# SOLVING RAYLEIGH-PLESSET EQUATION AND THOMAS A.IVEY'S EQUATION USING ADOMIAN DECOMPOSITION METHOD

**Yohannes Abdissa**

**A Thesis Submitted to Department of Applied Mathematics,**

**School of Applied Natural Science**

**Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Mathematics(Differential equation)**

**Office of Graduate Studies**

**Adama Science and Technology University**

February 19, 2023

Adama, Ethiopia

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**By**

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**Advisor: Feyissa Kebede(Assistant Professor)**

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# Declaration

I declare that this thesis proposal entitled "solving Rayleigh-plesset equatin and Thomas A.Iveys equation using Adomian decomposition method" is my own work and has not been submitted to any university for similar purpose. The references used in this proposal are duly recognized by proper citations.

| Name of Student | Signature | Date |
|---|---|---|

# Recommendation of Advisors

We, the advisors of this thesis, here by certify that we have read the revised version of the thesis entitled “solving Rayleigh-plesset equatin and Thomas A.Iveys equation using Adomian decomposition method ”prepared under our guidance by Yohannes Abdissa Abuna submitted in partial fulfillment of the requirements for the degree of Master’s of Science in Applied Mathematics. Therefore, we recommend that the submission of revised version of the thesis to the department following the applicable procedures.

| Major Advisor | Signature | Date |
| --- | --- | --- |

# Approval Page

We, the advisors of the thesis entitled “solving Rayleigh-plesset equatin and Thomas A.Iveys equation using Adomian decomposition method ”and developed by Yohannes abdissa, here by certify that the recommendation and suggestion made by the board of examiners are appropriately incorporated into the final version of the thesis..

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| Major Advisor | Signature | Date |

We, the undersigned, members of the Board of Examiners of the thesis by Yohannes Abdissa ,have read and evaluated the thesis entitled“solving second order nonlinear differential equation using Adomian decomposition method”and examined the candidate during the open defence. This is, therefore, to certify that the thesis is accepted for partial fulfilment of the requirement of the degree of Master of Science in Applied Mathematics.

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Finally approval and acceptance of the thesis proposal is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate Council (DGC) and School Graduate Committee (SGC).

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# ACKNOWLEDGEMENT

First of all I would like to thank to God who helped me to do every activity in every time. Second, I would like to express my deepest appreciation and sincere gratitude to my advisor, **Feyissa Kebede(Assistant profesor)** for his kind encouragement, guidance, support and supervisions.

My special thanks go to my wife Mrs. **Burtukan Agizu Bulto** her support was so great. And My special thanks go to my brother **Kassaye Abdissa (PhD Cand.)** for his all rounded support and encouragement. I like to express my gratitude to my beloved child **Senan Yohannes Abdissa** for her tolerance while missing my attention. again, I would like to thanks my school principals for them all rounded support and encouragement while missing me from school.I would like to thanks my brothers Yohannes Melese Kenete and Alamu Dhaba Dechasa for their encouragement, guidance and support.

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# LIST OF ACRONOMY

ADM-Adomian decomposition method.

IVPs-Initial value problem

RK4-Runge kutta fourth order

ODE-Ordinary differential Equation

PDE-Partial differential differential equation

RPE-Rayleigh-Plesset equation

# Abstract

In this study, We studied on solving Rayleigh-plesset equatin and Thomas A.Iveys equation using the proposed Adomian decomposition method. In the 1980's, George Adomian introduced a semi-analytical technique known as, Adomian decomposition method, for solving linear and nonlinear differential equations. The Adomian decomposition method is a powerful method which considers the approximate solution of a nonlinear equation as an infinite series which usually converges to the exact solution. The approximate analytic solution is found in the form of a convergent power series with easily computed components. Two examples will be taken to show the efficiency of the method.

**keywords:** Adomian decomposition method; Adomian polynomial; Initial value problem; Taylor series expansion.

Chapter 1

# INTODUCTION

## 1.1 Background of the study

Differential equations are among the most important mathematical tools used in producing models in the physical sciences, biological sciences, and engineering. Then the solution of the underlying problem lies in the solution of differential equation. Finding solution of the differential equation is then critical to that real world problem.
The Adomian decomposition method was firstly introduced by George Adomian in 1980 and developed in (Adomian, 1988). It has been paid much attention in the recent years in applied mathematics, and in the field of series solution particularly. This method has been applied to solve differential and integral equations of linear and nonlinear problems in mathematics, physics, biology and chemistry and up to now a large number of research papers have been published to show the feasibility of the decomposition method. Adomian decomposition method is a kind of algorithm, based on a technique of decomposition, to construct the approximate solutions and even exact solutions with suitable initial data for nonlinear systems.

The main advantage of this method is that it can be applied directly to all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant or variable coefficients. It is a very effective approach for solving wide classes of nonlinear partial and ordinary differential equations, with important applications in applied mathematics, engineering, physics etc. It can solve some nonlinear problems which cannot be solved easily by other numerical methods (iterative, etc.). Another advantage is that, the method is capable of reducing the size of computational work (Hamasalih, Ahmed, & Ahmed, 2020). The Adomian decomposition method decomposes a solution into an infinite series which converges rapidly to the exact solution. The convergence of the Adomian decomposition method has been investigated by a number of authors (Cherruault, 1989), (Inc, 2005). In the recent years, many researchers attracted to solve nonlinear differential equations by using Adomian decomposition method, because this method solve problems easily and elegantly without linearising. (Adomian, 1990; Hosseini, 2006; Adomian, 1990).

The Rayleigh-Plesset equation is a mathematical model for the dynamics of a gas–filled cavity in an incompressible liquid. The first principles of Rayleigh-Plesset equation was to describe the bubble cavitation in liquids in terms of macroscopic hydrodynamics. It is a nonlinear ordinary differential equation of second order that governs the dynamics of a spherical bubble and plays an essential role in interpreting many real world phenomena involving the presence of bubbles in engineering and medical fields.(Xia, 2018).

The Rayleigh-Plesset equation is the most popular nonlinear equation for description of the nonlinear response of a gas bubble in liquid to a driving pressure field .

The Thomas A. Ivey's equation was derived in 1982 to introduce the flow for studying integrable geometric evolution equations for Curves . The flow of curvature is a degenerate heat type equation on metrics (Ivey, 2001).An evolution equation for a curve is geometric if the velocity is expressed purely in terms of objects invariant under Euclidean motions . In this work, we use Adomain decomposition method to find approximate analytic solution of nonlinear differential equations; Rayleigh-Plesset equation and Thomas A.Ivey's equation

## 1.2 Statement of the problem

In this study, we consider two initial value problems.

$$\frac{d^2y}{dx^2} + \frac{kdy}{dx} + m(\frac{dy}{dx})^3 + n^2y = 0 \tag{1.1}$$

with initial condition: $y(0) = 0$ and $y'(0) = 1$.

and

$$\frac{d^2y}{dx^2} + \frac{1}{y}(\frac{dy}{dx})^2 + \frac{2}{x}\frac{dy}{dx} + ky^2 = 0 \tag{1.2}$$

with initial condition: $y(1) = 1$ and $y'(1) = 1$

The aim of this study is to apply Adomian decomposition method to find approximate analytic solution of the above two problems.

This study is intended to answer the following basic questions:

- how to use Adomian decomposition method to solve Rayleigh-Plesset equation and Thomas A.Ivey's equation?
- can we write an efficient algorithm of these equations using the method?
- is this method is efficient to solve nonlinear second order ordinary differential equatios?

## 1.3 Objectives of the study

### 1.3.1 General Objective of the study

The general objective of the study is to solve Rayleigh-Plesset equation and Thomas A. Ivey's equation using Adomian decomposition method.

### 1.3.2 Specific Objectives of the study

Specific objectives of the study are to:

- apply Adomian decomposition method to find an approximate analytic solution of Rayleigh-Plesset equation and Thomas A.Ivey's equation.
- write an efficient algorithm of Adomian decomposition methods for the problems.
- understand how the method works on Rayleigh-Plesset and Thomas A.Ivey's equation.

## 1.4 Significance of the study

The focus of this research will be on the solution of nonlinear second order ordinary differential equations.

The result of this study will help:

- the researchers how to use Adomian decomposition method to solve nonlinear second order differential equations.
- the researches by serving as a resource for those who need to conduct research in this field.
- scientific community for further investigation of similar problems.
- as a research reference

# Chapter 2

# LITERATURE REVIEW

In this section, we review some non-linear differential equations solved by researchers using Adomain Decomposition method. The Adomian decomposition method has been receiving much attention in recent years in the area of series solutions. A considerable research work has been invested recently in applying this method to a wide class of linear and nonlinear equations and integral equations.

The one dimensional nonlinear Burgers' equation is solved by using the Adomian decomposition method. They proved the convergence of this method applied to the nonlinear Burgers' equation. The approximate solution of this equations is calculated in the form of a series with easily computable components. The accuracy of the proposed numerical scheme is examined by comparison with other analytical and numerical results. The method performs well (Inc, 2005). that means, the method solves nonlinear second order ODEs with a simplest way without changing nonlinear.

(Wazwaz, 2000) developed a reliable technique for calculating Adomian polynomials for nonlinear operators. The new algorithm offers a promising approach for calculating Adomian polynomials for all forms of nonlinearity. The algorithm will be illustrated by studying suitable forms of nonlinearity. Here, a nonlinear evolution model was investigated.

(Li & Pang, 2020) this study, we have been concerned with the Adomian decomposition method. Adomian decomposition method is a kind of algorithm, based on a technique of decomposition, to construct the approximate solutions and even exact solutions with suitable initial data for nonlinear systems. The main advantages of the methods is that it is very easy to apply and can solve wide classes of nonlinear systems including algebraic equation, ordinary differential equations, partial differential equations, integral equations, integro-differential equations.

(Wazwaz, 1999) studied, how to find an analytic approximation solution for Volterra's model for population growth of a species in a closed system using ADM. The nonlinear integro-differential model includes an integral term that characterizes accumulated toxicity on the species in addition to the terms of the logistic equation. The series solution method and the decomposition method are implemented independently to the model and to a related ODE. The Padé approximants, that often show superior performance over series approximations, are effectively used in the analysis to capture the essential behavior of the population $u(t)$ of identical individuals.

In (Duan, Rach, Baleanu, & Wazwaz, 2012) this study, the Adomian decomposition method (ADM) is a well-known systematic method for practical solution of linear or nonlinear and deterministic operator equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations, etc. The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering.The accuracy of the analytic approximate solutions obtained can be verified by direct substitution.

The decomposition method can be an effective procedure for solution of nonlinear and/or stochastic continuous-time dynamical systems without usual restrictive assumptions (Adomian, 1990)

In (Holmquist, 2007) this study, George Adomian introduced a new method to solve nonlinear functional equa- tions [28]. This method has since been termed the Adomian decomposition method (ADM) and has been the subject of much investigation. The ADM involves separating the equation under investigation into linear and nonlinear portions. The linear operator representing the linear portion of the equation is inverted and the inverse operator is then applied to the equation. Any given conditions are taken into consideration. The nonlinear portion is decomposed into a series of

Adomian polynomials. This method generates a solution in the form of a series whose terms are determined by a recursive relationship using these Adomian polynomials.

(Marasi & Nikbakht, 2011) proposed ADM to solve an eigenvalue problems. Three test problems are considered to illustrate the proposed decomposition method. It is shown that the series solutions converge to the exact solutions for each problem.
All the above authors were apply ADM to solve some nonlinear ordinary differential equations and partial equation using ADM. However, no try to attempt to solve non-linear ordinary differential equations of Rayleigh-Plesset equation and Ivey's equation using the method. So, we will use Adomian decomposition methods to find an approximate analytic solution these equations.

# Chapter 3

# MATHEMATICAL PRELIMINARIES

We will gather some general preliminaries in this chapter, which we will use through out the thesis. We will not go into detail, but will provide sites where the interested reader can learn more.

## 3.1 Differential Equation

A differential equation is an equation containing the derivatives of one or more dependent variable, with respect to one or more independent variable. It may be noted that most of the fundamental laws of nature can be formulated as DE's, namely, the growth of a population, conduction of heat in a rod, vibration of structures, etc.

### Ordinary Differential Equation

**Definition 3.1.1** *An ordinary differential equation is a differential equation that contains only one independent variable so that all the derivatives occurring in it are ordinary derivatives.or An ordinary differential equation is a differential equation in which the unknown function's depends only on one variabls.*
*The general form of a $n^{th}$-order ODE is,*

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0$$

**Linear Differential Equations**

**Definition 3.1.2** *The linear differential equations are those in which, the dependent variable and its derivatives appear only in the first degree, and products of derivatives and the dependent variable do not occur. Linear differential equations can be expressed as:*

$$c_0\frac{d^ny}{dx^n} + c_1\frac{d^{n-1}y}{dx^{n-1}} + ... + c_{n-1}\frac{dy}{dx} + c_ny = f(x)$$

*where the coefficient* $c_0, ..., c_n$ *and* $f(x)$ *denote constants and the functions of* $x$*,respectively.*

**Nonlinear Differential Equations**

**Definition 3.1.3** *The nonlinear differential equations are Differential equations that are not linear. That is, the products of the unknown function and derivatives are allowed and degree of dependent variable is > 1.*

## 3.2 Initial Value Problem's

*Many problems in science and engineering can be modeled by ODE's or PDE's,along with one or more supplementary conditions. If these conditions are given at one point of the independent variable, the problem is called the initial value problem(IVP),and these conditions are known as initial conditions. Let us consider the first-order ODE*

$$\frac{dy}{dx} = f(x, y)$$

*which satisfies the initial condition* $y(xo) = yo$

*Similarly, for a second order ODE*

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

*subject to initial condition*

$$y(x_0) = \quad y_0, \quad y'(x_0) = \ y_0$$

*and the nth-order IVP with n initial conditions be written as*

$$\frac{d^ny}{dx^n} = f(x, y, \frac{dy}{dx}, ..., \frac{d^{n-1}y}{dx^{n-1}})$$

*with initial conditions*

$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$
$$y''(x_0) = y_2$$
$$y^{n-1}(x_0) = y_{n-1}$$

## 3.3 Taylor series expansion

***Definition 3.3.1*** *A Taylor series expansion is a representation of a function by an infinite series of polynomials around a point.(Wright, 1940) Mathematically, the Taylor series of a function, is defined as where is the function f and its $n^{th}$ derivatives:*

$$f', f'', f''', ...$$

*To get the solution of the differential equation using Taylor series expansion:*
*1.Taylor expansion of f(x) close to $x = x_0$ as follows:*

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + ... = \sum_{n=0}^{\infty} \frac{(x - x_0)^k}{2!} f^{k+1}(x_0)$$

*2.If we put $x_0$ = 0 with Taylor series expansion, then we can get Maclaurin Expansion as follows:*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... = \sum_{n=0}^{\infty} \frac{(x)^k}{2!} f^{k+1}(x_0).$$

## 3.4 Adomian polynomials

***Definition 3.4.1*** *Adomian polynomials decompose a function into a sum of components for a nonlinear operator .(Rach, 2008) The main part of Adomian dicomposition methods is calculating Adomian polynomials for nonlinear polynomials.*
*The decomposition method decomposes the solution $y(x)$ and the nonlinear $N(y)$ into series*

$$y(x) = \sum_{n=0}^{\infty} y_n \qquad , \qquad N(y) = \sum_{n=0}^{\infty} A_n$$

*To compute $A_n$ take $N(y) = F(y)$ to be a nonlinear function in y, where $y = y(x)$,and consider the Taylor series expansion of $f(y)$ around $y_0$*

$$F(y) = F(y_0) + F'(y_0)(y - y_0) + \frac{F''}{2!}(y_0)(y - y_0)^2 + \frac{F'''}{3!}(y_0)(y - y_0)^3 + ...,$$

*but y=$y_0 + y_1 + y_2 + y_3 \ldots$ , then*

$$F(y) = F(y_0) + F'(y_0)(y_1 + y_2 + y_3 + \ldots) + \frac{F''}{2!}(y_0)(y_1 + y_2 + y_3 + \ldots)^2 + \frac{F'''}{3!}(y_0)(y_1 + y_2 + y_3 + \ldots)^3 + \ldots,$$

*by expanding all terms we get*

$$F(y) = F(y_0) + F'(y_0)(y_1) + F'(y_0)(y_2) + F'(y_0)(y_3) + \ldots + \frac{F''}{2!}(y_0)(y_1)^2$$
$$+ \frac{2F''}{2!}(y_0)(y_1 y_2) + \frac{F''}{2!}(y_0)(y_1 y_3) + \ldots + \frac{F'''}{3!}(y_0)(y_1)^3 + \frac{3F'''}{3!}(y_0) y_1^2 y_2 + \frac{F'''}{3!}(y_0) y_1^1 y_3 + \ldots$$

*However, the nonlinear term $F(y)$ can be expressed by an infinite series of the so-called Adomian polynomials ($A_n$ ) given in the form*

$$F(y) = \sum_{n=0}^{\infty} A_n(A_0, A_1, A_2, \ldots, A_N) \qquad (3.1)$$

*$A_n$ can be evaluated for all forms of nonlinearity. Hence, the Adomian polynomials ($A_n$) are calculated by the general formula:*

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[F(\sum_{n=0}^{\infty} \lambda^n y_n)]|_\lambda = 0 \qquad (3.2)$$

*One can calculate the first few components of the Adomian polynomials as follows:*

$$A_0 = F(y_0)$$
$$A_1 = y_1 F'(y_0)$$
$$A_2 = y_2 F'(y_0) + \frac{1}{2!} y_1^2 F''(y_0)$$
$$A_3 = y_3 F'(y_0) + \frac{2}{2!} y_1 y_2 F''(y_0) + \frac{1}{3!} y_1^3 F''(y_0)$$
$$A_4 = y_4 F'(y_0) + [\frac{1}{2} y_2^2 + y_1 y_3] F''(y_0) + \frac{1}{2} y_1^2 y_2 F'''(y_0) + \frac{1}{4} y_1^4 F''''(y_0).$$

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## 3.5 Adomian decomposition method

The Adomian decomposition method involves separating the equation under investigation into linear and nonlinear part. The main part of Adomian decomposition method is calculating Adomian polynomials for nonlinear polynomials. This method generates a solution in the form of a series whose terms are determined by a recursive relationship using the Adomian Polynomials.

Consider a general nonlinear differential equation:

$$F(y(x)) = f(x) \tag{3.3}$$

where $F$ represents a general nonlinear differential operator involving linear and non-linear terms ,the linear term is decomposed in to $L + R$ , where L is invertable and R the remainder of linear operator, $N(y)$ represents the nonlinear term. Thus equation3.3 can be written as:

$$L(y) + R(y) + N(y) = f \tag{3.4}$$

Applying the inverse operator $L^{-1}$ on both sides of equation(3.4) , we obtain

$$L^{-1}Ly = L^{-1}f - L^{-1}R(y) - L^{-1}N(y) \tag{3.5}$$

Thus , we get

$$y(x) = g(x) - L^{-1}R(y) - L^{-1}N(y), \tag{3.6}$$

where $g(x)$ is the function generated from $f$ using initial conditions.

The ADM assumes that the unknown function $y(x)$ can be expressed by an infinite series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{3.7}$$

where the components $y_n(x)$ will be determined recursively. The ADM defines the nonlinear term into a series of polynomials called Adomian polynomials $(A_n)$. Also the ADM assumes that the nonlinear operator $N(y)$ can be decomposed by an infinite series of polynomials defined by

$$N(y) = \sum_{n=0}^{\infty} A_n \tag{3.8}$$

substituting equation (3.7) and (3.8) into equation (3.6), we get

$$\sum_{n=0}^{\infty} y_n(x) = g(x) - L^{-1}R(\sum_{n=0}^{\infty} y_n) - L^{-1}(\sum_{n=0}^{\infty} A_n), \tag{3.9}$$

where $A_n$ can be determined as,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{n=0}^{\infty}\lambda^n y_n(x))]|_\lambda = 0, where \quad n = 0, 1, 2, ... \tag{3.10}$$

$$y_0 + y_1 + y_2 + \cdots = g(x) - L^{-1}R(\sum_{n=0}^{\infty} y_n) - L^{-1}(\sum_{n=0}^{\infty} A_n). \tag{3.11}$$

Therefore, the recursive relation is found to be

$$\begin{cases} y_0 & = g(x) \\ \vdots & \\ y_{n+1} & = -L^{-1}R(y_n) - L^{-1}(An), \end{cases} \tag{3.12}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recursively and Adomian polynomials $A_n$ generates for each nonlinear terms.
One can write the first four Adomian polynomials as follows:

$$\begin{aligned} A_0 &= N(y_0) \\ A_1 &= y_1 N'(y_0) \\ A_2 &= y_2 N'(y_0) + \frac{y_1^2}{2!}N''(0) \\ A_3 &= y_3 N'(y_0) + y_1 y_2 N''(y_0) + \frac{y_1^3}{3!}N'''(y_0) \end{aligned} \tag{3.13}$$

$$\vdots$$

Using Adomian decomposition method, the components $y_n(x)$ can be determined as:

$$\begin{cases} y_0(x) & = g(x), \\ y_{n+1}(x) & = -L^{-1}R(y_n) - L^{-1}(An), \end{cases} \tag{3.14}$$

which gives

$$\begin{aligned} y_0(x) &= g(x) \\ y_1(x) &= -L^{-1}R(y_0) - L^{-1}(A_0) \\ y_2(x) &= -L^{-1}R(y_1) - L^{-1}(A_1) \\ y_3(x) &= -L^{-1}R(y_2) - L^{-1}(A_2) \end{aligned} \tag{3.15}$$

$$\vdots$$

Therefore the approximate analytic solution in a series form can be written as

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

## 3.6 Numerical methods for ODEs

Numerical techniques that listed below are summarized from (Lenhart and Work man,2007). When ODEs involves, equation that can not be solved analytically, numerical methods are used to compute the approximate solution. Fourth order Runge-Kutte methos(RK4) is widely used for solving initial value problems for ODEs which is given

$$\begin{cases} y' = \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases} \tag{3.16}$$

on the interval $t_0 < t < T$, we divided the interval into N small sepements of constant length h which is called the step size given by $h = \dfrac{T - t_0}{N}$. The first steps is discritizing the interval by defining the time steps given by $t_n = t_n + ah, h = 0, 1, 2, ..., N$ where $N$ is the number of steps. Again we use the notation $y_n$ to denote $(t, y_n)$.We can summarize the numerical techniques:

**The Runge-Kutta Fourth order**

Given a well-posed initial value problems for the above conditions (3.16) the Runge-Kutta methods of order four construct a sequense of approximate points $(t, y) \approx (t, y(t))$ to the exact solution of an ODE based on the following values of some slopes:

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h \tag{3.17}$$

where knowing the value of $y = y_i$ at $x_i$ , we can find the value of $y = y_{i+1}$ at $x_{i+1}$ , and h $= x_{i+1}$ - $x_i$.

Equation (3.17) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + y'(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!}y''(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!}y'''(x_i, y_i)(x_{i+1} - x_i)^3 + \frac{1}{4!}y''''(x_i, y_i)(x_{i+1} - x_i)^4 \tag{3.18}$$

Knowing that $h = x_{i+1} - x_i$ and $\dfrac{dy}{dx} = f(x, y)$

$$y_{i+1} = y_i + f'(x_i, y_i)(h) + \frac{1}{2!}f''(x_i, y_i)(h)^2 + \frac{1}{3!}f'''(x_i, y_i)(h)^3 + \frac{1}{4!}f''''(x_i, y_i)(h)^4. \tag{3.19}$$

Based on equating Equation(3.18) and Equation(3.19), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{3.20}$$

therefore,the Runge-Kutta 4th Order Method was calculated as

$$\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}h) \\
k_3 &= f(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}h) \\
k_4 &= f(x_i + \frac{h}{2}, y_i + \frac{k_3}{2}h)
\end{aligned} \tag{3.21}$$

Chapter 4

# RESULT AND DISCUSSION

In this section, we apply Adomian decomposition method as a powerful mechanism for initial value problems of Rayleigh-plesset equation and Thomas A. Ivey's equation which are nonlinear second order ordinary differential equations. To show the efficiency of the method described in the previous section, we present the following initial value problems. These problems are chosen such that there exist an approximate analytical solutions for them using the Adomian decomposition method.

**Problem-1**

Consider Rayleigh-Plesset equation:

$$\frac{d^2y}{dx^2} + \frac{kdy}{dx} + m(\frac{dy}{dx})^3 + n^2y = 0 \tag{4.1}$$

with initial condition: $y(0) = 0$ and $y'(0) = 1$.

Let us consider the nonlinear problem (4.1) in an operator form

$$\begin{aligned} Ly &= -y - y' - (y')^3 \\ &= -y - R(y) - N(y), \end{aligned} \tag{4.2}$$

where $R(y) = y'$, $N(y) = (y')^3$ and where k =m =n =1

Assuming that the inverse operator $L^{-1}$ exists and take as a double definite integral with respect to $x$ from 0 to $x$. That is

$$L^{-1}(.) := \int_0^x \int_0^x (.)dsds$$

Thus, applying the inverse operator $L^{-1}$ to (4.2) gives

$$L^{-1}Ly = -L^{-1}y - L^{-1}y' - L^{-1}(y')^3$$

Therefore it follows that

$$\int_0^x \int_0^x \frac{d^2y}{ds^2} ds ds = -L^{-1}y - L^{-1}y' - L^{-1}(y')^3$$
$$\int_0^x \frac{dy}{ds} \Big|_0^x ds = -L^{-1}y - L^{-1}y' - L^{-1}(y')^3$$
$$y(x) - y(0) - xy'(0) = -L^{-1}y - L^{-1}y' - L^{-1}(y')^3$$
$$y(x) = y(0) + xy'(0) - L^{-1}y - L^{-1}y' - L^{-1}(y')^3$$

Next decompose the unknown function $y(x)$ by a sum of components defined by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

Now using the assumption of the Adomian decomposition method, we have

$$\sum_{n=0}^{\infty} y_n(x) = y(0) + xy'(0) - L^{-1}(\sum_{n=0}^{\infty} y_n) - L^{-1}(\sum_{n=0}^{\infty} y_n') - L^{-1}(\sum_{n=0}^{\infty} A_n)$$
$$y_0 + y_1 + y_2 + ... = y(0) + xy'(0) - L^{-1}(\sum_{n=0}^{\infty} y_n) - L^{-1}(\sum_{n=0}^{\infty} y_n') - L^{-1}(\sum_{n=0}^{\infty} A_n)$$
$$y_0 + y_1 + y_2 + ... = x - L^{-1}(\sum_{n=0}^{\infty} y_n) - L^{-1}(\sum_{n=0}^{\infty} y_n') - L^{-1}(\sum_{n=0}^{\infty} A_n)$$

This gives the recursive relationship,

$$\begin{cases} y_0 & = x \\ \vdots & \\ y_{n+1} & = -L^{-1}y_n - L^{-1}y_n' - L^{-1}A_n. \end{cases} \tag{4.3}$$

By using Adomian polynomials formula the nonlinear terms are computed as follows.

$$\begin{aligned} A_0 &= (y_0')^3 \\ A_1 &= 3(y_0')^2(y_1') \\ A_2 &= 3[y_0'(y_1')^2 + (y_0')^2 y_2'] \\ A_3 &= (y_1')^3 + 4y_0'y_1'y_2' + 3y_0'y_3' \\ &. \qquad\qquad . \\ &. \qquad\qquad . \end{aligned} \tag{4.4}$$

Using the first few components of Adomian polynomial the recursive relation gives the first four terms as

$$\begin{aligned}
y_1 &= -L^{-1}[y_0 + y_0' + A_0] \\
y_2 &= -L^{-1}[y_1 + y_1' + A_1] \\
y_3 &= -L^{-1}[y_2 + y_2' + A_2] \\
y_4 &= -L^{-1}[y_3 + y_3' + A_3]
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&. \qquad\qquad . \\
&. \qquad\qquad .
\end{aligned}$$

$$\begin{aligned}
y_0 &= x \\
y_1 &= -L^{-1}[y_0 + y_0' + A_0] \\
&= -\int_0^x \int_0^x (y_0 + y_0' + A_0)dsds \\
&= -\int_0^x \int_0^x (s+2)dsds \\
&= -\int_0^x \frac{s^2}{2} + 2s) \mid_0^x ds \\
&= -\frac{x^3}{6} - x^2
\end{aligned}$$

$$\begin{aligned}
A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{n=0}^{\infty} \lambda^n y_n(x))]|_\lambda = 0 \\
A_1 &= \frac{1}{1!}\frac{d}{d\lambda^n} N(\sum_{n=0}^{\infty} \lambda^n y_n(x))]|_\lambda = 0 \\
&= \frac{1}{1!}\frac{d}{d\lambda} N(y_0' + y_1'\lambda)^3|_\lambda = 0 \\
&= 3(y_0' + y_1'\lambda)^2(y_1')|_\lambda = 0 \\
&= 3(y_0')^2 y_1' \\
&= \frac{-3x^2}{2} - 6x
\end{aligned}$$

$$
\begin{aligned}
y_2 &= -\int_0^x \int_0^x (y_1 + y_1' + A_1) dsds \\
&= -\int_0^x \int_0^x (\frac{-s^3}{6} - s^2) + (\frac{-s^2}{2} - 2s) + (\frac{-3s^2}{2} - 6s) dsds \\
&= -\int_0^x \int_0^x (\frac{-s^3}{6} - sx^2 - 8s) dsds \\
&= -\int_0^x (-4s^2 - s^3 - \frac{s^4}{24}) \Big|_0^x ds \\
&= \frac{4x^3}{3} + \frac{x^4}{4} + \frac{x^5}{120}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(x) &= y_0 + y_1 + y_2 + \cdots \\
&= x + (-\frac{x^3}{6} - x^2) + \frac{4x^3}{3} + \frac{x^4}{4} + \frac{x^5}{120} \qquad (4.6) \\
&= x - x^2 + \frac{7x^3}{6} + \frac{x^4}{4} + \frac{x^5}{120} + \cdots
\end{aligned}
$$

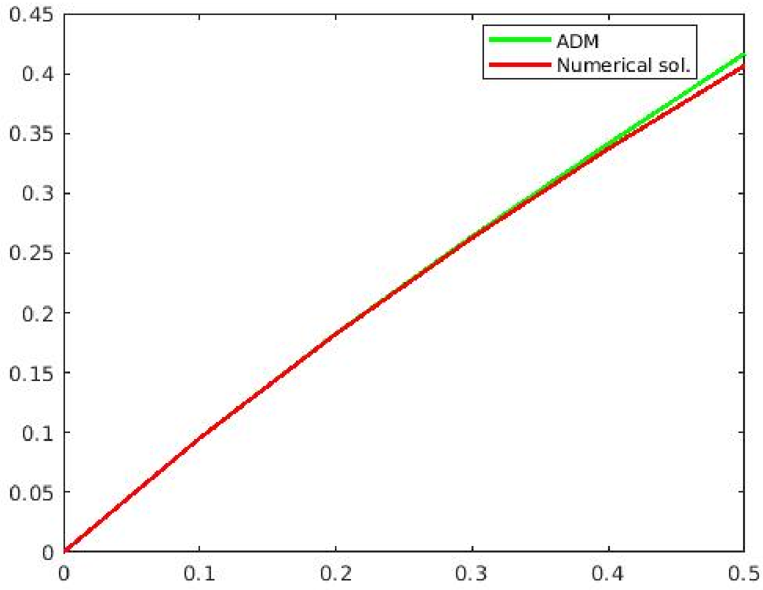


Figure 4.1: ADM vs RK-4 approximations

Table 4.1: ADM vs RK-4 approximations

| x | ADM | Rk-4 | Error |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.0953 | 0.0953 | 0 |
| 0.2 | 0.1824 | 0.1827 | 0.0003 |
| 0.3 | 0.2627 | 0.2640 | 0.0013 |
| 0.4 | 0.3372 | 0.3413 | 0.0042 |
| 0.5 | 0.4066 | 0.4167 | 0.0101 |

**Problem-2**

Consider nonlinear second order Thomas .A.Ivey's equation

$$y'' + \frac{1}{y}(y')^2 + \frac{2}{x}y' + ky^2 = 0 \tag{4.7}$$

with initial condition: $y(1) = 1$ and $y'(1) = 1$

Let us consider the nonlinear problem (4.7) in an operator form where $k = 1$.

$$\begin{aligned} Ly &= -\frac{2y'}{x} - y^2 - \frac{1}{y}(y')^2, \\ &= \frac{2R}{x} - N(y), \end{aligned} \tag{4.8}$$

where $R(y) = -\frac{2y'}{x}$ , $N(y) = -y^2 - \frac{1}{y}(y')^2$

The notation

$$L = \frac{d^2}{dx^2}$$

Assuming that the inverse operator $L^{-1}$ exists and it can conveniently be take as double definite integral with respect to $x$ from 1 to $x$. That is

$$L^{-1}(.) = \int_1^x \int_1^x (.)dsds$$

Thus, applying the inverse operator $L^{-1}$ to (4.8) gives

$$L^{-1}Ly = -L^{-1}\frac{2y'}{x} - L^{-1}y^2 - L^{-1}\frac{1}{y}(y')^2,$$

Therefore, it follows that

$$\begin{aligned} \int_1^x \int_1^x \frac{d^2y}{ds^2}dsds &= -L^{-1}\frac{2y'}{x} - L^{-1}y^2 - L^{-1}\frac{1}{y}(y')^2, \\ \int_1^x \frac{dy}{ds}\Big|_1^x ds &= -L^{-1}\frac{2y'}{x} - L^{-1}y^2 - L^{-1}\frac{1}{y}(y')^2, \\ y(x) - y(1) - xy'(1) &= -L^{-1}\frac{2y'}{x} - L^{-1}y^2 - L^{-1}\frac{1}{y}(y')^2 \\ y(x) &= y(1) + xy'(1) - L^{-1}\frac{2y'}{x} - L^{-1}y^2 - L^{-1}\frac{1}{y}(y')^2. \end{aligned}$$

Next decompose the unknown function $y(x)$ by a sum of components defined by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

Now using the assumption of the Adomian decomposition method we have

$$\sum_{n=0}^{\infty} y_n(x) = y(0) + xy'(0) - L^{-1}(\sum_{n=0}^{\infty} \frac{2y_n'}{x}) - L^{-1}(\sum_{n=0}^{\infty} y_n^2) - L^{-1}(\sum_{n=0}^{\infty} \frac{1}{y_n}(y_n')^2)$$

$$y_0 + y_1 + y_2 + ... = y(1) + xy'(1) - L^{-1}(\sum_{n=0}^{\infty} \frac{2y_n'}{x}) - L^{-1}(\sum_{n=0}^{\infty} A_n) - L^{-1}(\sum_{n=0}^{\infty} B_n)$$

$$y_0 + y_1 + y_2 + ... = 1 + x - L^{-1}(\sum_{n=0}^{\infty} \frac{2y_n'}{x}) - L^{-1}(\sum_{n=0}^{\infty} A_n) - L^{-1}(\sum_{n=0}^{\infty} B_n)$$

This gives the recursive relationship,

$$\begin{cases} y_0 & = 1 + x \\ \vdots & \\ y_{n+1} & = -L^{-1}\frac{2}{x}y_n' - L^{-1}A_n - L^{-1}B_n. \end{cases} \tag{4.9}$$

where

$$B_0 = N(y_0) = \frac{1}{y_0}(y_0')^2 \quad and \quad A_0 = N(y_0) = y_0^2$$

By using the Adomian polynomials formula the nonlinear terms computed as follows.

$$A_0 = (y_0)^2 \quad and \quad B_0 = \frac{1}{y_0}(y_0')^2$$

$$A_1 = 2(y_0)(y_1) \quad and \quad B_1 = \frac{-y_1(y_0')^2}{(y_0)^2} + \frac{2(y_0'y_1')}{y_0}$$

$$\begin{matrix} . & & & . \\ . & & & . \\ . & & & . \end{matrix}$$

Using the first few components of Adomian polynomial the recursive relation gives the first four terms as.

$$\begin{aligned}
y_1 &= -L^{-1}\frac{2}{x}y_0' - L^{-1}A_0 - L^{-1}B_0 \\
y_2 &= -L^{-1}\frac{2}{x}y_1' - L^{-1}A_1 - L^{-1}B_1 \\
y_3 &= -L^{-1}\frac{2}{x}y_2' - L^{-1}A_2 - L^{-1}B_2 \\
y_4 &= -L^{-1}\frac{2}{x}y_3' - L^{-1}A_3 - L^{-1}B_3
\end{aligned} \tag{4.10}$$

$$\begin{array}{cc}
. & . \\
. & . \\
. &
\end{array}$$

$$
\begin{aligned}
y_0 &= 1+x \\
y_1 &= -L^{-1}\frac{2}{x}y_0' - L^{-1}A_0 - L^{-1}B_0 \\
&= -L^{-1}(\frac{2}{x} + x^2 + 2x + 1 + \frac{1}{x+1}) \\
&= -\int_1^x \int_1^x (\frac{2}{x} + x^2 + 2x + 1 + \frac{1}{x+1})dsds \\
&= -\int_1^x 2lns + \frac{s^3}{3} + s^2 + s + ln(s+1)\,\Big|_1^x ds \\
&= -\int_1^x 2lns + \frac{s^3}{3} + s^2 + s + lns\,\Big|_1^x ds \\
&= -3xlnx + \frac{16x}{3} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{12} - \frac{53}{12} \\
A_1 &= -6x^2lnx - 6xlnx + \frac{11x}{6} + \frac{31x^2}{3} - \frac{5x^3}{6} - \frac{5x^4}{6} - \frac{x^5}{6} - \frac{53}{6} \\
B_1 &= (36xlnx - 36x + 6x^3 + 3x^4)(\frac{x^2}{2}) - \frac{6lnx}{x} - 3x - 2x^2) \\
y_2 &= -L^{-1}\frac{2}{x}y_1' - L^{-1}A_1 - L^{-1}B_1 \\
&= -L^{-1}[\frac{2}{x}y_1' + \frac{-y_1}{(y_0)^2}.(y_0')^2 + \frac{2}{y_0}(y_0'y_1') + 2y_0y_1] \\
&= -L^{-1}[\frac{2}{x}y_1' + 2y_0y_1 + \frac{-y_1}{(y_0)^2}.(y_0')^2 + \frac{2}{y_0}(y_0'y_1')] \\
&= -L^{-1}(\frac{14}{3x} - \frac{6lnx}{x} - 2 - 2x - \frac{2x^2}{3} - 6x^2lnx - 6xlnx + \frac{11x}{6} + \frac{31x^2}{3} - \frac{5x^3}{6} - \frac{5x^4}{6} - \frac{x^5}{6} \\
&- \frac{53}{6} + (36xlnx - 36x + 6x^3 + 3x^4)(\frac{x^2}{2}) - \frac{6lnx}{x} - 3x - 2x^2) \\
&= -\int_1^x \int_1^x (\frac{14}{3x} - \frac{-12lnx}{x} - 2 + x - \frac{2x^2}{3} + 5x^2 - 2x^2 - 6x^2lnx - \frac{5x^3}{3} - 18x^3 - \frac{5x^4}{6} - \frac{x^5}{6} + 3x^5 \\
&+ 18x^3lnx + \frac{3x^6}{2})dsds \\
&= -\int_1^x (\frac{14}{3}lns - 6(lns)^2 - s^2 - 2s - \frac{2s^3}{9} - 2s^3lns - \frac{37s^4}{24} - \frac{s^5}{6} + \frac{71s^6}{36} + \frac{9s^4lns}{2} - \frac{3s^7}{14} \\
&- \frac{219}{56})ds \\
&= \frac{14slns}{3} - \frac{14s}{3} + 2s^3lns - \frac{9s^5lns}{10} + \frac{s^4lns}{2} - \frac{2s^3}{3} + \frac{9s^5}{50} - \frac{35s^4}{72} - \frac{7s^3}{6} + s^2 + \frac{37s^5}{120} + \frac{s^6}{36} \\
&- \frac{71s^7}{252} + \frac{3s^8}{112} + \frac{219s}{56})\,\Big|_1^x \\
&= \frac{14xlnx}{3} - \frac{14x}{3} + 2x^3lnx - \frac{9x^5lnx}{10} + \frac{x^4lnx}{2} - \frac{2x^3}{3} + \frac{9x^5}{50} - \frac{35x^4}{72} - \frac{7x^3}{6} + x^2 + \frac{37x^5}{120} + \frac{x^6}{36} \\
&- \frac{71x^7}{252} + \frac{3x^8}{112} + \frac{219x}{56} - \frac{10531416}{1411200}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(x) &= y_0 + y_1 + y_2 + ... \\
&= 1 + \frac{527x}{52} + \frac{x^2}{2} - \frac{13x^3}{6} - \frac{41x^4}{72} + \frac{293x^5}{600} + \frac{x^6}{36} - \frac{71x^7}{252} + \frac{3x^8}{112} + ...
\end{aligned} \tag{4.11}
$$

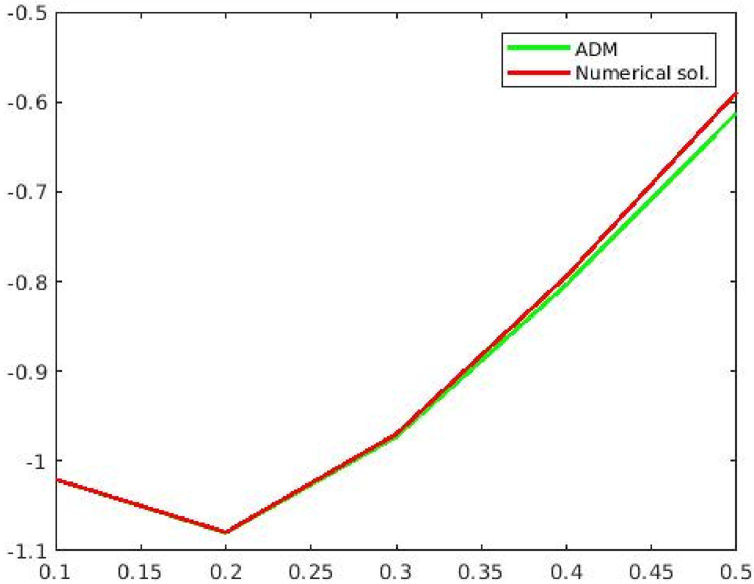


Figure 4.2: ADM vs RK-4 approximations

Table 4.2: ADM vs RK-4 solution approximations

| x | ADM | NRk-4 | Error |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 5.6842 | 5.6842 | 0 |
| 0.2 | 5.4383 | 5.4299 | 0.0084 |
| 0.3 | 5.1729 | 5.1687 | 0.0042 |
| 0.4 | 4.6945 | 4.6933 | 0.0012 |
| 0.5 | 4.0903 | 4.0886 | 0.0017 |

## 4.1 Discussion

In this work, Adomian decomposition method was used to obtain the approximate analytical solution of the second order Rayleigh-plesset equation and Thomas A. Ivey's equation. For illustration purpose two problems were selected to show how to apply Adomian decomposition method to find approximate analytic solution. The approximate solution and numerical solution are compared and shows small error.

## 4.2 Conclusion

In this study, the approximate analytical solutions of second order Rayleigh-plesset equation and second order Thomas.A.Ivey's equation are obtained using the Adomian decompositiom method. We successfully found an approximate analytic solution of two problems and compare the result with numerical method RK4. Thus comparison show that Adomian decomposition method is a very powerful method for solving nonlinear ordinary differential equations. Our interests in future work is to apply Adomian decomposition method to other nonlinear ordinary differential equations, which arises in the fields of other disciplines.

## 4.3 Recommendations

This thesis, specifically focused on application of Adomian decomposition method to get approximate analytic solution of second order Rayleigh-Plesset equation and second order Thomas.A.Ivey's equation.This method is very interesting and powerful to find analytic approximate solution of ordinary differential equation as well as partial differential equation. There are other equations which are more complex and more coupled than those we have considered and solved. In our future work, we will apply Adomian decomposition method to other nonlinear differential equations, which arises in the fields of other disciplines.

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