

**ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY
SCHOOL OF APPLIED NATURAL SCIENCE
APPLIED MATHEMATICS PROGRAM**



**A THESIS ON ANALYSIS OF FINITE ELEMENT METHOD AND
FINITE DIFFERENCE METHOD FOR SOLVING POISSON'S
EQUATION IN 2D**

**SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENT
FOR THE DEGREE OF MASTER OF SCIENCE IN APPLIED
MATHEMATICS**

**BY:
GETU MEKONNEN**

**ADAMA, ETHIOPIA
JUNE, 2017**

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Abstract

In this thesis, we focus on analysis of Finite Element Method and Finite Difference Method for the numerical solution of Poisson's equation in 2D with Dirichlet boundary conditions. The fundamental idea of the finite element method is the replacement of continuous functions by piecewise approximations, usually polynomials. The finite element method has now reached a very sophisticated level of development, so much so that it is applied routinely in a wide variety of application areas, a general description of the steps involved in obtaining a solution to a Poisson's equation is provided. This description includes discussion of the types of elements available for a finite element method solution such as triangulations. Weak formulation reduce the continuity requirements on the approximation (or basis functions) functions there by allowing the use of easy to construct and implement polynomials. Weak forms give relatively very accurate results with the mesh refinement, which are extremely good for engineering simulations. Improving the accuracy of a solution in weak formulations depend upon the type of problem we are solving. In poisson's equation only mesh refinement is good enough to get accurate results. The accuracy can also be improved by using higher-order shape functions. Finite difference methods are numerical methods for solving differential equations in which finite difference approximate the derivatives. Discretization, convergence, consistence, and stability analysis of the method for solving poisson's equation in 2D is discussed. Numerical examples are provided using basic Matlab codes. The study shows that the error decrease as the discretization of the element increases on its domain and convergence ensured by suitable mesh refinement and finite element method is better in terms of convergence than finite difference method for solving poisson's equation in 2D.

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Getu Mekonnen

Chapter 1

Introduction

1.1 Background of study

Numerical Analysis is the branch of mathematics that proposes, develops, analyzes and applies methods from scientific computing to several fields including analysis, linear algebra, geometry, approximation theory, functional equations, optimization and differential equations. Other disciplines such as physics, the natural sciences, engineering, economics and the financial sciences frequently give rise to problems that need scientific computing for their solutions. As such, numerical mathematics is the crossroad of several disciplines of great relevance in modern applied sciences, and can become a crucial tool for their qualitative and quantitative analysis. Hence, the finite element method (FEM) is one of the numerical techniques for solving problems which are described by partial differential equations or can be formulated as functional minimization. Finite difference methods are numerical methods for solving differential equations in which finite difference approximate the derivatives.

Finite elements began in the 1950s in the aircraft industry. Use of the techniques followed a paper by Turner, Clough, Martin, and Topp [TCMT] that was published in 1956. Wide spread application of the methods required large computer resources that were not available until the early 1970s. The modern use of finite elements really started in the field of structural engineering. Probably the first attempts were by Hrennikoff (1941) and McHenry (1943), who developed analogies between actual discrete elements for beams and bars.

The finite element method is a numerical method for solving problems of engineering and mathematical physics. Typical problem areas of interest in engineering and mathematical physics that are solvable by use of the finite element method include structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential. The fundamental idea of the finite element method is the replacement of continuous functions by piecewise approximations, usually polynomials.

For problems involving complicated geometries, loadings, and material properties, it

is generally not possible to obtain analytical mathematical solutions. Analytical solutions are those given by a mathematical expression that yields the values of the desired unknown quantities at any location in a body (here total structure or physical system of interest) and are thus valid for an infinite number of locations in the body. These analytical solutions generally require the solution of partial differential equations, which, because of the complicated geometries, loadings, and material properties, are not usually obtainable. Hence we need to rely on numerical methods, such as the finite element method, for acceptable solutions.

The finite element formulation of the problem results in a system of simultaneous algebraic equations for solution, rather than requiring the solution of differential equations. These numerical methods yield approximate values of the unknowns at discrete numbers of points in the continuum. Hence this process of modeling a body by dividing it into an equivalent system of smaller bodies or units (finite elements) interconnected at points common to two or more elements (nodal points or nodes) and/or boundary lines and/or surfaces is called discretization. In the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and combine them to obtain the solution of the whole body.

Briefly, the solution for structural problems typically refers to determining the displacements at each node and the stresses within each element making up the structure that is subjected to applied loads. In non structural problems, the nodal unknowns may, for instance, be temperatures or fluid pressures due to thermal or fluid fluxes. A domain of interest is represented as an assembly of finite elements. Approximating functions in finite elements are determined in terms of nodal values of a physical field which is sought. A continuous physical problem is transformed into a discretized finite element problem with unknown nodal values.

1.2 Statement of the problem

The term 'finite element method' was introduced by Clough (1960) in a paper describing applications in plane elasticity. The engineers had put the finite element method on the map as a practical technique for solving their elasticity problems, and although a rigorous mathematical basis had not been developed, the next few years saw an expansion of the method to solve a large variety of structural problems. In a paper published in 1943 but not widely recognized for many years, Courant proposed setting up the solution of stresses in a variational form. Then he introduced piecewise interpolation (or shape) functions over triangular subregions making up the whole region as a method to obtain approximate numerical solutions. Zienkiewicz noted that Courant had already developed some of the ideas in the 1920s without taking them further. Similar papers followed by Polya (1952) and Weinberger (1956). Greenstadt (1959) presented the idea of considering a continuous region as an assembly of several discrete parts and making assumptions about the variables in each region, variational principles being used to find values for these variables. We note here also that the work of Schoenberg (1946) was very much in the spirit of finite

elements, since the piecewise polynomial approximation led to the development of the theory of splines. Many physical phenomena in engineering and science can be described in terms of partial differential equations. The finite element method (FEM) is a numerical approach by which these partial differential equations can be solved approximately. The solution of the resulting partial differential equation for a complicated geometry, such as an engine block, is impossible by classical methods like separation of variables. Numerical methods such as finite difference methods are also quite awkward for arbitrary shapes; software developers have not marketed finite difference programs that can deal with the complicated geometries that are common place in engineering.

The governing equation of the Poisson's equation is given as :

$$-\nabla^2 u(x, y) = f(x, y) \text{ in } D$$

subject to the non-homogeneous Dirichlet boundary condition

$$u = g(s)$$

on some part C of the boundary.

Therefore, the study answer the following research questions:

- Does the domain discretization affect the quality of the approximate solution?
- Is FEM and FDM are applicable for Poisson's equation in 2D?
- How to estimate errors and how convergence was ensured by suitable mesh refinement?
- Which method is more stable and more convergent? FEM or FDM?

1.3 Objective of the study

1.3.1 General objective of the study

The general objective of the study is solving the Poisson's equation in 2D by using finite element method and compare with finite difference method.

1.3.2 Specific objective of the study

The specific objective is :

1. Be able to analysis a suitable finite element and finite difference model for Poisson's equation in 2D.
2. Derivation of the weak formulation of the Poisson's equation.
3. stability, consistence and convergence analysis of the two methods.
4. Comparison of the two methods.

1.4 Significant and Beneficiaries of the Study

The most important aspect in which the FEM is to divide the domain area into triangular and rectangular elements and other polygonal elements. Hence these research May provide:

- Useful result and information for computing Poisson's equation
- Understand the fundamental ideas of finite element method and finite difference method for Poisson's equation.
- Since there are a lot of studies have to be done on FEM for Poisson's equation , it will give a comprehensive starting point for more advanced researches. This will also develop awareness for the importance research center in this field.
- It may serve as references for those needs to conduct research on this area.

1.5 Delimitation of the Study

The Finite Element Method are perhaps the most important to solve differential equation in applied mathematics and engineering. So this study is delimited to a class of FEM and FDM for Poisson equation.

The governing equation of the Poisson equation are given by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x, y) \text{ in domain } D$$

Majority of the physical problems are represented by linear/non linear PDE. Out of these problems can be solved with the help of analytical methods. But some of non linear problems cannot be solved analytically. Thus it is not always possible to find the exact solution of such problems. Hence we have to determine an approximate numerical solution of these problems. Here we are using the specific finite element method and finite difference method.

Chapter 2

Literature Review

Probably the first attempts were by Hrennikoff(1941) and McHenry(1943), who developed analogies between actual discrete elements, for example bars and beams, and the corresponding portions of a continuous solid. These methods belonged to a class of semi-analytic techniques which were used in the 1940s for aircraft structural design. Matrix methods for the solution of such problems were developed at that time, and it is interesting to note that the work of Argyris(1955), in an engineering context, introduced a minimization process which is also the basis of the mathematical underpinning of the finite element method. With the development of high-speed, jet-powered aircraft, these semi-analytic methods were soon found to be inadequate and the quest began for a more reliable approach.

A direct approach, based on the principle of virtual work, was given by Argyris(1955), and in a series of papers he and his colleagues developed this work to solve very complex problems using computational techniques by Argyris and Kelsey(1960). At about the same time, Turner et al.(1956) presented the element stiffness matrix, based on displacement assumptions, for a triangular element, together with the direct stiffness method for assembling the elements. Solutions of three-dimensional problems required only simple extensions to the basic two dimensional theory by Argyris(1964). One area of application of plate elements was that of modelling thin shells, and some success was achieved by Clough and Johnson (1968).

Plasticity problems, involving non-linear material behaviour, were modelled at this time Gallagher et al.(1962) and the method was also applied to the solution of problems in visco elasticity Zienkiewicz et al.(1968). Besides the static analysis described above, dynamic problems were also being tackled, and Archer(1963) introduced the concept of the consistent mass matrix. Both vibration problems by Zienkiewicz et al.(1966) and transient problems by Koenig and Davids(1969) were considered. Thus the period from its conception in the early 1950s to the late 1960s saw the method being applied extensively by the engineering community. With the successes of these practical applications in the structural field, it was open for engineers in other disciplines (Silvester and Ferrari 1983) to get hold of the finite element method. An obvious candidate was fluid mechanics.

There are excellent accounts of applications from the mid 1970s onwards in the texts by Zienkiewicz and Taylor(2000a,b). Let us return to the early days of the developments: at the same time as the engineers were pushing forward with the practical aspects of the method, similar work was being carried out by applied mathematicians, each group apparently unaware of the work of the other. Courant (1943) gave a solution to the torsion problem, using piecewise linear approximations over a triangular mesh, formulating the problem from the principle of minimum potential energy.

Zienkiewicz noted that Courant had already developed some of the ideas in the 1920s without taking them further. Similar papers followed by Polya (1952) and Weinberger (1956). Greenstadt (1959) presented the idea of considering a continuous region as an assembly of several discrete parts and making assumptions about the variables in each region, variational principles being used to find values for these variables. We note here also that the work of Schoenberg (1946) was very much in the spirit of finite elements, since the piecewise polynomial approximation led to the development of the theory of splines. Similar work was being carried out in the physics community.

The variational principle for the beam equation is fundamental to the development of the finite element method and characterizes the solution to the beam equation as the function that minimizes an integral over all functions in the given domain with boundary conditions. In the late 1940s, Prager and Synge (1947) developed a geometric approach to a variational principle in elasticity which led to the so-called hypercircle method, which is also in the spirit of the finite element method. A three-dimensional problem in electrostatics was solved, using linear tetrahedral elements, by McMahon (1953).

At about the same time as Courant was working on variational methods for elliptic problems. By Trefftz (1926) developed a technique in which a partial differential equation, defined over a region, becomes an integral equation over the boundary of that region. The immediate advantage is in the reduction of the dimension of the problems. It was some time before Birkhoff et al. (1968) and Zlamal (1968) published a convergence proof and error bound for finite element and finite difference method applied in mathematics literature. However, the first convergence proof in the engineering literature had already been given by Melosh (1963), who used the principle of minimum potential energy, and this work was extended by Jones (1964) using Reissners variational principle. Once it was realized that the method could be interpreted in terms of variational methods, the mathematicians and engineers were brought together and many extensions of the method to new areas soon followed.

Error estimation is clearly an important aspect of any numerical approximation, and the first developments were by Babuska (1971, 1973) and Babuska and Rheinboldt (1978, 1979), who showed how to estimate errors and how convergence was ensured by suitable mesh refinement. Stability analysis was discussed by Martin (1965). The concept of piecewise polynomial approximation offered a simple and efficient procedure for the application of the classical Rayleigh Ritz method. The principles could be

clearly seen in the much earlier work of Lord Rayleigh (Strutt 1870) and Ritz (1909).

From a physical point of view, it meant that problems outside the structural area could be solved using standard structural packages by associating suitable meanings to the terms in the corresponding variational principles. This was just what was done by Zienkiewicz and Cheung (1965) in the application of the finite element method to the solution of Poissons equation and by Doctors (1970) in the application to potential flow. Similarly, transient heat conduction problems were considered by Wilson and Nickell (1966).

Currently, the terms finite element method, is used, and they are all really variations on a more general weighted residual theme. Zienkiewicz (1995) suggested that a more appropriate generic name would be generalized by Galerkin method Fletcher (1984). The phrase finite element was introduced by Clough (1960) when both triangular and rectangular elements were used for plane stress analysis. A flat, rectangular-plate bending-element stiffness matrix was developed by Melosh in 1961. This was followed by development of the curved-shell bending element stiffness matrix for axisymmetric shells and pressure vessels by Grafton and Strome (1963).

Chapter 3

Research Methodology

3.1 Study Design

The study involves numerical implementation with MATLAB. For the study, the domain is discretized into finite elements. Then, we add up the weak formulation for FEM over each element, select interpolation functions and we apply function value at each node by using the given boundary conditions. The numerical examples to the solution of our FEM and FDM are identified. Finally, the resulting linear system is solved using matrix for the unknown values on the node.

3.2 Study Procedures

In order to accomplish the stated objectives, the study will follow the following procedures:

- Discretization of the 2D domain.
- Derivation of the weak formulation of the governing differential equation.
- Proper choice of interpolation functions.
- Assembly of the stiffness matrix.
- Imposition of boundary conditions.
- Solution of the global matrix system.
- Validating the method using Numerical examples.
- Writing the Algorithm for FEM and FDM for Poisson's equation.

Chapter 4

Finite Element and Finite Difference Method for Solving Poisson's Equation in 2D

4.1 Weak Formulation for finite element method for Poisson's equation With Dirichlet problem

Consider the Poisson's equation in two dimension:

$$\nabla^2 u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x, y) \text{ in } \Omega, u = 0 \text{ on } \Gamma \quad (4.1.1)$$

In the development of the weak form we need only consider a typical element. We assume that is a typical $\Omega_e = [x_a, x_b]$ triangular element. We develop the weak form of equation (4.1.1) over the typical element. Douglas N. Arnold(2001). The first step is to multiply (4.1.1) with a weight function v , which is assumed to be differentiable once with respect to x and then integrate the equation over the Ω , we get that

$$\int_{\Omega} \Delta u v \, dx = - \int_{\Omega} f v \, dx \quad (4.1.2)$$

and conversely, if this equation is satisfied for all integrable functions v , then u satisfies Poisson's equation. In fact, it is sufficient that the equation be satisfied for all C^∞ functions with compact support inside Ω . In particular, if v is a C^1 function on Ω which vanishes on Γ , we may integrate by parts to get

$$\int_{\Omega} \text{grad } u \cdot \text{grad } v \, dx = - \int_{\Omega} f v \, dx \quad (4.1.3)$$

Thus it is evident that a C^2 function which vanishes on Γ satisfies Poisson's equation if and only if it satisfies (4.1.3) for all C^2 functions v which vanish on Γ . Navudy Sharma(2014). The use of C^2 functions is, however, not very natural for the formulation (4.1.3). A more natural space would be the *Sobolev Space*:

$$H^1(\Omega) = \{v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega)\}$$

This is a Hilbert space with inner product

$$\langle u, v \rangle_{H^1(\Omega)} = \int_{\Omega} (u v + \text{grad } u \cdot \text{grad } v) dx$$

and corresponding norm

$$\|v\|_1 = \|v\|_{H^1(\Omega)} := \sqrt{\|v\|_{L^2(\Omega)}^2 + \|\text{grad } v\|_{L^2(\Omega)}^2}$$

(Note: we are using the same notation $L^2(\Omega)$ both for real-valued and vector-valued square integrable functions.) We also define

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v|_{\Gamma} \equiv 0\}.$$

We then define the weak formulation of the Dirichlet problem for Poisson's equation to be: Find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} \int_{\Omega_e} v \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) + f(x, y) \right] dx dy &= 0 \\ \Rightarrow \int_{\Omega} \text{grad } u \cdot \text{grad } v \, dx &= - \int_{\Omega} f v \, dx \quad \text{for all } v \in H_0^1(\Omega) \end{aligned} \quad (4.1.4)$$

$$\Rightarrow \int_{\Omega_e} \left[v \frac{\partial F_1}{\partial x} + v \frac{\partial F_2}{\partial y} + v f \right] dx dy = 0$$

Where $F_1 = \frac{\partial u}{\partial x}$, $F_2 = \frac{\partial u}{\partial y}$

$$\Rightarrow \frac{\partial}{\partial x} (v F_1) = v \frac{\partial F_1}{\partial x} + \frac{\partial v}{\partial x} F_1$$

$$\Rightarrow \frac{\partial}{\partial y} (v F_2) = v \frac{\partial F_2}{\partial y} + \frac{\partial v}{\partial y} F_2$$

$$\int_{\Omega} \left[\left\{ \frac{\partial}{\partial x} (v F_1) - \frac{\partial v}{\partial x} F_1 \right\} + \left\{ \frac{\partial}{\partial y} (v F_2) - \frac{\partial v}{\partial y} F_2 \right\} + v f \right] dx dy = 0$$

$$\int_{\Omega} \left[\frac{\partial v}{\partial x} F_1 + \frac{\partial v}{\partial y} F_2 - v f \right] dx dy - \int_{\Omega} \left[\frac{\partial}{\partial x} (v F_1) + \frac{\partial}{\partial y} (v F_2) \right] dx dy = 0$$

$$\int_{\Omega} \left[\frac{\partial v}{\partial x} F_1 + \frac{\partial v}{\partial y} F_2 - v f \right] dx dy - \oint_{\Gamma} v (F_1 n_x + F_2 n_y) ds = 0$$

$$\int_{\Omega} \left[\frac{\partial v}{\partial x} F_1 + \frac{\partial v}{\partial y} F_2 - v f \right] dx dy - \oint_{\Gamma} v \left[n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} \right] ds = 0$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} - v f \right] dx dy - \oint_{\Gamma} [v q_n] ds = 0 \quad (4.1.5)$$

Where $q_n = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$

The weak formulation fits an abstract framework we shall see frequently. We have a Hilbert space V (namely $H_0^1(\Omega)$), a bilinear form $B : V \times V \rightarrow \mathfrak{R}$ (given by the left-hand

side of (4.1.4)), a linear functional $F : V \rightarrow \Re$ (given by the right-hand side of (4.1.4), and the weak formulation is

Find $u \in V$ such that $B(u, v) = F(v)$ for all $v \in V$.

Finite element models of time dependent problems can be developed in two alternative ways: Coupled formulation in which the time is treated as an additional coordinate along with the spatial coordinate, and Decoupled formulation where time and spatial variations are assumed to be separable. Chandhari and patel(2015)

We obtain solution in the form,

$$\begin{aligned} u(x, t_s) &= \sum_{j=1}^n u_j^e(t_s) \Psi_j^e(x, y) \\ &= \sum_{j=1}^n (u_j^s)^e \Psi_j^e(x, y), (s = 1, 2, 3, \dots) \end{aligned} \quad (4.1.6)$$

Where $(u_j^s)^e$ is the value of $u(x, y)$ at time $t = t_s$ and node j of the element Ω_e

Substituting $v = \Psi_i(x, y)$ (to obtain the solution i^{th} equation of the system) and equation (4.2.2) into equation (4.2.1).

We obtain

$$\begin{aligned} &\int_{\Omega} \left[\frac{\partial v}{\partial x} \left(\sum_{j=1}^n u_j^e \frac{\partial \Psi_j^e}{\partial x} \right) + \frac{\partial v}{\partial y} \left(\sum_{j=1}^n u_j^e \frac{\partial \Psi_j^e}{\partial y} \right) - v f \right] dx dy - \oint_{\Gamma} v q_n ds = 0 \\ &= \sum_{j=1}^n \left\{ \int_{\Omega} \left[\frac{\partial \Psi_i^e}{\partial x} \frac{\partial \Psi_j^e}{\partial x} + \frac{\partial \Psi_i^e}{\partial y} \frac{\partial \Psi_j^e}{\partial y} \right] dx dy \right\} u_j^e - \int_{\Omega} [\Psi_i^e f] dx dy - \oint_{\Gamma} \Psi_i^e q_n ds = 0 \quad (4.1.7) \\ &\sum_{j=1}^n k_{ij}^e u_j^e = f_i^e + Q_i^e \end{aligned}$$

Where

$$k_{ij}^e = \int_{\Omega} \left[\frac{\partial \Psi_i^e}{\partial x} \frac{\partial \Psi_j^e}{\partial x} + \frac{\partial \Psi_i^e}{\partial y} \frac{\partial \Psi_j^e}{\partial y} \right] dx dy$$

$$f_i^e = \int_{\Omega} [\Psi_i^e f] dx dy$$

$$Q_i^e = \oint_{\Gamma} \Psi_i^e q_n ds$$

Hence, we arrived on systems of algebraic equation:

$$KU = F$$

Where

$K = k_{ij}^e$, is the reduced stiffness matrix

$U = u_j^e$ is contains only the unknown nodal values and

$F = f_i^e + Q_i^e$ is the modified force vector.

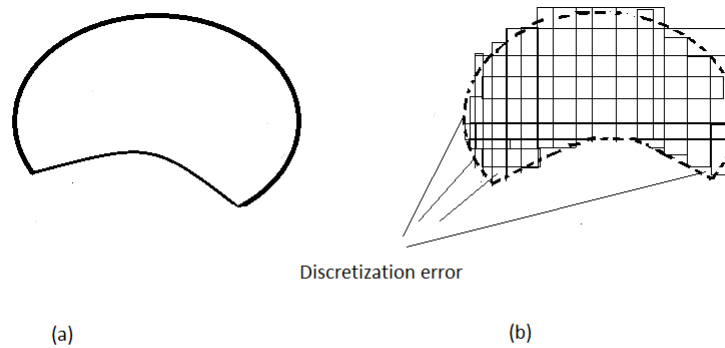


Figure 4.1: (a) Irregular 2D domain. (b) Finite element mesh using rectangular elements

4.2 The General Approach of Finite Element Method for Poisson's Equation

The implementation of the finite element approach usually follows a standard step-by-step procedure. The following provides a brief overview of each of these steps. (Steven C.Chapra and Raymond P.Canale(2010).

4.2.1 Domain Discretization

The domain of a 2D BVP usually has an irregular shape, as shown in Figure 4.2(a). Using the FEM, the first step is to accurately represent the physical domain of the problem by a set of basic shapes called the *finite elements*. The use of a rectangle, for example, as a basic finite element to discretize an irregular domain is certainly the simplest but not the most suitable choice because an assembly of rectangles cannot accurately represent the arbitrary geometrical shape of the domain. In such a case, the discretization error is significant, as shown in Figure 4.2(b), although it tends to decrease as the size of rectangles in the domain becomes smaller. On the other hand, if a triangle is used instead of a rectangle as the basic element for the meshing of the 2D domain, the discretization error would be effectively much smaller. This is illustrated graphically in Figure 4.2(a). The quadrilateral is another basic element that is commonly used in 2D finite element analysis. A coarse mesh of the irregular domain using quadrilateral elements is shown in Figure 4.2(b). As was the case with the triangular element, the quadrilateral element results in a smaller discretization error than the one caused by the use of the rectangular element. Anastasis C. Polycarpou(2006).

Mesh generation has certain rules that must be followed at all times:

- For a triangular mesh, the shape of triangles must be close to equilateral.
- For a quadrilateral mesh, the shape of quadrilaterals must be close to square.

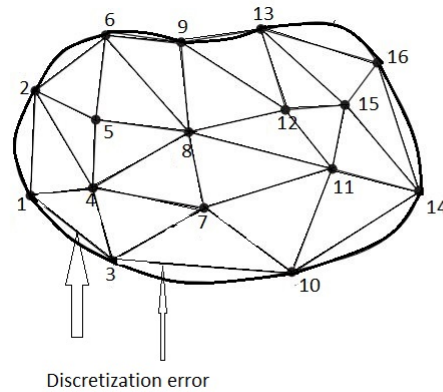


Figure 4.2: Finite element mesh using triangular elements.

- Nodes must appear at source points.
- The finite element mesh must accurately represent the geometrical domain of the problem.
- There must be no overlap of elements.
- An interior node (non boundary node) must belong to at least three elements.

4.2.2 Interpolation Functions

Interpolation functions are used to interpolate the field variables over the element. Often, polynomials are selected as interpolation functions. The degree of the polynomial depends on the number of nodes assigned to the element. Proper interpolation functions must be developed for triangular and quadrilateral elements since they are both widely used in the discretization of 2D domains. These interpolation functions must satisfy certain key requirements.

- They must guarantee continuity of the primary unknown quantity across interelement boundaries.
- They must be at least once differentiable since the governing differential equation at hand is of second order.
- They must be complete polynomials to provide sufficient representation of the solutions behavior in the finite element domain.

Initially, we will concentrate on the development of interpolation functions based on linear/bilinear elements and, then, move on to higher order elements based on the Lagrange polynomials.

				1					
				x		y			
			x^2	xy		y^2			
		x^3	x^2y		xy^2	y^3			
	x^4	x^3y	x^2y^2		xy^3	y^4			
x^5	x^4y	x^3y^2	x^2y^3		xy^4	y^5			

Table 4.1: Complete polynomials up to order 5

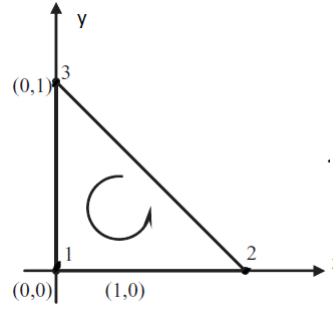


Figure 4.3: Linear triangular element in x y plane

Linear Triangular Element

The triangle consists of three vertices which correspond to the three nodes of the element. The nodes are locally numbered in a counter-clockwise direction to avoid having a negative area. A linear interpolation function spanning a triangle must be linear in two orthogonal directions. These could be the orthogonal axes defined by the natural coordinates x and y . Anastasis C. Polycarpou(2006). Each linear interpolation function corresponds to a triangle node. Denoting the three interpolation functions by $\Psi_1^e(x, y)$, $\Psi_2^e(x, y)$, and $\Psi_3^e(x, y)$. These are assigned to nodes 1, 2, and 3, respectively. The interpolation functions must satisfy the condition $\Psi_j^e = \delta_{jk}$ at all nodes $(j, k) = 1, 2, 3, \dots, n$ and $\sum_{j=1}^n \Psi_j^e = 1$ within Ω_e . The *Kronecker Delta* is defined as

$$\delta_{jk} = 1, \text{ for } j = k \text{ and}$$

$$\delta_{jk} = 0, \text{ for } j \neq k.$$

Starting from a linear representation of shape function

$$u(x, y) = c_1 + c_2x + c_3y \quad (4.2.1)$$

where $u(x, y)$ the dependent variable, the $c_i, i = 1, 2, 3$. is coefficients, and x and y are independent variables. This function must pass through the values of $u(x, y)$ at the triangle's nodes (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) .

Therefore,

$$u_1(x, y) = c_1 + c_2x_1 + c_3y_1$$

$$u_2(x, y) = c_1 + c_2x_2 + c_3y_2$$

$$u_3(x, y) = c_1 + c_2x_3 + c_3y_3$$

or in matrix form

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

which can be solved for

$$c_1 = \frac{1}{2A_e} [u_1(x_2y_3 - x_3y_2) + u_2(x_3y_1 - x_1y_3) + u_3(x_1y_2 - x_2y_1)] \quad (4.2.2)$$

$$c_2 = \frac{1}{2A_e} [u_1(y_2 - y_3) + u_2(y_3 - y_1) + u_3(y_1 - y_2)] \quad (4.2.3)$$

$$c_3 = \frac{1}{2A_e} [u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1)] \quad (4.2.4)$$

where A_e is the area of the triangular element,

$$A_e = \frac{1}{2} [(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)]$$

Equations (4.2.2) through (4.2.4) can be substituted into Eq. (4.2.1). After a collection of terms, the result can be expressed as

$$u = \Psi_1u_1 + \Psi_2u_2 + \Psi_3u_3 \quad (4.2.5)$$

Where

$$\Psi_1 = \frac{1}{2A_e} [(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$\Psi_2 = \frac{1}{2A_e} [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$\Psi_3 = \frac{1}{2A_e} [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

Equation (4.2.5) provides a means to predict intermediate values for the element on the basis of the values at its nodes. Steven C.Chapra and Raymond P.Canale(2010)

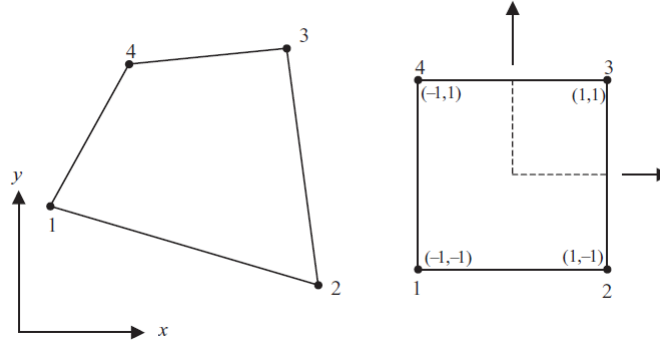


Figure 4.4: Quadrilateral element in the xy-plane.

Bilinear Quadrilateral Element

A bilinear quadrilateral element in the xy-plane is shown in Figure 4.5. The quadrilateral has four local nodes which are numbered in a counter-clockwise direction. Knowing the solution at the four nodes of the element, the primary unknown quantity can be evaluated at any point inside the element by using the appropriate interpolation functions. Anastasis C. Polycarpou(2006). A generic bilinear interpolation function for local nodes spanning the geometrical domain of the quadrilateral element has the form

$$u(x, y) = c_1 + c_2x + c_3y + c_4xy \quad (4.2.6)$$

Assuming an isoparametric quadrilateral element, the primary unknown quantity and the x and y space coordinates inside an element can be expressed in terms of basis functions given by:

$$u = u_1\Psi_1 + u_2\Psi_2 + u_3\Psi_3 + u_4\Psi_4 = \sum_{i=1}^4 u_i\Psi_i \quad (4.2.7)$$

And

$$x = x_1\Psi_1 + x_2\Psi_2 + x_3\Psi_3 + x_4\Psi_4 = \sum_{i=1}^4 x_i\Psi_i$$

$$y = y_1\Psi_1 + y_2\Psi_2 + y_3\Psi_3 + y_4\Psi_4 = \sum_{i=1}^4 y_i\Psi_i$$

4.2.3 Find the element properties

The matrix equation for the finite element should be established which relates the nodal values of the unknown function to other parameters. For this task different approaches can be used; the most convenient are: the variational approach and the Galerkin method.

4.2.4 Boundary conditions and Assembly

The incorporation of boundary conditions and the assembly of the system matrix also become more complicated when the finite-element technique is applied to two and three dimensional problems. However, as with the derivation of the element matrix, the difficulty relates to the mechanics of the process rather than to conceptual complexity. To

find the global equation system for the whole solution region we must assemble all the element equations. In other words we must combine local element equations for all elements used for discretization. Element connectivities are used for the assembly process. Before solution, boundary conditions (which are not accounted in element equations) should be imposed.

4.2.5 Solve the global equation system

. The finite element global equation system is typically sparse, symmetric and positive definite. Direct and iterative methods can be used for solution. The nodal values of the sought function are produced as a result of the solution.

4.3 Analysis of Finite Element Methods

Continuous Problem:

Let V, W be Hilbert Spaces, $B : V \times W \rightarrow \mathfrak{R}$ bounded bilinear form, $f \in W^*$, Find $u \in V$ such that $B(u, w) = \langle f, w \rangle, w \in W$
 $\Leftrightarrow Lu = f$, where $\langle Lu, w \rangle := B(u, w)$ Feng kang(2009)

Example

$$V = W = H_0^1(\Omega), B(v, w) = \int_{\Omega} \nabla v \nabla w \, ds, \langle f, v \rangle = \int_{\Omega} f v \, dx$$

Discrete Problem

Let V_h, W_h of equal finite dimension, $B_h : V_h \times W_h \rightarrow \mathfrak{R}$, bilinear, $f_h \in W_h^*$. Find $u_h \in V_h$ such that $B_h(u_h, w) = \langle f_h, w \rangle, w \in W_h$. This is a generalized Galerkin method. If V_h and W_h are piecewise polynomial spaces of a certain sort, it is a finite element method. Davies(2011).

Definition 4.3.1. We say that B is coercive on V if there exists a constant $\gamma \geq 0$ such that

$$B(u, u) \geq \gamma \|u\|_V^2, \text{ for all } u \in V$$

Definition 4.3.2. We say that the bilinear form B satisfies the inf-sup conditions on V if there exists $\gamma > 0$ such that

$$\sup_{v \in V} \frac{B(u, v)}{\|v\|_V} \geq \gamma \|u\|_V, \text{ for all } u \in V \quad (4.3.1)$$

and

$$\inf_{u \in V} \frac{B(u, v)}{\|u\|_V} \geq \gamma \|v\|_V, \text{ for all } v \in V \quad (4.3.2)$$

Remark 4.3.1. Clearly, if B is symmetric both conditions are the same.

Remark 4.3.2. Note that condition (4.3.1) (and analogously (4.3.2)) can be written as

$$\inf_{u \in V} \sup_{v \in V} \frac{B(u, v)}{\|u\|_V \|v\|_V} > 0$$

which justifies the usual terminology.

Remark 4.3.3. If B is coercive it satisfies the inf-sup conditions. In fact,

$$\sup_{v \in V} \frac{B(u, v)}{\|v\|_V} \geq \frac{B(u, u)}{\|u\|_V} \geq \gamma \|u\|_V$$

4.3.1 Stability Analysis

For a generalized Galerkin method, the *stability constant* can be expressed as the reciprocal of the *inf-sup constant*:

$$\|L_h^{-1}\| \tau(V_h, V_h) = \frac{1}{\gamma_h}, \text{ where } \gamma_h = \inf_{0 \neq v \in V_h} \sup_{0 \neq w \in V_h} \frac{B_h(v, w)}{\|v\|_V \|w\|_W}$$

So we need to bound γ_h below. This is easy if B_h is coercive:

$$B_h(v, v) \geq \gamma \|v\|^2,$$

but otherwise can be quite difficult. Feng kang(2009)

4.3.2 Consistency Analysis

For r_h take the V -orthogonal projection into V_h : $r_h u = \arg \min_{v \in V_h} \|u - v\|_V$. Feng kang(2009)

$$\text{consistency error} := \|L_h r_h u - f_h\|_{V_h^*} = \sup_{w \in V_h} \frac{|B_h(r_h u, w) - \langle f_h, w \rangle|}{\|w\|_V}$$

The simplest (and most important) case is a conforming FEM, i.e

$$B_h = B|_{v_h \times v_h}, f_h = f|_{v_h}$$

. Then the consistency error is $\sup_{v \in V_h} \frac{|B_h(r_h u, w) - \langle f_h, w \rangle|}{\|w\|_V}$

$$= \sup_{v \in V_h} \frac{|B_h(r_h u - f_h, w)|}{\|w\|_V}$$

$$\leq \|B\| \inf_{v \in V_h} \|u - v\|_V$$

For a conforming method the consistency error is bounded by the approximation error times $\|B\|$

In the general case

$$\text{Consistency error} \leq \|B_h\| \inf_{v \in V_h} \|u - v\|_V + \sup_{v \in V_h} \frac{|B_h(r_h u, w) - \langle f_h, w \rangle|}{\|w\|_V}$$

4.3.3 Convergence Analysis

By the fundamental theorem, for a conforming FEM we have

$$\|u_h - r_h u\|_V \leq \gamma_h^{-1} \|B\| \inf_{v \in V_h} \|u - v\|_V.$$

with the triangle inequality this becomes

$$\begin{aligned} \|u - u_h\|_V &\leq (1 + \gamma_h^{-1} \|B\|) \inf_{v \in V_h} \|u - v\|_V \\ \Rightarrow \|u - u_h\|_V &\leq c \inf_{v \in V_h} \|u - v\|_V \end{aligned} \quad (4.3.3)$$

Where $c = (1 + \gamma_h^{-1} \|B\|)$ This says that when measured in $H^1(\Omega)$, the error in the Galerkin approximation is no worse than a constant factor times the error in the best approximation to u from V_h (with the constant independent of the particular solution u and the particular subspace V_h). This property of the finite element solution is called *quasioptimal*. The fundamental quasioptimality result (4.3.3) reduces the error in a Galerkin approximation to a question of approximation theory. Davies(2011)

Theorem 4.3.1. *Let there be given a family of triangulations $\{\tau_h\}$ of a polygonal domain Ω and let $h = \max_{T \in \tau_h} \text{diam}(T)$. Let k be a positive integer. For each h let $r_h : C(\Omega) \rightarrow M_0^1(\tau_h)$ denote the nodal interpolant. Then there is a constant c such that*

$$\|u - r_h u\|_{L^\infty(\Omega)} \leq c h^{k+1} \|u^{(k+1)}\|_{L^\infty(\Omega)} \text{ for all } u \in C^{k+1}(\Omega)$$

$$\|u - r_h u\|_{L^2(\Omega)} \leq c h^{k+1} \|u^{(k+1)}\|_{L^2(\Omega)} \text{ for all } u \in H^{k+1}(\Omega)$$

If, moreover, the family of triangulations is shape regular, then there is a constant C such that

$$\|\text{grad}(u - r_h u)\|_{L^\infty(\Omega)} \leq C h^k \|u^{(k+1)}\|_{L^\infty(\Omega)} \text{ for all } u \in C^{k+1}(\Omega)$$

$$\|\text{grad}(u - r_h u)\|_{L^2(\Omega)} \leq C h^k \|u^{(k+1)}\|_{L^2(\Omega)} \text{ for all } u \in H^{k+1}(\Omega)$$

Douglas N. Arnold(2001). The combination of the quasioptimality estimate (4.3.3) and the bounds on interpolation error of the last subsection immediately gives an H^1 error estimate for the finite element method using piecewise linear elements:

$$\|u - u_h\|_1 \leq Ch \|D^2 u\|_{L^2(\Omega)},$$

or simply

$$\|u - u_h\|_1 \leq Ch \|u\|_2,$$

where the norm on the right-hand side is the *Sobolev 2 norm*. For elements of degree k the analogous result is

$$\|u - u_h\|_1 \leq Ch^k \|u\|_{k+1}$$

In these estimates the constant C doesn't grow as h tends to zero as long as the meshes remain shape regular. Thus the error in the finite element method with piecewise linear elements converges to 0 with order h in $H^1(\Omega)$ as the mesh is refined.

4.4 Finite Difference Method

In mathematics finite difference methods are numerical method for solving differential equations in which finite difference approximate the derivatives. Finite difference method is most commonly used method to solve a Ordinary Differential Equations and PDEs in a bounded domain. The basic idea of finite difference methods is simple: derivatives in differential equations are written in terms of discrete quantities of dependent and independent variables, resulting in simultaneous algebraic equations with all unknowns prescribed at discrete nodal points for the entire domain. In brief about FDM , the different unknowns are defined by their values on discrete (finite) grid and differential operators are replaced by difference operators using neighbouring points. Douglas N. Arnold(2000).

The basic idea of finite difference methods is to seek approximations to solutions of the PDE on a lattice. To approximate the derivatives appearing in the PDE, differences between lattice values at neighbouring points are used. A finite difference scheme is produced when the partial derivatives in the partial differential equation(s) governing a physical phenomenon are replaced by a finite difference approximation. The result is a single algebraic equation or a system of algebraic equations which, when solved, provide an approximation to the solution of the original partial differential equation at selected points of a solution grid. The solution grid (also referred to as computational grid or numerical grid) is originated by dividing the axes representing the independent variables in the solution domain into a number of intervals. The extreme points of the interval will represent points in the solution grid. If we draw lines perpendicular to a given axes passing through the extreme points of the intervals, the resulting grid is the computational grid.

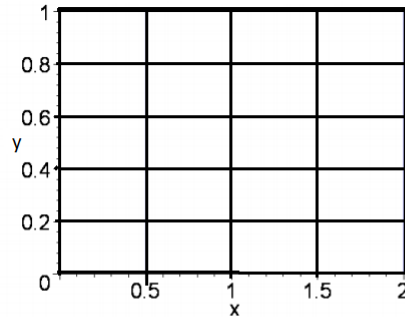


Figure 4.5: Discretization of finite difference method

Figure (4.6) shows the computational grid. Finite difference approximations result from replacing the partial derivatives in a governing equations by its finite-difference approximation. If the actual solution to a problem in an x-y computational domain is given by $u = u(x, y)$, the approximations in the nodes of a computational grid will be given by $u_{i,j} = u(x_i, y_j)$. The following are finite-difference approximations for the first derivative with respect to the spatial variable x , which we get from Taylor series expansion:

$$u_{i+1,j} = u_{i,j} + hu_x + ku_y + \frac{h^2}{2!}u_{xx} + \frac{k^2}{2!}u_{yy} + \frac{hk}{2!}u_{xy} + \dots, \quad \text{.....}^*$$

and

$$u_{i-1,j} = u_{i,j} - hu_x + ku_y + \frac{h^2}{2!}u_{xx} + \frac{k^2}{2!}u_{yy} - \frac{hk}{2!}u_{xy} + \dots, \quad \text{.....}^{**}$$

where $h = \Delta x, k = \Delta y$. From * and ** we obtain:

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h), \text{ Forward Difference}$$

$$u_x = \frac{u_{i,j} - u_{i-1,j}}{h} + O(h), \text{ Backward Difference}$$

$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2), \text{ Central Difference}$$

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2), \text{ Central difference.}$$

similarly,

$$u_{i,j+1} = u_{i,j} + hu_x + ku_y + \frac{h^2}{2!}u_{xx} + \frac{k^2}{2!}u_{yy} + \frac{hk}{2!}u_{xy} + \dots, \quad \text{.....}^{***}$$

and

$$u_{i,j-1} = u_{i,j} + hu_x - ku_y + \frac{h^2}{2!}u_{xx} + \frac{k^2}{2!}u_{yy} - \frac{hk}{2!}u_{xy} + \dots, \quad \text{.....}^{****}$$

from *** and **** we get:

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{h} + O(k), \text{ Forward Difference}$$

$$u_y = \frac{u_{i,j} - u_{i,j-1}}{k} + O(k), \text{ Backward Difference}$$

$$u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2), \text{ Central Difference}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + O(k^2)$$

The terms such as $O(\Delta x)$ or $O(\Delta x^2)$ indicate that the error incurred when using a particular finite difference approximation to replace the first derivative $\frac{\partial}{\partial x}u(x, y)$ by that approximation is proportional to either the spatial increment, Δx , or to its square. Typically Δx is a small quantity so that $\Delta x^2 < \Delta x$, therefore, a second order error represents a better approximation. Thus, the centered finite difference approximation shown above for the derivative $\frac{\partial}{\partial x}u(x, y)$ represents a better approximation than either the forward or backward finite difference approximations for the first derivative. The term $\Delta x = x_{i+1} - x_i = x_i - x_{i-1} = h$ represents a constant spatial interval. Thus, the spatial grid is said to be equally-spaced.

4.5 Analysis of Finite Difference Method

We will use the following partial differential equation (PDE) to illustrate the ideas of consistency, convergence, and stability of a particular finite-difference numerical scheme. Consider Poisson's equation:

$$u_{xx} + u_{yy} = -f(x, y)$$

and it's subject to the following initial and boundary conditions:

$$u(x, 0) = u_0(x), \text{ initial condition}$$

$$u(0, y) = u_1(y), \text{ boundary condition .}$$

The approximations that we will select for the derivatives will be the following:

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

similarly,

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Thus, the finite difference scheme for this governing equation will be given by the following algebraic equation

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = -f(x, y) \quad [A]$$

4.5.1 Consistency Analysis

A finite difference scheme or operator is consistent if the operator reduces to the original differential equation as the increments in the independent variables vanish. For the present case, the finite differences shown in [A], above, will reproduce the original equation as $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$. This can be shown by noticing that

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = u_{xx} + O(\Delta x^2)$$

and

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = u_{yy} + O(\Delta y^2)$$

Thus, equation [A] can be written as

$$u_{xx} + O(\Delta x^2) + u_{yy} + O(\Delta y^2)$$

Thus, as $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$, the scheme shown above reduces to the original equation, namely,

$$u_{xx} + 0 + u_{yy} + 0 = f(x, y)$$

and, therefore, the proposed scheme for the numerical solution of this equation is *consistent*.

It should be point out that any finite difference scheme based on reasonable approximations of the derivatives should be consistent. However, we should also check the scheme for convergence and stability, as shown next.

4.5.2 Convergence and Stability (Von Neumann) analysis

These two characteristics of a numerical scheme in the solution of partial differential equations can be analyzed simultaneously as illustrated next.

Stability means that the error caused by a small perturbation in the numerical solution remains bound. Stability is harder to prove than consistency. Stability can be proven using either

- Eigenvalue analysis of the matrix representation of the FDM.
- Fourier analysis on the grid (von Neumann analysis)
- Computing the domain of dependence of the numerical method.
- *Lax-Equivalence Theorem* : A consistent approximation to a well-posed problem is convergent if and only if it is stable.

Convergence means that the finite-difference solution approaches the true solution to the partial differential equation as the increments $\Delta x, \Delta y$ go to zero.

Example: The analytical solutions of the Poisson's equation $u_{xx} + u_{yy} = -f$ can be found in the form

$$u(x, y) = \sum_{-\infty}^{\infty} e^{i\alpha x} e^{\beta y}$$

with $(\beta^2 - \alpha^2) \neq 0$. Here $e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$

To analyze the growth of different Fourier modes as they evolve under the numerical scheme we consider each frequency separately, namely

$u(x, y) = \sum_{-\infty}^{\infty} e^{i\alpha x} e^{\beta y}$. In the discrete case we assume that $U_j^k = G^k e^{i\alpha \Delta x}$ with $G = e^{\beta \Delta y}$. Any growth in the solution will be due to the presence of terms involving G . Thus requiring that this amplification factor G is bounded by one as $k \rightarrow \infty$ gives rise to a relation between Δy and Δx called the *von Neumann stability condition*.

Chapter 5

Result and Discussions

5.1 Numerical Examples

1) Example for finite element method

Consider the problem

$$-\nabla^2 u = 2(x + y) - 4 \quad \text{on} \quad [0, 1]^2$$

With homogeneous boundary conditions $u = 0$

Find the finite element solution using four triangular elements.(Davies(2011),Exercise page 116.)

SOLUTION

For this discretization $h = 1/2$ and $k = 1/2$.

We define

$$\sum_{j=1}^n k_{ij}^e u_j^e = f_i^e + Q_i^e$$

Where

$$k_{ij}^e = \int_{\Omega} \left[\frac{\partial \Psi_i^e}{\partial x} \frac{\partial \Psi_j^e}{\partial x} + \frac{\partial \Psi_i^e}{\partial y} \frac{\partial \Psi_j^e}{\partial y} \right] dx dy$$

$$f_i^e = \int_{\Omega} [\Psi_i^e f] dx dy$$

$$Q_i^e = \oint_{\Gamma} \Psi_i^e q_n ds$$

Where

$$\Psi_1 = \frac{1}{2A_e} [(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$\Psi_2 = \frac{1}{2A_e} [(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$\Psi_3 = \frac{1}{2A_e} [(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

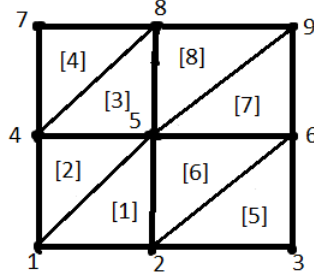


Figure 5.1: Finite element discretization with four triangular elements

and the stiffness matrix for each elements becomes

$$K^e = \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix}$$

for element 1

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 0), (1/2, 0), (1/2, 1/2)$

Area of element 1

$$A_1 = \frac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)] = \frac{1}{8}$$

Shape functions of element 1:

$$\begin{aligned} \Psi_1^1 &= \frac{1}{2A_1}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] = 1 - 2x \\ \Psi_2^1 &= \frac{1}{2A_1}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] = 2x - 2y \\ \Psi_3^1 &= \frac{1}{2A_1}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] = 2y \end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^1}{\partial x} = -2, \quad \frac{\partial \Psi_1^1}{\partial y} = 0, \quad \frac{\partial \Psi_2^1}{\partial x} = 2, \quad \frac{\partial \Psi_2^1}{\partial y} = -2, \quad \frac{\partial \Psi_3^1}{\partial x} = 0, \quad \frac{\partial \Psi_3^1}{\partial y} = 2$$

Stiffness matrix for element 1

$$k_{ij}^e = \int_{\Omega} \left[\frac{\partial \Psi_i^1}{\partial x} \frac{\partial \Psi_j^1}{\partial x} + \frac{\partial \Psi_i^1}{\partial y} \frac{\partial \Psi_j^1}{\partial y} \right] dx dy$$

and

Note that $k_{ij}^e = k_{ji}^e$

$$\begin{aligned}
k_{11}^1 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_1^1}{\partial x} \frac{\partial \Psi_1^1}{\partial x} + \frac{\partial \Psi_1^1}{\partial y} \frac{\partial \Psi_1^1}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(-2)(-2) + (0)(0)] dx dy = 1 \\
k_{12}^1 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_1^1}{\partial x} \frac{\partial \Psi_2^1}{\partial x} + \frac{\partial \Psi_1^1}{\partial y} \frac{\partial \Psi_2^1}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(-2)(2) + 0 * (-2)] dx dy = -1 \\
k_{13}^1 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_1^1}{\partial x} \frac{\partial \Psi_3^1}{\partial x} + \frac{\partial \Psi_1^1}{\partial y} \frac{\partial \Psi_3^1}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(-2)(0) + 0 * (0)] dx dy = 0 \\
k_{22}^1 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_2^1}{\partial x} \frac{\partial \Psi_2^1}{\partial x} + \frac{\partial \Psi_2^1}{\partial y} \frac{\partial \Psi_2^1}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(2)(2) + (-2)(-2)] dx dy = 2 \\
k_{23}^1 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_2^1}{\partial x} \frac{\partial \Psi_3^1}{\partial x} + \frac{\partial \Psi_2^1}{\partial y} \frac{\partial \Psi_3^1}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(2)(0) + (-2)(2)] dx dy = -1 \\
k_{33}^1 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_3^1}{\partial x} \frac{\partial \Psi_3^1}{\partial x} + \frac{\partial \Psi_3^1}{\partial y} \frac{\partial \Psi_3^1}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(0)(0) + (2)(2)] dx dy = 1
\end{aligned}$$

$$K^1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

for element 2

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 0), (1/2, 1/2), (0, 1/2)$

Area of element 2

$$A_2 = \frac{1}{2}[(x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)] = \frac{1}{8}$$

Shape functions of element 2:

$$\begin{aligned}
\Psi_1^2 &= \frac{1}{2A_2}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y] = 1 - 2y \\
\Psi_2^2 &= \frac{1}{2A_2}[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y] = 2x \\
\Psi_3^2 &= \frac{1}{2A_2}[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y] = -2x + 2y
\end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^2}{\partial x} = 0, \quad \frac{\partial \Psi_1^2}{\partial y} = -2, \quad \frac{\partial \Psi_2^2}{\partial x} = 2, \quad \frac{\partial \Psi_2^2}{\partial y} = 0, \quad \frac{\partial \Psi_3^2}{\partial x} = -2, \quad \frac{\partial \Psi_3^2}{\partial y} = 2$$

Stiffness matrix for element 1

$$\begin{aligned}
k_{11}^2 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_1^2}{\partial x} \frac{\partial \Psi_1^2}{\partial x} + \frac{\partial \Psi_1^2}{\partial y} \frac{\partial \Psi_1^2}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(0)(0) + (-2)(-2)] dx dy = 1 \\
k_{12}^2 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_1^2}{\partial x} \frac{\partial \Psi_2^2}{\partial x} + \frac{\partial \Psi_1^2}{\partial y} \frac{\partial \Psi_2^2}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(0)(2) + (-2)(0)] dx dy = 0 \\
k_{13}^2 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_1^2}{\partial x} \frac{\partial \Psi_3^2}{\partial x} + \frac{\partial \Psi_1^2}{\partial y} \frac{\partial \Psi_3^2}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(0)(-2) + (-2)(2)] dx dy = -1 \\
k_{22}^2 &= \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_2^2}{\partial x} \frac{\partial \Psi_2^2}{\partial x} + \frac{\partial \Psi_2^2}{\partial y} \frac{\partial \Psi_2^2}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(2)(2) + (0)(0)] dx dy = 1
\end{aligned}$$

$$k_{23}^2 = \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_2^2}{\partial x} \frac{\partial \Psi_3^2}{\partial x} + \frac{\partial \Psi_2^2}{\partial y} \frac{\partial \Psi_3^2}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(2)(-2) + (0)(2)] dx dy = -1$$

$$k_{33}^2 = \int_0^{1/2} \int_0^{1/2} \frac{\partial \Psi_3^2}{\partial x} \frac{\partial \Psi_3^2}{\partial x} + \frac{\partial \Psi_3^2}{\partial y} \frac{\partial \Psi_3^2}{\partial y} dx dy = \int_0^{1/2} \int_0^{1/2} [(-2)(-2) + (2)(2)] dx dy = 2$$

$$K^2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

for element 3

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 1/2), (1/2, 1/2), (1/2, 1)$

Area of element 3

$$A_3 = \frac{1}{2}[(x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)] = \frac{1}{8}$$

Shape functions of element 3:

$$\begin{aligned} \Psi_1^3 &= \frac{1}{2A_3}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y = 1 - 2x \\ \Psi_2^3 &= \frac{1}{2A_3}[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y = 1 + 2x - 2y \\ \Psi_3^3 &= \frac{1}{2A_3}[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y = -1 + 2y \end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^3}{\partial x} = -2, \quad \frac{\partial \Psi_1^3}{\partial y} = 0, \quad \frac{\partial \Psi_2^3}{\partial x} = 2, \quad \frac{\partial \Psi_2^3}{\partial y} = -2, \quad \frac{\partial \Psi_3^3}{\partial x} = 0, \quad \frac{\partial \Psi_3^3}{\partial y} = 2$$

Stiffness matrix for element 3

$$k_{11}^3 = \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_1^3}{\partial x} \frac{\partial \Psi_1^3}{\partial x} + \frac{\partial \Psi_1^3}{\partial y} \frac{\partial \Psi_1^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/2} [(-2)(-2) + (0)(0)] dx dy = 1$$

$$k_{12}^3 = \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_1^3}{\partial x} \frac{\partial \Psi_2^3}{\partial x} + \frac{\partial \Psi_1^3}{\partial y} \frac{\partial \Psi_2^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/2} [(-2)(2) + (0)(-2)] dx dy = -1$$

$$k_{13}^3 = \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_1^3}{\partial x} \frac{\partial \Psi_3^3}{\partial x} + \frac{\partial \Psi_1^3}{\partial y} \frac{\partial \Psi_3^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/2} [(-2)(0) + (0)(2)] dx dy = 0$$

$$k_{22}^3 = \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_2^3}{\partial x} \frac{\partial \Psi_2^3}{\partial x} + \frac{\partial \Psi_2^3}{\partial y} \frac{\partial \Psi_2^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/2} [(2)(2) + (-2)(-2)] dx dy = 2$$

$$k_{23}^3 = \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_2^3}{\partial x} \frac{\partial \Psi_3^3}{\partial x} + \frac{\partial \Psi_2^3}{\partial y} \frac{\partial \Psi_3^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/2} [(2)(0) + (-2)(2)] dx dy = -1$$

$$k_{33}^3 = \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_3^3}{\partial x} \frac{\partial \Psi_3^3}{\partial x} + \frac{\partial \Psi_3^3}{\partial y} \frac{\partial \Psi_3^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/2} [(0)(0) + (2)(2)] dx dy = 1$$

$$K^3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

for element 4

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 1/2), (1/2, 1), (0, 1)$

Area of element 3

$$A_4 = \frac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)] = \frac{1}{8}$$

Shape functions of element 4:

$$\begin{aligned}\Psi_1^4 &= \frac{1}{2A_4}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] = 2 - 2y \\ \Psi_2^3 &= \frac{1}{2A_4}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] = 2x \\ \Psi_3^4 &= \frac{1}{2A_4}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] = -1 - 2x + 2y\end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^4}{\partial x} = 0, \quad \frac{\partial \Psi_1^4}{\partial y} = -2, \quad \frac{\partial \Psi_2^4}{\partial x} = 2, \quad \frac{\partial \Psi_2^4}{\partial y} = 0, \quad \frac{\partial \Psi_3^4}{\partial x} = -2, \quad \frac{\partial \Psi_3^4}{\partial y} = 2$$

Stiffness matrix for element 4

$$\begin{aligned}k_{11}^4 &= \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_1^4}{\partial x} \frac{\partial \Psi_1^4}{\partial x} + \frac{\partial \Psi_1^4}{\partial y} \frac{\partial \Psi_1^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/2} [(0)(0) + (-2)(-2)] dxdy = 1 \\ k_{12}^4 &= \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_1^4}{\partial x} \frac{\partial \Psi_2^4}{\partial x} + \frac{\partial \Psi_1^4}{\partial y} \frac{\partial \Psi_2^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/2} [(0)(2) + (-2)(-2)] dxdy = 0 \\ k_{13}^4 &= \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_1^4}{\partial x} \frac{\partial \Psi_3^4}{\partial x} + \frac{\partial \Psi_1^4}{\partial y} \frac{\partial \Psi_3^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/2} [(0)(-2) + (-2)(2)] dxdy = -1 \\ k_{22}^4 &= \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_2^4}{\partial x} \frac{\partial \Psi_2^4}{\partial x} + \frac{\partial \Psi_2^4}{\partial y} \frac{\partial \Psi_2^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/2} [(2)(2) + (0)(0)] dxdy = 1 \\ k_{23}^4 &= \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_2^4}{\partial x} \frac{\partial \Psi_3^4}{\partial x} + \frac{\partial \Psi_2^4}{\partial y} \frac{\partial \Psi_3^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/2} [(2)(-2) + (0)(2)] dxdy = -1 \\ k_{33}^4 &= \int_{1/2}^1 \int_0^{1/2} \frac{\partial \Psi_3^4}{\partial x} \frac{\partial \Psi_3^4}{\partial x} + \frac{\partial \Psi_3^4}{\partial y} \frac{\partial \Psi_3^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/2} [(-2)(-2) + (2)(2)] dxdy = 2\end{aligned}$$

$$K^4 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Similarly, we obtain

$$K^5 = K^7 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

And

$$K^6 = K^8 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Finding the load vector for each elements

Where $f = 2(x + y) - 4$, then

$$f_i^e = \int_{\Omega} f \Psi_i^e + Q_i^e$$

We only find the load vector f_3^3, f_2^4 and f_1^5 since they are only the load force at node 5.

Hence,

$$f_3^3 = \int_{1/2}^1 \int_0^{1/2} \Psi_3^3 f dx dy = \int_{1/2}^1 \int_0^{1/2} [(-1 + 2y)(2(x + y) - 4)] dx dy = \frac{7}{48}$$

$$f_2^4 = \int_{1/2}^1 \int_0^{1/2} \Psi_2^4 f dx dy = \int_{1/2}^1 \int_0^{1/2} [(-1 + 2x - 2y)(2(x + y) - 4)] dx dy = 1$$

$$f_1^5 = \int_0^{1/2} \int_0^{1/2} [\Psi_1^5 f] dx dy = \int_0^{1/2} \int_0^{1/2} [(2 - 2x)(2(x + y) - 4)] dx dy = -\frac{5}{6}$$

Assembly of the elements

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 5 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 5 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 3 \end{bmatrix}$$

And

$$F = \begin{bmatrix} F^1 & F^2 & F^3 & F^4 & F^5 & F^6 & F^7 & F^8 \end{bmatrix}^T$$

The global system of matrix becomes $KU = F$.

Hence

$$-u_4 + 3u_5 - 2u_6 = F^5$$

Where $F^5 = f_3^3 + f_2^4 + f_1^5$ and the given boundary conditions $u_1 = u_2 = u_3 = u_4 = u_6 = u_7 = u_8 = 0$ and $F^5 = 0.312506$

$$3u_5 = 0.312506$$

$$u_5 = \frac{0.312506}{3}$$

$$u_5 = 0.1041687$$

The exact solution of the given pde is:

$$u(x, y) = (1 - x)y^2 + (1 - y)x^2$$

$$u(1/2, 1/2) = 0.25$$

Node	x	y	FEM	Exact	Error
1	1/2	1/2	0.1041687	0.25	0.1458313

Table 5.1: Comparison of finite element method with exact value

2) Example for Finite Difference Method

Consider the problem

$$-\nabla^2 u = 2(x + y) - 4 \quad \text{on} \quad [0, 1]^2$$

With homogeneous Dirichlet boundary conditions $u = 0$

$$-\nabla^2 u = 2(x + y) - 4$$

SOLUTION

Domain discretization

Divide the rectangle into 2 by 2 rectangles as figure below. $h = 1/2, k = 1/2$

Replace the PDE by the difference equation

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

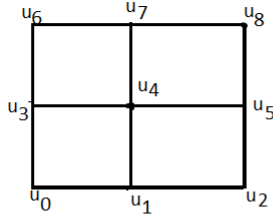


Figure 5.2: FDM discretization with one inner node

$$u_{xx} + u_{yy} = -f \implies \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = -f_{ij}$$

$$u_{ij} = \frac{1}{4(h^2+k^2)}[k^2(u_{i+1,j} + u_{i-1,j}) + h^2(u_{i,j+1} + u_{i,j-1}) + h^2k^2f_{ij}]$$

where, $f_{ij} = 2(x_i + y_j) - 4$, $h = \frac{1}{2}$, $k = \frac{1}{2}$

Hence,

$$u_4 \approx u_{11} = \frac{1}{4(h^2+k^2)}[k^2(u_{21} + u_{01}) + h^2(u_{12} + u_{10}) + h^2k^2f_{11}]$$

$$u_4 \approx u_{11} = \frac{1}{4(1/4+1/4)}[1/4(u_{21} + u_{01}) + 1/4(u_{12} + u_{10}) + 1/16(2(1/2 + 1/2) - 4)]$$

By using the given boundary conditions we get,

$$u_4 \approx u_{11} = 1/2[1/4(0 + 0) + 1/4(0 + 0) + 1/16(2 - 4)]$$

$$u_4 \approx u_{11} = 1/2[-2/16]$$

$$u_4 \approx u_{11} = -\frac{1}{16} = -0.0625$$

Node	x	y	FDM	Exact	Error
1	1/2	1/2	-0.0625	0.25	0.3125

Table 5.2: Comparison of finite difference method with exact value

3) Example for finite element method

Consider the problem

$$-\nabla^2 u = 2(x + y) - 4 \quad \text{on} \quad [0, 1]^2$$

With homogeneous Dirichlet boundary conditions $u = 0$

Find the finite element solution using six triangular elements.

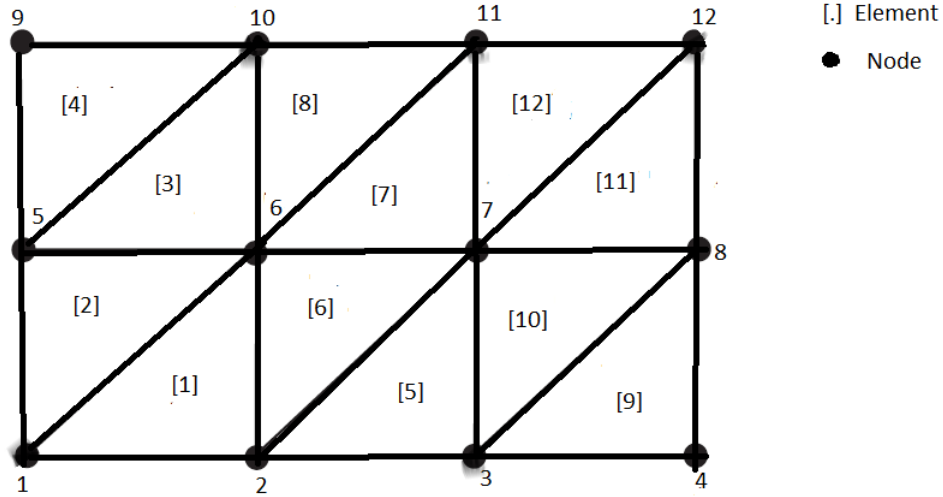


Figure 5.3: Finite element discretization with six triangular elements

For this discretization $h = 1/3$. and $k = 1/2$

SOLUTION

We define

$$\sum_{j=1}^n k_{ij}^e u_j^e = f_i^e + Q_i^e$$

Where

$$k_{ij}^e = \int_{\Omega} \left[\frac{\partial \Psi_i^e}{\partial x} \frac{\partial \Psi_j^e}{\partial x} + \frac{\partial \Psi_i^e}{\partial y} \frac{\partial \Psi_j^e}{\partial y} \right] dx dy$$

$$f_i^e = \int_{\Omega} [\Psi_i^e f] dx dy$$

$$Q_i^e = \oint_{\Gamma} \Psi_i^e q_n ds$$

Where

$$\Psi_1 = \frac{1}{2A_e} [(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$\Psi_2 = \frac{1}{2A_e} [(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$\Psi_3 = \frac{1}{2A_e} [(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

and the stiffness matrix for each elements becomes

$$K^e = \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix}$$

Finding the stiffness matrix for each elements

for element 1

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 0), (1/3, 0), (1/3, 1/2)$.

Area of element 1

$$A_1 = \frac{1}{2}[(x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)] = \frac{1}{12}$$

Shape functions of element 1:

$$\begin{aligned} \Psi_1^1 &= \frac{1}{2A_1}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y = 1 - 3x \\ \Psi_2^1 &= \frac{1}{2A_1}[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y = 3x - 2y \\ \Psi_3^1 &= \frac{1}{2A_1}[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y = 2y \end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^1}{\partial x} = -3, \quad \frac{\partial \Psi_1^1}{\partial y} = 0, \quad \frac{\partial \Psi_2^1}{\partial x} = 3, \quad \frac{\partial \Psi_2^1}{\partial y} = -2, \quad \frac{\partial \Psi_3^1}{\partial x} = 0, \quad \frac{\partial \Psi_3^1}{\partial y} = 2$$

Stiffness matrix for element 1

$$k_{ij}^e = \int_{\Omega} \left[\frac{\partial \Psi_i^1}{\partial x} \frac{\partial \Psi_j^1}{\partial x} + \frac{\partial \Psi_i^1}{\partial y} \frac{\partial \Psi_j^1}{\partial y} \right] dx dy$$

$$k_{11}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_1^1}{\partial x} \frac{\partial \Psi_1^1}{\partial x} + \frac{\partial \Psi_1^1}{\partial y} \frac{\partial \Psi_1^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(-3)(-3) + (0)(0)] dx dy = \frac{3}{2}$$

$$k_{12}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_1^1}{\partial x} \frac{\partial \Psi_2^1}{\partial x} + \frac{\partial \Psi_1^1}{\partial y} \frac{\partial \Psi_2^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(-3)(3) + (0)(-2)] dx dy = -\frac{3}{2}$$

$$k_{13}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_1^1}{\partial x} \frac{\partial \Psi_3^1}{\partial x} + \frac{\partial \Psi_1^1}{\partial y} \frac{\partial \Psi_3^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(-3)(0) + (0)(2)] dx dy = 0$$

$$k_{21}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_2^1}{\partial x} \frac{\partial \Psi_1^1}{\partial x} + \frac{\partial \Psi_2^1}{\partial y} \frac{\partial \Psi_1^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(3)(-3) + (-2)(0)] dx dy = -\frac{3}{2}$$

$$k_{22}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_2^1}{\partial x} \frac{\partial \Psi_2^1}{\partial x} + \frac{\partial \Psi_2^1}{\partial y} \frac{\partial \Psi_2^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(3)(3) + (-2)(-2)] dx dy = \frac{13}{6}$$

$$k_{23}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_2^1}{\partial x} \frac{\partial \Psi_3^1}{\partial x} + \frac{\partial \Psi_2^1}{\partial y} \frac{\partial \Psi_3^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(3)(0) + (-2)(2)] dx dy = -\frac{2}{3}$$

$$k_{31}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_3^1}{\partial x} \frac{\partial \Psi_1^1}{\partial x} + \frac{\partial \Psi_3^1}{\partial y} \frac{\partial \Psi_1^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(0)(-3) + (2)(0)] dx dy = 0$$

$$k_{32}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_3^1}{\partial x} \frac{\partial \Psi_2^1}{\partial x} + \frac{\partial \Psi_3^1}{\partial y} \frac{\partial \Psi_2^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(0)(3) + (2)(-2)] dx dy = -\frac{2}{3}$$

$$k_{33}^1 = \int_0^{1/2} \int_0^{1/3} \left[\frac{\partial \Psi_3^1}{\partial x} \frac{\partial \Psi_3^1}{\partial x} + \frac{\partial \Psi_3^1}{\partial y} \frac{\partial \Psi_3^1}{\partial y} \right] dx dy = \int_0^{1/2} \int_0^{1/3} [(0)(0) + (2)(2)] dx dy = \frac{2}{3}$$

$$K^1 = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{13}{6} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Note that $k_{ij}^e = k_{ji}^e$

For element 2

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 0), (1/3, 1/2), (0, 1/2)$

Area of element 2:

$$A_2 = \frac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)] = \frac{1}{12}$$

Shape functions of element 2:

$$\begin{aligned} \Psi_1^2 &= \frac{1}{2A_2}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] = 1 - 2y \\ \Psi_2^2 &= \frac{1}{2A_2}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] = 3x \\ \Psi_3^2 &= \frac{1}{2A_2}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] = -3x + 2y \end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^2}{\partial x} = 0, \quad \frac{\partial \Psi_1^2}{\partial y} = -2, \quad \frac{\partial \Psi_2^2}{\partial x} = 3, \quad \frac{\partial \Psi_2^2}{\partial y} = 0, \quad \frac{\partial \Psi_3^2}{\partial x} = -3, \quad \frac{\partial \Psi_3^2}{\partial y} = 2$$

Stiffness matrix for element 2:

$$\begin{aligned} k_{11}^2 &= \int_0^{1/2} \int_0^{1/3} \frac{\partial \Psi_1^2}{\partial x} \frac{\partial \Psi_1^2}{\partial x} + \frac{\partial \Psi_1^2}{\partial y} \frac{\partial \Psi_1^2}{\partial y} dxdy = \int_0^{1/2} \int_0^{1/3} [(0)(0) + (-2)(-2)] dxdy = \frac{2}{3} \\ k_{12}^2 &= \int_0^{1/2} \int_0^{1/3} \frac{\partial \Psi_1^2}{\partial x} \frac{\partial \Psi_2^2}{\partial x} + \frac{\partial \Psi_1^2}{\partial y} \frac{\partial \Psi_2^2}{\partial y} dxdy = \int_0^{1/2} \int_0^{1/3} [(0)(3) + (-2)(0)] dxdy = 0 \\ k_{13}^2 &= \int_0^{1/2} \int_0^{1/3} \frac{\partial \Psi_1^2}{\partial x} \frac{\partial \Psi_3^2}{\partial x} + \frac{\partial \Psi_1^2}{\partial y} \frac{\partial \Psi_3^2}{\partial y} dxdy = \int_0^{1/2} \int_0^{1/3} [(0)(-3) + (-2)(2)] dxdy = -\frac{2}{3} \\ k_{21}^2 &= k_{12}^2 = 0 \\ k_{22}^2 &= \int_0^{1/2} \int_0^{1/3} \frac{\partial \Psi_2^2}{\partial x} \frac{\partial \Psi_2^2}{\partial x} + \frac{\partial \Psi_2^2}{\partial y} \frac{\partial \Psi_2^2}{\partial y} dxdy = \int_0^{1/2} \int_0^{1/3} [(3)(3) + (0)(0)] dxdy = \frac{3}{2} \\ k_{23}^2 &= \int_0^{1/2} \int_0^{1/3} \frac{\partial \Psi_2^2}{\partial x} \frac{\partial \Psi_3^2}{\partial x} + \frac{\partial \Psi_2^2}{\partial y} \frac{\partial \Psi_3^2}{\partial y} dxdy = \int_0^{1/2} \int_0^{1/3} [(3)(-3) + (0)(-2)] dxdy = -\frac{3}{2} \\ k_{31}^2 &= k_{13}^2 = -\frac{2}{3} \\ k_{32}^2 &= k_{23}^2 = -\frac{3}{2} \\ k_{33}^2 &= \int_0^{1/2} \int_0^{1/3} \frac{\partial \Psi_3^2}{\partial x} \frac{\partial \Psi_3^2}{\partial x} + \frac{\partial \Psi_3^2}{\partial y} \frac{\partial \Psi_3^2}{\partial y} dxdy = \int_0^{1/2} \int_0^{1/3} [(-3)(-3) + (2)(2)] dxdy = \frac{13}{6} \end{aligned}$$

$$K^2 = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ -\frac{2}{3} & -\frac{3}{2} & \frac{13}{6} \end{bmatrix}$$

For element 3

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 1/2), (1/3, 1/2), (1/3, 1)$

Area of element 3:

$$A_3 = \frac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)] = \frac{1}{12}$$

Shape functions of element 3:

$$\begin{aligned} \Psi_1^3 &= \frac{1}{2A_3}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] = 1 - 3x \\ \Psi_2^3 &= \frac{1}{2A_3}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] = 1 + 3x - 2y \\ \Psi_3^3 &= \frac{1}{2A_3}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] = -1 + 2y \end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^3}{\partial x} = -3, \quad \frac{\partial \Psi_1^3}{\partial y} = 0, \quad \frac{\partial \Psi_2^3}{\partial x} = 3, \quad \frac{\partial \Psi_2^3}{\partial y} = -2, \quad \frac{\partial \Psi_3^3}{\partial x} = 0, \quad \frac{\partial \Psi_3^3}{\partial y} = 2$$

Stiffness matrix for element 3:

$$\begin{aligned} k_{11}^3 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_1^3}{\partial x} \frac{\partial \Psi_1^3}{\partial x} + \frac{\partial \Psi_1^3}{\partial y} \frac{\partial \Psi_1^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/3} [(-3)(-3) + (0)(0)] dx dy = \frac{3}{2} \\ k_{12}^3 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_1^3}{\partial x} \frac{\partial \Psi_2^3}{\partial x} + \frac{\partial \Psi_1^3}{\partial y} \frac{\partial \Psi_2^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/3} [(-3)(3) + (0)(-2)] dx dy = -\frac{3}{2} \\ k_{13}^3 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_1^3}{\partial x} \frac{\partial \Psi_3^3}{\partial x} + \frac{\partial \Psi_1^3}{\partial y} \frac{\partial \Psi_3^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/3} [(-3)(0) + (0)(2)] dx dy = 0 \\ k_{21}^3 &= k_{12}^3 = -\frac{3}{2} \\ k_{22}^3 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_2^3}{\partial x} \frac{\partial \Psi_2^3}{\partial x} + \frac{\partial \Psi_2^3}{\partial y} \frac{\partial \Psi_2^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/3} [(3)(3) + (-2)(-2)] dx dy = \frac{13}{6} \\ k_{23}^3 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_2^3}{\partial x} \frac{\partial \Psi_3^3}{\partial x} + \frac{\partial \Psi_2^3}{\partial y} \frac{\partial \Psi_3^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/3} [(3)(0) + (-2)(2)] dx dy = -\frac{2}{3} \\ k_{31}^3 &= k_{13}^3 = 0 \\ k_{32}^3 &= k_{23}^3 = -\frac{2}{3} \\ k_{33}^3 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_3^3}{\partial x} \frac{\partial \Psi_3^3}{\partial x} + \frac{\partial \Psi_3^3}{\partial y} \frac{\partial \Psi_3^3}{\partial y} dx dy = \int_{1/2}^1 \int_0^{1/3} [(0)(0) + (2)(2)] dx dy = \frac{2}{3} \end{aligned}$$

$$K^3 = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{13}{6} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

for element 4

nodes: $(x_1, y_1), (x_2, y_2), (x_3, y_3) = (0, 1/2), (1/3, 1), (0, 1/2)$.

Area of element 4

$$A_4 = \frac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)] = \frac{1}{12}$$

Shape functions of element 1:

$$\begin{aligned} \Psi_1^4 &= \frac{1}{2A_4}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y = 1 - 3x \\ \Psi_2^4 &= \frac{1}{2A_4}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y = 3x - 2y \\ \Psi_3^4 &= \frac{1}{2A_4}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y = 2y \end{aligned}$$

The derivative of shape functions:

$$\frac{\partial \Psi_1^1}{\partial x} = 0, \quad \frac{\partial \Psi_1^1}{\partial y} = -2, \quad \frac{\partial \Psi_2^1}{\partial x} = 3, \quad \frac{\partial \Psi_2^1}{\partial y} = 0, \quad \frac{\partial \Psi_3^1}{\partial x} = -3, \quad \frac{\partial \Psi_3^1}{\partial y} = 2$$

Stiffness matrix for element 4

$$\begin{aligned} k_{11}^4 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_1^4}{\partial x} \frac{\partial \Psi_1^4}{\partial x} + \frac{\partial \Psi_1^4}{\partial y} \frac{\partial \Psi_1^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/3} [(0)(0) + (-2)(-2)] dxdy = \frac{2}{3} \\ k_{12}^4 &= \int_0^{1/2} \int_0^{1/3} \frac{\partial \Psi_1^4}{\partial x} \frac{\partial \Psi_2^4}{\partial x} + \frac{\partial \Psi_1^4}{\partial y} \frac{\partial \Psi_2^4}{\partial y} dxdy = \int_{1/2}^1 8 \int_0^{1/3} [(0)(3) + (-2)(0)] dxdy = 0 \\ k_{13}^4 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_1^4}{\partial x} \frac{\partial \Psi_3^4}{\partial x} + \frac{\partial \Psi_1^4}{\partial y} \frac{\partial \Psi_3^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/3} [(0)(-3) + (-2)(2)] dxdy = -\frac{2}{3} \\ k_{21}^4 &= k_{12}^4 = 0 \\ k_{22}^4 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_2^4}{\partial x} \frac{\partial \Psi_2^4}{\partial x} + \frac{\partial \Psi_2^4}{\partial y} \frac{\partial \Psi_2^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/3} [(3)(3) + (0)(0)] dxdy = \frac{3}{2} \\ k_{23}^4 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_2^4}{\partial x} \frac{\partial \Psi_3^4}{\partial x} + \frac{\partial \Psi_2^4}{\partial y} \frac{\partial \Psi_3^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/3} [(3)(-3) + (0)(-2)] dxdy = -\frac{3}{2} \\ k_{31}^4 &= k_{13}^4 = -\frac{2}{3} \\ k_{32}^4 &= k_{23}^4 = -\frac{3}{2} \\ k_{33}^4 &= \int_{1/2}^1 \int_0^{1/3} \frac{\partial \Psi_3^4}{\partial x} \frac{\partial \Psi_3^4}{\partial x} + \frac{\partial \Psi_3^4}{\partial y} \frac{\partial \Psi_3^4}{\partial y} dxdy = \int_{1/2}^1 \int_0^{1/3} [(-3)(-3) + (2)(2)] dxdy = \frac{13}{6} \end{aligned}$$

$$K^4 = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ -\frac{2}{3} & -\frac{3}{2} & \frac{13}{6} \end{bmatrix}$$

Similarly, we get

$$K^5 = K^7 = K^9 = K^{11} = K^3 = K^1 = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{13}{6} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

and

$$K^6 = K^8 = K^{10} = K^{12} = K^4 = K^2 = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ -\frac{2}{3} & -\frac{3}{2} & \frac{13}{6} \end{bmatrix}$$

Finding the load vector for each elements

$f = 2(x + y) - 4$, then

$$f_i^e = \int_{\Omega} f \Psi_i^e + Q_i^e$$

Hence,

$$f_1^1 = \int_0^{1/2} \int_0^{1/3} f \Psi_1^1 dx dy + Q_1^1$$

$$f_1^1 = \int_0^{1/2} \int_0^{1/3} (1 - 3x)(2(x + y) - 4) dx dy + Q_1^1$$

$$f_1^1 = \int_0^{1/2} \int_0^{1/3} (14x - 6x^2 - 6xy + 2y - 4) dx dy + Q_1^1$$

$$f_1^1 = -\frac{59}{216} + Q_1^1$$

$$f_2^1 = \int_0^{1/2} \int_0^{1/3} f \Psi_2^1 dx dy + Q_2^1$$

$$f_2^1 = \int_0^{1/2} \int_0^{1/3} (3x - 2y)(2(x + y) - 4) dx dy + Q_2^1$$

$$f_2^1 = \int_0^{1/2} \int_0^{1/3} (6x^2 - 12x + 2xy - 4y^2 + 8y) dx dy + Q_2^1$$

$$f_2^1 = -\frac{1}{216} + Q_2^1$$

$$f_3^1 = \int_0^{1/2} \int_0^{1/3} f \Psi_3^1 dx dy + Q_3^1$$

$$f_3^1 = \int_0^{1/2} \int_0^{1/3} (2y)(2(x + y) - 4) dx dy + Q_3^1$$

$$f_3^1 = \int_0^{1/2} \int_0^{1/3} (4xy + 4y^2 - 8y) dx dy + Q_3^1$$

$$f_3^1 = -\frac{1}{4} + Q_3^1$$

$$F^1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \end{bmatrix} = \begin{bmatrix} -\frac{59}{216} \\ -\frac{1}{216} \\ -\frac{1}{4} \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \end{bmatrix}$$

Similarly, we get

$$\begin{aligned} F^2 &= \begin{bmatrix} \frac{5}{36} \\ -\frac{73}{216} \\ -\frac{11}{216} \end{bmatrix} + \begin{bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \end{bmatrix}, F^3 = \begin{bmatrix} -\frac{59}{72} \\ -\frac{87}{72} \\ \frac{78}{72} \end{bmatrix} + \begin{bmatrix} Q_1^3 \\ Q_2^3 \\ Q_3^3 \end{bmatrix}, F^4 = \begin{bmatrix} -\frac{1}{6} \\ -\frac{37}{216} \\ -\frac{13}{216} \end{bmatrix} + \begin{bmatrix} Q_1^4 \\ Q_2^4 \\ Q_3^4 \end{bmatrix}, \\ F^5 &= \begin{bmatrix} -\frac{47}{216} \\ \frac{17}{72} \\ -\frac{14}{72} \end{bmatrix} + \begin{bmatrix} Q_1^5 \\ Q_2^5 \\ Q_3^5 \end{bmatrix}, F^6 = \begin{bmatrix} -\frac{2}{9} \\ \frac{67}{216} \\ -\frac{191}{216} \end{bmatrix} + \begin{bmatrix} Q_1^6 \\ Q_2^6 \\ Q_3^6 \end{bmatrix}, F^7 = \begin{bmatrix} -\frac{29}{216} \\ -\frac{1}{216} \\ -\frac{1}{9} \end{bmatrix} + \begin{bmatrix} Q_1^7 \\ Q_2^7 \\ Q_3^7 \end{bmatrix}, \\ F^8 &= \begin{bmatrix} -\frac{5}{36} \\ -\frac{25}{216} \\ \frac{1}{216} \end{bmatrix} + \begin{bmatrix} Q_1^8 \\ Q_2^8 \\ Q_3^8 \end{bmatrix}, F^9 = \begin{bmatrix} -\frac{35}{216} \\ -\frac{1}{216} \\ -\frac{5}{36} \end{bmatrix} + \begin{bmatrix} Q_1^9 \\ Q_2^9 \\ Q_3^9 \end{bmatrix}, F^{10} = \begin{bmatrix} -\frac{1}{6} \\ -\frac{229}{216} \\ -\frac{107}{216} \end{bmatrix} + \begin{bmatrix} Q_1^{10} \\ Q_2^{10} \\ Q_3^{10} \end{bmatrix}, \\ F^{11} &= \begin{bmatrix} -\frac{71}{216} \\ -\frac{61}{216} \\ -\frac{1}{18} \end{bmatrix} + \begin{bmatrix} Q_1^{11} \\ Q_2^{11} \\ Q_3^{11} \end{bmatrix}, F^{12} = \begin{bmatrix} -\frac{1}{12} \\ -\frac{13}{216} \\ \frac{1}{216} \end{bmatrix} + \begin{bmatrix} Q_1^{12} \\ Q_2^{12} \\ Q_3^{12} \end{bmatrix}, \end{aligned}$$

Assembly of the element

$$K = \begin{bmatrix} K^1 & K^2 & K^3 & K^4 & K^5 & K^6 & K^7 & K^8 & K^9 & K^{10} & K^{11} & K^{12} \end{bmatrix}$$

$$F = \begin{bmatrix} F^1 & F^2 & F^3 & F^4 & F^5 & F^6 & F^7 & F^8 & F^9 & F^{10} & F^{11} & F^{12} \end{bmatrix}^T$$

$$U = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} \end{bmatrix}^T$$

Columns 1 through 12

$$K = \begin{bmatrix} 1.5 & -1.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.5000 & 2.8333 & -0.6667 & -0.6667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.6667 & 3.6667 & -3.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.6667 & -3.0000 & 5.0000 & -0.6667 & -0.6667 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6667 & 2.1667 & -1.5000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6667 & -1.5000 & 4.3333 & -1.5000 & -0.6667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.5000 & 5.8333 & -3.6667 & -0.6667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.6667 & -3.6667 & 6.5000 & -2.1667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.6667 & -2.1667 & 5.0000 & -1.5000 & -0.6667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.5000 & 5.8333 & -3.6667 & -0.6667 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6667 & -3.6667 & 6.5000 & -2.1667 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6667 & -2.1667 & 2.8333 \end{bmatrix}$$

And

$$F = \begin{bmatrix} -0.2731 \\ 0.1343 \\ -1.4070 \\ -1.4259 \\ 0.9120 \\ 0.5000 \\ 0.2731 \\ -1.1991 \\ -0.4352 \\ -1.4769 \\ -0.8611 \\ -0.1204 \end{bmatrix}$$

We arrived on the global system of equations of the form of

$$KU = F$$

Imposition of boundary condition

The given homogeneous Dirichlet boundary conditions:

$$U_1 = U_2 = U_3 = U_4 = U_5 = U_8 = U_9 = U_{10} = U_{11} = U_{12} = 0.$$

and assume Q_i^e closes to zero.

We find U_6 and U_7

$$-0.6667U_4 - 1.5000U_5 + 4.3333U_6 - 1.5000U_7 - 0.6667U_8 = 0.5$$

$$-1.5000U_6 + 5.8333U_7 - 3.6667U_8 - 0.6667U_9 = 0.2731$$

$$4.3333U_6 - 1.5000U_7 = 0.5$$

$$-1.5000U_6 + 5.8333U_7 = 0.2731$$

we solve this system of equations by using any direct method or iterative method, we get

$$U_6 = 0.1444$$

and

$$U_7 = 0.0839$$

The exact solution of the given pde is:

$$u(x, y) = (1 - x)y^2 + (1 - y)x^2$$

$$u(1/3, 1/2) = 0.22222222$$

$$u(2/3, 1/2) = 0.30555556$$

Node	x	y	FEM	Exact	Error
1	1/3	1/2	0.1444	0.22222222	0.077822
2	2/3	1/2	0.0839	0.30555556	0.22165556

Table 5.3: Comparison of finite element method with exact value in 2 inner node.

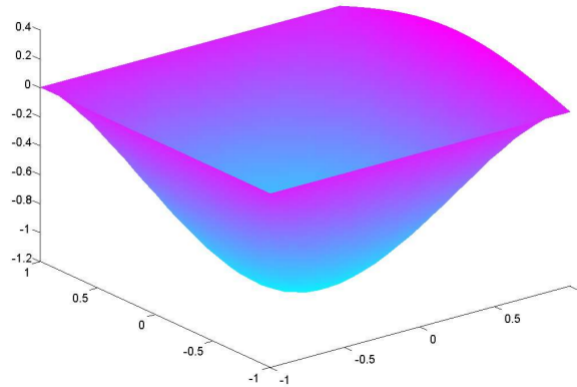


Figure 5.4: Graph of finite element solution.

4) Example for Finite Difference Method

Consider the problem

$$-\nabla^2 u = 2(x + y) - 4$$

in the square whose vertices are at $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. With homogeneous Dirichlet boundary conditions $u = 0$

$$-\nabla^2 u = 2(x + y) - 4$$

SOLUTION

Domain discretization

Divide the rectangle into 2 by 3 rectangles as figure below.

Replace the PDE by the difference equation

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

$$u_{xx} + u_{yy} = -f$$

$$\implies \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = -f_{ij}$$

$$u_{ij} = \frac{1}{4(h^2 + k^2)} [k^2(u_{i+1,j} + u_{i-1,j}) + h^2(u_{i,j+1} + u_{i,j-1}) + h^2 k^2 f_{ij}]$$

where,

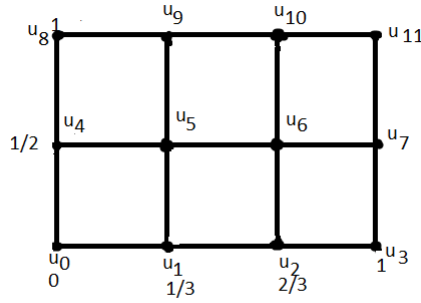


Figure 5.5: FDM discretization of two inner node

$$f_{ij} = 2(x_i + y_j) - 4, h = \frac{1}{3}, k = \frac{1}{2}$$

Hence,

$$u_5 \approx u_{11} = \frac{1}{4(h^2+k^2)}[k^2(u_{21} + u_{01}) + h^2(u_{12} + u_{10}) + h^2k^2f_{11}]$$

By using the given boundary conditions, we get that:

$$= \frac{1}{4(1/4+1/9)}[1/4(u_{21} + 0) + 1/9(0 + 0) + 1/36(2(1/3 + 1/2) - 4)]$$

$$u_5 \approx u_{11} = \frac{9}{52}u_{21} - 0.0449 \dots\dots\dots 1$$

And,

$$u_6 \approx u_{21} = \frac{1}{4(h^2+k^2)}[k^2(u_{31} + u_{11}) + h^2(u_{22} + u_{20}) + h^2k^2f_{21}]$$

$$= \frac{1}{4(1/9+1/4)}[1/4(0 + u_{11}) + 1/9(0 + 0) + 1/36(2(2/3 + 1/2) - 4)]$$

$$u_6 \approx u_{21} = \frac{9}{52}u_{11} - 0.032052 \dots\dots\dots 2$$

from equation 1 and 2 we get:

$$u_5 \approx u_{11} = -0.052$$

$$u_6 \approx u_{21} = -0.410$$

Node	x	y	FDM	Exact	Error
1	1/3	1/2	-0.0520	0.22222222	0.27422
2	2/3	1/2	-0.0410	0.30555556	0.346556

Table 5.4: Comparison of finite difference method with exact value in 2 inner node.

5.1.1 Comparison of FEM and FDM with exact value

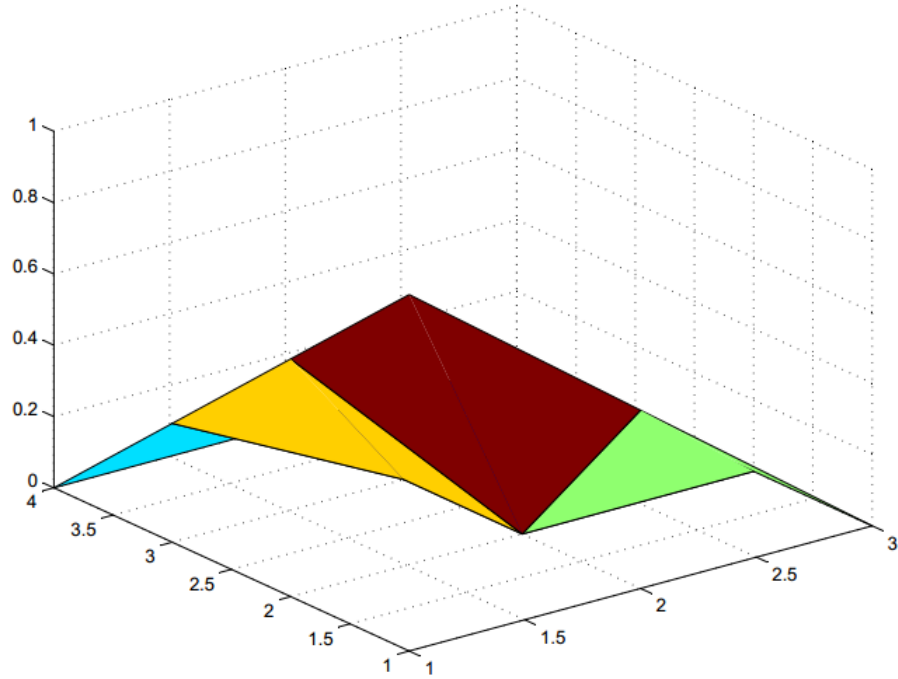


Figure 5.6: Graoh of finite difference solution

Node	x	y	FEM	FDM	Exact	Error with FEM	Error with FDM
1	1/2	1/2	0.1041687	-0.0625	0.25	0.1458313	0.3125

Table 5.5: Comparison of finite element method and finite difference method in one inner node

Node	x	y	FEM	FDM	Exact	Error with FEM	Error with FDM
1	1/3	1/2	0.1444	-0.0520	0.222222	0.077822	0.27422
2	2/3	1/2	0.0839	-0.0410	0.3555556	0.2216556	0.346556

Table 5.6: Comparison of finite element method and finite difference method

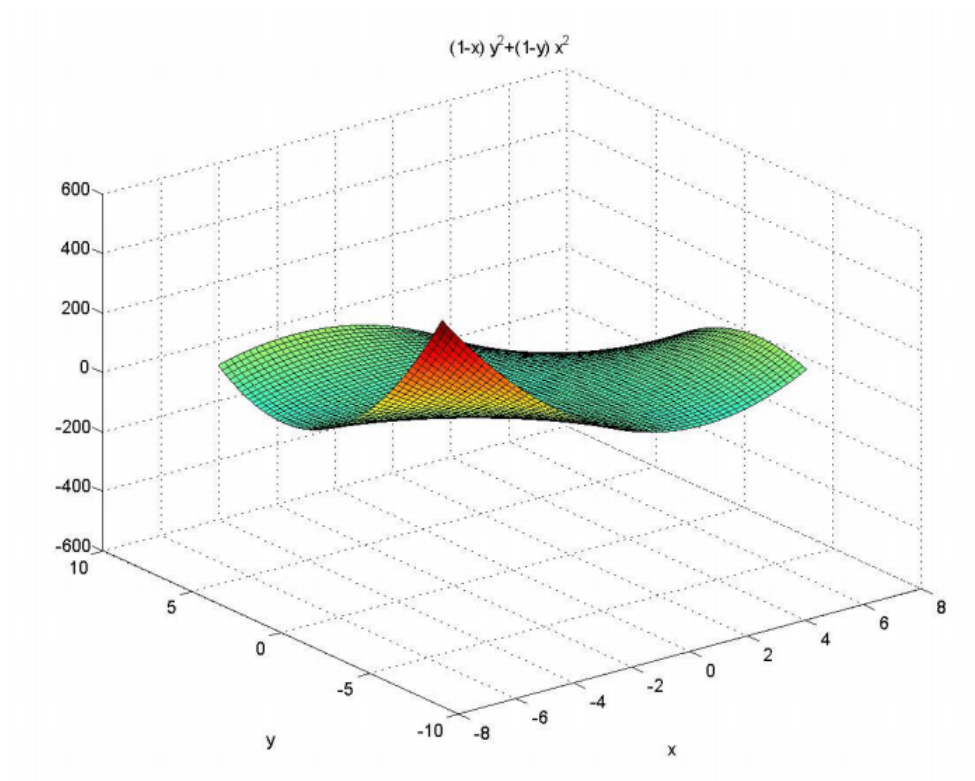


Figure 5.7: Graph of Exact solution.

Discussions

From the above worked example in the case of finite element method, the result obtained showed that as the number of triangular elements increases the approximate solution approaches to the exact solution with decreasing error. It has been observed that an increase in the number of grid points gives rise to an increase in the accuracy of the finite element method solution. The error decrease as the discretization of the element increases with mesh refinement on its domain and convergence ensured by suitable mesh refinement. Hence discretization affects the quality of the results. The graph of analytical solution and numerical solution of the 2D Poisson's equation with Dirichlet boundary condition also almost resembles and the only difference occur due to the error presents. In the case of finite element method the approximate solution more approaches to the exact solution than finite difference method. As we see in the above table(4.5) and table(4.6) the error in the finite element method is less than the error in the finite difference method. Hence finite element method is converges and stable much faster than finite difference method. Finite element method is more accurate than finite difference method. Finite element method is better in terms of convergence and stability than finite difference method for solving Poisson's equation in two dimension. Higher order accuracy is achieved by using higher order basis (i.e) shape functions. Extending to higher order accuracy is relatively complex than finite difference method. Higher order accurate calculations are expensive in computation and Mathematical formulation. The finite difference method is easiest to understand when the physical grid is Cartesian, but through the use of curvilinear transforms the method can be extended to domains that are not easily represented by brick-shaped elements. The discretization results in a system of equation of the variable at nodal points, and once a solution is found, then we have a discrete representation of the solution. Finite difference methods are generally considered easier to program than finite element method. As a general finite element method is best method rather than finite difference method to solve the Poisson's equation in 2D with less errors.

Chapter 6

Conclusion and Recommendation

6.1 Conclusion

This thesis presents the basic understanding of the Analysis of Finite Element Method and Finite Difference Method and the methodology to solving Poisson's equation in 2D. The fundamental idea of the finite element method is the replacement of continuous functions by piecewise approximations, usually polynomials. Stiffness matrix is generated and solution is obtained. The result of this study showed that increasing the number of element increases accuracy of the solution on the boundary and within the domain. That is error decreases when the number of element increases, so that the numerical solution converges to the analytical solution. The boundary conditions for this problem are of the Dirichlet type because values for the primary variables are specified on the boundary. The assembly of the element equations will require a little more bookkeeping in order to balance the source and flux terms. By introducing a little more complexity in formulating the Finite Element model, many useful and practical problems could be solved. FDM is created from basic definition of differentiation that is "h" tends to zero. In numerical analysis, its not possible to divide a number by "0" so "zero" means a small number. So FDM is similar to differential calculus but it has killed the heart that is limit tend to "zero". So in most of the cases accuracy of FDM increases with refining grid. Easy method but not reliable for conservative differential equations and solutions having shocks. Tough to implement in complex geometry where it needs complex mapping and mapping makes governing equation even tougher. Extending to higher order accuracy is very simple. FEM is a numerical tool that is borrowed from calculus of variation. Here finite element assume some trial function and multiply that trial function with weighting function . In Galerkin's method the trial function itself weighting function. Different methods follow different ways in weighting. Then this weighting functionis multiplied with trial function then integrated over the control volume (weak form) and equated to zero (This procedure will differ for different types of FEM but theme is same). Then we get one set of algebraic equations. Solving that will give solution. FEM method is more accurate than FDM. Ideal for linear PDEs, expensive and complex for non-linear PDEs. Here higher order accuracy is achieved by using higher order basis (i.e) shape functions. Extending to higher order accuracy is relatively complex than FDM. Higher

order accurate calculations are expensive in computation and Mathematical formulation. With FEM it is easier to make more finer subdivisions of the space in regions where we need more accuracy. This is not so easy to do with FDM. Thus the power of the Finite Element method becomes more evident, because the Finite Difference method will have much more difficulty in solving problems in a domain with complex geometries. .

6.2 Recommendation

The Finite Element Method (FEM) has been developed into a key indispensable technology in the modelling and simulation of various engineering systems. In the development of an advanced engineering system, engineers have to go through a very rigorous process of modelling, simulation, visualization, analysis, designing, testing, and finally, fabrication/construction. As such, techniques related to modelling and simulation in a rapid and effective way play an increasingly important role in building advanced engineering systems, and therefore the application of the FEM has multiplied rapidly. So its a better area of the researches.

The following recommendation should be considered.

- 1) Apply Finite Element Method on irregular geometries on complex domain in the future.

- 2) In this Thesis, we try to analysis and compare FEM with FDM for solving Poisson's equation in 2D. Hence the researchers may compare FEM with other methods such as Boundary Element Method.

Appendixes

Matlab code for finite element method and finite difference method

Matlab code for FDM for Poisson equation

```
k=1/100;
h=1/100;
x=0:h:1;
y=0:k:1;
n=length(x);
m=length(y);
u=zeros(n,m);
u(1,:)=f1;
u(n,:)=f2;
u(:,1)=g1;
u(:,m)=g2;
exact=(1-x).*y.^2+(1-y).*x.^2;
for j=1:m;
for i=1:n;
t=1/(4.*(h.^2+k.^2));
f(i,j)=2.*(x+y)-4;
a=k.^2.*(u(i+1,j)+u(i-1,j));
b=h.^2.*(u(i,j+1)+u(i,j-1));
u(i,j)=t.*(a+b+h.^2.*k.^2.*f(i,j));
e(i,j)=abs(u(i,j)-exact);
end
end
u;
surf(u)
title('Poisson equation')
xlabel('x')
ylabel('y')
zlabel('u')
```

Matlab code for exact value

```

x = [0 : 1/100 : 1];

y = [0 : 1/50 : 1];

[X,Y] = meshgrid(x,y);

z = (1 - x). * y.^2 + (1 - y). * x.^2;

ezsurf(z)

```

Matlab code for FEM for solving Poisson equation

```

% Poisson's Equation on Rectangular Domain Using a Fast Poisson Solver
% This example solves a Poisson's equation using the |poisolv| function in
% the Partial Equation.
% The specific Poisson's problem is
%

$$-\Delta u = 2(x + y) - 4$$

% on a square domain with Dirichlet boundary conditions,
% A Fast Poisson Solver, like the one found in |poisolv|, is able to
% numerically solve a Poisson's Equation on a regular grid is less time
% than the more general solver found in |asempde|.
% Problem Definition
% The following variables will define our problem:
%
% * |g|: A specification function that is used by |initmesh|.
% * |b|: A boundary file used by |asempde|.
% * |c|, |a|, |f|: The coefficients of the PDE.
g='squareg';
b='squareb4';
c=1;
b=0;
a=1;
f='2*x+2*y-4';
% Mesh for Fast Poisson Solver
% The fast Poisson solver requires a regular rectangular grid. A mesh
% meeting this requirement that has been generated by |poimesh| is shown below.
[p,e,t] = poimesh(g,2);
figure;
pdemesh(p,e,t); axis equal
% Solve Using Both Fast Poisson Solver and Standard Solver

```

```

% We solve using both the fast Poisson solver that is implemented in
% |poisolv| and the standard solver that is implemented in |asempde|.
fprintf (,'Fast Poisson','Standard');
fprintf('-----|-----|----- \ n');
for n = [2 4 8 16 32 ]
[p, e, t] = poimesh(g, n);
tic;
u=poisolv(b,p,e,t,f);
tfast = toc;
tic;
u1=asempde(b,p,e,t,c,a,f);
tstandard = toc;
fprintf('end
% Plot FEM Solution on Finest Mesh
figure;
pdesurf(p,t,u);
displayEndOfDemoMessage(mfilename)

```

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