Fredholm integro-differential equations.

1.3.2 Specific Objectives

The specific objectives of this study are:

- to construct a non-standard finite difference method together with a composite Simpson's $1/3$ rule on uniform mesh for the numerical solution of linear second-order singularly perturbed Fredholm integro-differential equations,
- to establish a stability analysis of the scheme,
- to establish the parameter-uniform convergence of the scheme.

1.4 Significance of the Study

The result obtained from this study may:

- used as a reference material for scholars who works on this area,

- provide a numerical method for solving singularly perturbed Fredholm integro-differential equations.

1.5 Delimitation of the Study

This study is delimited to a linear second-order singularly perturbed Fredholm integro-differential equation (SPFIDE) of form:

$$\begin{aligned} Lu := L_1 u + \lambda \int_0^l K(x,s)u(s)ds = f(x), \qquad & x \in (0,l), \\ u(0) = A, \qquad & u(l) = B, \end{aligned} \tag{1.1}$$

where $L_1 u = -\varepsilon u''(x) + a(x)u(x)$, ε (i.e. $0 < \varepsilon \leq 1$) is perturbation parameter and λ, A, B are given constants. It was presumed that $a(x) \geq \alpha > 0$, $f(x)$, $(x \in [0,l])$, and kernel function $K(x,s)$, $((x,s) \in [0,l]^2)$ are the sufficiently smooth, bounded functions. Furthermore, kernel $K(x,s)$ is nonnegative, monotone in each variable, and satisfying certain triangle inequality, such that there exist $M > 0$ i.e. $K(x,s) \leq M[K(x,z) + K(z,s)]$ if $s < z < x$. Under these conditions, the solution $u(x)$ of the problem (1.1) has in general boundary layers at $x = 0$ and $x = l$ (Amiraliyev et al., 2021).

CHAPTER 2

LITERATURE REVIEW

The numerical treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions. As a result, in recent years, numerous works on the numerical treatment of Fredholm/Volterra integral equations have been recorded in the literature.

Durmaz et al. (2022) developed a fitted second-order difference scheme on Shishkin mesh by using interpolating quadrature rules and an exponential basis function for the numerical treatment of the singularly perturbed Fredholm integro-differential equation with mixed boundary conditions. In their work, the stability and convergence of the scheme were analyzed in terms of the discrete maximum norm.

In order to solve the initial-value problem for a singularly perturbed Fredholm integro-differential equation, Amiraliyev et al. (2018) proposed a fitted finite difference scheme on a uniform mesh. The difference scheme was via the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. They discussed the convergence of the method and showed that the method exhibits a first-order uniform convergent in terms of perturbation parameter.

Amiraliyev and Yapman (2019) analyzed the convergence of the fitted mesh method applied to singularly perturbed Volterra delay-integro-differential equation. Their mesh comprises a special nonuniform mesh on the first sub-interval and uniform mesh on another part. In that work, the difference scheme was constructed based on the method of integral identity by using appropriate interpolating quadrature rules with remainder terms in integral form. Error estimates are obtained using difference analogue of Gronwall's inequality with delay. In order to test convergence of the method, they implemented the present method on two particular problems the numerical results show that the proposed method is first order uniformly accurate.

Kudu et al. (2016) constructed a parameter uniform numerical method for initial value problems of first-order singularly perturbed delay integro-differential equations. They used implicit difference rules to discretize the differential part and composite numerical quadrature rules for the integral part. The error estimation for an approximate solution was established on a layer-adapted mesh. They considered some numerical examples to test the convergence of the method and showed that the method is first-order convergence.

Cimen and Cakir (2021) presented a new difference scheme to solve boundary value problems of singularly perturbed Fredholm integro-differential equations. The difference scheme was constructed via the method of integral identities using interpolating quadrature rules with remainder terms in integral form. They discussed the convergence of the method and showed that the proposed method is first-order convergence.

Karim et al. (2018) used the Adomian Decomposition Method (ADM) to solve linear second-order Fredholm Integro-differential equations. They considered some numerical examples to illustrate the ADM for solving the problems, and the results were compared with the existing exact solution. The obtained results showed that the method converges to the exact solution quickly and, at the same time, reduces the computational work for solving the problem.

Xue et al. (2018) developed an improved reproducing kernel method to find the numerical solution of Fredholm integro-differential equations. The solution representation was obtained based on the good properties of the reproducing kernel function and the conjugate operator. They discussed the convergency of the method and showed that the proposed method is first-order convergence.

In Jackiewicz et al. (2008), several approaches to the numerical solution of a new Fredholm integro-differential equation model for neural networks were proposed. A solution strategy was presented based on the expansion of standard cardinal basis functions and collocations. The numerical experiments show that the method based on rational approximation on the uniform grid is most accurate for the number of grid points $n \geq 30$ and that the method based on polynomial approximation on the uniform or Chebyshev grid becomes unstable and leads to the loss of accuracy for large values of n.

Karimi and Jozi (2015) presented a new method for solving systems of linear Fredholm integral equations of the second kind. The new method they proposed was based on Taylor series expansion, which degenerates the kernel and reduces the system of integral equations into a block algebraic linear system. Under certain conditions, it was shown that the approximate solution they obtained by the proposed method converges to the exact solution.

In Jalilian and Tahernezhad (2020), the exponential spline method for approximating solutions of Fredholm integro-differential equations of the second kind was proposed, and it was shown that the proposed method is very accurate with respect to some mentioned methods and needs fewer computational efforts. Also, the convergence analysis of the method was proved.

In Chen et al. (2020), a fast multi-scale Galerkin method was developed for solving second order linear Fredholm integro-differential equations with Dirichlet boundary conditions. The method proposed was based on a matrix truncation strategy, which leads to generating a cost-efficient matrix rapidly. In their work, they proved that the method was stable and had an optimal convergence order and nearly linear computational complexity (up to a logarithmic factor). Numerical examples were presented to illustrate its computational efficiency, approximation accuracy, and theoretical results, and to compare the computed results with those of the original. The Multi-scale Galerkin method was recently proposed by the same authors.

A general overview of several techniques to integrate Fredholm/Volterra itegral or integro-differential equations has been recorded in the past half century. Pandey (2015) used the non-standard finite difference method along with the repeated/composite trapezoidal quadrature rule to develop a numerical method for linear Fredhom integro-differential equations. He considered some numerical experiments on some linear model problems and showed that the proposed method is second-order accurate.

Jalius and Abdul Majid (2017) applied the quadrature-difference method with the Gauss elimination method to solve second-order linear Fredholm integro-differential equations. In order to drive approximation equations, they combined composite Simpson's $1/3$ rule and the second-order finite difference method. Some numerical experiments were carried out to examine the accuracy of the proposed method.

Two numerical methods for solving the linear integro-differential equations are presented and compared in (Saadati et al., 2008). The first method is based upon the Trapezoidal rule and numerical differentiation, which transforms the integro-differential equation into a system of linear algebraic equations, and the second method is the variational iteration method.

Amiraliyev et al. (2021) proposed a fitted finite difference scheme on a uniform mesh to develop a numerical solution for (1.1). The difference scheme was constructed via the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder terms in integral form. They discussed the convergence of the method and showed that the proposed method has first-order convergence. On the other hand, Durmaz and Amiraliyev (2021) proposed an exponentially fitted difference scheme on shishkin mesh for the numerical solution of the same problems. The fitting factor was introduced via the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder terms in integral form. Their method was second-order accurate. However, up to the best of our knowledge, except the work in Amiraliyev et al. (2021) and Durmaz and Amiraliyev (2021), not much work has been done to solve the problem under consideration.

In this study, motivated by the studies of Amiraliyev et al. (2021) and Durmaz and Amiraliyev (2021), we construct and analyze an accurate and uniformly convergent numerical scheme for solving linear second-order singularly perturbed Fredholm integro-differential equations with Dirichlet boundary condition. The proposed scheme uses nonstandard finite difference scheme to approximate the differential part and composite Simpson's $1/3$ rule for integral part.

CHAPTER 3

DESCRIPTION AND ANALYSIS OF THE METHOD

3.1 Properties of Continuous Problems

In the study of the numerical aspects of singularly perturbed problems, their analytical aspects play an important role. Problems (1.1) satisfy the following lemma.

Lemma 3.1. *If $a, f \in C^2[0,l]$, $\frac{\partial^s K}{\partial x^s} \in C^2[0,l]^2$, $(s = 0,1,2)$ and*

$$|\lambda| < \frac{\alpha}{\max_{1\leq x\leq l} \int_0^l |K(x,s)|ds}, \tag{3.1}$$

then the solution $u(x)$ of (1.1) satisfies the following estimates

$$\|u\|_\infty \leq C, \tag{3.2}$$

$$|u'(x)| \leq C\left\{1 + \frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(1-x)}{\sqrt{\varepsilon}}}\right)\right\}. \tag{3.3}$$

Proof. Using the maximum principle for the operator $L_1 u = -\varepsilon u'' + a(x)u$ we obtain the estimate

$$\|u\|_\infty = |A| + |B| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}|\lambda| \max_{1\leq x\leq l} \int_0^l |K(x,s)||u(s)|ds,$$

which after taking into account (3.1), leads to (3.2). Next, we prove the estimate (3.3). Using (3.2) on (1.1) we have

$$|u''(x)| = \frac{1}{\varepsilon}|f(x) - a(x)u(x) - \lambda \int_0^l K(x,s)u(s)ds| \leq \frac{C}{\varepsilon},\ 0 \leq x \leq l.$$

Moreover, we now proceed with the estimation of $|u'(0)|, |u'(l)|$. Here we use the following relation which holds for any function $y \in C^2[0,l]$:

$$y'(x) = y(\alpha_0, \alpha_1) - \int_{\alpha_0}^{\alpha_1} K_0(\xi, x)y''(\xi)d\xi,\ \alpha_0 < \alpha_1, \tag{3.4}$$

where

$$y(\alpha_0,\alpha_1)=\frac{y(\alpha_1)-y(\alpha_0)}{\alpha_1-\alpha_0},$$
$$K_0(\xi,x)=T_0(\xi-x)-(\alpha_1-\alpha_0)^{-1}(\xi-\alpha_0),$$

and

$$T_0(\lambda)=\begin{cases}1, & \lambda\geq 0,\\ 0, & \lambda<0.\end{cases}$$

Equality (3.4) with the values $y(x)=u(x), x=0, \alpha_0=0$, and $\alpha_1=\sqrt{\varepsilon}$ yields

$$|u'(0)|\leq\frac{u(\sqrt{\varepsilon})-u(0)}{\sqrt{\varepsilon}}-\int_0^{\sqrt{\varepsilon}}K_0(\xi,0)u''(\xi)d\xi\leq\frac{C}{\varepsilon}. \tag{3.5}$$

Similarly, using (2.4) for $y(x)=u(x), x=l, \alpha_0=l-\sqrt{\varepsilon}$, and $\alpha_1=l$ we confirm that

$$|u'(l)|\leq\frac{u(l)-u(l-\sqrt{\varepsilon})}{\sqrt{\varepsilon}}-\int_{l-\sqrt{\varepsilon}}^{l}K_0(\xi,l)u''(\xi)d\xi\leq\frac{C}{\varepsilon}. \tag{3.6}$$

Next, differentiating (1.1), according to (3.5) and (3.6), we get

$$-\varepsilon v''(x)+a(x)v=F(x),\ v(0)=O(\frac{1}{\sqrt{\varepsilon}}),\ v(l)=O(\frac{1}{\sqrt{\varepsilon}}), \tag{3.7}$$

with

$$v(x)=u'(x),\ F(x)=f'(x)-a'(x)u(x)-\lambda\int_0^l K(x,s)u(s)ds.$$

By virtue of (3.2) evidently

$$|F(x)|\leq C. \tag{3.8}$$

In order to estimate the solution of the problem (3.7), we present it in the form

$$v(x)=v_1(x)+v_2(x),$$

where the functions $v_1(x)$ and $v_2(x)$ are the solutions of the following problems respectively:

$$\begin{aligned}-\varepsilon v_1''+a(x)v&=F(x),\\ v_1(0)=v_1(l)&=0,\end{aligned} \tag{3.9}$$

$$\begin{aligned} -\varepsilon v_2 + a(x)v_2 = F(x), \\ v_2(0) = O(\frac{1}{\sqrt{\varepsilon}}),\ v_2(l) = O(\frac{1}{\sqrt{\varepsilon}}). \end{aligned} \tag{3.10}$$

For the solution of the problem (3.9), using the maximum principle and (3.8), we have

$$|v_1(x)| \leq \alpha^{-1}\|F\|_\infty \leq C \qquad 0 \leq x \leq l. \tag{3.11}$$

According to the maximum principle, from the problem (3.10), we also conclude that

$$|v_2(x)| \leq w(x), \tag{3.12}$$

where the function $w(x)$ is the solution of the following problem:

$$\begin{aligned} -\varepsilon w'' + \alpha w = 0, \\ w(0) = |v_2(0)|,\ w(l) = |v_2(l)|. \end{aligned} \tag{3.13}$$

The solution of problem (3.13) is given by

$$w(x) = \frac{1}{sinh(\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}})}\left(w(0)sinh\left(\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}\right) + w(l)sinh\left(\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}\right)\right).$$

Hence, taking into consideration (3.10) we obtain

$$w(x) \leq \frac{C}{\sqrt{\varepsilon}}\left(e^{\frac{-\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{\frac{-\sqrt{\alpha}(1-x)}{\sqrt{\varepsilon}}}\right). \tag{3.14}$$

Finally, the use bounds (3.11), (3.12) and (3.14) in the inequality

$$|u'(x)| \leq |v_1(x)| + |v_2(x)|$$

immediately leads to (3.3). □

3.2 Construction of the Difference Scheme

In this section, the nonstandard finite-difference method together with composite Simpson's $1/3$ rule is applied to discretize the second-order singularly perturbed Fredholm integro-differential equations to construct approximation equations.

The theoretical basis of non-standard discrete finite difference method (NSFDM) is based on the development of exact finite difference method. Mickens (2005) presented techniques and rules for developing non-standard FDMs for different problem types. According to Mickens rule 2, to develop a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedure. These complicated denominator functions constitutes a general property of these schemes, which is useful for constructing the reliable scheme for such problems. In line to this, we need to derive a nonstandard finite difference method which captures the layer behaviour of the problem on a uniform mesh. Mickens (2005) gave a novel approach of nonstandard finite difference method, and the basic idea of this method is to replace the denominator function h^2 of the second-order derivative with a suitable function in the differential equation, which comes under the category of the fitted operator finite difference method. On the domain $[0,l]$, we introduce the equidistant meshes such that

$$\Omega_N = \left\{ x_0 = 0, x_i = ih, i = 1(1)N-1,\ x_N = l,\ h = \frac{l}{N} \right\},$$

where h is step size and N is the number of mesh points.

For the problem of the form in (1.1), in order to construct the exact finite difference scheme, we follow the procedures used in Gobena and Duressa (2021). Consider the constant coefficient sub-equations given in (3.15) by ignoring the integral part as

$$-\varepsilon u''(x) + \alpha u(x) = 0, \tag{3.15}$$

where $a(x) \geq \alpha > 0$. Thus the equation in (3.15) has two linearly independent solutions namely $e^{(\lambda_1 x)}$ and $e^{(\lambda_2 x)}$ with

$$\lambda_{1,2} = \pm\sqrt{\frac{\alpha}{\varepsilon}}.$$

We denote the approximate solution of $u(x)$ at x_i by U_i. Now our main motive is to calculate a difference equation which has the same general solution as the differential equation in (3.15) has at the grid point x_i given by $U_i = A_1 e^{(\lambda_1 x_i)} + A_2 e^{(\lambda_2 x_i)}$. Using the procedure used in Gobena and Duressa (2021), we have

$$\begin{vmatrix} U_{i-1} & e^{(\lambda_1 x_{i-1})} & e^{(\lambda_2 x_{i-1})} \\ U_i & e^{(\lambda_1 x_i)} & e^{(\lambda_2 x_i)} \\ U_{i+1} & e^{(\lambda_1 x_{i+1})} & e^{(\lambda_2 x_{i+1})} \end{vmatrix} = 0,$$

then we have

$$U_{i-1} \begin{vmatrix} e^{(\lambda_1 x_i)} & e^{(\lambda_2 x_i)} \\ e^{(\lambda_1 x_{i+1})} & e^{(\lambda_2 x_{i+1})} \end{vmatrix} - U_i \begin{vmatrix} e^{(\lambda_1 x_{i-1})} & e^{(\lambda_2 x_{i-1})} \\ e^{(\lambda_1 x_{i+1})} & e^{(\lambda_2 x_{i+1})} \end{vmatrix} + U_{i+1} \begin{vmatrix} e^{(\lambda_1 x_{i-1})} & e^{(\lambda_2 x_{i-1})} \\ e^{(\lambda_1 x_i)} & e^{(\lambda_2 x_i)} \end{vmatrix} = 0,$$

$$U_{i-1}\left(e^{(\lambda_2 h)} - e^{(\lambda_1 h)}\right) - U_i\left(e^{((\lambda_2 - \lambda_1)h)} - e^{((\lambda_1 - \lambda_2)h)}\right) + U_{i+1}\left(-e^{(-\lambda_2 h)} + e^{(-\lambda_1 h)}\right) = 0$$

substituting the values of λ_1 and λ_2 we obtain:

$$U_{i-1}\left(e^{(\sqrt{\frac{\alpha}{\varepsilon}}h)} - e^{(-\sqrt{\frac{\alpha}{\varepsilon}}h)}\right) - U_i\left(e^{(2\sqrt{\frac{\alpha}{\varepsilon}}h)} - e^{(-2\sqrt{\frac{\alpha}{\varepsilon}}h)}\right) + U_{i+1}\left(-e^{(-\sqrt{\frac{\alpha}{\varepsilon}}h)} + e^{(\sqrt{\frac{\alpha}{\varepsilon}}h)}\right) = 0$$

dividing both sides by $e^{(\sqrt{\frac{\alpha}{\varepsilon}}h)} - e^{(-\sqrt{\frac{\alpha}{\varepsilon}}h)}$ we obtain:

$$U_{i-1} - \left(e^{(\sqrt{\frac{\alpha}{\varepsilon}}h)} + e^{(-\sqrt{\frac{\alpha}{\varepsilon}}h)}\right) U_i + U_{i+1} = 0,$$

$$U_{i-1} - 2\cosh\left(\sqrt{\frac{\alpha}{\varepsilon}}h\right) U_i + U_{i+1} = 0. \tag{3.16}$$

Using the identity

$$2\cosh\left(\sqrt{\frac{\alpha}{\varepsilon}}h\right) = 2 + 4\sinh^2\left(\frac{\sqrt{\frac{\alpha}{\varepsilon}}h}{2}\right),$$

equation in (3.16) can be

$$U_{i-1} - \left(2 + 4\sinh^2\left(\frac{\sqrt{\frac{\alpha}{\varepsilon}}h}{2}\right)\right) U_i + U_{i+1} = 0,$$

which implies

$$U_{i-1} - 2U_i + U_{i+1} = 4\sinh^2\left(\frac{\sqrt{\frac{\alpha}{\varepsilon}}h}{2}\right) U_i,$$

dividing both sides by $4\sinh^2\left(\frac{\sqrt{\frac{\alpha}{\varepsilon}}h}{2}\right)$ we have

$$\frac{U_{i-1}-2U_i+U_{i+1}}{4\sinh^2\left(\frac{\sqrt{\frac{\alpha}{\varepsilon}}h}{2}\right)}-U_i=0,$$

multiplying both sides by $-\varepsilon\gamma^2$, where $\gamma=\sqrt{\frac{\alpha}{\varepsilon}}$ we obtain

$$-\varepsilon\frac{U_{i-1}-2U_i+U_{i+1}}{\frac{4}{\gamma^2}\sinh^2(\frac{\gamma h}{2})}+\alpha U_i=0,$$

which is an exact difference scheme for equation in (3.15). The denominator function becomes $\psi^2=\frac{4}{\gamma^2}\sinh^2(\frac{\gamma h}{2})$. Adopting this denominator function for the variable coefficient problem we can write as:

$$\psi_i^2=\frac{4}{\left(\sqrt{\frac{a(x_i)}{\varepsilon}}\right)^2}\sinh^2\left(\frac{\sqrt{\frac{a(x_i)}{\varepsilon}}h}{2}\right). \tag{3.17}$$

By using the denominator function ψ_i^2 in to the main discrete scheme, we obtain

$$L^N U_i\equiv-\varepsilon\frac{U_{i-1}-2U_i+U_{i+1}}{\psi_i^2}+a_iU_i+\lambda\int_0^l K_i(s)U(s)ds=f_i, \tag{3.18}$$

Now, the truncation error of scheme (3.18) is given by

$$\begin{aligned}
L^N(U_i-u_i)&=f_i-L^N u_i,\\
&=(L-L^N)u_i,\\
&=\left(-\varepsilon\frac{d^2u_i}{dx^2}+a_iu_i+\lambda\int_0^l K_i(s)u(s)ds\right)\\
&-\left(-\varepsilon\frac{u_{i-1}-2u_i+u_{i+1}}{\psi_i^2}+a_iu_i+\lambda\int_0^l K_i(s)u(s)ds\right),\\
&=-\varepsilon\frac{d^2u_i}{dx^2}+\frac{\varepsilon}{\psi_i^2}(u_{i-1}-2u_i+u_{i+1}),\ i=1(1)(N-1).
\end{aligned}$$

Taylor series expansions of the terms u_{i+1} and u_{i-1} are given as follows:

$$\begin{aligned}u_{i+1} &= u_i + hu_i' + \frac{h^2}{2!}u_i'' + \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i^{(iv)} + \frac{h^5}{5!}u_i^{(v)} + \ldots,\\ u_{i-1} &= u_i - hu_i' + \frac{h^2}{2!}u_i'' - \frac{h^3}{3!}u_i''' + \frac{h^4}{4!}u_i^{(iv)} - \frac{h^5}{5!}u_i^{(v)} + \ldots.\end{aligned}$$

Using the truncated Taylor series expansions of the terms u_{i+1} and u_{i-1} yields

$$L^N(U_i - u_i) = -\varepsilon u_i'' + \frac{\varepsilon}{\psi_i^2}\left(h^2 u_i'' + \frac{h^4}{12}u_i^{(iv)}(\xi)\right) = R_1,\ \xi \in (u_{i-1}, u_{i+1}).$$

Moreover applying the composite Simpson $1/3$ rule to the integral term in (3.18). In order to drive composite Simpson's $1/3$ rule substitute $n = 2$ in Newton-Cotes quadrature formulae in Tuemay and Dessalegn (2019) obtain the following result

$$\int_{s_0}^{s_2} K(x_i,s)u(s)ds = \frac{h}{3}\left[K(x_i,s_0)u(s_0) + 4K(x_i,s_1)u(s_1) + K(x_i,s_2)\right],$$

where $K(x_i,s_j) = K_{ij}\ u(s_j) = u_j$ for $j = 0,\ 1,\ldots,\ n$. Now, the error of this method can be approximated in the following way.

Let the function $y = K(x_i,s)u(s)$ be continuous and possess a continuous derivatives in $[s_0,s_2]$. Expanding y about $s = s_0$ we obtain

$$y(x) = y_0 + (s-s_0)y_0' + \frac{1}{2}(s-s_0)^2 y_0'' + \frac{1}{3!}(s-s_0)^3 y_0''' + \frac{1}{4!}(s-s_0)^4 y_0^{(iv)} + \ldots.$$

$$\int_{s_0}^{s_2} K(x_i,s)u(s)ds = \int_0^2 h\left[y_0 + phy_0' + \frac{(ph)^2}{2}y_0'' + \frac{(ph)^3}{3!}y_0''' + \frac{(ph)^4}{4!}y_0^{(iv)} + \ldots\right]dp,$$

where $s = s_0 + ph$.

$$\begin{aligned}&= h\left[py_0 + \frac{p^2h}{2}y_0' + \frac{(ph)^2}{6}py_0'' + \frac{(ph)^3}{24}py_0''' + \frac{(ph)^4}{120}py_0^{(iv)} + \ldots\right]_0^2,\\ &= 2hy_0 + 2h^2y_0' + \frac{4h^3}{3}y_0'' + \frac{2h^4}{3}y_0''' + \frac{4h^5}{15}y_0^{(iv)} + \ldots.\end{aligned} \tag{3.19}$$

Therefore

$$y_0 = y_0, \tag{3.20}$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(iv)} + \ldots, \tag{3.21}$$

$$y_2 = y_0 + 2hy_0' + 2h^2y_0'' + \frac{4h^3}{3}y_0''' + \frac{2h^4}{3}y_0^{(iv)} + \dots. \tag{3.22}$$

From (3.20) (3.21)and (3.22), we get:

$$\begin{aligned} \frac{h}{3}[y_0 + 4y_1 + y_2] &= \frac{h}{3}\left[6y_0 + 6hy_0' + 4h^2y_0'' + 2h^3y_0''' + \frac{5h^4}{6}y_0^{(iv)} + \dots,\right] \\ &= 2hy_0 + 2h^2y_0' + \frac{4h^3}{3}y_0'' + \frac{2h^4}{3}y_0''' + \frac{5h^5}{18}y_0^{(iv)} + \dots. \end{aligned} \tag{3.23}$$

From (3.19) and (3.23) we obtain,

$$\int_{s_0}^{s_2} yds - \frac{h}{3}[y_0 + 4y_1 + y_2] = \left(\frac{4}{15} - \frac{5}{18}\right)h^5y_0^{(iv)} + \dots = \frac{-1}{90}h^5y_0^{(iv)} + \dots.$$

This is the error committed in the interval $[s_0, s_2]$. Generally, the composite Simpson's $1/3$ rule we need an even number of sub-divisions. Let $[0, l]$ be sub-divided into N even number of sub-divisions, $0 = s_0 < s_1 < s_2 < \dots < s_N = l$, then from the previous results we have,

$$\int_{s_0}^{s_2} yds = \frac{h}{3}[y_0 + 4y_1 + y_2],$$

$$\int_{s_2}^{s_4} yds = \frac{h}{3}[y_2 + 4y_3 + y_4],$$

And similarly, we obtain

$$\int_{s_{N-2}}^{s_N} yds = \frac{h}{3}[y_{N-2} + 4y_{N-1} + y_N].$$

Therefore, the integral over the whole interval is found by adding these integrations and is equal to

$$\int_{s_0}^{s_N} yds = \frac{h}{3}\left[y_0 + 4\sum_{j=1}^{N/2} y_{2j-1} + 2\sum_{j=1}^{(N/2)-1} y_{2j} + y_N\right].$$

In similar manner, we obtain the errors in the remain sub-intervals we have:

$$\begin{aligned} R_2 &= \frac{-1}{90}h^5\left[y_0^{(iv)} + y_2^{(iv)} + \dots + y_{N-2}^{(iv)}\right], \\ &= \frac{-l}{180}h^4u(\xi)^{(iv)},\ \xi \in [0, l], \end{aligned}$$

where $u(\xi)^{(iv)}$ is the largest value of the N-quantities on 4^{th} derivatives. Therefore , the integral term in (3.18) is approximated as:

$$\int_0^l K(x_i,s)u(s)ds = \frac{h}{3}\left(4\sum_{j=1}^{N/2} K(x_i,s_{2j-1})u(s_{2j-1}) + 2\sum_{j=1}^{N/2-1} K(x_i,s_{2j})u(s_{2j})\right) + \frac{h}{3}\left(K(x_i,s_0)u(s_0)+K(x_i,s_N)u(s_N)\right)+R_2. \tag{3.24}$$

Clearly, from (3.18) and (3.24) we have the following exact relation for $u(x_i)$

$$L_\varepsilon^N u_i := -\varepsilon\frac{u_{i-1}-\ 2u_i+\ u_{i+1}}{\psi_i^2} +\ a_iu_i+\lambda h\left(\sum_{j=0}^{N}\eta_j K_{ij}u_j\right)+R = f_i, \quad 1\le i\le N-1, \tag{3.25}$$

where

$$R = R_1 + R_2 \quad and$$

$$\eta_j = \begin{cases} \frac{1}{3}, & \text{for } j=0,N, \\ \frac{4}{3}, & \text{for } j=1,3,5,\dots,N-1, \\ \frac{2}{3}, & \text{for } j=2,4,6,\dots,N-2. \end{cases}$$

Based on (3.25) we propose the following difference scheme for approximating (1.1).

$$L_\varepsilon^N y_i := -\varepsilon\frac{y_{i-1}-\ 2y_i+\ y_{i+1}}{\psi_i^2} +\ a_iy_i+\lambda h\left(\sum_{j=0}^{N}\eta_j K_{ij}y_j\right) = f_i, \quad 1\le i\le N-1, \quad y_0=A, \quad y_N=B. \tag{3.26}$$

Lastly, from (3.26) the linear system equations for $u_1,u_2,u_3,\dots,u_{N-1}$ are generated. Therefore, the generated system of linear algebraic equations can be written in matrix form of

$$(M+S)U = F, \tag{3.27}$$

where M and S are coefficient matrix, F is a given function and U is an unknown function which is to be determined. The entries of M, S and F are given as:

$$M = \begin{cases} m_{ii} = \frac{2\varepsilon}{\psi_i^2} + a_i & \text{for } i = 1, 2, \ldots, N-1, \\ m_{ii+1} = \frac{-\varepsilon}{\psi_i^2} & \text{for } i = 1, 2, \ldots, N-2, \\ m_{ii-1} = \frac{-\varepsilon}{\psi_i^2} & \text{for } i = 2, 3, \ldots, N-1, \end{cases}$$

$$S = \begin{cases} \frac{4h\lambda}{3} K_{i,2j-1}, & \text{for } j = 1 : \frac{N}{2}, \\ \frac{2h\lambda}{3} K_{i,2j}, & \text{for } j = 1 : \frac{N}{2} - 1, \end{cases}$$

$$F = \begin{cases} f_1 - \left((\lambda h \eta_0 K_{1,0} - \frac{\varepsilon}{\psi_1^2}) A + \lambda h \eta_N K_{1,N} B \right), \\ f_i - \lambda h (\eta_0 K_{i,0} A + \eta_N K_{i,N} B), \quad for\ i = 2, 3, \ldots, N-2, \\ f_{N-1} - \left(\lambda h \eta_0 K_{N-1,0} A + (\lambda h \eta_N K_{N-1,N} - \frac{\varepsilon}{\psi_N^2}) B \right). \end{cases}$$

3.3 Stability and Uniform Convergence Analysis

In this section, we need to show the discrete scheme in (3.26) satisfy the discrete maximum principle, uniform stability estimates, and uniform convergence.

Lemma 3.2 (Discrete Maximum Principle)**.** *Assume that* $\lambda h \left(\sum_{j=1}^{N-1} \eta_j K_{ij} \right) \leq a_i$*. Let the difference operator*

$$L_\varepsilon^N y_i = -\varepsilon \frac{y_{i+1} - 2y_i + y_{i-1}}{\psi_i^2} + a_i y_i + \lambda h \left(\sum_{j=1}^{N-1} \eta_j K_{ij} y_j \right), \quad 1 \leq i \leq N-1,$$

be given. Then for all mesh function Ψ_i *such that* $\Psi_0 \geq 0$ *and* $\Psi_N \geq 0$*.Then* $l_\varepsilon^N \Psi_i \geq 0$*, for all* $1 \leq i \leq N-1$*, implies that* $\Psi_i \geq 0$*, for all* $0 \leq i \leq N$*.*

Proof. Let k be such that $\Psi_k = \min_{0 \leq i \leq N} \Psi_i$ and suppose that $\Psi_k < 0$. Evidently, $k \notin \{0, N\}$, $\Psi_k \leq \Psi_{k+1}$ and $\Psi_k \leq \Psi_{k-1}$. If $\psi_k^2 = \frac{4}{\sqrt{\frac{a(x_k)}{\varepsilon}}^2} \sinh^2(\sqrt{\frac{a(x_k)}{\varepsilon}} \frac{h}{2})$, it follows that

$$\begin{aligned} L_\varepsilon^N \Psi_k &= -\varepsilon \frac{\Psi_{k+1} - 2\Psi_k + \Psi_{k-1}}{\psi_k^2} + a_k \Psi_k + \lambda h (\eta_k K_{i,k}) \Psi_k, \\ &= -\varepsilon \frac{(\Psi_{k+1} - \Psi_k) + (\Psi_{k-1} - \Psi_k)}{\psi_k^2} + (a_k + \lambda h (\eta_k K_{i,k})) \Psi_k, \\ &< 0, \end{aligned}$$

which is a contradiction. It follows that $\Psi_k \geq 0$, and thus $\Psi_i \geq 0, \quad \forall\, 0 \leq i \leq N$. □

The uniqueness of the solution is guaranteed by this discrete maximum principle. The existence follows easily since, as for linear problems, the existence of the solution is implied by its uniqueness (Iragi & Munyakazi, 2020). The discrete maximum principle enables us to prove the following lemma which provides the boundedness of the solution.

Lemma 3.3. *Let linear difference operator $l_\varepsilon^N y_i$ be defined as*

$$l_\varepsilon^N y_i = -\varepsilon \frac{y_{i+1} - 2y_i + y_{i-1}}{\psi_i^2} + a_i y_i, \quad 1 \le i \le N-1. \tag{3.28}$$

For the solution of the difference boundary value problem

$$l_\varepsilon^N y_i = F_i, \quad 1 \le i \le N-1,$$
$$y_0 = \mu, \quad y_N = \beta$$

the following inequality holds

$$\|y_i\|_\infty \le \alpha^{-1} \max_{0 \le i \le N} |l_\varepsilon^N y_i| + max\{| \mu |, | \beta |\}.$$

Proof. We consider the functions $\Psi^\pm$ defined by

$$\Psi_i^\pm = p \pm y_i,$$

where

$$p = \alpha^{-1} \max_{0 \le i \le N} |l_\varepsilon^N y_i| + max\{| \mu |, | \beta |\}.$$

At the boundaries we have

$$\Psi_0^\pm = p \pm y_0 = p \pm \mu \ge 0,$$
$$\Psi_N^\pm = p \pm y_N = p \pm \beta \ge 0.$$

Now for Ω_N we have

$$
\begin{aligned}
l_\varepsilon^N \Psi_i^{\pm} &= -\varepsilon\left(\frac{p\pm y_{i+1}-2(p\pm y_i)+p\pm y_{i-1}}{\psi_i^2}\right)+a_i(p\pm y_i),\\
&= a_i p \pm l_\varepsilon^N y_i,\\
&= a_i\left(\alpha^{-1}\max_{0\le i\le N}|l_\varepsilon^N y_i| + max\{|\,\mu\,|,|\,\beta\,|\}\right)\pm F_i,\\
&\ge 0 \; since \; a_i\ge\alpha.
\end{aligned}
$$

From Lemma 3.2 it follows that $\Psi_i^{\pm}\ge 0,\ \forall x_i\in[0,l]$, this completes the proof. □

Lemma 3.4 (Stability for the difference problem(3.26)). *Let the difference operator $l_\varepsilon^N y_i$ is given by (3.28). Then for the difference problem (3.26) we get*

$$
l_\varepsilon^N y_i \le C\lambda h\sum_{j=1}^{N-1}\eta_j|y_j| + \|f\|_\infty, \quad 1\le i\le N-1. \tag{3.29}
$$

Proof. From (3.26) we have

$$
|l_\varepsilon^N y_i| \le |f_i| + \left|\lambda\frac{h}{3}(K_{i0}\,A + K_{iN}\,B)\right| + \left|4\lambda\frac{h}{3}\sum_{j=1}^{N/2}K_{i,2j-1}y_{2j-1}\right| + \left|2\lambda\frac{h}{3}\sum_{j=1}^{(N/2)-1}K_{i,2j}y_{2j}\right|.
$$

Using the assumption that the kernel $K(x,s)$ is bounded, this leads to (3.29). □

Lemma 3.5. *For the solution of the difference scheme (3.26) we have the following estimate*

$$
|\,y_i\,| \le \left(\alpha^{-1}\|f\|_\infty + \max\{|A|,|B|\}\right)\left(\exp(C\alpha^{-1}x_i)+\exp(C\alpha^{-1}(1-x_i))\right), \quad 1\le i\le N-1.
$$

Proof. Let $v_i = \lambda h\sum_{j=1}^{N-1}\eta_j\mid y_j\mid$. Thus from lemma (3.4) we have following difference inequality:

$$
\begin{aligned}
l_\varepsilon^N y_i &\le Cv_i + \|f\|_\infty,\\
y_0 = A,& \quad y_N = B.
\end{aligned}
$$

Using discrete maximum principle we have $|y_i|\le w_i$, where w_i is the solution of the problem

$$
\begin{aligned}
l_\varepsilon^N w_i &\le Cv_i + \|f\|_\infty,\\
w_0 =|\,A\,|,& \quad w_N =|\,B\,|\,.
\end{aligned}
$$

From here by virtue of lemma (3.3), it follows that

$$|y_i| \leq |w_i| \leq \alpha^{-1}Cv_i + \alpha^{-1}\|f\|_\infty + \max\{|A|,|B|\}. \tag{3.30}$$

Application of the difference analogue of the differential inequality then gives (Amiraliyev & Şevgin, 2006)

$$v_i \leq (\alpha^{-1}\|f\|_\infty + \max\{|A|,|B|\})\alpha C^{-1}(\exp(C\alpha^{-1}x_i) + \exp(C\alpha^{-1}(1-x_i))),$$

which together with (3.30) complete the proof. □

Lemma 3.6. *For all integers k on a fixed mesh, we have that*

$$\lim_{\varepsilon\to 0}\max_{1<i<N-1}\frac{exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0$$

and

$$\lim_{\varepsilon\to 0}\max_{1<i<N-1}\frac{exp(-C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0,$$

where $x_i = ih, h = 1/N, \forall i = 1 : N-1.$

Proof. The proof is by like approach as in Mbroh and Munyakazi (2021). □

Next, we analyze the uniform convergence of the method. From (3.25) and (3.26) for the error of the approximate solution $z_i = y_i - u_i$ we have

$$\begin{gathered} L_\varepsilon^N z_i := -\varepsilon\frac{z_{i-1} - 2z_i + z_{i+1}}{\psi_i^2} + a_i z_i + \lambda h\left(\sum_{j=0}^{N}\eta_j K_{ij} z_j\right) = R,\ 1\leq i\leq N-1, \\ z_0 = 0, \quad z_N = 0. \end{gathered} \tag{3.31}$$

Theorem 3.1. *If* $a,\ f \in C^2[0,l]$, $\frac{\partial^s K}{\partial x^s} \in C^2[0,l]^2$, $(s = 0,1,2)$ *and*

$$|\lambda| < \frac{\alpha}{\max_{1\leq i\leq N}\sum_{j=0}^{N} h\eta_j |K_{ij}|},$$

the solution of (3.26) converges uniformly to the solution of (1.1). For the error of approximate solution the following estimate holds

$$\|y-u\|_{\infty,\bar{\Omega}_N} \leq Ch^2.$$

Proof. Applying the maximum principle, from (3.31) we have

$$\begin{aligned}
\|z\|_{\infty,\bar{\Omega}_N} &\leq \alpha^{-1}\|R-\lambda h\sum_{j=0}^{N}\eta_j K_{ij}z_j\|_{\infty,\Omega_N},\\
&\leq \alpha^{-1}\|R\|_{\infty,\Omega_N}+|\lambda|\alpha^{-1}\max_{1\leq i\leq N}\sum_{j=0}^{N}h\eta_j|K_{ij}|\|z\|_{\infty,\bar{\Omega}_N},
\end{aligned}$$

hence

$$\|z\|_{\infty,\bar{\Omega}_N} \leq \frac{\alpha^{-1}\|R\|_{\infty,\Omega_N}}{1-|\lambda|\alpha^{-1}\max_{1\leq i\leq N}\sum_{j=0}^{N}h\eta_j|K_{ij}|}$$

which implies of

$$\|z\|_{\infty,\bar{\Omega}_N} \leq C\|R\|_{\infty,\Omega_N}. \tag{3.32}$$

Further we estimate for $\|R\|_{\infty,\Omega_N}$. Since $R=R_1+R_2$, we have

$$\begin{aligned}
|R| &= |R_1+R_2|,\\
&= \left|-\varepsilon u_i''+\frac{\varepsilon}{\psi_i^2}\left(h^2u_i''+\frac{h^4}{12}u^{(iv)}(\xi)\right)+\frac{l}{180}h^4u^{(iv)}(\xi)\right|.
\end{aligned}$$

Using truncated Taylor series expansion of the denominator function

$$\frac{1}{\psi_i^2}=\frac{1}{h^2}-\frac{\zeta}{(12\varepsilon)}+\frac{\zeta^2h^2}{(240\varepsilon^2)} \quad \text{(Mbroh \& Munyakazi, 2021)}$$

and this result into

$$\begin{aligned}
|R| &= \left|\left(\frac{\varepsilon}{h^2}-\frac{\zeta}{12}+\frac{\zeta^2h^2}{240\varepsilon}\right)\left(h^2u_i''+\frac{h^4}{12}u^{(iv)}(\xi)\right)-\varepsilon u_i''+\frac{l}{180}h^4u^{(iv)}(\xi)\right|,\\
&\leq \left|\left(\varepsilon-\frac{\zeta h^2}{12}+\frac{\zeta^2h^4}{240\varepsilon}\right)\left(u_i''+\frac{h^2}{12}u^{(iv)}(\xi)\right)-\varepsilon u_i''+\frac{l}{180}h^4u^{(iv)}(\xi)\right|,\\
&\leq \left|\left(\frac{\varepsilon}{12}u^{(iv)}(\xi)-\frac{\zeta}{12}u_i''\right)h^2+\left(\frac{\zeta^2}{240\varepsilon}u_i''+\left(\frac{l}{180}-\frac{\zeta}{144}\right)u^{(iv)}(\xi)\right)h^4\right|\\
&+\left|\left(\frac{\zeta^2}{2880\varepsilon}u^{iv}(\xi)\right)h^6\right|.
\end{aligned}$$

Using the bounds on the derivatives and Lemma 3.6 gives

$$|R| \leq \left|\left(\frac{\varepsilon}{12}-\frac{\zeta}{12}\right)h^2+\left(\frac{l}{180}-\frac{\zeta}{144}+\frac{\zeta^2}{240\varepsilon}\right)h^4+\frac{\zeta^2}{2880\varepsilon}h^6\right|,$$

where we have also used the relation $h^2 > h^4 > h^6 > \ldots$. Therefore

$$\|R\|_{\infty,\Omega_N} \leq Ch^2. \tag{3.33}$$

The bound (3.33) together with (3.32) completes the proof. □

CHAPTER 4

RESULTS AND DISCUSSIONS

To verify the established theoretical results in this thesis, we perform the experiments using the proposed numerical scheme on the problem of the form given in (1.1). We use the double mesh principle to estimate the errors.

Example 1. *Consider the singularly perturbed problem (Amiraliyev et al., 2021)*

$$Lu := -\varepsilon u'' + u + \frac{1}{2}\int_0^1 x\, u(s)ds = x - \varepsilon + \varepsilon e^{\frac{-x}{\varepsilon}},$$

$$u(0) = 1, \qquad u(1) = 2 - \varepsilon + \varepsilon e^{\frac{-1}{\varepsilon}}.$$

In this Example $a(x) = 1$, $\lambda = \frac{1}{2}$, $K(x,s) = x$ and $f(x) = x - \varepsilon + \varepsilon e^{\frac{-x}{\varepsilon}}$. Corresponding to the difference scheme given in (3.26), the denominator function ψ_i^2 becomes

$$\begin{aligned} \psi_i^2 &= \frac{4}{\left(\sqrt{\frac{a(x_i)}{\varepsilon}}\right)^2} \sinh^2\left(\frac{\sqrt{\frac{a(x_i)}{\varepsilon}}h}{2}\right), \\ &= 4\varepsilon \sinh^2\left(\frac{h}{2\sqrt{\varepsilon}}\right). \end{aligned}$$

Therefore, the difference scheme for Example 1 becomes

$$-\varepsilon\frac{u_{i-1} - \ 2u_i + \ u_{i+1}}{4\varepsilon\sinh^2\left(\frac{h}{2\sqrt{\varepsilon}}\right)} + \ u_i + \frac{h}{2}\sum_{j=0}^{N}\eta_j x_i u_j = x_i - \varepsilon + \varepsilon e^{\frac{-x_i}{\varepsilon}},$$

$$1 \le i \le N-1, \quad u_0 = 1, \quad u_N = 2 - \varepsilon + \varepsilon e^{\frac{-1}{\varepsilon}},$$

where η_j defined as in (3.26).

Example 2. *Consider singularly perturbed problem (Durmaz & Amiraliyev, 2021)*

$$Lu := -\varepsilon u'' + (2 - e^{-x})u + \frac{1}{2}\int_0^1 \left(e^{x\cos(\pi s)} - 1\right) u(s)ds = \frac{1}{1+x},$$

$$u(0) = 1, \qquad u(1) = 0.$$

In this Example $a(x) = 2 - e^{-x}$, $\lambda = \frac{1}{2}$, $K(x,s) = e^{x\cos(\pi s)} - 1$ and $f(x) = \frac{1}{1+x}$. Corresponding to the difference scheme given in (3.26), the denominator function ψ_i^2 becomes

$$\psi_i^2 = \frac{4\varepsilon}{(2-e^{-x_i})}\sinh^2\left(\frac{\sqrt{2-e^{-x_i}}h}{2\sqrt{\varepsilon}}\right).$$

Therefore, the difference scheme for Example 2 becomes

$$-\varepsilon\frac{u_{i-1} - 2u_i + u_{i+1}}{\frac{4\varepsilon}{(2-e^{-x_i})}\sinh^2\left(\frac{\sqrt{2-e^{-x_i}}h}{2\sqrt{\varepsilon}}\right)} + (2-e^{-x_i})u_i + \frac{h}{2}\sum_{j=0}^{N}\eta_j\left(e^{x_i\cos(\pi s_j)} - 1\right)u_j = \frac{1}{1+x_i},$$

$$1 \le i \le N-1, \quad u_0 = 1, \quad u_N = 0,$$

where η_j is defined in similar to (3.26).

Example 3. *Consider the particular problem:*

$$Lu := -\varepsilon u'' + (2-e^{-x})u + \frac{1}{2}\int_0^1 \left(e^{xcos(\pi s)} - 1\right)u(s)ds = x,$$

$$u(0) = 1, \qquad u(1) = 2.$$

In this Example $a(x) = 2 - e^{-x}$, $\lambda = \frac{1}{2}$, $K(x,s) = e^{x\cos(\pi s)} - 1$ and $f(x) = x$. Corresponding to the difference scheme given in (3.26), the denominator function ψ_i^2 becomes

$$\psi_i^2 = \frac{4\varepsilon}{(2-e^{-x_i})}\sinh^2\left(\frac{\sqrt{2-e^{-x_i}}h}{2\sqrt{\varepsilon}}\right).$$

Therefore, the difference scheme for Example 3 becomes

$$-\varepsilon\frac{u_{i-1} - 2u_i + u_{i+1}}{\frac{4\varepsilon}{(2-e^{-x_i})}\sinh^2\left(\frac{\sqrt{2-e^{-x_i}}h}{2\sqrt{\varepsilon}}\right)} + (2-e^{-x_i})u_i + \frac{h}{2}\sum_{j=0}^{N}\eta_j\left(e^{x_i\cos(\pi s_j)} - 1\right)u_j = x_i,$$

$$1 \le i \le N-1, \quad u_0 = 1, \quad u_N = 2,$$

where η_j defined as in (3.26).

The maximum absolute error is calculated using the formula (Doolan et al., 1980)

$$E_\varepsilon^N = \max_{0\le i\le N}|U^N(x_i) - U^{2N}(x_{2i})|,$$

and rate of convergence by formula

$$Roc^N = \log 2\left(\frac{E_r^N}{E_r^{2N}}\right).$$

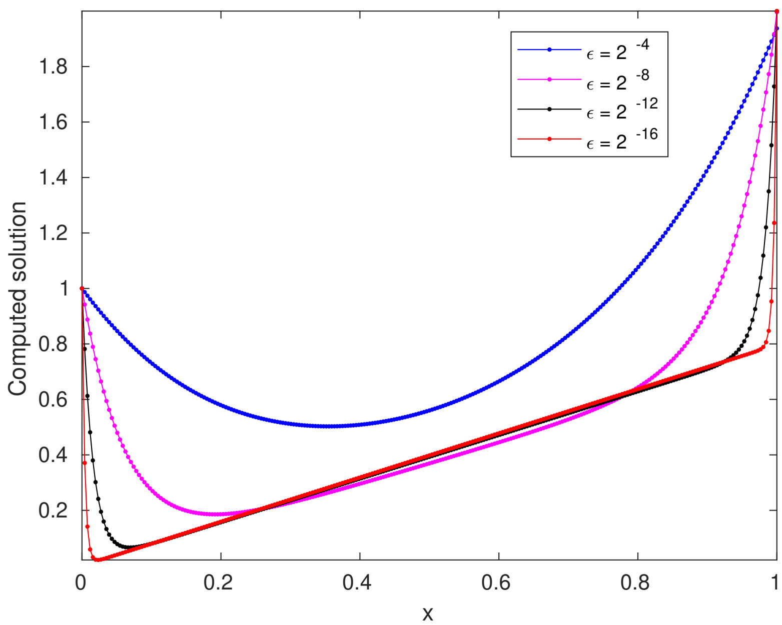

Figure 4.1: Solutions profile for Example 1 with boundary layer formation as ε goes small with mesh size $N = 2^8$.

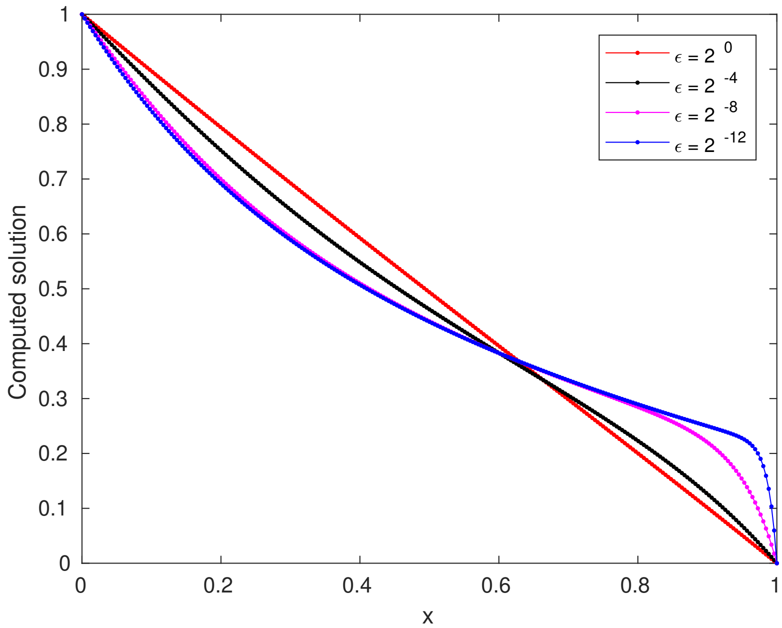

Figure 4.2: Solutions profile for Example 2 with boundary layer formation as ε goes small with mesh size $N = 2^8$.

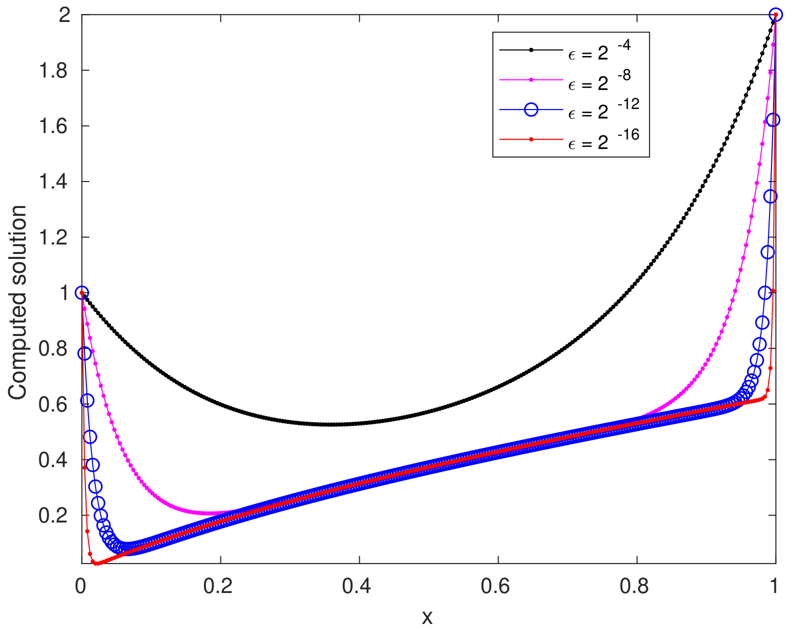

Figure 4.3: Solutions profile for Example 3 with boundary layer formation as ε goes small with mesh size $N = 2^8$.

Table 4.1: Example 1's maximum absolute error and rate of convergence for the proposed method.

$N \rightarrow$ $\varepsilon \downarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
2^0	4.2307e-06	1.0587e-06	2.6468e-07	6.6170e-08	1.6505e-08	4.0694e-09
	1.9986e+00	2.0000e+00	2.0000e+00	2.0033e+00	2.0200e+00	
2^{-2}	1.7772e-05	4.4479e-06	1.1124e-06	2.7811e-07	6.9524e-08	1.7294e-08
	1.9984e+00	1.9994e+00	1.9999e+00	2.0001e+00	2.0072e+00	
2^{-4}	2.9647e-05	7.5264e-06	1.8872e-06	4.7238e-07	1.1812e-07	2.9529e-08
	1.9779e+00	1.9957e+00	1.9982e+00	1.9997e+00	2.0000e+00	
2^{-6}	3.1288e-05	9.3513e-06	2.4711e-06	6.2516e-07	1.5690e-07	3.9252e-08
	1.7424e+00	1.9200e+00	1.9829e+00	1.9944e+00	1.9990e+00	
2^{-8}	1.3865e-05	4.9014e-06	2.2887e-06	7.0094e-07	1.8356e-07	4.6429e-08
	1.5002e+00	1.0987e+00	1.7072e+00	1.9330e+00	1.9832e+00	

Table 4.2: Comparison of results for Example 1's maximum absolute error.

N $\rightarrow$ $\varepsilon \downarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Proposed scheme						
2^0	4.2307e-06	1.0587e-06	2.6468e-07	6.6170e-08	1.6505e-08	4.0694e-09
2^{-4}	2.9647e-05	7.5264e-06	1.8872e-06	4.7238e-07	1.1812e-07	2.9529e-08
2^{-8}	1.3865e-05	4.9014e-06	2.2887e-06	7.0094e-07	1.8356e-07	4.6429e-08
2^{-12}	6.9416e-04	5.9134e-05	4.0396e-06	2.5841e-07	1.6151e-08	2.3062e-08
2^{-16}	4.0163e-03	1.2458e-03	1.8212e-04	1.5518e-05	1.0605e-06	6.7831e-08
Result obtaianed in Amiraliyev et al. (2021)						
2^0	0.00343868	0.00198874	0.00110332	0.00060368	0.00030394	0.00015197
2^{-4}	0.01032126	0.00605257	0.00338123	0.00185003	0.00094445	0.00047551
2^{-8}	0.01125894	0.00660244	0.00368841	0.0020181	0.00103025	0.00051871
2^{-12}	0.011200979	0.00656845	0.00366942	0.00200771	0.00102495	0.00051604
2^{-16}	0.0112049	0.00657075	0.00367071	0.00200842	0.00102531	0.00051622

Table 4.3: Example 2's maximum absolute error and rate of convergence for the proposed method.

N $\rightarrow$ $\varepsilon \downarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
2^0	1.4021e-05	3.5078e-06	8.7725e-07	2.1932e-07	5.4820e-08	1.3692e-08
	1.9990e+00	1.9995e+00	2.0000e+00	2.0003e+00	2.0014e+00	
2^{-4}	8.3355e-05	2.0870e-05	5.2203e-06	1.3052e-06	3.2632e-07	8.1577e-08
	1.9978e+00	1.9992e+00	1.9999e+00	1.9999e+00	2.0001e+00	
2^{-8}	2.0102e-04	5.1032e-05	1.2800e-05	3.2036e-06	8.0104e-07	2.0027e-07
	1.9779e+00	1.9953e+00	1.9984e+00	1.9997e+00	1.9999e+00	
2^{-12}	2.5085e-04	7.6354e-05	1.9909e-05	5.0336e-06	1.2619e-06	3.1570e-07
	1.7160e+00	1.9393e+00	1.9838e+00	1.9960e+00	1.9990e+00	
2^{-16}	2.0675e-04	7.6047e-05	1.9640e-05	5.7644e-06	1.4992e-06	3.7847e-07
	1.4429e+00	1.9531e+00	1.7686e+00	1.9430e+00	1.9859e+00	
2^{-20}	2.1327e-04	1.0804e-04	5.2959e-05	1.9586e-05	3.9418e-06	9.8186e-07
	9.8112e-01	1.0286e+00	1.4351e+00	2.3129e+00	2.0053e+00	

Table 4.4: Comparison of results for Example 2's maximum absolute error.

N $\rightarrow$ $\varepsilon \downarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Proposed scheme						
2^0	1.4021e-05	3.5078e-06	8.7725e-07	2.1932e-07	5.4820e-08	1.3692e-08
2^{-2}	4.0894e-05	1.0233e-05	2.5588e-06	6.3973e-07	1.5993e-07	3.9962e-08
2^{-4}	8.3355e-05	2.0870e-05	5.2203e-06	1.3052e-06	3.2632e-07	8.1577e-08
2^{-6}	1.3785e-04	3.4622e-05	8.6671e-06	2.1673e-06	5.4188e-07	1.3547e-07
2^{-8}	2.0102e-04	5.1032e-05	1.2800e-05	3.2036e-06	8.0104e-07	2.0027e-07
Result obtained in Durmaz and Amiraliyev (2021)						
2^0	0.02882363	0.00729132	0.00183933	0.00046239	0.00011608	0.00002906
2^{-2}	0.02860477	0.00725102	0.0018317	0.00046143	0.00015067	0.00003785
2^{-4}	0.04001304	0.01015697	0.00257469	0.00065085	0.0001643	0.00004139
2^{-6}	0.04331213	0.01100204	0.00279084	0.00070696	0.00017896	0.00004527
2^{-8}	0.04342876	0.01104697	0.00280418	0.00071231	0.00018094	0.00004593

Table 4.5: Example 3's maximum absolute error and rate of convergence for the proposed method.

N $\rightarrow$ $\varepsilon \downarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
2^0	4.5386e-06	1.1365e-06	2.8421e-07	7.1060e-08	1.7782e-08	4.4008e-09
	1.9976e+00	1.9996e+00	1.9998e+00	1.9986e+00	2.0146e+00	
2^{-4}	4.7504e-05	1.1867e-05	2.9661e-06	7.4148e-07	1.8537e-07	4.6342e-08
	2.0011e+00	2.0003e+00	2.0001e+00	2.0000e+00	2.0000e+00	
2^{-8}	2.6857e-04	6.9095e-05	1.7495e-05	4.3794e-06	1.0957e-06	2.7394e-07
	1.9586e+00	1.9816e+00	1.9981e+00	1.9989e+00	1.9999e+00	
2^{-12}	6.6845e-04	3.0691e-04	8.1741e-05	2.0780e-05	5.2171e-06	1.3072e-06
	1.1230e+00	1.9087e+00	1.9759e+00	1.9939e+00	1.9968e+00	
2^{-16}	1.0737e-03	2.5600e-04	1.9205e-04	8.1349e-05	2.1584e-05	5.4813e-06
	2.0684e+00	4.1466e-01	1.2393e+00	1.9142e+00	1.9774e+00	
2^{-20}	1.2196e-03	6.5715e-04	3.0150e-04	6.8690e-05	4.9737e-05	2.0647e-05
	8.9211e-01	1.1241e+00	2.1340e+00	4.6578e-01	1.2684e+00	

Discussions

The error analyses performed in this work reveal that the proposed method is second-order uniformly convergent. This result, summarized in Theorem 3.1, is confirmed by numerical results displayed in tables 4.1, 4.3 and 4.5, where we computed the maximum absolute errors and rates of convergence for different values of mesh size N and perturbation parameter ε for

Examples 1, 2 and 3, respectively. On these tables one can observe that, as ε goes small the maximum absolute error of developed scheme becomes stable and bounded. This indicates that maximum absolute error of the scheme is independent of the perturbation parameter ε, implying that the scheme is uniformly convergent. In Table 4.2, we compared the result of the proposed scheme with the results in Amiraliyev et al. (2021) for Example 1 and in Table 4.4, we compared the result of the proposed scheme with the results in Durmaz and Amiraliyev (2021) for Example 2. As one can observes in these tables results by this method is better than that obtained in Amiraliyev et al. (2021) and Durmaz and Amiraliyev (2021).

In order to show the physical behavior of the given problem, we give plots of the computed solutions for different values of ε. In Figures 4.1, 4,2 and 4.3, the profile of the solutions is given for different values of the perturbation parameter ε as ε goes small.

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In this study, a linear second-order boundary-value problem for SPFIDE has been considered. We have presented a second-order uniformly convergent numerical method for solving this problem, which comprises the non-standard finite difference for differential part and composite Simpson's $1/3$ rule for integral part of the problem on uniform mesh. Stability and convergence analysis of the proposed scheme is proved. The applicability of the proposed scheme is investigated by taking three examples. The effect of perturbation parameter on the solution of the problem are shown by using figures. The method is shown to be uniformly convergent with order of convergence two. The performance of the proposed scheme is investigated by comparing the results with prior studies. It has been found that the proposed method gives more accurate and stable numerical results. Matlab has been used for computations in this thesis.

Recommendations

This thesis specifically focused on the application of nonstandard finite difference method together with composite Simpson's $1/3$ rule to solve linear second-order singularly perturbed Fredholm integro-differential equations on uniform mesh. Nonstandard finite difference method is very interesting and applicable for obtaining stable numerical solution for singularly perturbed problems. Hence, we recommended that, future researchers can extend the nonstandard finite difference method together with appropriate quadrature rule to solve non-linear SPFIDEs.

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