ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

SCHOOL OF APPLIED NATURAL SCIENCE

PROGRAM OF APPLIED MATHEMATICS

PRESSURE POISSEN METHOD FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS USING GALERKIN FINITE ELEMENT

Thesis submitted for the fulfillment of Master of Science in Mathematics

by

YESHIGETA MENGISTE ID:-GSU/0451/06

December, 2017

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APPROVAL SHEET

ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY SCHOOL OF APPLIED NATURAL SCIENCES PROGRAM OF APPLIED MATHEMATICS

As thesis research advisors, we here by certify that we have read and evaluated this thesis proposal under our guidance, by Yeshigeta Mengiste, entitled with **pressure Poisson Method for the incompressible Navier-stokes Equations using Galerkin Finite Element**. We recommended that it can be submitted as fulfillment of the thesis requirement.

Name of student	Signature	Date

As member of the board of examiners of the MSC thesis open defense examination, we certify that we have read, evaluated the thesis prepared by Yeshigeta Mengiste and examined the candidate. We recommend that the thesis is accepted as fulfilling the thesis requirement for the degree of Master of Science in applied mathematics.

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List of Symbols

Re	Reynolds number
u ,v	Component of fluid velocity acting in the x and y direction
$[K_e]$	Element stiffness matrix
K	Stiffness coefficient
$\iint_{\Omega}.$	Integration over a region, Ω
$\iint_{\partial\Omega}.$	Integration over a region, $\partial\Omega$
μ	Fluid dynamic viscosity
P	fluid pressure
τ_{xx}	Normal stresses acting on the fluid particles
τ_{xy}	Shear stresses acting on the fluid particles
N_i	Interpolation function at node i
N_i^s	Shape function at node i
W_i	Arbitrary weighting function at node i
R_{xi} ,R_{yi}	Nodal loading term acting in the x and y direction at node i
n_x ,n_y	Directional cosine acting in x and y direction
∇	Gradient operator
$\nabla.$	Divergence operator
Δ	Laplacian operator
$L^2\boldsymbol{\Omega}$	Space of square integrable function
$H^1\boldsymbol{\Omega}$	Sobolev space of order 1
H_0^1	Space of $H^1\boldsymbol{\Omega}$ function with trace on $\boldsymbol{\Omega}$

List of Abbreviation

MAC	Marker and cell scheme
NSE	Navier-Stokes Equations
PDE	Partial Differential equation
PPE	Pressure Poisson Equation
IBVP	initial boundary value problem

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Abstract

Most of real situations in fluid flow are characterized by the Navier-Stokes equations that are the model of nonlinear partial differential equation. These nonlinearities make most problems difficult or impossible to solve. In this work we examined the incompressible isothermal Navier-Stokes equations based on a primitive variable formulation in which the incompressibility constraint replaced by a pressure Poisson equation. For this the Galerkin finite element method is used to obtain the solution for the incompressible Navier-Stokes equations. It is a numerical technique which represents an equation over a particular region as linear combination of finite number of equation variable values within the region. In this work, Galerkin finite element method is proposed to solve the two dimensional incompressible Navier-Stokes equations with appropriate boundary conditions. The element is formulated directly from the Stokes equation of motion, a special case of the Navier-Stokes equations where the inertia terms are dropped, using the method of weight residuals with Galerkin's criterion applied velocities in two directions and pressure are solved for at all three nodes and solved simultaneously at three corner nodes. For this NSE solved with the help of MATLAB.The numerical results were compared with analytical result and also the velocity contour and pressure contour was also presented .As the result numerical result obtained with minimum numerical error.

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CHAPTER ONE

1. Introduction

Fluid flows are present everywhere in nature and are widely experienced by those people who observe with open eyes. Hence, without fluid motion, life in the form we know it on the earth could not exist. Fluids flows are vital to plants, animals and people living on the earth. Because of this reason, fluids flows are studied by physists. The branch of physics that studies the mechanics of fluid (liquid, gases and plasma) and force on them is called fluid mechanics. It is one of the most complex branches of mechanics. Fluid mechanics has [Jewet,2004] a wide range of applications, including mechanical engineering , geophysics ,astrophysics and biology .Fluid mechanics can be divided in to fluid statics (the study of the fluid at rest) and fluid dynamics (the study of the effect of force on fluid motion). Fluid dynamics is one of the most complex branches of mechanics and the active field of research with many problems that are partly or wholly unsolved. Fluid mechanics can be mathematically complex; can be best solved by numerical method. The solution of fluid dynamics typically involves calculating various properties of fluid such as velocity, pressure, density and temperature as a functional of space and time. To describe this various properties the Navier-Stokes equation are very important. The study of incompressible equations is practical interest and helps researches gain a better understanding of the processes which occur in many industrial applications .Therefore, this work focus on approximation solution of incompressible Navier-stokes equations with continuity equation replaced by pressure Poisson equation. Appropriate boundary conditions are developed for a fully decoupled numerical scheme to recover the pressure. The variational form of the Navier-Stokes equations with pressure Poisson equation will be derived and used in the Galerkin finite element method discretization. The Galerkin finite element is then used to solve the Navier-Stokes equations with Pressure Poisson equation. For this control surface is defined as a closed surface which bounds a volume. The control volume may be fixed in space within the fluid moving through it. Fundamental principles such as mass conservation, conservation of momentum are applied to the fluid inside the control volume and to the fluid crossing the control surface. Therefore, instead of looking at the whole field at once with control volume model we limit our attention to just to the fluid in finite region of the volume itself. Thus computational fluid dynamics is the art of replacing the integral or partial derivatives in Navier-Stokes equation with discretized algebraic forms, which in turn are solved to obtain numbers for the flow field values at discrete points in time and space.

1.1 Back ground of the study

Fluid mechanics has a history of erratically occurring early achievements, then an intermediate era of study fundamental discovers in the 18^{th} and 19^{th} centuries. Archimedes [285-212 B.C] formulated the law of buoyancy and applied them to floating and submerged bodies [Yung,1998], actually deriving a form of the differential calculus as part of analysis .Then Leonard Da. Vinci [1452-1519] derived the equation of conservation of mass in one dimension steady flow. Leonardo was an excellent experimentalist. In 1687, Isaac Newton [1642-1727] postulated his laws of motion and the laws of viscosity of linear fluids now called Newtonian. The theory first yielded to the assumption frictionless fluid, and 18^{th} century mathematicians produced many beautiful solution of frictionless flow problem. Euler developed both differential equation of motion and the integrated form. Rapid advancement in fluid mechanics began with introduction of mathematical fluid dynamics. Inviscid flow was further analyzed by varies mathematician and viscous flow was explored multitude of engineers .Further mathematical justification was proved by Clouded- Louis , Navier and George Gabriel stokes in the Nervier- stokers equation. The NSE named by them to describe the motion of viscous fluid substances and used to determine the velocity vector field that applies to fluid , given some initial conditions [A.segel,Steveenhoven,1986] .They arise from the application of Newton's second law in combination with a fluid stress (due to viscosity)and pressure term .The NSE are the fundamental PDEs that describes the flow of incompressible fluid .Due to non- linear and coupled nature it is difficult to get the analytical solution NSE .The numerical method are important to understand the behavior of Navier-stokes equation. The numerical result can be obtained by applying theory and numerical method.

1.2 Statements of the problem

When we consider the terms in incompressible Navier-Stokes equation a special difficulty arises due to the weak coupling of the velocity and pressure fields. For incompressible fluids, the continuity equation is only function of velocity and not a function of pressure. Only the momentum equations contain pressure terms. Since most the terms in the momentum equation are the function the velocity component it is natural to use these equations to produce to the solutions for the velocity components. The problem is how to obtain the pressure solution since continuity does not contain pressure. A number of different computational methods have been developed to numerically solve the Navier-stokes equations. Several of the procedures have been successful to some extent, and a number of difficulties remain up to now. In the review literature [Gersho, Sani, 2003] the research conducted shows that the incompressible isothermal Navier-stokes equation for Newtonian fluids has difficulty in determining the proper boundary condition. This causes weak coupling between the pressure and the velocity. Therefore, by determining the proper boundary condition and using pressure Poisson equation (PPE) we are able to overcome the weak coupling between velocity and pressure. The study will fill this gab by adding boundary condition for studying the Navier-Stokes equation with continuity equation replaced by pressure Poisson equation in order to fully decouple and also we applying Galerkin Finite Element Method to solve the Navier-Stokes equation with suitable primitive variable formulation. The study answers the following questions:

1, To what extent is the result accurate by implementing proper boundary condition for pressure Poisson Equation?

2, Can it be possible to obtain the solution of the velocity and the pressure by implementing Galerkin finite element method to solve Navier-Stokes equations?

3, what strategies have been used to overcome the weak coupling between velocity and pressure when we implement proper boundary condition for pressure Poisson equation?

1.3 Objective of the Study

1.3.1 General Objective

- ❖ To obtain the numerical solution of the incompressible Navier- stokes equation using Galerkin finite element method with minimum numerical error.

1.3.2 Specific objective

- ❖ To get moderate accuracy by implementing proper boundary condition.

- To solve the solution of the velocity and pressure given some initial condition with numerical technique by implementing the Galerkin finite element method to solve Navier-Stokes equations.

- Using proper boundary condition to overcome the weak coupling between pressure and velocity.

1.4 Significant of the study

- To understand the distribution of pressure and velocity within flow fields.
- Used to predict and control the fluid motion.
- For the application in industrial fluid flow.
- For the application of the Galerkin finite element method.
- For future progress in the study of the research area.

1.5 The scope of the study

To limit the number of unknowns and to minimize the computational requirements for carrying out this study, the method of this study is restricted to incompressible isothermal Navier-Stokes equation, stokes equation, using Galerkin Finite Element in primitive variable flow in two-dimension.

1.6 Limitation of the study

This study did not come to an end without limitation. Some of the possible limitation of the study will be:

- If insufficient resolution of mesh is used the result will be incorrect. The mesh should also consist of high quality elements which mean that the triangle should be close to equilateral as possible.
- The mathematical analysis performed to obtain numerical estimates required numerical assumptions to be made. The assumption include homogeneous material properties (viscosity, density, etc) that the model domain Ω can be completely resolved by the element with zero discretization error. There is zero error in associated with boundary condition .These, we emphasize very strongly that the result of this work will be by no

means completely universal. However, violating these assumption results lower error estimates.

- The size of memory of the PC required will affect the result
- If the problem becomes highly convective stability problem occur and also if the mesh doesn’t resolve the smallest vortices, the solution will be incorrect.

CHAPTER TWO

2 Review of Related Literature

2.1 Derivation of Navier-Stokes Equations

2.1.1 Conservation of Mass

Consider a fixed domain occupied by a fluid inside a closed surface $\partial\boldsymbol{\Omega}$ enclosing $\boldsymbol{\Omega}$. The velocity of the fluid can be described by the velocity field u(x,t) and the density $\boldsymbol{\rho}$ (x, t) . The total mass of the fluid inside $\boldsymbol{\Omega}$ is$\int_{\Omega} \rho d_{\Omega}$ .The rate of change of mass inside $\boldsymbol{\Omega}$ is given by

$$\frac{d}{dt}\int_{\Omega} \rho d_{\Omega}=\int_{\Omega} \frac{\partial\rho}{\partial t} d_{\Omega} \tag{1}$$

The change of mass due to the fact that some of the fluid parcel flow into or out of the domain which is characterized by surface integral of flux and given by :

$$\int_{\partial\Omega} \rho u.\boldsymbol{n} d_{s} \tag{2}$$

Were **n** is the out word unit norm vector surface. The negative sign chosen since **n** is the out ward norm direction and u **.n**> 0 means the loss of mass .The principle of conservation of mass state that:

The time rate of change of mass in fixed domain $\boldsymbol{\Omega}$ equals that amount of fluid flowing through $\boldsymbol{\partial\Omega}$.If there are no sources or losses of fluid with in sub domain of the flow under consideration mass remains constant. Mass conservation mathematically expressed in integral from for fixed domain

$$\int_{\Omega} \frac{\partial\rho}{\partial t} d_{\Omega}= -\int_{\partial\Omega} \rho u.\boldsymbol{n} d_{s} \tag{3}$$

Applied to the control volume, the fundamental physical principle that the mass is conserved means net mass flow out of the control volume through the surface is equal to the time rate of decrease of mass inside the control volume.

When the divergence theorem may be applied to surface integral [Henningson, Berggren, 2009] changing it into volume integral

$$\int_{\Omega}\left(\frac{\partial \rho}{\partial t} + \nabla\,.\,(\rho u)\right) = 0 \qquad (4)$$

Since this is valid for arbitrary domain **Ω** it follows that

$$\frac{\partial \rho}{\partial t} + \nabla.\,(\boldsymbol{\rho}\, \boldsymbol{u}) = \mathbf{0} \qquad \mathbf{(5)}$$

Application of the principle to an element of the region (called control volume) results in the above equation is called the continuity equation and expresses the differential form of principle of conservation of mass. When density change following a fluid particle is negligible, the continuum is termed incompressible and $\frac{\partial \rho}{\partial t} = 0$. for steady-state condition the continuity Equation reduced to;

$$\nabla\,.\,u = 0 \qquad (6)$$

This is often referred to as the incompressibility condition. The incompressibility condition expresses the fact that volume change is zero for an incompressible fluid during its deformation and these flows are therefore usually called incompressible flows.

2.1.2 Conservation of momentum

Principle of conservation of linear momentum commonly known as Newton's second law .According to this law, the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all external forces sum acting on the particles of the fluid. Newton's second law can be written as F=ma. This is a vector relation, and hence can be split into three scalar relations along x, y and z axes. Considering only the x- component of Newton's second law is:.

$$F_X = m_x a_x \qquad (7)$$

Where F_X and a_x are scalar x- components of the force and acceleration, respectively. This law states that when force is applied to the moving fluid element equals its mass times the acceleration of the element. Consider a general flow field as streamlines and imagine a closed volume drawn within a finite region of the flow. This volume may be moving with the fluid such that the same fluid particles are always inside it as shown in the fig.1 below. The fundamental physical principles are applied to the fluid inside the control volume and the fluid crossing the control surface.

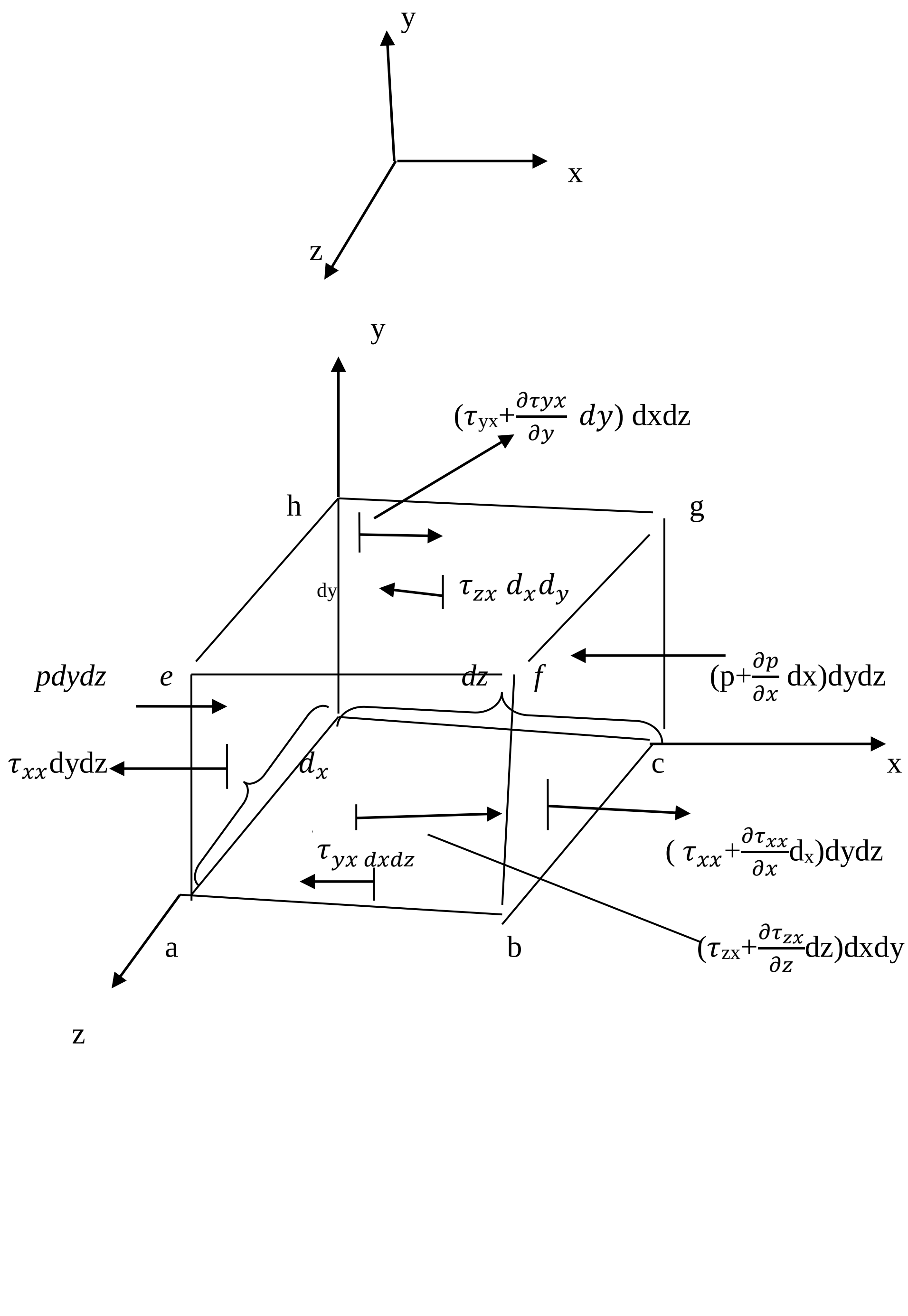

Figure 1.Infinitesimally small, moving fluid element. Only the force in the x-direction is shows model used for the derivation of the x-component of the momentum equation

Moving fluid experiences a force in the x- direction as shown in the fig – 1 above; there are two sources for this force. The first is the body force, which are acting on the volumetric mass of the fluid element. The Second is the surface force, which act directly on the surface of the fluid element. They are due to only two sources: (a) the pressure distribution acting on the surface, imposed by the outside fluid surrounding the fluid element, and (b) the shear and normal stress distributions acting on the surface, also imposed by outside fluid pulling or pushing on the surface by means of friction. If we denote the body force per unit mass acting on the fluid element is f with f_x as its x-component .The volume of the fluid element is $(d_x d_y d_z)$.Hence

$$\text{The body force acting in the x-direction} = f_x(d_x d_y d_z) \tag{8}$$

The shear and normal stresses in fluid are related to the time rate of change of the deformation of the fluid elements [J.Chorin, 1987].The shear stress denoted by τ_{xy} ,is related to the time rate of change of the shearing deformation of the fluid element, where as the normal stress, denoted by τ_{xx} ,is related to the time rate of change of volume of the fluid element. On face $abcd$, the only force in the x- direction is that due to shear stress,$\tau_{yx} d_x d_y$.Face $efgh$ is a distance d_y above face $abcd,$:hence shear force in x-direction on face $efgh$ is $[\tau_{yx} + (\partial \frac{\tau_{yx}}{\partial y}) d_y]\ d_x d_z$. On the bottom face τ_{yx} is to the left(the negative x-direction) whereas on the top face $\tau_{yz} + (\partial \frac{\tau_{yx}}{\partial y}) d_y$ is to the right(to positive x-direction).The direction of all the other viscous stresses including τ_{xx} ,can be justified in a like fashion. Specifically on face $dcgh$, τ_{zx} acts in the negative x-direction whereas on face $abfe$ $[\tau_{zx} + (\partial \frac{\tau_{zx}}{\partial z}) d_z$ acts on positive x direction .On face $adhe$,which in perpendicular to the x-axis ,the only force in the x- direction are the pressure force $\mathrm{P} d_y d_z$,which in always acts in the direction into the fluid element, and $\tau_{xx} d_y d_z$, which is in the negative x-direction. For moving fluid element the equation can be written as:

$$\text{Net surface force in x-direction} = [\mathrm{P}- (\mathrm{P} + \frac{\partial P}{\partial x} d_x\ )] d_y d_z + \left[\left(\tau_{xx}\ +\ \partial \frac{\tau_{xx}}{\partial x} d_x\right) - \tau_{xx}\ \right] d_y d_z + [\left(\tau_{yx} + \frac{\tau_{yx}}{\partial y} d_y\right) - \tau_{yx}] d_x d_z + \left[(\tau_{zx} + \partial \frac{\tau_{zx}}{\partial z} d_z\right) - \tau_{zx}\ ] d_x d_y \tag{9}$$

The total force in the x-direction F_x, is given by the same Esq.[8] and[9].Adding, and cancelling terms, we obtain:

$$F_x = [-\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}] d_x d_y d_z + \mathrm{P} f_x \quad (10)$$

Equation [10] represents the left hand side of Eq.[7]. The mass of the fluid is fixed and is equal to:

$$\mathrm{m}= \rho d_x d_y d_z \quad (11)$$

And the component of the acceleration in the x-direction, denoted by a_x, is simply the time rate of velocity u and equal to:

$$a_x = \frac{Du}{Dt} \quad (12)$$

Combining Esq.[7] and [10] to [12], can be obtain:

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \mathrm{P} f_x \quad (13a)$$

This is the x-component of the momentum equation for viscous flow. In similar fashion, the y and z components can obtain as:

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \mathrm{P} f_y \quad \text{.............................} \quad (13b)$$

And $$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \mathrm{P} f_z \quad \text{................................} \quad (13c)$$

Equations [13a] to [13b] are the x, y, and z component respectively, of the momentum equation. They are partial differential equation obtain directly from an application of the fundamental physical principle to an infinitesimal fluid element. These equations are called the Navier-Stokes equations in honor of two men -- the Frenchman M.Navier and Englishman G.Stokes – who independently obtained the equation in the half of nineteenth century [Johnston,2007]

2.1.3Compressible Navier-Stokes Equations

The Navier-Stokes equations can be derived from the basic conservation of the momentum and continuity equations applied to properties of fluid. Writing the left hand side of Eq.[13a] in terms of the definition of the substantial derivative, we have :

$$\rho \frac{Du}{Dt} = \rho \frac{\partial u}{\partial t} + \rho V \,.\nabla u \qquad (14)$$

Also expanding the following derivative,

$$\frac{\partial(\rho u)}{\partial t} = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

And rearranging the equation we get:

$$\rho \frac{\partial u}{\partial t} = \frac{\partial(\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t} \qquad (15)$$

The vector identity for the divergence of product of scalar times a vector, we have:

$$\nabla.(\rho u V) = u\nabla.(\rho V) + (\rho V).\nabla u$$

$$\text{Or} \quad V\rho.\nabla \mathrm{u} = \nabla.(\rho \mathrm{u} \mathrm{V}) - \mathrm{u}\nabla.(\rho \mathrm{V}) \qquad (16)$$

Substituting Esq.[15] and [16] into [14]

$$\rho \frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t} - u\nabla.(\rho V) + \nabla.(\rho \mathrm{u} \mathrm{V}) = \frac{\partial(\rho u)}{\partial t} - u\left[\frac{\partial \rho}{\partial t} - \nabla.(\rho V)\right] + \nabla.(\rho u V) \qquad (17)$$

The term in the bracket in equ.[17] is simply the left hand side of continuity equation as equ.[5]; hence it the term in the brackets is zero. Thus equ.[17] reduces to:

$$\rho \frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} + \nabla.(\rho u V) \qquad (18)$$

Substituting Equation [18] into equation [13a]

$$\rho \frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} + \nabla.(\rho u V) = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \mathrm{P} f_x \qquad (19a)$$

Similarly, Esq. [13b] and [13c] can be expressed

$$\rho \frac{Dv}{Dt} = \frac{\partial(\rho v)}{\partial t} + \nabla.(\rho v V) = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \mathrm{P} f_y \qquad (19b)$$

And

$$\rho \frac{Dw}{Dt} = \frac{\partial(\rho w)}{\partial t} + \nabla.(\rho w V) = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial yz}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \mathrm{P} f_z \tag{19c}$$

For Newtonian fluid, Stokes in 1845 obtained:

$$\tau_{xx} = \lambda(\nabla.V) + 2\mu \frac{\partial u}{\partial x} \tag{20a}$$

$$\tau_{yy} = \boldsymbol{\lambda}(\nabla.V) + 2\mu \frac{\partial v}{\partial x} \tag{20b}$$

$$\tau_{zz} = \lambda(\nabla.V) + 2\mu \frac{\partial w}{\partial x} \tag{20c}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{20d}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{20e}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{20f}$$

where μ the molecular viscosity coefficient and $\boldsymbol{\lambda}$ is the second viscosity coefficient. Stokes made the hypothesis that $\boldsymbol{\lambda}=-\frac{2}{3}\mu$ which is frequently used but which has still not been definitely confirmed to the present day [Aderson, 2012].Substituting equation [20] into equation [19] we obtain the x-component of compressible Navier-Stokes equation become:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\rho u^2}{\partial x} + \frac{\rho uv}{\partial y} + \frac{\rho uw}{\partial z} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}(\lambda(\nabla.V) + 2\mu \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}[\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}[\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})] + \rho f_x \tag{21}$$

2.1.4 Incompressible Navier-Stokes Equations

The incompressible Navier-Stokes equations can be obtained from compressible form simply by setting density equal to a constant and also with further assumption that the fluid dynamics viscosity is constant throughout the flow ,Esq.[13a] to [13c] combined with Esq.[20a] to [20f] becomes:

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho \frac{Du}{Dt} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \rho f_x \tag{22}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu\frac{\partial}{\partial x}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + 2\mu\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial}{\partial z}(\frac{\partial w}{\partial y} + \frac{\partial y}{\partial z}) + \rho f_y \tag{23}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu\frac{\partial}{\partial x}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}) + \mu\frac{\partial}{\partial y}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) + 2\mu\frac{\partial^2 w}{\partial x^2} + \rho f_z \tag{24}$$

The fact that $\nabla.V = 0$ for incompressible flow allows a further reduction of Esq.[22] to [24] as follows:

$$\nabla.V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{25}$$

Rearranging Eq.[25],we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \tag{25a}$$

Differentiating Eq.[25a] with respect to x ,we obtain:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 w}{\partial x \partial z} \tag{26}$$

Adding $\frac{\partial^2 u}{\partial x^2}$ to both side of Eq.[26] and multiply by μ we obtain:

$$\partial\mu\frac{\partial^2 u}{\partial x^2} = \mu\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial^2 v}{\partial x \partial y} - \mu\frac{\partial^2 w}{\partial x \partial z} \tag{27}$$

Substituting Eq.[27] for the second term on the right of Eq.[22] and cancelling terms the x-momentum equation for viscous, incompressible flow becomes:

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial w^2}\right) + \rho f_x$$

Or

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\nabla^2 u + \rho f_x \tag{28}$$

Where $\nabla^2 u$ is the Laplacian of x- component of velocity, u equation.[23] and [26] can be obtain in similar fashion .

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu\nabla^2 v + \rho f_y \quad (29)$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu\nabla^2 w + \rho f_z \quad \text{…………………………………..} \quad (30)$$

The vector notation of incompressible Navier-Stokes Equations from Equations [28] to [30] can be written in the form:

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \mu \nabla^2 \vec{v} + \rho \vec{f}$$

Or $$\rho(\frac{\partial \vec{u}}{\partial t} + \vec{u}.\nabla\vec{u}) = -\nabla P + \mu \nabla^2 \vec{v} + \rho \vec{f}$$

Dividing both side by ρ we obtain

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}.\nabla\vec{u} = -\frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2 \vec{v} + \vec{f}$$

Thus Navier-Stokes equations composed of four terms. These are:

1, The left side of the equation is the force on each fluid particles

2, The pressure term ($-\nabla P$) It is pressure gradients of surface forces, analogous to stresses in solids. The pressure gradient arises from normal stress that turn up in almost all situations. It prevents motion due to normal stresses. The fluid presses against itself and keeps it from shrinking in volume .This may be thought of as how the particles move as pressure changes, specially the tendency to move away from areas of higher pressure.

3, Viscous term ($\mu\nabla^2 u$)

Viscous force causes motion due to horizontal friction and shear stresses. Conventionally represents the shear effect for incompressible flow. The shear stress causes turbulence and viscous flow. The turbulence is the result of shear stress.

4, The body force (f)

The so called body forces are those which apply to entire mass of the fluid element such forces are usually due to external fields such gravity, centrifugal force .These force act on every single fluid particle.

The continuity equation and the Navier-Stokes equation are self-contained; they are four equations for the four dependent variable u, v, w and P. Though the assumption of density is constant and μ is constant, the energy equation has been completely decoupled from the analysis.

The implication here is that the continuity and momentum equations are all that are necessary to solve for velocity and pressure fields in an incompressible flow, and that is a given problem involves heat transfer and hence temperature gradients exists in the flow .The temperature field can be obtained directly from energy equation after the velocity and pressure fields are obtained. This study does not deal with a temperature field. By assuming temperature is constant.

2.2 viscosity

Viscosity is a measure of the resistance of a fluid to a stress and differs among materials that may have similar properties such as density and weight. It is a fluid property that relates the shearing stress or pressure per area to the velocity gradient. If the shearing stress is linearly related to the velocity gradient of the fluid is known as Newtonian where as if the shear stress is not linearly related to the velocity gradient, the fluid is known as non-Newtonian. Shear stress which always opposes the motion tend to retard the fluid elements..So does the pressure gradient when it is adverse. If on the other hand that pressure gradient is favorable and sufficiently large it can overcome the retarding shear stress and causes the fluid element to accelerate. For Newtonian fluid the stress is proportional to the rate of deformation (the change velocity in the direction of stress) .

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad .(31)$$

The proportionality constant μ is called the viscosity of the fluid and defines how easily the fluid flow when subjected to the body force.

2.3 The Navier-Stoke equations

The motion of fluid is governed by the global laws of the conservation of mass momentum, and energy. These equations consist of a set of nonlinear partial differential equations in terms of the velocity components temperature, and pressure. The equations of motion resulting from the application of linear momentum principle are known as the Navier-Stokes equations. When the temperature effects are not important the energy equation is uncoupled from Navier-Stokes equations. These for isothermal flows, only the Navier-Stocks equations and the continuity equation are solved. These equations are an incompressible, Newtonian fluid and have been written almost one hundred eighty years ago. In fact, they were proposed in 1822 by French engineer C.M.L.H.Navier upon the base of a suitable molecular model [K.Jain, 2014]. It is interesting to observe ,however, the law of interaction between the molecules postulated by Navier were shortly recognized to be totally inconsistent from physical point of view for several materials and, in particular, for liquid. It was only more than twenty years later that the same equations were re derived by 26 years old G.H.Stokes [1845] in a quite general way, by means of the theory of continua. In the case was the fluid subject to the action of the body force f. The equations governing the steady, viscous flow of an incompressible Stokesian fluid can be written in alternative forms having respectively, stream function and vorticity, and velocity and pressure, as dependent variables [D.J.Ascheson, 1990]. The primitive variable formulation consists in two dimensions of the two momentum equations and a zero velocity- divergence constraint representing mass conservation. When a strait forward finite elements solution is attempted for these problems a singular matrix is frequently encountered [Brasses, 2007]. The standard procedure for avoiding this problem is to reduce the order of the pressure interpolation to one less than that used for the velocity components. Although this is mathematically expedient, the effect has not been clearly established. It has been argued [Chiang,2007] that the reduced order for pressure is, in fact, consistent with the equation of motion but these arguments are based on a linearization of the acceleration terms in the momentum equation. An alternative approach to overcome the pressure interpolation problem by utilizing the pressure Poisson equation is discussed in [Rempfer, 2005]. This offers the potential for overcoming the problems associated with the weak coupling between the pressure and velocity fields by placing pressure in a more dominant role the equation is derived by taking the divergence of vector momentum equation. The incompressible Navier-Stokes equation needs boundary and initial conditions in order for a solution to be possible [Henningson, 2009]. In primitive variable (u, p), the initial value problem for incompressible Navier- stokes equation is using vector notation;

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}.\nabla\vec{u} = -\frac{\nabla P}{\rho} + \nu\nabla^2\vec{u} + \vec{f} \qquad \text{for } \mathbf{x}\in \boldsymbol{\Omega} \tag{32}$$

$$\nabla.\vec{u} = 0 \qquad , \text{for } \mathbf{x}\in \boldsymbol{\Omega} \ , \mathbf{t} > 0 \qquad .(33)$$

With the following boundary conditions and initial conditions

$$\text{B}\,(\vec{u},\text{p}) = \text{g} \qquad \text{for} \quad \mathbf{x} \in \partial\boldsymbol{\Omega} \ ,\text{t} > o \tag{34}$$

DQ(x,o)=q_0(x) for **x**∈ $\boldsymbol{\Omega}$,t >0

Where zero net flux through the boundary $\int_{\Omega} n.gd_A = 0$ **n** is the outward unit normal on the boundary and d_A is the area element on the boundary .This equation is the consistent condition for g ,since an incompressible fluid must have zero net flux through the boundary. In this IBVP, **x**=(x,y,z) is vector containing the Cartesian coordinates in physical space $\mathcal{P}, \boldsymbol{\Omega}$ is a bounded domain in $\mathcal{P}\in \mathfrak{R}^N$ (N=1,2,3) , $\partial\mathcal{D}$ is the boundary of the domain $\boldsymbol{\Omega}$, t is physical time, $\vec{u}$=(u,v,w) is the vector containing the velocity field in $\mathcal{P}$, p is the pressure, .ν is kinematic viscosity which is equal to $\mathcal{V}=\frac{\mu}{\rho}$, B is boundary operator, g is boundary data and q_0 formulation of the governing equation in primitive variables. The system of equation from [32] up to [34] is called the velocity divergence. Several mathematical properties for system have been deeply investigated over the years and is still the object of profound researcher. However after more than one hundred seventy years from their formulation the fundamental problem related to them remain still unsolved [Niyogs,Chakrabartty,2013] .So far, this problem is only partially solved, in three dimensions it is known that smooth solutions to the incompressible equations exists for short time and depends continuously on the initial data .It is a major open problem in fluid mechanics to prove or to disprove that the solutions for incompressible equation exist for all time..In fact there is a $ 1,000,000 prize for the person that can develop a theory that can satisfyingly explain their behavior in 3 space dimensions. In two dimension solutions are known to exist for all time, for smooth viscous and Inviscid flow. Beginning with Pioneer Marker and Cell(MAC) scheme [R.Temam,1983] ,enforcement of the incompressibility constraint equ.[33] has always been a right central theme in development of numerical methods for Navier-Stokes equation MAC scheme is the direct discretization of equ.[32]using second order finite differences implemented on a staggered grid with explicit time treatment of non linear convection and viscous terms and implicit time treatment of the pressure term in addition to enforcing incompressibility. The key success of the scheme is its efficient, which result from the decoupling of the computation of the momentum and kinematic equation .This can be accomplished by applying the incompressibility constraint to the discretized momentum equations, resulting in discrete Poisson equation for the pressure. However beside the restriction of to the second order finite difference method for simple domains the limitation to the MAC scheme in the original form if the viscous term treated implicitly the decoupling process no longer possible .Of course in the intervening years much work has gone into addressing the shortcoming of MAC scheme .another well-known method to solve to system Navier-Stokes equations is the projection method ,introduced in late 60s by [Chorin, Temam,1993],which allows implicit treatment of viscous term while retaining the efficiency of the splitting scheme in which an intermediate velocity is first computed and then projected on to the space of incompressible vector fields by solving a Poisson equation for the pressure. In case of explicit treatment of convective and implicit treatment of the viscous terms, which introduce a splitting error .The other method for solving the Navier-Stokes equation is vortex-stream function method. This method decouples the equations, but has dimensional limitations.

2.4 The Pressure Poisson Equation

In this approach the incompressibility constraint for the flow velocity is replaced by a Poisson equation for the pressure. This then allows an extra boundary condition which must be selected so that incompressibility is maintained by the resulting system .Adding $\nabla . u = 0$ as boundary condition yields a system of equation that is equivalent to the Navier-Stokes equations. Therefore, Pressure Poisson Equation (PPE) is derived from the momentum equation by taking divergence of momentum and applies the incompressibility condition to eliminate the viscous term and the time derivative term .Thus taking divergence of momentum equation [32] and divergence equation [33] we derive the Pressure Poisson Equation as follow:

$$\boldsymbol{\rho} \left(\frac{\partial u}{\partial t} + \overrightarrow{v} . \nabla\vec{v}\right) = -\nabla P + \mu \nabla^2 u + f$$

$$\boldsymbol{\rho}\nabla . \left(\frac{\partial u}{\partial t} + \overrightarrow{v} . \nabla\vec{v}\right) = \nabla . (-\nabla P + \mu \nabla^2 u + f)$$

$$\boldsymbol{\rho} \left(\frac{\partial \nabla . u}{\partial t} + \nabla . (\overrightarrow{u} . \nabla \mathrm{u}\right) = -\nabla^2 P + \mu \nabla^2 (\nabla . u) + \nabla f$$

$$\rho\nabla . (\overrightarrow{u} . \nabla\vec{u}) = - \nabla^2 P + \nabla f$$

$$\nabla^2 P = \nabla f + \rho\nabla . (\overrightarrow{u} . \nabla\vec{u})$$

$$\nabla^2 P = \nabla (f + \rho (\overrightarrow{u} . \nabla \mathrm{u})$$

$$\Delta p = \nabla . (\vec{f} - \rho(\vec{u} . \nabla)\vec{u})$$

This gives the new system of equations

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\,.\nabla\vec{u} = -\,\frac{\nabla P}{\rho} + \nu\,\nabla^2\vec{u} \qquad , \text{for} \quad \mathbf{x} \in \Omega \quad , t > 0 \qquad (35)$$

$$\Delta\, P = \nabla.\,(\vec{f} - (\vec{u}.\nabla)\vec{\mathrm{u}}) \qquad , \text{for} \quad \mathrm{x} \quad \in \Omega \quad , t > 0$$

With the following boundary and initial conditions;

$$\mathrm{B}\,(\vec{u},\mathrm{p})=\mathrm{g} \qquad , \text{ for} \quad \mathrm{x} \quad \in \partial\boldsymbol{\Omega} \quad , \mathrm{t}>0 \qquad (36)$$

where ,

$$\nabla.\vec{u} = 0 \qquad , \text{for x} \in \partial\boldsymbol{\Omega} \;, \mathrm{t}>0 \qquad (37)$$

$$\mathcal{D}\mathrm{Q}(\mathrm{x},\,0) = q_0\,(\mathrm{x}) \qquad \text{for} \quad \mathbf{x}\in \boldsymbol{\Omega} \quad , \mathrm{t} >0$$

The system of equation [35-37] is called velocity pressure formulation or pressure Poisson equation of the governing in primitive variable. Equation [35] implies that the pressure can be calculated provide the velocity field is known. This is the form of the equation that will be discretized in this thesis by adding extra boundary condition using equation [37]. Due to the complex nature of NSE analytical solution range from difficult to practically impossible.

2.5 Galerkin Finite Element method

The finite element method is very powerful and flexible numerical approach for solving partial differential equation. It was first proposed in 1943 by German mathematician Richard Courant who solved the Poisson equation based on minimization of functional over piecewise linear approximations on sub regions. The finite element method may be considered as a general discretization tool [Reday,2008] for partial differential equation .The main reason to use FEM is its ability to tackle relatively easily as a constructive way to create basic function, the integral, in a relative simple way. And the region $\mathcal{D}$ is subdivided into simple elements. In $\mathfrak{R}^1$ the elements are lines, in $\mathfrak{R}^2$ usually triangles or quadrilaterals, in $\mathfrak{R}^3$ tetrahedral and hexahedral are very popular. The subdivision of the region in elements is performed by a so-called mesh generator. In each element a number of nodal points are chosen and the unknown function is approximated by polynomial. These polynomial approximations implicitly define the basic function. The finite element method is a Galerkin approximation in which the space is given by piecewise polynomial. In this study the Navier-Stokes equations will be examined with the continuity equation replaced by pressure Poisson equation. Appropriate boundary conditions will be developed for the pressure Poisson equation, which allow for fully decoupled numerical scheme to recover the pressure. The variational form of the Navier-Stokes equations with the pressure Poisson equation will be derived and used in Galerkin finite element method discretization. The Galerkin finite element method will be used to solve the Navier-Stokes equation with the pressure Poisson equation. In the Galerkin's method, approximating or trial function are considered to be the weighting functions. Consider a system with homogeneous boundary conditions

$$L(\emptyset)=g \quad \text{in } \boldsymbol{\Omega} \tag{38}$$

An approximate function that satisfies these condition can be used such that

$$\emptyset \approx \overline{\emptyset} = \sum_{i=1}^{n} N_i \, \Psi_i \tag{39}$$

Here N_i are the assumed functions and $\Psi_i$are either the unknown parameters or unknown functions of the independent variables. Substituting this approximating function in Esq.[38] may produce residual as

$$R = L(\overline{\emptyset}) - g \neq 0 \tag{40}$$

In Galerkin's method, the error is made orthogonal to the same trial function Ψ_i i.e.

$$\int R w_i \, d_\Omega = 0 \quad \text{where } w_i = N_i \tag{41}$$

This method yields n linear equations to determine n values of Ψ_i.Galerkin's method may produce symmetrical coefficients in many cases since the weighting function and approximating function are the same.

2.6 Discretization of incompressible Navier-Stokes Equation

Since we are developing Galerkin finite element method the choice of the weight functions is restricted to the spaces of the approximation functions used for pressure and velocity fields. Thus, it is clear that the dependent variable of u, v, and P are approximated by expansion of the form:

$$u(x,y,t) = \sum_{i=1}^{n} \psi(x,y) u_i t \qquad (42)$$

$$v(x,y,t) = \sum_{i=1}^{n} \psi(x,y) v_i \mathrm{t} \qquad (43)$$

$$P(x,y,t) = \sum_{i=1}^{m} \emptyset(x,y) P_i \qquad (44)$$

Where **ψ** and Ø are vectors of interpolation functions and v are vector of nodal values of velocity components and P are the vectors of nodal values of pressure.

2.6.1 Stokes Equations in finite element matrix from

It is necessary at this point to express the Stokes and continuity differential equations as linear combination of u_i, v_i, p_i. the general form of these approximations has been given by equations [45] and [46]

$$\emptyset(x,y) \approx \emptyset_A(x,y) = \sum_{i=1}^{n} N_i \emptyset_i \qquad (45)$$

$$\frac{\partial \emptyset}{\partial x}(x,y) \approx \frac{\partial \emptyset_A}{\partial x}(x,y) = \sum_{i=1}^{n} \frac{\partial N_i}{\partial x} \emptyset_i \qquad (46a)$$

$$\frac{\partial \emptyset}{\partial x}(x,y) \approx \frac{\partial \emptyset_A}{\partial y}(x,y) = \sum_{i=1}^{n} \frac{\partial N_i}{\partial x} \emptyset_i \qquad (46b)$$

In order to apply the method of weighted residuals W_i^I and $W_i^{II}$are assumed to be two weight functions. The method of weighted residuals applied to the Stokes and continuity equations yields.

$$\int_\Omega {W_i}^{II} \left[\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial p}{\partial x}\right] d_x d_y \qquad (47a)$$

$$\int_\Omega {W_i}^{II} \left[\mu \ \frac{\partial v}{\partial x^2} + \frac{\partial^2 v}{\partial y} - \frac{\partial p}{\partial y}\right] d_x d_y \qquad (47b)$$

$$\int_\Omega {W_i}^{I} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] d_x d_y \qquad (49)$$

Integrating by parts over the domain, Ω, and around the boundary $\partial\Omega$ for equation [47] yields.

$$\int_\Omega \left[\mu \ \left(\frac{\partial {W_i}^{I}}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial {W_i}^{II}}{\partial y}\frac{\partial u}{\partial y}\right) - \frac{\partial {W_i}^{II}}{\partial x} p\right] d_x d_y - \int_\Lambda {W_i}^{II} I_x \, d_{\partial\Omega} = 0 \qquad (50a)$$

$$\int_\Omega \left[\mu \ \left(\frac{\partial {W_i}^{II}}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial {W_i}^{II}}{\partial y}\frac{\partial v}{\partial y}\right) - \frac{\partial {W_i}^{II}}{\partial x} p\right] d_x d_y - \int_\Lambda {W_i}^{II} I_y \, d_{\partial\Omega} = 0 \qquad (50b)$$

$$I_x = \mu\left(n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}\right) - n_x p \qquad (51a)$$

$$I_y = \mu\left(n_x \frac{\partial v}{\partial x} + n_y \frac{\partial v}{\partial y}\right) - n_y p \qquad (52b)$$

And n_x and n_y represent direction cosines to the surface $\partial\Omega$. Integration by part is not carried out over the continuity equation at this point. The following numerical approximations are made.

$$u = [N_1^{II} + N_2^{II} + \cdots + N_n^{II}]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = N^{II}\{u\} \qquad (53a)$$

$$v = [N_1^{II} + N_2^{II} + \cdots + N_n^{II}]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = N^{III}\{v\} \qquad (53b)$$

$$\mathrm{p} = [N_1^{I} + N_2^{I} + \cdots + N_n^{I}]\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = [N^{I}]\{p\} \qquad (53c)$$

Following that the derivatives are expressed as:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[N^{II}]\{u\} \qquad (54a)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[N^{II}]\{u\} \qquad (54b)$$

$$\frac{\partial x}{\partial x} = \frac{\partial}{\partial x}[N^{II}]\{v\} \qquad (54c)$$

$$\frac{\partial x}{\partial y} = \frac{\partial}{\partial y}[N^{II}]\{v\} \qquad (54d)$$

Where $[N^{I}]$ and $[N^{II}]$ may be shape functions of different order. Along with making these substitutions in equations [49] and [50] Galerkin's criterion is also applied so that

$$W_i^{\,I}(x,y) = N_i^{\,I}(x,y) \qquad (55a)$$

$$W_i^{II}(x,y) = N_i^{II} \tag{55b}$$

The resulting sets of equation are:

$$\int_{\Omega}\left[\mu\left(\frac{\partial N_i^{II}}{\partial x}\frac{\partial[N^{II}]}{\partial x}+\frac{\partial N_i^{II}}{\partial y}\frac{\partial[N^{II}]}{\partial y}\right)\{u\}-\frac{\partial N_i^{II}}{\partial x}[N^I]\{p\}\right]d_x d_y$$
$$-\int_{\partial\Omega} N_i^{I} I_x d_{\partial\Omega} = 0 \tag{56a}$$

$$\int_{\Omega}\left[\mu\left(\frac{\partial N_i^{II}}{\partial x}\frac{\partial[N^{II}]}{\partial x}+\frac{\partial N_i^{II}}{\partial y}\frac{\partial N^{II}}{\partial y}\right)\{v\}-\frac{\partial N_i^{II}}{\partial x}[N^I]\{p\}\right]d_x d_y$$
$$-\int_{\partial\Omega} N_i^{I} I_y d_{\partial\Omega} = 0 \tag{56b}$$

$$\int_{\Omega} N_i^{I}\left(\frac{\partial}{\partial x}[N^{II}]\{u\}+\frac{\partial}{\partial y}[N^{II}]\{v\}\right)d_x d_y = 0 \tag{56c}$$

Let

$$\int_{\partial\Omega} N_i^{II} I_x d_{\partial\Omega} = R_{xi} \tag{57a}$$

$$\int_{\partial\Omega} N_i^{II} I_y d_{\partial\Omega} = R_{yi} \tag{57b}$$

The resulting set of equations in matrix form is :

$$\begin{bmatrix} \underset{nxn}{k_1} & 0 & \underset{nxm}{k_2} \\ 0 & \underset{nxn}{k_1} & \underset{nxm}{k_3} \\ \underset{mxn}{k_2^T} & \underset{mxn}{k_3^T} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ v_n \\ \vdots \\ P_i \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} R_{x1} \\ \vdots \\ R_{xn} \\ R_{y1} \\ \vdots \\ R_{yn} \\ 0 \end{bmatrix} \tag{59}$$

Where

$$K_{1\,ij} = \int_{\Omega}\mu[\frac{\partial N_i^{II}}{\partial x}\frac{\partial N_j^{II}}{\partial x}+\frac{\partial N_i^{II}}{\partial y}\frac{\partial N_j^{II}}{\partial y} d_x d_y \tag{60a}$$

$$K_{2\,ij} = -\int_{\Omega}\mu\frac{\partial N_i^{II}}{\partial x}N_j^{I} d_x d_y. \tag{60b}$$

$$K_{3\,ij} = -\int_{\Omega}\mu\frac{\partial N_i^{II}}{\partial y}N_j^{I} d_x d_y \tag{60c}$$

$$G = [u_1, u_2, u_3\ ,\ \ \dots\ \ u_n]^T \tag{61a}$$

$$H = \left[v_1, v_{2,} v_3\ \ ,\ \ \dots\ \ v_n\right]^T \tag{61b}$$

$$I = [p_1, p_2, p_3 \quad \dots \quad p_n]^T \tag{61c}$$

In matrix form the indices i and j represent the row and column number, respectively. The continuity equation can be expressed in a different, but equivalent manner,

Chapter Three

3 Research Method and procedures

3.1 Numerical method

In this study we will study isothermal incompressible Navier-Stokes equations with the continuity equation replaced by pressure Poisson equation. Typically to solve this fluid dynamics a researcher is free to build a method in much way by choosing different discretization technique for resulting discrete system and setting a lot of parameters to some particular values. There are different methods to treat problem numerically for example finite difference method, finite element method, finite volume method. Each of these has their own strength and weakness. For this thesis the method we apply is the Galerkin finite element method. If it is applied properly, it gives the possibility of rigorous error control and it is suitable for complex geometries. For this the domain we consider is the Stokes equation that describe incompressible flow in which advective inertial forces are neglect able compared to viscous force. In order to solve these equations we consider the weak formulation to discretize the domain with Galerkin finite element with the help of MATLAB to solve the NSE with PPE.

3.2 Study procedures

To examine the study, the following procedures are involved to apply Galerkin Finite Element Method. These are:

Step-1 Generate a triangulation over the domain

Step-2 Construct basic function over triangulation

Step-3 Assemble the stiffness matrix and load vector element by element using Galerkin Method (weak form)

Step-4 Solve the system of equation

Step-5 Do the error analysis

Chapter 4

4. Description of the method, Numerical Result and Discussion

4.1 Incompressible Navier-Stokes Equations

In this work we test the incompressible Navier-Stokes equations under constant speed. These equations are the conservative form of continuity equation, momentum equation, and energy equation. The momentum equation consists of non-linear partial differential equation in terms of velocity components. Under isothermal condition, energy equation is uncoupled from momentum equation Thus only incompressible Navier-Stokes equation and continuity equation needed to be solved. In this study the non-linear term of the Navier-Stokes equation and the time derivative are omitted for simplicity by setting the body force zero the equations reduced to in vector form

$$-\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial \mathrm{p}}{\partial \mathrm{x}} + \frac{\partial p}{\partial y} = 0 \qquad \mathrm{x} \in \boldsymbol{\Omega} \tag{62}$$

$$-\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial p}{\partial y} = 0 \qquad \mathrm{x} \in \boldsymbol{\Omega}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \mathrm{x} \in \boldsymbol{\Omega} \tag{63}$$

For the simplicity we have taken no slip boundary condition. The velocity is set to zero at the boundary. There is no flow across the wall boundary and the flow does not slip along the wall. In this study we introduce a class of numerical method for Navier-Stokes equations based on the following equivalent pressure Poisson equations.

$$-\mu\nabla^2 u + \nabla P = 0 \qquad , \quad \mathrm{x} \in \boldsymbol{\Omega} \tag{64a}$$

$$-\mu\nabla^2 v + \nabla P = 0 \qquad , \quad \mathrm{x} \in \boldsymbol{\Omega}$$

$$\Delta P = 0 \qquad , \quad \mathrm{x} \in \boldsymbol{\Omega} \tag{64b}$$

With the following initial and boundary conditions

$$\nabla . u = 0 \qquad x \in \partial\boldsymbol{\Omega} \tag{64c}$$

$$u = 0 \qquad\qquad x \in \partial\boldsymbol{\Omega}$$

Here the incompressibility constraint $\nabla . u = 0$ has been replaced by pressure Poisson equation and a Neumann boundary condition for pressure This study uses the following exact solution for two dimensional tests we have chosen $\Omega = (0, 1)^2$, $\mathcal{V}$ =1, and prescribed velocity field u = (**u**, v) and pressure P where

$$u(x, y) = -\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y)$$

$$v(x, y) = \sin 2\pi x \cos 2\pi y - \sin(2\pi x) \qquad [65]$$

$$p(x, y) = 2\pi(\cos(2\pi y) - \cos(2\pi x)$$

In Ω = [0, 1] x [0, 1] find {u, p} solving with $f = 0$ and u (x, 1) =1, v (x, 1) = 0 and u and v are zero on the remainder of the boundary

4.2 Flow on square domain

A two-dimensional lid-driven cavity is common test or bench mark problem in computational fluid dynamics. It models a plane flow of an isothermal fluid in square lid-cavity. The upper side of the cavity moves in its plane at unit speeds while the other fixed. The boundary conditions are indicated in the figure Motion of the lid put the fluid inside the cavity into motion a large clockwise rotating vortex forms, with possible smaller vortices at the bottom and top left corners. Therefore, the domain consists of a unit square with prescribed velocity at the upper boundary, fixed at the other boundaries and prescribed pressure P = 0 at the bottom left corner, which is the no slip and no flux condition. There is discontinuity in the boundary conditions at the two upper corners of the cavity. Here the corner points are assumed to belong to the fixed vertical wall. All velocities components are zero at the domain boundary except that of the lid. Since the velocity components along the wall are constant, there derivatives along the wall must be zero. The moving lid will cause motion in the fluid because of the motion is viscous. The task is to compute the velocity field and pressure distribution in the cavity using simplified Navier-Stokes equations to solve numerically. The domain is discretized as shown in fig.3 by finite element method into a net work of discrete finite elements .These elements are continuously joined along their common boundaries and related to each other by their common nodal points. For this purpose, the functional spaces for the velocity and pressure are introduced as $H^1\boldsymbol{\Omega}$ and $L^2\boldsymbol{\Omega}$ **.**The linear space defined is:-

X c $H^1\boldsymbol{\Omega}$, v = {v ∈ X; v = 0 on $\partial\boldsymbol{\Omega}$ }

Q c $L^2\boldsymbol{\Omega}$, v = {w ∈ X; v = g on $\partial\boldsymbol{\Omega}$ }

The velocity u belongs to the closed subspace v of the vector fields that satisfy the prescribed boundary conditions. The pressure P belongs to the closed subspace Q of the functions with zero average in **Ω**.A weak formulation of NSE is obtained by requiring that the integral of this PDE, against all functions in appropriate space V, Q of test functions, is satisfied so we multiply the equation by test functions v ∈ V and q ∈ Q respectively and then we integrate it on the domain and we apply Green's formula to obtain a weak formulation.

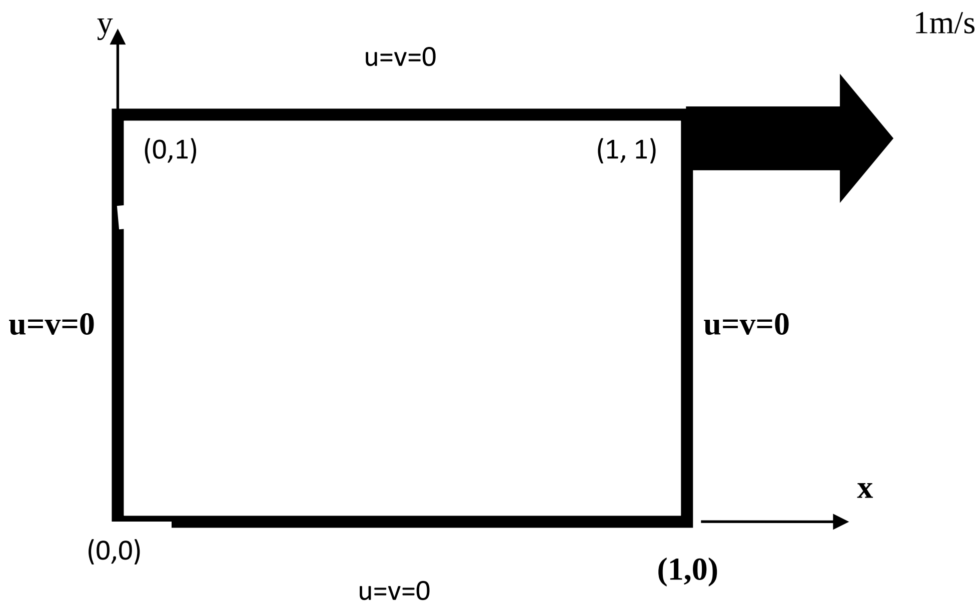

Figure 2. Flow on square domain with no slip and no flux

For x-momentum become $\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} = 0$

The y-momentum become $\mu\frac{\partial^2 v}{\partial x^2} + \mu\frac{\partial^2 v}{\partial y^2} - \frac{\partial P}{\partial x} = 0$

The continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Using Galerkin's approach, the unknown exact solution **ϕ** is approximated as

$$\boldsymbol{\phi} = \sum N_i\,(x, y)\phi_i$$

where N is the shape function and ϕ_i is the array of the field variable at specified nodes of elements of the discretized domain.Applying Galerkin's approach on x-momentum equation the following equation is obtained

$$\iint_{\Omega} [N]^T \left(\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} \right) d_{\Omega} = 0$$

After integration by parts the first term can be converted as

$$\iint_{\Omega} [N]^T \left(\frac{\partial^2 u}{\partial x^2}\right) d_{\Omega} = \int_{\partial\Omega} [N]^T \frac{\partial u}{\partial x} n_x d_{\partial\Omega} - \iint_{\Omega} \frac{\partial [N\}^T}{\partial x} \left(\frac{\partial u}{\partial x}\right) d_{\Omega}$$

Similarly the second term can also be written as

$$\mu \int_{\partial\Omega} [N]^T \left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y\right) d_{\partial\Omega} - \iint_{\Omega} \mu \left(\frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right) d_{\Omega} \{\mathrm{u}\} - \iint_{\Omega} [N]^T \frac{\partial [N]}{\partial x} d_{\Omega} \{\mathrm{P}\} = 0$$

Similarly the y-momentum can be written

$$\mu \int_{\partial\Omega} [N]^T \left(\frac{\partial v}{\partial x} n_x + \frac{\partial v}{\partial y} n_y\right) d_{\partial\Omega} - \iint_{\Omega} \mu \left(\frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right) d_{\Omega} \{\mathrm{v}\} - \iint_{\Omega} [N]^T \frac{\partial [N]}{\partial y} d_{\Omega} \{\mathrm{P}\} = 0$$

Using the Galerkin's approach the continuity equation can be written as

$$\iint_{\Omega} [N]^T \frac{\partial [N]}{\partial x} d_{\Omega} \{u\} + \iint_{\Omega} [N]^T \frac{\partial N}{\partial y} d_{\Omega} \{v\} = 0$$

The domain and the boundary conditions are shown in the fig.2.The domain is divided into elements as shown in the fig.3,. The shape function is given by:-

$$N_{i=} \frac{c_i + b_i x + a_i}{2A} \qquad \text{Where} \quad i = 1, 2, 3$$

And the area is given by:-

$$2\mathrm{A} = \det\begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$$

And we use the following notations:-

$$x_{ij} = x_i - x_j \qquad\qquad y_{ij} = y_i - y_j$$

And $$x^T = \begin{bmatrix} x_{32} \\ x_{13} \\ x_{21} \end{bmatrix} \quad , y^T = \begin{bmatrix} y_{23} \\ y_{31} \\ y_{21} \end{bmatrix}$$

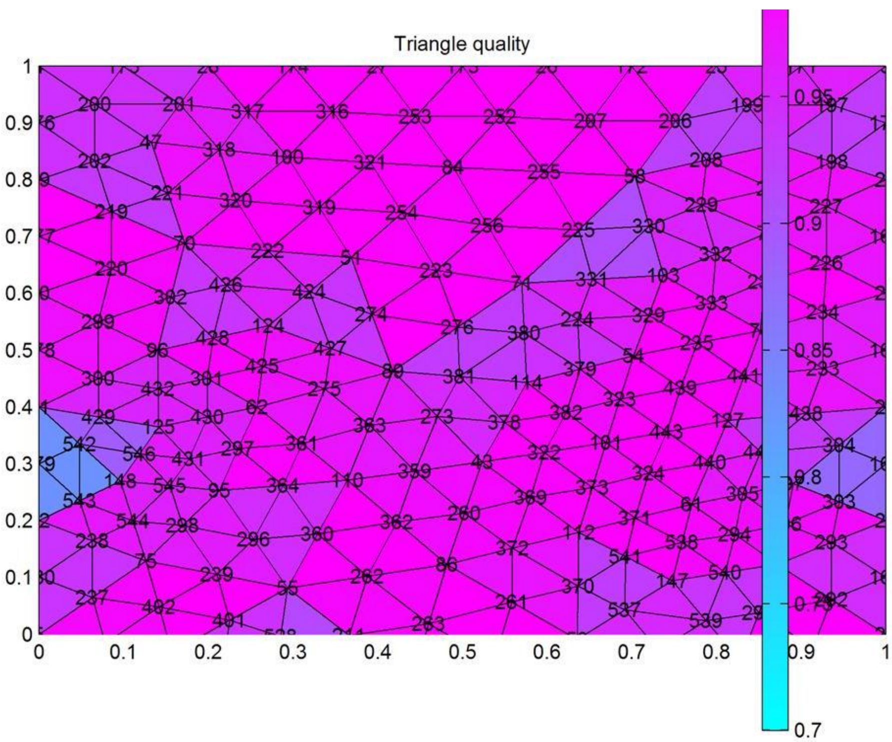

Figure 3. Finite element discretization of the problem domain

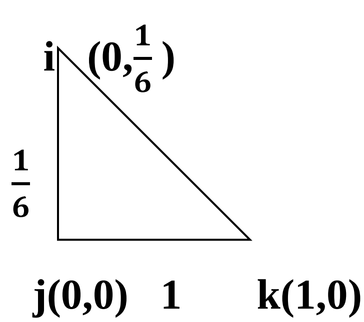

Figure 4. triangular element

From the shape shown in the figure 4 with one unit as the side length of the triangle and $\frac{1}{6}$ units as the height of the triangle, the various properties can be written as follows.

$$\text{Area} = A = \frac{1}{2} b\, h = \frac{1}{2}(1 \text{x} \frac{1}{6}) = \frac{1}{12}$$

$$\boldsymbol{b_i = y_{kj} = y_k - y_j = 0} \qquad \boldsymbol{b_j = y_{ij} = y_i - y_k = \frac{1}{6}} \qquad \boldsymbol{b_k = y_{ji} = y_j - y_i = -\frac{1}{6}}$$

$$\boldsymbol{a_i = x_{kj} = x_k - x_j = 1} \qquad \boldsymbol{a_j = x_{ik} = x_i - x_k = -1} \qquad \boldsymbol{a_k = x_{ji} = x_j - x_i = 0}$$

$$\boldsymbol{\frac{\partial N}{\partial x} = \frac{1}{2A}[b_i + b_j + b_k] = 6[\,0 \quad \frac{1}{6} \quad -\frac{1}{6}\,] = [\,0 \quad 1 \quad -1\,]}$$

$$\boldsymbol{\frac{\partial N}{\partial y} = \frac{1}{2A}[a_i + a_j + a_k] = 6[1 \quad -1 \quad 0]}$$

$$\boldsymbol{\frac{\partial [N]^T}{\partial x}\frac{\partial N}{\partial x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}}$$

$$\boldsymbol{\frac{\partial [N]^T}{\partial y}\frac{\partial N}{\partial y} = 36\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = 36\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\boldsymbol{\iint_\Omega \frac{\partial [N]^T}{\partial x}\frac{\partial N}{\partial x} d_\Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{-1}{12} \\ 0 & \frac{-1}{12} & \frac{1}{12} \end{bmatrix}}$$

$$\boldsymbol{\iint_\Omega \frac{\partial [N]^T}{\partial y}\frac{\partial N}{\partial y} d_\Omega = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\boldsymbol{\iint_\Omega \left(\frac{\partial [N]^T}{\partial x}\frac{\partial N}{\partial x} + \frac{\partial [N]^T}{\partial y}\frac{\partial N}{\partial y}\right) d_\Omega = \begin{bmatrix} 3 & -3 & 0 \\ -3 & \frac{37}{12} & \frac{-1}{12} \\ 0 & \frac{-1}{12} & \frac{1}{12} \end{bmatrix}}$$

$$\iint_{\Omega}[N]^T \frac{\partial N}{\partial x} d_{\Omega} = \begin{bmatrix} \frac{A}{3} \\ \frac{A}{3} \\ \frac{A}{3} \end{bmatrix} \frac{1}{2A} [b_i \quad b_j \quad b_k]$$

$$= \frac{1}{36}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [0 \quad 1 \quad -1] = \frac{1}{36}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\iint_{\Omega}[N]^T \frac{\partial N}{\partial x} d_{\Omega} = \begin{bmatrix} \frac{A}{3} \\ \frac{A}{3} \\ \frac{A}{3} \end{bmatrix} \frac{1}{2A} [a_i \quad a_j \quad a_k]$$

$$= \frac{1}{36}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 6[1 \quad -1 \quad 0] = \frac{1}{6}\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

For the following element shape

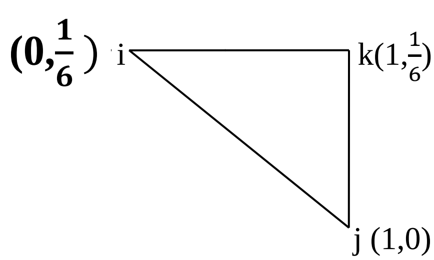

Figure 5. triangular element

$b_i = y_{kj} = y_k - y_j = \frac{1}{6}$ $\quad b_j = y_{ik} = y_i - y_k = 0$ $\quad b_k = y_{ji} = y_j - y_i = -\frac{1}{6}$

$a_i = x_{kj} = x_k - x_j = 0$ $\quad a_j = x_{ik} = x_i - x_k = -1$ $\quad a_k = x_{ji} = x_j - x_i = 1$

$$\frac{\partial[N]}{\partial x} = \frac{1}{2A}[b_i + b_j + b_k] = [\,1 \quad 0 \quad -1\,]$$

$$\frac{\partial[N]}{\partial y} = \frac{1}{2A}[a_i + a_j + a_k] = 6[\,0 \quad -1 \quad 1\,]$$

$$\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} [1 \quad 0 \quad -1] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\iint_{\Omega} \frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} d_{\Omega} = \begin{bmatrix} \frac{1}{12} & 0 & \frac{-1}{12} \\ 0 & 0 & 0 \\ \frac{-1}{12} & 0 & \frac{1}{12} \end{bmatrix}$$

$$\iint_{\Omega} \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} d_{\Omega} = \frac{36}{12}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} [0 \quad -1 \quad 1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\iint_{\Omega} \left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right) d_{\Omega} = \begin{bmatrix} \frac{1}{12} & 0 & \frac{-1}{12} \\ 0 & 3 & -3 \\ \frac{-1}{12} & -3 & \frac{37}{12} \end{bmatrix}$$

$$\iint_{\Omega} [N]^T \frac{\partial N}{\partial x} d_{\Omega} = \begin{bmatrix} \frac{A}{3} \\ \frac{A}{3} \\ \frac{A}{3} \end{bmatrix} \frac{1}{2A} [b_i \quad b_j \quad b_k]$$

$$= \frac{1}{36}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \quad 0 \quad -1] = \frac{1}{36}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\iint_{\Omega} [N]^T \frac{\partial[N]}{\partial y} d_{\Omega} = \begin{bmatrix} \frac{A}{3} \\ \frac{A}{3} \\ \frac{A}{3} \end{bmatrix} \frac{1}{2A} [a_i \quad a_j \quad a_k]$$

$$= \frac{6}{36}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 6[1 \quad -1 \quad 1] = \frac{1}{6}\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

From the first elemental shape shown in fig.4 the following equation can be obtained:

$$\mu\begin{bmatrix} \mathbf{3} & \mathbf{-3} & \mathbf{0} \\ \mathbf{-3} & \frac{\mathbf{37}}{\mathbf{12}} & \frac{\mathbf{-1}}{\mathbf{12}} \\ \mathbf{0} & \frac{\mathbf{-1}}{\mathbf{12}} & \frac{\mathbf{1}}{\mathbf{12}} \end{bmatrix}\{\boldsymbol{u}\} \quad + \frac{\mathbf{1}}{\mathbf{36}}\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{bmatrix}\{\mathbf{P}\} = \boldsymbol{\mu}\int_{\partial\Omega}[N]^T \frac{\partial u}{\partial n} d_{\partial\Omega}$$

$$\mu\begin{bmatrix} \mathbf{3} & \mathbf{-3} & \mathbf{0} \\ \mathbf{-3} & \frac{\mathbf{37}}{\mathbf{12}} & \frac{\mathbf{-1}}{\mathbf{12}} \\ \mathbf{0} & \frac{\mathbf{-1}}{\mathbf{12}} & \frac{\mathbf{1}}{\mathbf{12}} \end{bmatrix}\{\boldsymbol{v}\} \quad + \frac{\mathbf{1}}{\mathbf{6}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \end{bmatrix}\{\mathbf{P}\} = \boldsymbol{\mu}\int_{\partial\Omega}[N]^T \frac{\partial v}{\partial n} d_{\partial\Omega}$$

$$\frac{\mathbf{1}}{\mathbf{36}}\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{bmatrix}\{\mathbf{u}\} + \frac{\mathbf{1}}{\mathbf{6}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \end{bmatrix}\{\mathbf{v}\} = \mathbf{0}$$

Where $\frac{\partial u}{\partial n} = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial n} = n_x \frac{\partial v}{\partial x} + n_y \frac{\partial v}{\partial y}$

For the second elemental shape in fig 5 the following equation can also be obtained:

$$\mu\begin{bmatrix} \frac{\mathbf{1}}{\mathbf{12}} & \mathbf{0} & \frac{\mathbf{-1}}{\mathbf{12}} \\ \mathbf{0} & \mathbf{3} & \mathbf{-3} \\ \frac{\mathbf{-1}}{\mathbf{12}} & \mathbf{-3} & \frac{\mathbf{37}}{\mathbf{12}} \end{bmatrix}\{\mathbf{u}\} + \frac{\mathbf{1}}{\mathbf{36}}\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} \end{bmatrix}\{\mathbf{P}\} = \boldsymbol{\mu}\int_{\partial\Omega}[N]^T \frac{\partial u}{\partial n} d_{\partial\Omega}$$

$$\mu\begin{bmatrix} \frac{\mathbf{1}}{\mathbf{12}} & \mathbf{0} & \frac{\mathbf{-1}}{\mathbf{12}} \\ \mathbf{0} & \mathbf{3} & \mathbf{-3} \\ \frac{\mathbf{-1}}{\mathbf{12}} & \mathbf{-3} & \frac{\mathbf{37}}{\mathbf{12}} \end{bmatrix}\{\mathbf{v}\} + \frac{\mathbf{1}}{\mathbf{6}}\begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}\{\mathbf{P}\} = \boldsymbol{\mu}\int_{\partial\Omega}[N]^T \frac{\partial v}{\partial n} d_{\partial\Omega}$$

$$\frac{\mathbf{1}\boldsymbol{\mu}}{\mathbf{36}}\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} \end{bmatrix}\{\mathbf{u}\} \quad + \frac{\mathbf{1}}{\mathbf{6}}\begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}\{\mathbf{v}\} = \mathbf{0}$$

Using the above two equations, the system of the elemental equations can be assembled in the matrix form

$$[\ k\ ]\{\boldsymbol{\phi}\} = \{R\}$$

Where $\{\boldsymbol{\phi}\}$ represent the field variables {u, v, P}.After substitution of the boundary conditions and the value of μ the system of equations can be iteratively solved to find the unknown velocities and pressure.

4.3 Numerical Result

The domain first discretized by uniform mesh of size h = $\frac{1}{16}$(289 nodes 512 triangles in the fine mesh).This initial mesh is successively refined to produce meshes with sizes $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, and $\frac{1}{256}$ respectively 1089,4225,16641,and 66049 nodes. The discretization errors obtained for velocity and pressure of laminar flow inside a square cavity and the results are summarized in table 1. The result shows the exact correlation to the exact solution. This shows how the finite element solution approaches the exact solution as the mesh size increase. The possible cause of the error is the circulatory pattern inside the cavity while the flow remains steady and laminar at moderate Reynolds number (Re).It does exhibit vortices, which grow with Reynolds number, at lower sharp corner. Results for the cavity flow are displayed in countor plot from figure 6 up to 9 which allow us to visualize the streamlines and pressure response

Mesh size	$\|\|u - u_h\|\|_{L^2}$	$\|\|v - v_h\|\|_{H^1}$	$\|\|p - p_h\|\|_{L^2}$
$\frac{1}{16}$	4.9936 x 10^{-2}	3.8961 x 10^{-1}	3.2636 x 10^{-1}
$\frac{1}{32}$	1.2817 x 10^{-2}	1.0218 x 10^{-1}	8.6462 x 10^{-2}
$\frac{1}{64}$	3.2238 x 10^{-3}	2.7591 x 10^{-2}	2.3699 x 10^{-2}
$\frac{1}{128}$	8.0688 x 10^{-4}	7.5234 x 10^{-3}	6.8586 x 10^{-3}
$\frac{1}{256}$	2.017 x 10^{-4}	2.1973 x 10^{-3}	2.1075 x 10^{-3}

Table 1. Numerical errors

The result shows as the number of discretization increases the approximate solution converges to the exact solution, which is an error decrease. This shows the accuracy of the solution increases one can notice that the error approach to zero as the mesh size h goes to zero and the result is convergent

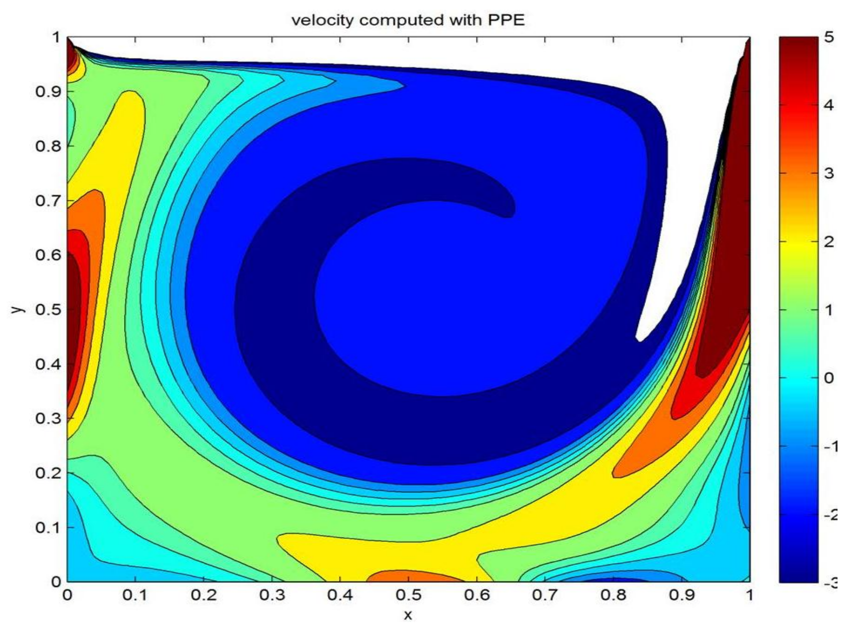

Figure 6. The streamline of the fluid with in the lid-cavity

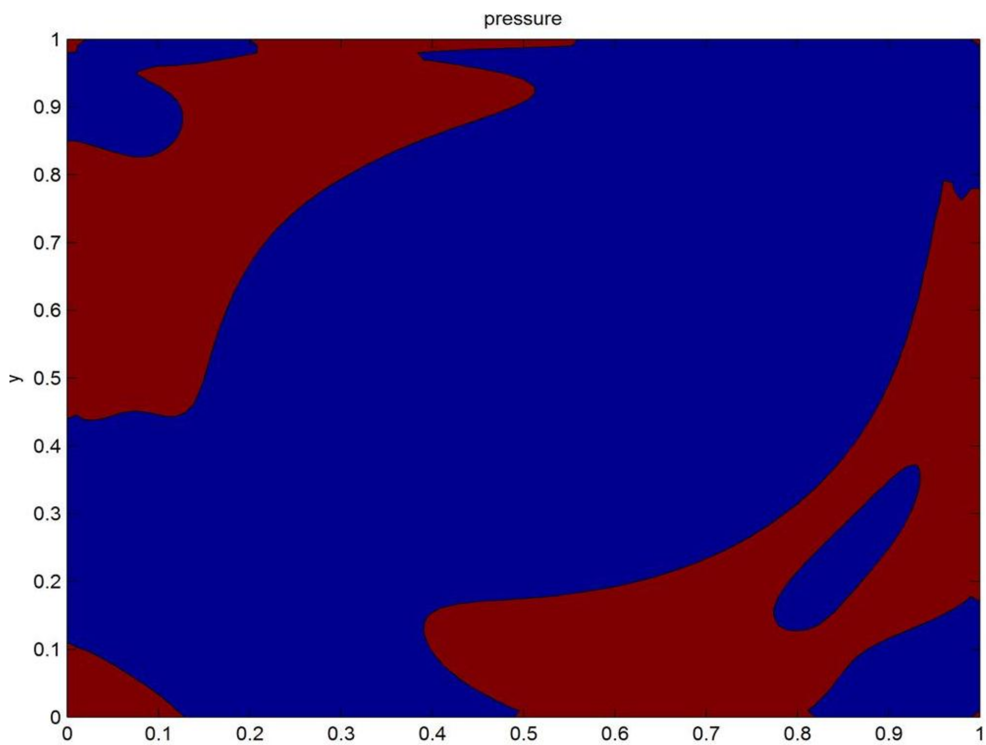

Figure 7. Shows the Pressure counter of the lid cavity

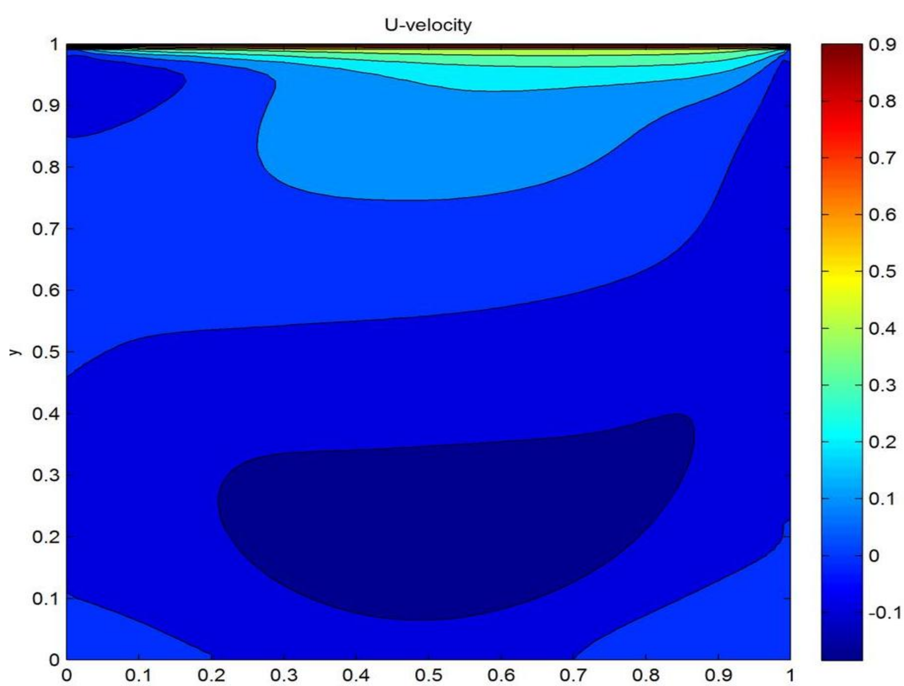

Figure 8 The horizontal velocity counter of the lid cavity

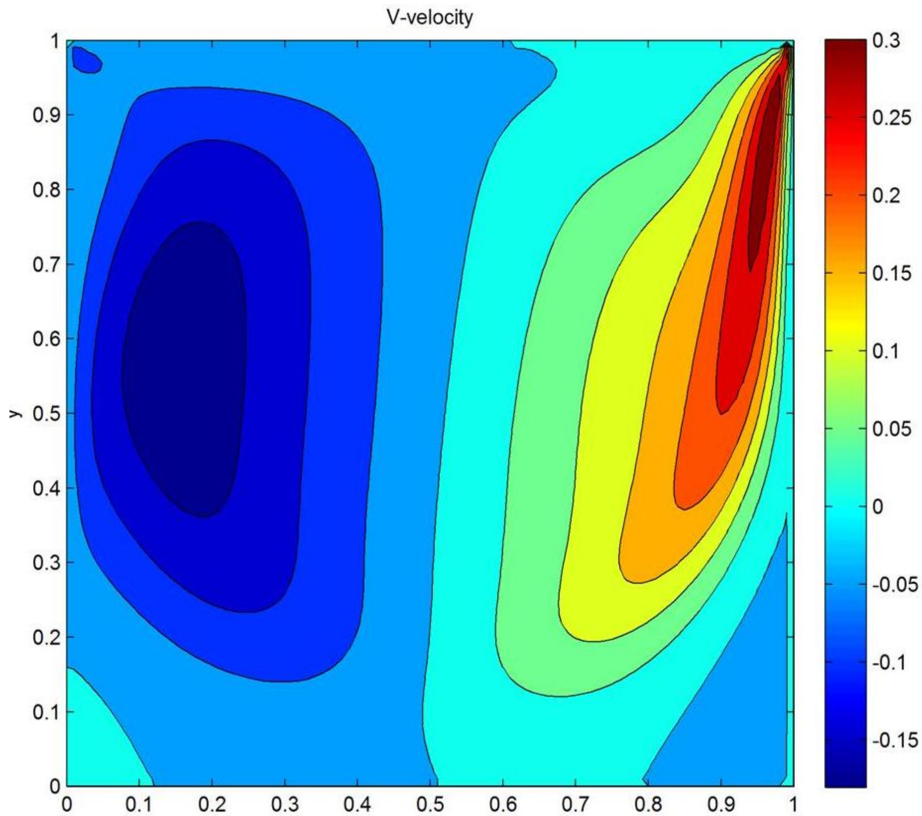

Figure 9 The vertical velocity velocity counter of the lid-cavity

Mesh size	Order of convergency of horizontal velocity	Order of convergency vertical velocity	Order of convergency for pressure
$\frac{1}{16}$	–	–	–
$\frac{1}{32}$	1.9620	1.9309	1.9163
$\frac{1}{64}$	1.9982	1.8888	1.8672
$\frac{1}{128}$	1.9983	1.8747	1.7888
$\frac{1}{256}$	2.0001	1.7756	1.7023

Table 2. Rate of convergence

Chapter 5

5. Conclusion and Recommendation

5.1 Conclusion

In this thesis we have investigated a linear steady incompressible fluid flow in lid-driven cavity based on the Galerkin finite element method. The main idea of the finite element method is to choose the appropriate basis functions and then expressing the unknown as the combination of the basis function. Finally a stiffness matrix is generated and the solution is obtained. The performance of the tool has been validating against an analytical function. The stability and accuracy of the method demonstrated using normal mode analysis of steady stokes flow. From the obtain results the conclusions are us follow:

- The program was successful in solving for velocity and for the pressure in the lid-cavity. The fact that both velocity and pressure are solved simultaneously and this shows the advantage of the direct finite element formulation.
- The accuracy of the solution increases as the mesh size decrease because of the norm of the error goes to zero.
- The velocity and the pressure are fully decoupled. This shows the boundary condition is well implemented

5.2 Recommendation

1, Addition of inertia term and compressibility effects

Although Stokes flow does not provide a practical model of flow through the lid cavity this practical model provides a bench mark for any further development. Thus, viscous flow model should include inertia and compressibility effect to be developed in order to describe the reality phenomenon in the future work.

2, Development of 3 dimensional elements this would best left until after addition of inertia and compressibility effects.

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Appendix

```
%LAPLACE2D Assembly of the Laplacian matrix of the Navier-Stokes equations

% Kh=laplace2d(p,t,cx,cy) or Kh=laplace2d(p,t,cx)

% p(1:np,1:2) nodes coordinates
% t(1:nt,1:3) triangle
np=size(p,1);% p=the pressure vector of nodal values
if (nargin==3), cy=cx;
 end
% Triangles area
x21=p(t(:,2),1)-p(t(:,1),1); y21=p(t(:,2),2)-p(t(:,1),2);
x32=p(t(:,3),1)-p(t(:,2),1); y32=p(t(:,3),2)-p(t(:,2),2);
x31=p(t(:,3),1)-p(t(:,1),1); y31=p(t(:,3),2)-p(t(:,1),2);
tarea=(x21.*y31-y21.*x31)/2;
% Derivatives of shape functions
dx=[-y32 y31 -y21];
dy=[ x32 -x31 x21];
ccx=0.25*cx./tarea;
ccy=0.25*cy./tarea;
% Assembly of sub diagonal elements
Kh=sparse(np,np);
Kh=Kh+sparse(t(:,2),t(:,1),ccx.*dx(:,2).*dx(:,1)+ccy.*dy(:,2).*dy(:,1),np,np)
;
Kh=Kh+sparse(t(:,3),t(:,1),ccx.*dx(:,3).*dx(:,1)+ccy.*dy(:,3).*dy(:,1),np,np)
;
Kh=Kh+sparse(t(:,3),t(:,2),ccx.*dx(:,3).*dx(:,2)+ccy.*dy(:,3).*dy(:,2),np,np)
;
% Transposition
Kh=Kh+Kh.';
% Assembly of diagonal elements
for i=1:3
Kh=Kh+sparse(t(:,i),t(:,i),ccx.*dx(:,i).^2+ccy.*dy(:,i).^2,np,np);
end
function [fh1,fh2]=stokesrhs(p,t,f1,f2)
% STOKESRHS: Assembly of the right-hand side of the Stokes equation
%
p=size(p,1);
% Area of triangles
x21=p(t(:,2),1)-p(t(:,1),1); y21=p(t(:,2),2)-p(t(:,1),2);
x32=p(t(:,3),1)-p(t(:,2),1); y32=p(t(:,3),2)-p(t(:,2),2);
x31=p(t(:,3),1)-p(t(:,1),1); y31=p(t(:,3),2)-p(t(:,1),2);
area=(x21.*y31-y21.*x31)/2;
nh=length(b1); ne2h=size(e2h,1); n2h=nh-ne2h;
lnh=(1:size(e2h,1))+n2h;
ie1=e2h(:,1); ie2=e2h(:,2);
% Initial velocity
u1=zeros(nh,1); u1(s)=R1\(R1'\b1(s));
u2=zeros(nh,1); u2(s)=R2\(R2'\b2(s));
% Initial gradient
rh=B1*u1+B2*u2; r2h=rh(1:n2h);
r2h=r2h+full(sparse(ie1,1,rh(lnh)/2,n2h,1)+sparse(ie2,1,rh(lnh)/2,n2h,1));
r2h(ibcp)=0; gz=zeros(n2h,1); gz(ss)=RKp\(RKp'\r2h(ss));
bp=alpha*Mp*gz+nu*r2h; bp(ibcp)=0; g=zeros(n2h,1); g(ss)=RMp\(RMp'\bp(ss));
```

```
gg=r2h'*g; gg0=gg;
% Initial direction
d=zeros(nh,1); d(1:n2h)=g;
tol=1; iter=0;
while (tol>epscg && iter<n2h)
iter=iter+1;
% Sensitivity problems
d(lnh)=(d(ie1)+d(ie2))/2;
b1=B1'*d; b1(ibc1)=0; w1=zeros(nh,1); w1(s)=R1\(R1'\b1(s));
b2=B2'*d; b2(ibc2)=0; w2=zeros(nh,1); w2(s)=R2\(R2'\b2(s));
%
rh=B1*w1+B2*w2; r2hk=rh(1:n2h);
r2hk=r2hk+full(sparse(ie1,1,rh(lnh)/2,n2h,1)+sparse(ie2,1,rh(lnh)/2,n2h,1));
r2hk(ibcp)=0; gz=zeros(n2h,1); gz(ss)=RKp\(RKp'\r2hk(ss));
bp=alpha*Mp*gz+nu*r2hk;bp(ibcp)=0; gk=zeros(n2h,1); gk(ss)=RMp\(RMp'\bp(ss));
% Step size
tk=gg/(rh'*d);
% Update
p=p-tk*d; g=g-tk*gk;
u1=u1-tk*w1; u2=u2-tk*w2;
% New conjugate gradient direction
r2h=r2h-tk*r2hk; gg1=r2h'*g; beta=gg1/gg;
d(1:n2h)=g+beta*d(1:n2h);
% Tolerance
tol=min(sqrt(gg1),sqrt(gg1/gg0)); gg=gg1;
end
%%---------- Test case with exact solution
% constants
alpha=0; nu=1;
Pen=10^10; %  for Dirichlet boundary conditions
epscg=10^-6; % Tolerance for  algorithm
% mesh
load carre289 ph t2h e2h ibc1 ibc2 ibcp
nph=size(p,1); np2h=nph-size(e2h,1); nth=size(th,1);
fprintf('Mesh : nph=%4d nth=%4d \n',nph,nth)
% %
Matrices generation / velocity
K=laplace2d(p,th,nu); [B1,B2]=div2d(p,th);
A1=K; A2=K; s=symamd(A1);
% %
Matrices generation / pressure
Kp=laplace2d(p(1:np2h,:),t2h,1); Mp=mass2d(p(1:np2h,:),t2h);
MMp=Mp; ss=symamd(Kp);
% %
Right hand side/ exact solution
x=p(:,1); y=p(:,2); pi2=2*pi;
pre=pi2*(cos(pi2*y)-cos(pi2*x));
u1e=-cos(pi2*x).*sin(pi2*y)+sin(pi2*y);
u2e= sin(pi2*x).*cos(pi2*y)-sin(pi2*x);
[b1,b2]=stokesrhs(p,th);
% Boundary condition
A1(ibc1,ibc1)=A1(ibc1,ibc1)+Pen*speye(length(ibc1));
A2(ibc2,ibc2)=A2(ibc2,ibc2)+Pen*speye(length(ibc2));
Kp(ibcp,ibcp)=Kp(ibcp,ibcp)+Pen*speye(length(ibcp));
MMp(ibcp,ibcp)=MMp(ibcp,ibcp)+Pen*speye(length(ibcp));
b1(ibc1)=0; b2(ibc2)=0;
% Visualization of the results
```

```
figure(1)
%velocity function
contourf(x2d,y2d,u),xlabel('x'),ylabel('y'),...
title('U-velocity');axis('square','tight');colorbar
figure(2)
contourf(x2d,y2d,v),xlabel('x'),ylabel('y'),...
title('V-velocity');axis('square','tight');colorbar
figure(3)
contourf(x2d,y2d,p,([-2.0:.01:2])),xlabel('x'),ylabel('y'),...
title('pressure');
figure(4)
quiver (x2d,y2d,u,v)...
,xlabel('x'),ylabel('y'),title('Velocity  function Plot'); axis([0 1 0
1]),axis('square')
% L^2 and H^1 errors
M=masse2d(p,th,1);
du1=u1-u1e; du2=u2-u2e; dp=pr-pre;
errL2u=du1'*M*du1+du2'*M*du2; errL2p=sqrt(dp'*M*dp);
errH1u=errL2u+du1'*K*du1+du2'*K*du2;
fprintf(iter )  %4d Err:u/L2 %15.8e /H1 %15.8e Err:p/L2 %15.8e \n’,iter,...
sqrt(errL2u),sqrt(errH1u),errL2p
```