

Numerical Treatment for a Class of Singularly Perturbed Parabolic Partial Differential Equations



Sisay Ketema Tesfaye

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Doctor of Philosophy in
Applied Mathematics (Numerical Analysis)

Office of Graduate Studies
Adama Science and Technology University

June, 2024
Adama, Ethiopia

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DEDICATION

*This work is dedicated to
my late father Ketema Tesfaye*

Declaration

I hereby declare that this dissertation entitled “**Numerical Treatment for a Class of Singularly Perturbed Parabolic Partial Differential Equations**” is my work and has not been submitted to any university for a similar purpose. The references used in this dissertation are duly recognized by proper citations.

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We, the supervisors of this dissertation, hereby certify that we have read and revised the dissertation entitled “**Numerical Treatment for a Class of Singularly Perturbed Parabolic Partial Differential Equations**” prepared under our guidance by **Sisay Ketema Tesfaye** submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics (Numerical Analysis). Therefore, we recommend the submission of the dissertation to the department for further review and defense.

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We hereby certify that the recommendation and suggestion given by the dissertation review committee are appropriately incorporated into the final dissertation entitled “**Numerical Treatment for a Class of Singularly Perturbed Parabolic Partial Differential Equations**” by **Sisay Ketema Tesfaye**.

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We, the undersigned, members of the Board of Examiners of the dissertation open defense by **Sisay Ketema Tesfaye** have read and evaluated the dissertation entitled “**Numerical Treatment for a Class of Singularly Perturbed Parabolic Partial Differential Equations**” and examined the candidate during the open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in Applied Mathematics (Numerical Analysis).

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LIST OF SYMBOLS, ACRONYMS AND ABBREVIATIONS

$z(s, t)$	Solution of continuous problem
$z_0(s, t)$	Solution of reduced problem
N, M	Number of mesh intervals in space and time directions, respectively
$Z^{N,M}$	Numerical solution
ε	Singular perturbation parameter
τ	Delay parameter
s, s_m	Continuous and discrete space variables
t, t_n	Continuous and discrete time variables
$\Omega, \bar{\Omega}$	Space domain, closed space domain
$\Lambda, \bar{\Lambda}$	Time domain, closed time domain
$\ \cdot\ $ or $\ \cdot\ _{\mathfrak{D}}$	Supremum norm over the domain
$C^{(k)}(\mathfrak{D}), C^{(k)}(\bar{\mathfrak{D}})$	k times continuously differentiable functions in the respective domain
C (or the sub-scripted)	Generic positive constants, independent of ε and mesh point
Ω^M, Λ^t	Discrete domain of space and time directions, respectively
T	Maximum/Final time
Δt	Time mesh size
$H, h, \Delta s,$	Space mesh size
$\delta^+, \delta^-, \delta^0, \delta^2$	Forward, backward, central difference, second-order central difference operator, respectively
$\mathfrak{L}, \mathfrak{L}_\varepsilon$	Differential operators
$\mathfrak{L}^{\Delta t}, \mathfrak{L}^{\Delta t, h}, \mathfrak{L}_\sigma^{\Delta t, h}$	Difference operators
$E_\varepsilon^{N, \Delta t}, E^{N, \Delta t}$	Maximum pointwise absolute error and ε -uniform error, respectively
$r_\varepsilon^{N, \Delta t}, r^{N, \Delta t}$	Rate of convergence and uniform rate of convergence, respectively
DEs	Differential equations
FDM	Finite difference method
FMFDMs	Fitted mesh finite difference methods
FMMs	Fitted mesh methods
FOFDMs	Fitted operator finite difference methods
FOMs	Fitted operator methods
PPDEs	Parabolic partial differential equations
PDPs	Parameter-dependent problems
PDEs	Partial differential equations
RPP	Regular perturbation problem
SPP	Singularly perturbed problem
SPPs	Singularly perturbed problems
SPDDEs	Singularly perturbed delay differential equations
SPPPDEs	Singularly perturbed parabolic partial differential equations

ABSTRACT

This dissertation focused on the development of efficient numerical methods for singularly perturbed parabolic partial differential equations. A singularly perturbed parabolic partial differential equation is a partial differential equation in which the highest space derivative of the differential equation is multiplied by a small parameter ε . When such a differential equation contains at least one delay parameter on the term different from the highest space derivative, then the differential equation is said to be a singularly perturbed delay differential equation. On the other hand, when the convection coefficient of the differential equation vanishes at a point in the domain, the problem is said to be a turning point problem. Due to the presence of the parameter ε , the solution of such differential equations exhibits a narrow region in which the solution varies rapidly near the layer region while changing slowly and smoothly away from it as ε approaches zero. This multi-scale nature of the solution makes it difficult to solve such differential equations analytically or standard numerical methods. As a result, it is essential to develop ε -uniform numerical methods to overcome the limitation. The main aim of this study is to develop, analyze and improve the ε -uniform numerical methods for solving a class of singularly perturbed parabolic partial differential equations having a time delay and a degenerating coefficient. Specifically, the two numerical approaches such as fitted operator finite difference method or fitted mesh finite difference method were used to develop the ε -uniform numerical methods. In the first approach, the construction of a fitted operator ε -uniform numerical scheme involves introducing an exponential fitting factor using the standard finite difference operator on a uniform mesh. The other successful approach of formulating a ε -uniform numerical scheme involves the use of standard finite difference operators on special piecewise uniform meshes condensed around the layer region. The stability and uniform convergence of these numerical methods were thoroughly analyzed and established. Extensive numerical computations were conducted to validate the theoretical findings. The results obtained using the proposed methods demonstrate higher accuracy compared to existing methods in the literature. Additionally, graphical representations were used to depict the boundary layer behaviour of the solutions.

Keywords: *Singular perturbation problem, Turning point problem, Time delay differential equations, Fitted numerical scheme, ε -uniform convergence, Boundary layer.*

CHAPTER ONE

INTRODUCTION

1.1 Singular perturbation problems

In real life, most physical phenomenon is described as a variant of causes and effects. In science and engineering, these variant relations are modeled by a class of differential equations (DEs). Such differential equations can be characterized by parameter-dependent problems (PDPs). PDPs are said to be singularly perturbed when arbitrary small parameter ε ($0 < \varepsilon \ll 1$) multiplies the highest order derivative term of the differential equations and ε is known as the singular perturbation parameter.

The concept of “singular perturbation” was introduced at the 3rd International Congress of Mathematicians held in Heidelberg in 1904, Ludwig Prandtl delivered a presentation on “fluid motion with small fraction.” His seven-page report [Prandtl \(1904\)](#) was published in the proceedings, which revolutionized the theoretical understanding of fluid and gas dynamics through introducing the concept of the boundary layer. Prandtl discussed the crucial role played by even a small quantity, such as the viscosity of common fluids such as water and air in determining their flow. Before 1904, the presence of boundary layers in fluid and gas dynamics was not well known. Then the notion of boundary layer became the foundation for the substantial work of ([Wasow, 1942](#)). The terminology of “singular perturbation” was introduced by Friedrichs and Wasow in their study on nonlinear vibration, which was presented at New York University. The term refers to a mathematical technique employed in the analysis of systems which exhibit multiple time scales. This technique is particularly valuable in the study of systems that exhibit rapid changes in behaviour and applies to a broad range of fields including physics, engineering, and finance. The work of Friedrichs and Wasow is an important contribution to the field of mathematical analysis, and their insights continue to inform cutting-edge research in a variety of domains ([Friedrichs and Wasow, 1946](#)).

Singularly perturbed problems (SPPs) are widely used mathematical models in various fields of study such as in fluid dynamics, in chemical reactions, in financial mathematics, in population dynamics, in semiconductor current, in transportation systems, and in others. Notably, in fluid and gas dynamics, Navier-stokes equations are the most prominent examples of SPPs. In two dimensions, these equations take on a specific form and represent a fundamental tool for understanding the behaviour of fluids and gases that are given by

$$\frac{\partial(z^2 + p)}{\partial s} + \frac{\partial(zv)}{\partial y} = \frac{1}{Re} \left(\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial y^2} \right), \quad (1.1)$$

subjected to the initial and boundary conditions. The components z and v are dependent variables referred to as the velocity of the materials (fluid or gas) and p is the pressure. The parameter Re is known as Reynolds number which is proportional to the length scale, and velocity scale and inversely proportional to the kinematic viscosity of the fluid. When Re is sufficiently much larger than one, the equation stated in (1.1) is converted into a singularly perturbed problem (SPP), and the corresponding small parameter, $\frac{1}{Re}$ is a perturbation parameter. Taking the perturbation parameter equal to zero, reduce (1.1) to the Euler equation (Miller et al., 1996). Thus, the singular perturbation nature of the problems occurs when the magnitude of the first-order derivative term, known as a convection term, is significantly greater than that of the higher-order derivative term. This is due to ε multiplies the diffusion term or highest-order derivative term. Hence, ignoring the perturbation parameter removes the crucial nature of the problem and leads to incorrect results and can hide important features of the solution (Kokotović et al., 1999).

Based on the impact of ε , perturbation problems can be classified as either regular or singular perturbation problems. A regular perturbation problem (RPP) is one in which the perturbed problem for small and non-zero values of ε is qualitatively identical to the unperturbed problem for $\varepsilon = 0$. A convergent expansion of the solution relative to ε , including the unperturbed solution and higher-order corrections can be obtained. Conversely, a singular perturbation problem is one in which the perturbed problem is qualitatively distinct from the unperturbed problem. One can obtain an asymptotic, but possibly divergent expansion of the solution, that depends on ε (Hunter, 2004).

SPPs are characterized by the presence of ε which causes a boundary layer in the solution. The boundary layer is a narrow region near the domain where the solution changes rapidly as ε approaches zero. As a result, SPPs exhibit a multi-scale nature of solutions, where the solution varies rapidly in the layer region and behaves regularly away from it. So, the behaviour of the solution in the layer region is different from that of outside the layer region. Hence, it is important to study SPPs in these regions, separately. Such an approach is critical for identifying the unique characteristics of each type of boundary layer, which ultimately enables the development of effective solutions to problems that stem from various practical applications. It is important to note that a boundary layer either regular or parabolic may occur away from any corner of the domain. A boundary layer is considered to be of parabolic type if the characteristics of the reduced problem, corresponding to $\varepsilon = 0$, are parallel to the boundary, whereas a regular type if this characteristic is not parallel to the boundary. On the other hand, a boundary layer that is near a corner is referred to as a corner type (Miller et al., 1996).

Boundary layers are commonly observed in various physical phenomena. For instance, in fluid mechanics, the effect of viscosity plays a crucial role in creating a fluid layer near a bounding surface for large Reynold's number (Landau and Lifshitz, 2013). In the water-pipe

problem, boundary layers are formed for cellular flows at high Peclet numbers (Novikov et al., 2005). On aircraft wings, viscous force distorts the surrounding non-viscous flow and forms a boundary layer on the part of the flow close to the wing (Duan et al., 2013). In the Earth's atmosphere, the atmospheric boundary layer is the air layer near the ground affected by diurnal heat, moisture, or momentum transfer from the surface (Chen et al., 1997).

Therefore, it is important to identify the location and behaviour of the boundary layer. The locations and behaviours of the boundary layer of SPPs may depend on ε and the nature of the perturbation problem. One common approach to identify the location of the boundary layer is to use asymptotic analysis, which involves expanding the solution in a series of terms that capture the dominant behaviour at different length scales. By matching these terms across the boundary or interfacing regions, one can determine the location and thickness of the boundary layer. Another approach is to use numerical methods, such as finite difference techniques, to solve the problem and then examine the behaviour of the solution across the boundary or interfacing regions. In particular, one can look for regions where the solution changes rapidly or exhibits oscillations, which may indicate the presence of a boundary layer. Ultimately, the best approach will depend on the specific problem at hand and the available tools and techniques for the computation and analysis. Due to ε multiplies the highest order derivative term of second order singularly perturbed convection-diffusion equation the convection term becomes the dominant term which leads to the solution of the problem to exhibits a boundary layer either on the left or the right end of the domain depending on the sign of the coefficient of the first order derivative term (O'Malley, 1991; Ross et al., 2008; Kaushik et al., 2010).

1.2 Singularly perturbed parabolic partial differential equations

Singularly perturbed parabolic partial differential equations (SPPPDEs) are a class of partial differential equations (PDEs) where the highest order derivative term is multiplied by ε . It can be classified as either SPPPDEs with turning point or without turning point. SPPPDEs with a turning point, when the convection coefficient of the problems vanishes at a point in the domain. SPPPDEs without a turning point, when the convection coefficient is non-zero throughout the domain (Ross et al., 2008).

SPPPDEs with turning point can be either problems with boundary turning point Vulanović and Farrell (1993); Gupta and Kadalbajoo (2011); Dunne et al. (2003) or problems with interior turning point (Geng et al., 2014; Munyakazi et al., 2019). The problem in which the convection coefficient vanishes inside the domain by changing signs is a turning point problem with the interior layer, whereas the problem in which the convection coefficient vanishes at the boundary of the domain is a turning point problem with a boundary layer. The point in the domain where the convection coefficient vanishes is known as the turning point. Turning point may be classified as simple or multiple, depending on whether the convection coefficient has single or

multiple zero values, respectively ([Singh et al., 2022](#); [Mbroh et al., 2021](#)).

SPPPDEs with turning points arise as mathematical models for various physical phenomena. Among these, the problems with the interior turning point represent the one-dimensional version of stationary convection-diffusion problems with a dominant convective term and a speed field that changes its sign in the catch basin. On the other hand, boundary turning point problems arise in geophysics [Hanks \(1971\)](#) and multiple turning point problems arise in the modeling of thermal boundary layers in laminar flow ([Vulanović and Farrell, 1993](#); [Schlichting and Gersten, 2016](#)). A single boundary turning point problem because of the assumption that the velocity distribution is linear, whereas multiple boundary turning point problems because of the assumption is higher order of velocity distribution ([Schlichting and Gersten, 2016](#)).

Many dynamical systems' behaviour depends not only on the current state of the system but also on its history. In science and engineering, this phenomenon is described by time-delay differential equations. If these equations include at least one delay parameter on the term different from the highest derivative term, they are called singularly perturbed delay differential equations (SPDDEs). Otherwise, they are neutral types ([Wang, 1963](#); [Kumar and Kumari, 2019](#)).

In a physical system, delay represents time lag or after effect ([Kumar et al., 2019](#)). Thus, such phenomenon arises from many biological, chemical, and physical systems, which are characterized by both spatial and temporal variables. For instance, the Britton-model (for population dynamics) and a physical system of numerical control of furnace processing in metal sheets are realistic examples that represent time-delayed phenomena. So, a mathematical model that represents control of furnace processing in metal sheets with time delay is given by

$$z_t(s, t) - \varepsilon z_{ss}(s, t) - v(g(z(s, t - \tau)))z_s(s, t) = c[h(z(s, t - \tau)) - z(s, t)], \quad (1.2)$$

where z denoted the temperature distribution in a metal sheet, and h and v denoted the heat source and the velocity with which the metal sheet is moving respectively. Both h and v are dependent on the term $z(s, t - \tau)$. The time delay of fixed length τ is induced because of the finite speed of the controlling device. When $\tau = 0$, the problem is said to be a thermal problem without time delay ([Wang, 1963](#)). Thus, the convection-diffusion-reaction equation is found in the application of three distinct processes. The first process, referred to as convection, arises due to the movement of materials from one medium to another medium. The second process, diffusion, is due to the movement of materials from a medium of high concentration to a medium of low concentration. The last process, reaction, occurs as a result of decay, adsorption, and the reaction of substances with other components. Qualitatively, these three processes collectively form the convection-diffusion-reaction model, which describes how the disturbed quantity is studied when the medium changes under the influence of these processes

([Mphephu, 2013](#)).

The classification of SPDDEs depends on the magnitude of their delay parameter to their perturbation parameter. Two distinct classifications are observed: SPDDEs with a small time delay, and SPDDEs with a large time delay. The first is characterized by a delay parameter magnitude smaller than the perturbation parameter, while the second is defined by a delay parameter magnitude higher than the perturbation parameter. It is important to observe these distinctions as they provide a clear understanding of the properties of SPDDEs and treated differently ([Kadalbajoo and Kumar, 2008](#)).

SPDDEs find a wide range of applications in various fields of science and engineering. Some of these applications include simulating the extraction of oil from underground reservoirs, modeling chemical processes, fluid flows, water quality problems in river networks, chemical systems, and transmission lines, and studying heat/mass transfer processes in composite materials with small heat conduction/diffusion. Additionally, they are used in control theory, the drift-diffusion model of semiconductor devices, industrial processes, chemical kinetics, transport delays, and calculating incubation periods by considering only the time required for the process to occur ([Das and Natesan, 2015](#); [Sharma, 2017](#); [Kumar and Kumari, 2019](#)). Over the last three decades, time-delay differential equations have become an increasingly popular area of study and a highly active field of research. For further details, interested individuals can refer to some of the articles on the topic [Ansari et al. \(2007\)](#); [Gowrisankar and Natesan \(2014\)](#); [Kumar and Kumar \(2014\)](#); [Sharma \(2017\)](#); [Govindarao et al. \(2020\)](#); [Singh et al. \(2018\)](#); [Wang et al. \(2017\)](#) and the references therein.

1.3 Numerical schemes to solve singularly perturbed problems

There are two main techniques to solve SPPs: numerical methods and asymptotic methods. Mathematicians use asymptotic methods to study the analytical solution of SPPs as the value of ε approaches zero. Asymptotic methods provide approximate solutions, but they don't give quantitative information. This technique provides qualitative information, which can be used to obtain differential equations of lower order than the governing equations. Usually, these lower-order differential equations do not satisfy all of the boundary or initial conditions. The techniques for overcoming such difficulty constructed the straight-forward expansion known as the outer expansion valid away from the boundary layer region. Within the boundary layer region, inner expansion is formed by using the stretched variable rather than the original independent variable, which takes account of the scale of certain derivative terms. Then, the inner and outer expansions are matched by using the method of matched asymptotic expansions. Multiple scales and matched asymptotic expansions are the most widely used asymptotic methods to approximate the solution of SPPs ([Bender and Orszag, 1999](#); [Bush, 1992](#); [O'Malley, 1991](#)). On the other hand, numerical methods provide quantitative results. When applied to

SPPs, the asymptotic methods yield a single solution that is valid throughout the domain of interest, while the numerical methods yield two solutions, known as inner and outer solutions. Therefore, numerical methods are necessary to obtain accurate results (Miller et al., 1996).

Classical numerical methods such as finite difference, finite element, or finite volume methods on uniform meshes fail to provide satisfactory numerical solutions and are usually unstable as ε tends to zero. This means that the pointwise error in the numerical solution increases until the mesh parameter and the perturbed parameter have the same order of magnitude. On the other hand, errors of the numerical solution depend on the distribution of mesh points and become small only when the effective mesh size in the layer is much less than the value of the parameter ε . This is contrary to the intuitive expectation that the pointwise error should decrease when the mesh is refined. Thus, to get satisfactory results, the mesh is further refined, which is impractical, especially in higher dimensions since it requires an extremely large number of mesh points. This leads to a massive computational cost (Miller et al., 1995,9; Ross et al., 2008; Das and Natesan, 2015). This drawback motivates us to have reliable numerical methods that can produce solutions with a pointwise error that is independent of ε , and that decreases as the mesh is refined at a rate of the order of convergence independent of ε . These requirements are important for achieving ε -uniformly convergent numerical schemes.

In the construction of ε -uniform numerical methods two approaches have been used now a day. The first approach is the fitted operator finite difference methods (FOFDMs), and usually, it is carried out on a uniform mesh. In this approach, the construction of the ε -uniform numerical method involves introducing an exponential fitting factor (artificial viscosity) along with the diffusion coefficient to control the rapid growth or decay of the solution in the boundary layer region. The exponentially fitted operator method was introduced by (Allen and Southwell, 1955). In a later study, Doolan et al. (1980) investigated the exponentially fitted operator method and developed an appropriate fitting factor. This method incorporates the denominator function (non-standard finite difference) of classical finite differences with positive function, which is derived using rules introduced by (Mickens, 1994). The advantage of these methods are easy to implement because the meshes used are uniform, and this contributed to their more rapid progress compared with condensing mesh methods. However, they are difficult to extend to higher dimensions SPPs (Miller et al., 1996; Patidar, 2005).

The second approach to develop ε -uniform numerical methods involves the use of mesh that is adapted to the singular perturbation parameter. Such methods are known as fitted mesh finite difference methods (FMFDMs) and can be incorporated into both finite difference and finite element methods (Miller et al., 1996). FMFDMs meshes lie under two classes: graded and piecewise uniform meshes. Bakhvalov and Shishkin meshes are the most well-known layer-adapted meshes. Bakhvalov was the first to use layer-adapted meshes for reaction-diffusion problems Bakhvalov (1969) and the mesh is designed as a graded mesh so that it is fine at

one end of the layer region and gradually becomes coarser as it moves outside the layer region, where the mesh is uniform. Thereafter, [Vulanovic \(1983\)](#) constructed a mesh which can capture the boundary layer. [Gartland \(1988\)](#) used a graded mesh in the domain divided into three regions. These regions are the inner region (boundary region), where the mesh is graded exponentially with respect to ε . The transition region in which the mesh is changing geometrically from a fine to a coarse position and the outer region in which the mesh is uniform. The generation of this mesh is more complicated to extend to higher dimensions.

In 1988, Shishkin proposed a relatively simple mesh that can be extended for higher dimensions and nonlinear SPPs. [Shishkin \(1988\)](#) constructed piecewise uniform meshes is a mesh that is a union of a finite number of uniform meshes having different mesh parameters, such as a fine mesh in the boundary layer regions and a coarse mesh outside the layer regions. A standard finite difference operator is applied to the Shishkin mesh to obtain a system of finite difference equations, which is then solved in the usual way to approximate solutions. Shishkin mesh has attracted much attention and is widely used for numerical approximation because of its simplicity in constructing piecewise uniform mesh. Both approaches can be enhanced by having a priori information about the layer region in SPPs ([Miller et al., 1996](#); [Linß, 2009](#); [Chakravarthy et al., 2018](#)).

If sufficient priori information about the solution is available, then an appropriate mesh can be constructed, where the solution accuracy of these methods depends only on the number of meshes and is independent of the perturbation parameter. Otherwise, a newly developed adaptive strategy can be implemented to determine the appropriate grading for obtaining a layer-adapted mesh. This technique uses the equidistribution of the positive monitor function, which defines some error measurement of the numerical solution. This simply means the rearrangement of the meshes so that the error quantity is equally distributed. Here, the importance of the monitored function is to automatically detect the presence of the boundary layers, along with their locations and width, and distribute mesh points accordingly. The mesh adaptation techniques, which study the proper movement of a fixed number of mesh points by the equidistribution of a positive monitor function, are known as moving mesh methods ([Linß, 2009](#); [Kumar et al., 2022](#)).

1.4 Statement of the problem

The study of SPPs has led to the development of powerful analytical and computational tools, revolutionizing the approach of scientists and engineers to solve complex problems. There are two main approaches available in the literature to solve SPPs, one is the analytical approach (or the asymptotic approximation approach) and the other one is the numerical approach (or the numerical method). The asymptotic approximation provides qualitative information about a family of problems, whereas numerical methods provide quantitative informa-

tion about particular problems. In the asymptotic approximation method, multiple scales and matched asymptotic expansions are the most widely used methods. Since the asymptotic provides qualitative information about a family of problems. So, to get quantitative information about a particular problem numerical methods are needed.

It is known that SPPPDEs are typically characterized by the presence of ε . This parameter causes the existence of a boundary layer which results in a multi-scale character of the solution that varies rapidly in the boundary layer region and behaves regularly away from the layer region. As a result of this inherent nature of the solution, treating such problems with classical numerical methods loses its stability (Jiang et al., 2017). To avoid this instability and improve the accuracy, an unacceptably large number of mesh points are required. Due to the limited size of computer memory, rounding off errors, and massive computational cost it is not practical (Miller et al., 1995; Ross et al., 2008; Kumar and Kumari, 2020; Woldaregay et al., 2021). These difficulties can be overcome by the use of fitted numerical methods whose accuracy does not depend on the perturbation parameter, i.e., methods that are ε -uniformly convergent. This context compels the authors to look out for an efficient ε -uniform numerical scheme. The schemes that are uniformly convergent regardless of ε , stable, and whose accuracy has a small error in the boundary layer region.

Consequently, to develop ε -uniform numerical scheme, there are two general approaches (that are: the fitted operator methods and the fitted mesh methods) are used. Ansari et al. (2007) initiated ε -uniform numerical scheme to solve a singularly perturbed parabolic PDE with a time delay of the form of reaction-diffusion types. Then, several authors sought sound numerical schemes. For treating SPPPDEs with large time delay Negero and Duressa (2021); Babu and Bansal (2021); Negero and Duressa (2022a); Woldaregay et al. (2021); Hassen and Duressa (2023) developed different numerical methods using the fitted operator finite difference methods, and Kumar and Kumari (2019); Das and Natesan (2018,0); Govindarao and Mohapatra (2018) used the fitted mesh finite difference methods. For treating SPPPDEs with a small time delay Chakravarthy and Kumar (2019); Rao and Chakravarthy (2014); Ayele et al. (2023) formulated numerical schemes using fitted numerical approaches.

Dunne et al. (2003); Gupta and Kadalbajoo (2011); Singh et al. (2022); Kumar et al. (2022); Ku Sahoo and Gupta (2021); Majumdar and Natesan (2017); Majumdar and Natesan (2019) considered singularly perturbed parabolic convection-diffusion equations with degenerate coefficients using fitted mesh finite difference methods and Mbroh et al. (2021); Gelu and Duressa (2023) used the fitted operator finite difference methods. Rai and Yadav (2021) treated a singularly perturbed delay parabolic convection-diffusion problem with degenerate coefficient using fitted mesh finite difference method.

From the literature, we observed that the numerical schemes to solve singularly perturbed parabolic PDEs have received much attention. Even though different numerical schemes are

developed in the area, the numerical solution of the equation under consideration has limitations in terms of accuracy and the order of convergence. To overcome this gap, we are motivated to formulate numerical schemes that give more accurate, stable, and ε -uniform convergence results.

1.5 Objectives of the study

1.5.1 General objective

The main objective of this research is to develop numerical methods for solving singularly perturbed parabolic partial differential equations.

1.5.2 Specific objectives

The specific objectives of this study are to:

1. formulate uniformly convergent numerical methods for solving singularly perturbed parabolic convection-diffusion equations.
2. establish the stability of the formulated numerical methods.
3. investigating the convergence of the formulated numerical methods.

1.6 Significance of the study

This dissertation provided fitted numerical methods for solving a class of singularly perturbed parabolic partial differential equations. The methods provides accurate, stable and uniformly convergent numerical schemes and allow to simulation and study of the behaviour of the solution. The outcome of this dissertation provides valuable insight to students, scholars, and researchers who are actively engaged in this research area. Furthermore, the study can be used as a reference to extend knowledge for further investigations in the area.

1.7 Scopes of the study

Numerous SPPPDEs with and without time delay have been extensively studied by several authors. Despite the considerable research on SPPs in the form of PDEs, investigations on a class of singularly perturbed delay differential equations and a class of singularly perturbed degenerate parabolic differential equations are still in the early stages. To explore further investigation into these areas, this dissertation focuses on the following governing problems:

Governing problem-I

A class of singularly perturbed parabolic partial differential equation having time delay on the rectangular domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ and the boundary $\eta = \eta_l \cup \eta_b \cup \eta_r$ of the

form

$$\begin{cases} z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -\kappa(s, t)z(s, t - \tau) + \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & (s, t) \in \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), & t \in \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t), & t \in \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (1.3)$$

where $\mathfrak{L}_\varepsilon z(s, t) = -\varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) + \chi(s, t)z(s, t)$.

For the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\tau > 0$ and the functions $\beta(s, t)$, $\chi(s, t)$, $\kappa(s, t)$, $\gamma(s, t)$, $\psi_b(s, t)$, $\psi_l(t)$, and $\psi_r(t)$ are assumed to be smooth and bounded on the domain.

Governing problem-II

A class of singularly perturbed degenerate parabolic convection-diffusion equation on the rectangle domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ and the boundary $\eta = \eta_l \cup \eta_b \cup \eta_r$ of the form

$$\begin{cases} -b(s, t)z_t(s, t) + \varepsilon z_{ss}(s, t) + \beta(s)z_s(s, t) - \chi(s, t)z(s, t) = \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t) \text{ on } \eta_b = \{(s, t) : t = 0, s \in \bar{\Omega}\}, \\ z(0, t) = \psi_l(t) \text{ on } \eta_l = \{(s, t) : s = 0, t \in \bar{\Lambda}\}, \\ z(1, t) = \psi_r(t) \text{ on } \eta_r = \{(s, t) : s = 1, t \in \bar{\Lambda}\}. \end{cases} \quad (1.4)$$

The functions $\gamma(s, t)$, $b(s, t)$, $\beta(s)$, $\chi(s, t)$, $\psi_b(s, t)$, $\psi_l(t)$, and $\psi_r(t)$ are assumed to be smooth and bounded on the domain. Moreover, the coefficient functions satisfy the conditions $\beta(s) = \beta_0(s)s^p$, $\beta_0(s) \geq \mu > 0$, $p \geq 1$, $0 < \alpha \leq b(s, t)$, $0 < \varpi \leq \chi(s, t)$ on $\mathfrak{D}$.

Governing problem-III

A class of singularly perturbed degenerate parabolic convection-diffusion equation having a time delay on the rectangular domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ and the boundary $\eta = \eta_l \cup \eta_b \cup \eta_r$ of the form

$$\begin{cases} \mathfrak{L}_\varepsilon z(s, t) - b(s, t)z_t(s, t) = \kappa(s, t)z(s, t - \tau) + \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t) \text{ on } \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t) \text{ on } \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t) \text{ on } \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (1.5)$$

where $\mathfrak{L}_\varepsilon z(s, t) = \varepsilon z_{ss}(s, t) + \beta(s)z_s(s, t) - \chi(s, t)z(s, t)$.

We assume the functions $\gamma(s, t)$, $b(s, t)$, $\beta(s)$, $\chi(s, t)$, $\psi_b(s, t)$, $\psi_l(t)$, and $\psi_r(t)$ are smooth and bounded on the domain. Moreover, the coefficient functions satisfy the conditions $\beta(s) =$

$\beta_0(s, t)s^p$, $\beta_0(s, t) \geq \mu > 0$, $p \geq 1$, $0 < \varpi \leq \chi(s, t)$, $0 < \alpha \leq b(s, t)$, $\chi(s, t) + \kappa(s, t) > 0$ on $\mathfrak{D}$. Problems of type (1.4) and (1.5) are known as singularly perturbed turning point (or degenerate coefficient) problems since the coefficient of the convective term vanishes at $s = 0$ and the point $s = 0$ is the turning point. Depending on the value of p we can have either a simple turning point or multiple turning points (Mbroh et al., 2021).

1.8 Mathematical preliminaries

This section introduce fundamental definitions, notations, and terms that are pertinent to this dissertation. Within the dissertation, the symbol $C > 0$ represents a generic positive constant that remains unaffected by mesh and perturbation parameters. It is important to note that C may take on different values at different points throughout the dissertation.

The space and time domain $\mathfrak{D} = \Omega \times \Lambda$, where Ω is given as $(0, 1)$ and Λ is given as $(0, T]$. Let $C^{i,l}(\mathfrak{D})$ denote the space of all functions that are continuously differential up to order i in the space direction and l in the time direction on $\mathfrak{D}$, respectively, where i and l are non-negative integers.

Throughout this dissertation for the analysis we used the standard supremum norm $\|\cdot\|_{\mathfrak{D}}$ is defined by

$$\|\gamma\| = \sup_{(s,t) \in \mathfrak{D}} |\gamma(s, t)|,$$

for the function $\gamma(s, t)$ on the domain of $\mathfrak{D}$. Alternatively, used the notation $\|\cdot\|$ when the domain is obvious (Miller et al., 1996).

We consider difference approximation on the domains $\Omega^N = \{s_m\}_{m=0}^N$, $s_0 = 0$, $s_N = 1$, $s_m = mh$, and $\Lambda^M = \{t_n = n\Delta t, \Delta t = T/M, 0 \leq n \leq M\}$ in the subsequent chapters.

The finite difference scheme

Given a mesh function $z(s_m, t_n) = z_m^n$, defined the forward, backward, central difference and second-order central difference operators δ_s^+ , δ_s^- , δ_s^0 and δ_s^2 in space for non-uniform mesh size are given by

$$\delta_s^+ z_m^n = \frac{z_{m+1}^n - z_m^n}{h_{m+1}}, \quad \delta_s^- z_m^n = \frac{z_m^n - z_{m-1}^n}{h_m}, \quad \delta_s^0 z_m^n = \frac{z_{m+1}^n - z_{m-1}^n}{\hat{h}_m}, \quad \delta_s^2 z_m^n = \frac{2(\delta_s^+ z_m^n - \delta_s^- z_m^n)}{\hat{h}_m}$$

, respectively, and define the backward difference operator δ_t^- in time is given by

$$\delta_t^- z_m^n = \frac{z_m^n - z_m^{n-1}}{\Delta t},$$

where $\hat{h}_m = h_m + h_{m+1}$, $m = 1, \dots, N-1$.

Definition 1.8.1. *Linß (2009); Ross et al. (2008)* A matrix A is said to be an M -matrix if its entries $a_{i,j} \leq 0$ for $i \neq j$ and its inverse A^{-1} exist with $A^{-1} \geq 0$.

Definition 1.8.2. *Ross et al. (2008)* A numerical method is said to be **stable** if the errors at any stage of the computation do not grow in the course of the computational processes.

Definition 1.8.3. *Ross et al. (2008)* A numerical scheme is **consistent** if the method approximating differential equation becomes exact as the step size tends to zero. For a method to be consistent, the truncation error should vanish as the step size approaches zero.

Definition 1.8.4. *Ross et al. (2008)* A numerical method is said to be **convergent** if the solution of the numerical method approaches the exact solution of the original differential equation in the course of the computation. A stable and consistent process of discretization leads to convergence of the solution.

Definition 1.8.5. *Roos et al. (2008); Miller et al. (1996)* (ε - **uniform convergence**) Consider a family of SPPs parameterized by a singular perturbation parameter ε ($0 \leq \varepsilon \ll 1$). Assume that each problem in the family exact solution is denoted by z and that each z is approximated by a sequence of numerical solutions $\{(Z, \bar{\Omega}^{N,\Delta t})\}$, where Z is defined on the mesh $\bar{\Omega}^{N,\Delta t}$ and N and Δt are discretization parameters. Then, the numerical solutions Z are said to converge ε -uniformly to the exact solution z , if there exist positive integers N_0, M_0 and positive numbers C, p and q , such that for all $N \geq N_0$ and $M \geq M_0$, where N_0, M_0, C, p , and q are all independent of ε , we have

$$\sup_{\varepsilon \in (0,1]} \|Z - z\| \leq C(N^{-p} + (\Delta t)^q),$$

where p and q are the ε -uniform order of convergence of the space and time directions respectively, and C is the ε -uniform error constant.

The choice of the maximum norm, for the measurement of the error, is due to the need to measure the error in the very small domains in which the boundary or interior layers occur. Other norms, such as the root mean square, involve averages of the error, which smooth out rapid changes in the solutions and therefore may fail to capture the local behaviour of the error in these layers. The unbounded behaviour of terms in the differential equation and their difference approximations give rise to difficulties in the proof of ε -uniform convergence of the relevant schemes.

Fitted (also called layer-adapted) meshes lie under two classes: graded and piecewise uniform meshes. The most successful and popular ones are those of Bakhvalov and Shishkin. A Bakhvalov mesh is designed in such a way that in the layer region, the mesh is fine at one end and gradually becomes coarse, and outside the layer region the mesh is uniform. A

Shishkin mesh is a union of two or more uniform meshes with different discretization parameters. Shishkin mesh is the most frequently studied mesh because of its simplicity and piecewise uniform mesh. Below is some of the brief explanation of Shishkin mesh.

Shishkin meshes

This part elaborates the notion of constructing Shishkin meshes for problems featuring right boundary layer. What sets Shishkin apart from other piecewise uniform meshes is the selection of the transition parameter, which denotes the point at which the mesh size changes abruptly. Suppose that the problem exhibits a boundary layer along the side $s = 1$; in that case, the mesh ought to be dense in the vicinity of $s = 1$. To establish the piecewise uniform mesh, partitioned the domain $[0, 1]$ into two sub-domains such that $[0, 1] = [0, 1 - \sigma] \cup (1 - \sigma, 1]$. Then divide each of the sub-domains into $N/2$. Denote the spatial grid by equal intervals

$$\Omega^N = \{0 = s_0, s_1, \dots, s_{N/2} = 1 - \sigma, \dots, s_N = 1\},$$

where

$$s_m = \begin{cases} m \frac{2(1-\sigma)}{N}, & m = 0, 1, 2, \dots, N/2 \\ (1 - \sigma) + (m - N/2) \frac{2\sigma}{N}, & m = (N/2) + 1, \dots, N. \end{cases}$$

The transition point $1 - \sigma$, which separates the coarse and the fine points on the mesh, is obtained by taking $\sigma = \min\{\frac{1}{2}, \frac{\sigma_0 \varepsilon}{\beta} \ln N\}$. The analysis has been done by assuming that $\sigma = \sigma_0 \frac{\varepsilon}{\beta} \ln N$, as otherwise N is exponentially large compared to ε . From the definition of s_m , the spatial step size can be denoted by

$$H := h_m = \frac{2(1 - \sigma)}{N}, \quad m = 0, 1, 2, \dots, N/2$$

$$h := h_m = \frac{2\sigma}{N}, \quad m = (N/2) + 1, \dots, N.$$

H and h are the mesh widths in $[0, 1 - \sigma]$ and $(1 - \sigma, 1]$, respectively. From the above equation it is true that $N^{-1} \leq H \leq 2N^{-1}$, $h = \sigma_0 \varepsilon N^{-1}$ and the uniform mesh can be obtained by choosing $\sigma = 1/2$ (Linß, 2009).

1.9 Organization of the dissertation

This dissertation contains nine chapters and conclusion and recommendation section. **Chapter 1**, deals with singularly perturbed problems in the introduction section, statements of the problem, objectives of the study, the scope of the study, and mathematical preliminaries that are considered in this study. **Chapter 2**, deals with a brief survey of the most related and recently published work on the area of singularly perturbed problems, the numerical methods formulated to treat the problems and the obtained results are discussed in detail. Identify the research

gap and the area in which little attention is given to conduct further study on the area of SPP-PDEs. **Chapter 3**, deals with the methodology, study design, source of information, methods applied, and general mathematical procedures of this study.

The rest of the chapters included the main work of this study and are organized in terms of the considered governing problems: In **Chapter 4**, **Chapter 5**, **Chapter 6**, and **Chapter 7** the governing problem (1.3) is considered. In **Chapter 4** and **Chapter 5**, the fitted operator finite difference method is used with a small delay and with a large delay, respectively. Fitted mesh finite difference method and fitted tension spline method is used in **Chapter 6** and **Chapter 7**, respectively. The time variable of the problem is discretized using the backward Euler scheme.

In **Chapter 8**, the governing problem (1.4) is considered. The fitted operator finite difference method is used. The time variable of the problem is discretized using the Crank-Nicolson scheme. In **Chapter 9**, the governing problem (1.5) with a small time delay is used. The fitted operator finite difference method is used. The time variable of the problem is discretized using the Crank-Nicolson scheme.

Finally, a summary of the findings, conclusions and recommendations and the scope for further investigations are presented.

CHAPTER TWO

LITERATURE REVIEW

Over the past thirty years, significant progress has been made in developing numerical methods for solving SPPDEs and various types of delay differential equations (DDEs) in the context of PDEs. This dissertation provides a survey of some of the research that has been conducted on SPPDEs with a time delay, singularly perturbed degenerate parabolic partial differential equations, and singularly perturbed degenerate parabolic partial differential equations with a time delay.

Notably, the study of numerical methods for SPPDEs with time delay has been an active area of research for many authors in the past three decades. The initial study on the topic was conducted by [Ansari et al. \(2007\)](#), who restated the mathematical model presented in (1.2) as a time-delayed reaction-diffusion type problem as described below. On the domain $\mathfrak{D} = (0, 1) \times (0, T]$

$$\begin{cases} z_t(s, t) - \varepsilon z_{ss}(s, t) + a(s, t)z(s, t) = -b(s, t)z(s, t - \tau) + f(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \phi_b(s, t), & (s, t) \in \Gamma_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \phi_l(t), & t \in \Gamma_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \phi_r(t), & t \in \Gamma_r = \{(1, t) : 0 \leq t \leq T\}. \end{cases} \quad (2.1)$$

For the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\tau > 0$ and the functions $a(s, t)$, $b(s, t)$, $f(s, t)$, $\phi_b(s, t)$, $\phi_l(t)$, and $\phi_r(t)$ are assumed to be smooth and bounded on the domain which meets $a(s, t) \geq \beta \geq 0$. They treated a problem using a central finite difference operator in the space direction on piecewise uniform mesh and an implicit Euler scheme in the time direction. They proved that the errors are bounded in the maximum norm by $C(N^{-2} \ln^2 N + \Delta t)$. Following this work, many authors have developed different numerical methods to solve Eq. (2.1). To provide more details, [Bashier and Patidar \(2011\)](#) considered a fitted operator finite difference method, while [Gowrisankar and Natesan \(2014\)](#) used an implicit Euler scheme for the time derivative and a classical central difference scheme for the spatial derivative via equidistribution grids. [Kumar and Kumar \(2014\)](#) developed a hybrid scheme that uses a generalized Shishkin mesh in spatial direction and the implicit Euler scheme on a uniform mesh in time direction. [Singh et al. \(2018\)](#) employed domain decomposition technique. [Govindarao et al. \(2020\)](#) developed a numerical scheme that uses an implicit Euler scheme on a uniform mesh for temporal discretization and a central difference scheme on the Shishkin mesh for spatial discretization. Subsequently, numerous authors extended this problem into the

convection-diffusion type of problem to seek an accurate enough, high rate of convergence and ε -uniformly convergent numerical scheme.

In the subsequent sections, the scheme developed to treat singularly perturbed parabolic partial differential equations with time delay and singularly perturbed degenerate parabolic partial differential equations are presented. Despite the numerous numerical methods developed to solve SPPPDEs with time delay and degenerating coefficient, the review focused on the most relevant and recent published works. This review advances further understanding and highlights the areas which need more investigation.

2.1 Singularly perturbed partial differential equations with time delay

[Kaushik et al. \(2010\)](#) treated a class of singularly perturbed parabolic partial differential equation of the convection-diffusion type, with a time delay in the reaction term on the domain $\mathfrak{D} = \Omega \times (0, T] = (0, 1) \times (0, T]$ of the form

$$\begin{cases} L_\varepsilon z(s, t) \equiv z_t(s, t) - \varepsilon z_{ss}(s, t) + a(s, t)z_s(s, t) + b(s, t)z(s, t - \tau) + c(s, t)z(s, t) = f(s, t), \\ z(s, t) = \varphi(s, t), \quad s \in \Omega \equiv (0, 1), \quad t \in [-\tau, 0), \\ z(s, 0) = \varphi(s, 0) = \varphi(s); \quad s \in \Omega \equiv (0, 1), \\ z(0, t) = \Phi, z(1, t) = \Psi; \quad t \in (0, T], \end{cases} \quad (2.2)$$

where the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\tau > 0$ and the functions $a(s, t)$, $b(s, t)$, $c(s, t)$, and $f(s, t)$ are assumed to be sufficiently smooth on the domain which meets $0 < \beta \leq a(s, t)$, $0 < \eta \leq b(s, t)$ and $c(s, t) \geq 0$. For the considered equation the authors developed a numerical scheme using the standard finite difference upwind operator on piecewise uniform mesh in the spatial direction and uniform mesh in the temporal direction and obtained first-order accurate a uniformly convergent numerical method. So, it needs to increase the rate of convergence.

[Kaushik and Sharma \(2012\)](#) considered the class of singularly perturbed parabolic partial differential equation of convection-diffusion type having a delay in the convection term on the domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ of the form

$$\begin{cases} p(s, t)z_t(s, t) - \varepsilon z_{ss}(s, t) + a(s, t)z_s(s, t) + b(s, t)z(s, t - \tau) = f(s, t), \quad (s, t) \in \mathfrak{D}, \\ z(s, t) = z_0(s, t), \quad t \in [-\tau, 0], \\ z(0, t) = z(1, t) = 0, \quad t \in (0, T]. \end{cases} \quad (2.3)$$

For the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\tau > 0$ and the functions $a(s, t)$, $p(s, t)$, $b(s, t)$, and $f(s, t)$ are assumed to be sufficiently smooth on the domain which meets $p(s, t) \leq \alpha < 0$, $b(s, t) \geq \theta > 0$ and $a(s, t) \neq 0$. For Eq. (2.3) the authors employed

implicit Euler for the time derivative and upwind finite difference method on a uniform mesh. The authors solved the system of difference equations using the discrete invariant embedding algorithm. The developed scheme is not accurate enough.

Hereafter, [Das and Natesan \(2015\)](#) extended the work of the preceding authors treating a class of singularly perturbed parabolic partial differential equation having time delay on the rectangular domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ and the boundary $\eta = \eta_l \cup \eta_b \cup \eta_r$ of the form

$$\begin{cases} z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -c(s, t)z(s, t - \tau) + f(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & (s, t) \in \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), & t \in \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t), & t \in \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (2.4)$$

where $\mathfrak{L}_\varepsilon z(s, t) = -\varepsilon z_{ss}(s, t) + b(s, t)z_s(s, t) + a(s, t)z(s, t)$. For the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\tau > 0$ and the functions $a(s, t), b(s, t), c(s, t), f(s, t), \psi_b(s, t), \psi_l(t)$, and $\psi_r(t)$ are assumed to be smooth and bounded on the domain which meets $b(s, t) \geq \beta > 0, a(s, t) \geq \alpha > 0, c(s, t) \geq \mu > 0$ with a boundary layer near $s = 1$ of width $O(\varepsilon)$. The authors developed the scheme which comprises the implicit Euler for the time derivative and the hybrid finite difference scheme on a piecewise uniform Shishkin mesh. The hybrid scheme is developed by a proper combination of the midpoint upwind scheme in the outer region and the central difference scheme in the inner layer region for the spatial derivative. The proposed scheme is a parameter-uniform almost first-order in the time direction and second-order up to a logarithm factor in the spatial direction, but this is not an optimal second-order accuracy. They showed that the rate of convergence is almost two up to a logarithm factor. To obtain accurate enough and develop a higher order uniformly convergent numerical scheme which provides the optimal second-order convergence or above, this governing equation is initiated by numerous authors and for (1.3) too. Hence, the most recently published works on Eq. (2.4) are given below.

[Gowrisankar and Natesan \(2017\)](#) developed a backward Euler scheme for the time derivative on uniform mesh and an upwind finite difference scheme for the spatial derivatives on piecewise uniform mesh for Eq. (2.4). They proved that the proposed scheme was ε -uniformly convergent of the first-order in time and the first-order up to logarithmic factor in space. i.e., $O(N^{-1} \ln N + \Delta t)$. The obtained method is not accurate enough and the rate of convergence needs to improve.

[Salama and Al-Amery \(2017\)](#) used a numerical method comprised of the Crank-Nicolson method on uniform mesh for temporal discretization and the spatial discretization operator compact implicit (OCI) method on piecewise uniform Shishkin mesh for the considered aforementioned Eq. (2.4). Obtained ε - uniformly convergent results with a higher order accuracy

of second-order in time and of second-order outside the region of the boundary layer, and almost fourth order inside the layer region in the spatial direction. They increased the rate of convergence, but not accurate enough.

Das and Natesan (2018) applied a backward Euler scheme for the time derivative on a uniform mesh and an upwind finite difference scheme for the spatial derivatives on a piecewise uniform Shishkin mesh for Eq. (2.4). Using the Richardson extrapolation technique, they improved the order of convergence from $O(N^{-1} \ln N + \Delta t)$ to $O(N^{-2} \ln^2 N + \Delta t^2)$, but not optimal second-order convergent and not accurate enough.

Govindarao and Mohapatra (2018) discretized the spatial direction using a hybrid scheme on Shishkin-type meshes (standard Shishkin mesh, Bakhvalov–Shishkin mesh) and the temporal direction using an implicit trapezoidal scheme on uniform mesh for Eq. (2.4). The hybrid finite difference scheme is a proper combination of the midpoint upwind scheme in the outer region and the central difference scheme in the inner region. They proved that the proposed scheme was parameter-uniformly convergent and second-order accurate to space and time. Numerical results are carried out to verify the theoretical estimates.

Kumar and Kumari (2019) considered the scheme comprises the Crank–Nicolson method and midpoint upwind finite difference scheme on a piecewise uniform mesh for Eq. (2.4). They proved that the considered difference scheme is second-order accurate in the temporal direction and first-order (up to a logarithmic factor) accurate in the spatial direction. The obtained result is not accurate enough and it needs to increase the first-order rate of convergence.

Chakravarthy and Kumar (2019) used Eq. (2.4) under the condition $a(s) \leq \beta < 0$, $b(s, t) + c(s, t) \geq 0$. Applied Taylor's series expansion to approximate the delayed argument. Employed backward Euler for time derivative and developed variable mesh using the concept of entropy function following the algorithm defined in (Chakravarthy et al., 2018). An adaptive mesh (non-uniform mesh) is generated using the concept of entropy function. Proposed the central finite difference scheme on nonuniform mesh for spatial direction. The proposed method is uniformly convergent and converges quadratically, but not accurate enough.

Kumar and Kumari (2020) suggested an implicit Euler scheme for the time derivative and the extended cubic B-spline, which is an extension of classical B-splines consisting of a free parameter on a piecewise uniform mesh on the spatial direction for Eq. (2.4). They proved that the proposed scheme is first-order accurate in the time direction and second-order (up to a logarithm factor) accurate in the space direction, but not optimal second-order convergent.

Kumar et al. (2021) applied backward Euler for the time derivative on uniform mesh and developed adaptive graded mesh using the concept of entropy function in conjunction with the classical difference schemes following the algorithm defined in Chakravarthy et al. (2018) for Eq. (2.4). The applied approach produces an oscillation-free boundary layer adaptive solution by starting with a small number of uniform mesh points. However, the obtained result is not

accurate enough.

Babu and Bansal (2021) proposed the scheme comprises of implicit Euler scheme for the time derivative on uniform mesh and the non-standard finite difference method following the procedures developed in Mickens (1994) for Eq. (2.4). Using the Richardson extrapolation technique, they improved the order of convergence in the space direction from $O(CN^{-1})$ to $O(CN^{-2})$, but not accurate enough.

Sharma (2017) also considered the class of singularly perturbed parabolic partial differential equation of convection-diffusion type having a delay in the reaction term on the domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ of the form

$$\begin{cases} z_t(s, t) - \varepsilon z_{ss}(s, t) + a(s)z_s(s, t) + b(s)z(s, t - \tau) = f(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = z_0(s, t), & t \in [-\tau, 0], \\ z(0, t) = z(1, t) = 0, & t \in (0, T]. \end{cases} \quad (2.5)$$

For the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\tau > 0$ and the functions $a(s)$, $b(s)$, and $f(s, t)$ are assumed to be sufficiently smooth on the domain which meets $a(s) \geq \beta > 0$ and $b(s) \geq \theta > 0$. The author considered backward Euler for the time derivative on uniform mesh and the standard finite difference method. The standard finite difference method was developed by approximating the second-order derivative by the central difference method and the first-order derivative by the forward difference method. They proved that the scheme is stable and ε -uniformly convergent. Further, Selvi and Ramanujam (2017) treats the aforementioned singularly perturbed parabolic delay differential equation with Robin-type boundary condition. The authors discretized by using a standard finite difference scheme on a fitted piecewise uniform Shishkin mesh in space direction and a backward difference quotient in time direction. The scheme proved that the error is of order $O(N_x^{-2} \ln^2 N_x + N_t^{-1})$, but not accurate enough and not optimal second-order convergent.

Negero and Duressa (2021) considered convection-diffusion problem having a large time delay on the rectangular domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ of the form

$$\begin{cases} z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -c(s, t)z(s, t - \tau) + f(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & (s, t) \in \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), & t \in \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t), & t \in \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (2.6)$$

where $\mathfrak{L}_\varepsilon z(s, t) = -\varepsilon z_{ss}(s, t) + a(s)z_s(s, t) + b(s, t)z(s, t)$. For the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\tau > 0$ and the functions $a(s)$, $b(s, t)$, $c(s, t)$, $f(s, t)$, $\psi_b(s, t)$, $\psi_l(t)$, and $\psi_r(t)$ are assumed to be smooth and bounded on the aforementioned domain which

meet $a(s) \geq \alpha > 0, b(s, t) \geq \beta > 0, c(s, t) \geq \mu > 0$ with a boundary layer near $s = 1$ of width $O(\varepsilon)$. They used the backward Euler method for the time derivative on uniform mesh and the non-standard finite difference method following the procedures developed in [Mickens \(1994\)](#) for space discretization. Using the Richardson extrapolation technique, they improved the order of convergence from $O(h + \Delta t)$ to $O(h^2 + (\Delta t)^2)$. Numerical results are carried out to validate the theoretical estimates. However, the obtained result is not accurate enough.

[Negero and Duressa \(2022a\)](#) developed the Crank-Nicolson method for the time derivative on uniform mesh and an exponentially fitted spline-based difference method for the space derivative for Eq. (2.6) with a large time delay. They proved that the developed scheme converges with second-order accuracy in both the temporal and the spatial directions. Numerical results are carried out to validate the theoretical estimates. The obtained result is not accurate enough.

[Negero and Duressa \(2022b\)](#) employed implicit Euler scheme in the time direction on uniform mesh and B-splines collocation nonstandard finite difference scheme in the space direction for Eq. (2.6) having general temporal delay. They proved that the presented scheme is unconditionally stable and ε -uniform convergent with an order of accuracy $O(h^2 + \Delta t)$. Numerical results are carried out to validate the theoretical estimates. The obtained result is not accurate enough.

[Woldaregay et al. \(2021\)](#) used the Crank-Nicolson method in temporal discretization and the exponentially fitted operator finite difference method in spatial discretization for Eq. (2.6) having a large time delay. Obtained ε - uniformly convergent with first-order accuracy in space and second-order accuracy in time. The obtained result is not accurate enough and it needs to increase the first-order rate of convergence.

[Ayele et al. \(2023\)](#) considered the Crank-Nicolson method in temporal discretization and a hybrid scheme with a proper combination of the cubic spline method in the boundary layer region and the midpoint upwind scheme in the outer layer region on a piecewise Shishkin mesh in the spatial direction for Eq. (2.6) having a small time delay. They proved that the considered scheme provides second-order convergence, but not accurate enough.

[Hassen and Duressa \(2023\)](#) employed the Crank-Nicolson method in the time direction and a fitted operator upwind finite difference method in the spatial direction for Eq. (2.6) having large time delay. They proved that the scheme is uniformly convergent with the first-order in space and the second-order in time. The obtained result is not accurate enough and it needs to increase the first-order rate of convergence.

[Sahoo and Gupta \(2023\)](#) applied higher-order difference with an identity expansion (HODIE) scheme over a piecewise uniform mesh in the spatial direction and the backward Euler method on a uniform mesh for discretization in the temporal direction for Eq. (2.6). The present scheme converges ε -uniformly with the order of convergence one in time and almost second-order in

space direction. Further, to increase the rate of convergence in the time variable, they implemented the Richardson extrapolation technique and obtained ε -uniformly convergent almost second-order in space and second-order in time. The obtained result is not accurate enough and not optimal second-order convergent.

[Raji and Mohapatra \(2023\)](#) considered delay parabolic differential-difference equation having a large delay in time on the domain $F = (0, 1) \times (0, T]$ and $\partial F = \Gamma = \Gamma_d \cup \Gamma_l \cup \Gamma_r$ of the form

$$\begin{cases} z_s + L_\varepsilon z(x, s) = -f(x, s)z(x, s - \delta) + g(x, s), & (x, s) \in F, \\ z|_{\Gamma_d} = \phi_d(x, s), & \Gamma_d = [0, 1] \times [-\delta, 0], \\ z|_{\Gamma_l} = \phi_l(x, s), & \Gamma_l = [-\xi, 0] \times (0, T], \\ z|_{\Gamma_r} = \phi_r(x, s), & \Gamma_r = [1, 1 + \eta] \times (0, T], \end{cases} \quad (2.7)$$

where $L_\varepsilon z(x, s) = -\varepsilon z_{xx}(x, s) + a(x)z_x(x, s) + b(x)z(x, s) + c(x)z(x - \xi, s) + d(x)z(x + \eta, s)$, $\xi = \eta = o(\varepsilon)$ are the delay and the advance parameters in space, respectively. For the perturbation parameter $0 < \varepsilon \ll 1$, with the delay parameter $\delta > 0$ and the functions $a(x), b(x), c(x), d(x), f(x, s), g(x, s), \phi_d(x, s), \phi_l(x, s)$, and $\phi_r(x, s)$ are assumed to be smooth and bounded. The authors developed the scheme in combination with the implicit trapezoidal in time direction on uniform mesh and hybrid scheme in space direction. A hybrid scheme is developed by a proper combination of the mid-upwind and the central difference scheme in space direction on standard Shishkin mesh.

From the above reviews, one can observe that the literature for SPPPDE with time delay is quite large. The problem with small time delay has little attention in the literature. So, it needs further investigation by applying different numerical methods. However, most of the formulated schemes are not accurate enough and most authors developed schemes that have low convergence rates. This all the gap that motivated us to develop an alternative numerical scheme in the area which provides more accurate and increases the rate of order of convergence.

2.2 Singularly perturbed degenerate parabolic partial differential equations

Singularly perturbed degenerate problems received systematic attention in the late 1960s ([Sharma et al., 2013](#)). Hereafter, several authors developed different numerical schemes to solve singularly perturbed degenerate parabolic partial differential equations without time or with time delay using the fitted numerical method. In this section, we present a review of singularly perturbed degenerate parabolic convection-diffusion-reaction type problems.

[Dunne et al. \(2003\)](#) considered singularly perturbed parabolic convection-diffusion problem with a boundary turning point on the domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ and the boundary

$\eta = \eta_l \cup \eta_b \cup \eta_r$ of the form

$$\begin{cases} \mathcal{L}_\varepsilon z(s, t) \equiv \varepsilon z_{ss}(s, t) + a(s, t)z_s(s, t) - b(s, t)z_t(s, t) - d(s, t)z(s, t) = f(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & \eta_b = \{(s, t) : t = 0, s \in \bar{\Omega}\}, \\ z(0, t) = \psi_l(t), & \eta_l = \{(s, t) : s = 0, t \in \bar{\Lambda}\}, \\ z(1, t) = \psi_r(t), & \eta_r = \{(s, t) : s = 1, t \in \bar{\Lambda}\}. \end{cases} \quad (2.8)$$

The functions $f(s, t)$, $a(s, t)$, $b(s, t)$, $d(s, t)$ are smooth and bounded. Moreover, the coefficient function satisfy the conditions $a(s, t) = a_0(s, t)s^p$, $a_0(s, t) \geq \alpha > 0$, $p \geq 1$, $b(s, t) \geq \beta > 0$, $d(s, t) \geq \delta > 0$ on $\mathfrak{D}$. The authors developed a method consisting of a standard upwind finite difference method on a fitted piecewise uniform mesh in the space direction and an implicit Euler scheme on a uniform mesh in the time direction. They analyzed for uniform convergence and validated the developed method using the test example. The obtained method is not accurate enough and low order of convergence rate. To obtain accurate enough and develop a higher order uniformly convergent numerical scheme, this governing equation is initiated by numerous authors and for (1.4) too. The subsequent reviews are based on the preceding Eq. (2.8).

[Gupta and Kadalbajoo \(2011\)](#) suggested the scheme comprises the implicit Euler scheme in the time direction on uniform mesh and the B-splines collocation method in the space direction on a piece-wise uniform Shishkin mesh. They proved that the presented scheme is almost second-order accurate in space and first-order accurate in time. Numerical results are carried out to validate the theoretical estimates.

[Majumdar and Natesan \(2017\)](#) applied the scheme consisting of the implicit Euler scheme for the time derivative on uniform mesh and the upwind finite difference scheme for the spatial derivatives on piece-wise uniform Shishkin mesh. Applying the Richardson extrapolation technique, they improved the accuracy of the numerical solution from $O(N^{-1} \ln^2 N + \Delta t)$ to $O(N^{-2} \ln^2 N + \Delta t^2)$. Numerical results are carried out to verify the theoretical estimates.

[Majumdar and Natesan \(2019\)](#) used the scheme comprises the backward Euler finite difference scheme for the time derivative on a uniform mesh and the hybrid finite difference scheme, which is a combination of the central difference scheme and the midpoint upwind scheme on a piece-wise uniform Shishkin mesh for the space derivative. They proved that the proposed scheme is ε -uniformly convergent of almost second-order (up to a logarithmic factor) in space and first-order in time. Numerical results are carried out to verify the theoretical results.

[Yadav et al. \(2020\)](#) applied a scheme that combines an implicit finite difference method for time discretization on a uniform mesh with a hybrid scheme for space discretization on a generalized Shishkin mesh. They applied the Richardson extrapolation technique to increase the order of convergence in the time direction. The resulting scheme has second-order convergence (up to a logarithmic factor) in space and second-order convergence in time. Numerical

experiments are conducted to verify the theoretical results.

[Mbroh et al. \(2021\)](#) developed a numerical scheme via Rothe's method. By using the Crank-Nicolson finite difference scheme, the time derivative is discretized to obtain a set of semi-discrete boundary value problems. Using a fitted operator finite difference scheme based on the midpoint downwind scheme, the system of boundary value problems is discretized and analyzed for convergence. Second-order accuracy is established for each discretization. Numerical results are given in support of theoretical findings.

[Ku Sahoo and Gupta \(2021\)](#) suggested a scheme consisting of an implicit Euler method on uniform mesh in time and a simple upwind scheme on piecewise uniform mesh in space. Then, they applied the Richardson extrapolation scheme in both space and time directions. Theoretically, they proved that the proposed scheme is almost a second-order parameter uniformly convergent and verified it by doing some numerical experiments.

[Kumar et al. \(2022\)](#) considered a numerical method using the implicit Euler scheme on a uniform mesh in the time direction and the upwind finite difference scheme on a layer adaptive non-uniform mesh in the spatial direction. The adaptive non-uniform mesh in the spatial direction is generated through the equidistribution of a suitably chosen monitor function. They proved that the method is uniformly convergent with first order in both time and space. Numerical results are given in support of theoretical findings.

[Singh et al. \(2022\)](#) semi-discretized the problem using the Crank-Nicolson scheme for the time, and then the quadratic spline basis functions were used to discretize the semi-discrete problem. A rigorous error analysis shows that the proposed method is boundary layer resolving and second-order parameter uniformly convergent. They devised some numerical experiments to support the theoretical findings.

[Rai and Yadav \(2021\)](#) extended the preceding Eq. (2.8) by considering singularly perturbed parabolic convection-diffusion with a time delay on $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ and the boundary $\eta = \eta_l \cup \eta_b \cup \eta_r$ of the form

$$\begin{cases} \varepsilon z_{ss}(s, t) + a(s, t)z_s(s, t) - b(s, t)z_t(s, t) - c(s, t)z(s, t) = e(s, t)z(s, t - \tau) + f(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), & \eta_l = \{(s, t) : s = 0, t \in \bar{\Lambda}\}, \\ z(1, t) = \psi_r(t), & \eta_r = \{(s, t) : s = 1, t \in \bar{\Lambda}\}. \end{cases} \quad (2.9)$$

The functions $f(s, t), a(s, t), b(s, t), c(s, t)$ and $e(s, t)$ are smooth and bounded. Moreover, the functions satisfy the conditions $a(s, t) = a_0(s, t)s^p, a_0(s, t) \geq \alpha > 0, p \geq 1, b(s, t) \geq \beta > 0, c(s, t) \geq \delta > 0, e(s, t) > 0$ on $\mathfrak{D}$. The authors proposed a numerical method which consists of a backward Euler scheme for time discretization on uniform mesh and a hybrid scheme by proper combination of midpoint upwind and central difference scheme for the spatial

discretization on modified Shishkin mesh. Further, the order of convergence is increased in time direction via implementing Richardson extrapolation. This governing equation is in little attention further investigation is needed on the area.

From this literature, we observed that different numerical methods have been developed. Even though different numerical methods were investigated, for singularly perturbed degenerate parabolic partial differential equations less detail attention is given. Especially a delay degenerate problem has been studied by a few authors. So, it needs further investigation by applying different numerical methods. However, most of the formulated schemes are not accurate enough and most authors developed schemes that have low convergence rates. Moreover, numerical schemes like cubic spline, fitted tension spline, and so forth are not considered well in these areas. So, it needs further investigation in the area. In this context, we are motivated to develop alternative numerical schemes in these areas which provides more accurate and increases the rate of order of convergence.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Study design

The dissertation is designed to present an extensive review of literature. Theoretical analysis of the time and the space derivatives were conducted. To validate the applicability of the schemes, numerical experiments were carried out using MATLAB software. The findings of the study were demonstrated in table and graphical form.

3.2 Source of information

In this dissertation, used relevant sources of information such as books, up-to-date articles, conference papers, seminar papers, and supplementary papers that are related to the governing problem under consideration.

3.3 Research method

To solve singularly perturbed partial differential equations (SPPDEs) using numerical methods that converge uniformly and applied common techniques such as fitted mesh and fitted operator numerical methods to discretize the spatial derivative. In this dissertation, the fitted operator method, which was described in [Kellogg and Tsan \(1978\)](#) on singularly perturbed theory was employed. Additionally, the simplest and the most commonly fitted mesh method, known as Shishkin meshes was used.

3.4 Mathematical procedures

During this research utilized various mathematical procedures to design the framework of this dissertation and achieve its intended objective.

1. Describing the governing problem: Describe the governing problem to be investigated along with the initial and boundary conditions and ensure the smoothness and bounded of the coefficient functions through all consequential research steps. The conditions imposed on the problem's data, necessary for ensuring solution smoothness and designing and justifying uniformly convergent schemes were also described.
2. Analyzing the properties of the continuous solution: Imposing appropriate compatibility conditions and guaranteeing the existence of unique solutions to the described problems. Using the maximum principle the stability of the solution is established. The solutions and derivatives depend on the value of the parameter and the behavior of the derivatives

in a neighborhood of the layer region is estimated. Using these estimates uniformly convergent numerical schemes are constructed.

3. Formulating numerical schemes: The numerical schemes were formulated following the time and space discretization procedures. These schemes were developed to maintain ε -uniform convergence and were constructed using FOFDM and FMFDM.
4. Establish the stability and uniform convergence of the discrete problem: The uniform stability of the finite difference operator is established using a discrete maximum principle. The uniform convergence analysis of the constructed discrete scheme is established in norms corresponding to the applied numerical methods.
5. Developing algorithms and writing MATLAB codes: The development of efficient computational algorithms often requires an understanding of the basic properties of the solutions to the model examples to be solved numerically. This procedure involves two steps:
 - Expressing the numerical problem to be solved and the methods to be used in a complete algorithm.
 - Translating the algorithm into a computer code using a specific programming language or implementing the numerical methods in a computer program. The numerical experiments in this study are conducted on Matlab.
6. Validate the applicability of the developed numerical schemes: Different model examples are considered to validate the applicability and efficiency of the developed method. The computed numerical results were described in terms of maximum pointwise error, rate of convergence and proved the theoretical finding of the developed scheme.
7. Presenting the numerical results using tables and graphs: The results of the numerical experiment on stability, convergence, and accuracy properties are presented using tables and graphs. The computed numerical experiment using the developed MATLAB codes is presented in a table form in terms of maximum pointwise absolute error, ε -uniform error and ε -uniform rate of convergence. The accuracy of the schemes is maintained in terms of the pointwise maximum errors. The solution profile is displayed graphically.
8. Compare the result of the developed scheme with existing published work: To demonstrate the performance and efficiency of the schemes developed, comparisons are made with earlier published research works. Comparison takes place in terms of maximum pointwise error and ε -uniform convergence.

CHAPTER FOUR

FITTED OPERATOR METHOD FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATION WITH SMALL TIME DELAY

In this chapter, the exponentially fitted operator method is developed to solve a singularly perturbed convection-diffusion problem with a small time delay which has a boundary layer near the left side of the domain. The continuous solution's properties are discussed. The backward Euler scheme in the temporal direction on a uniform mesh is used. Non-symmetric finite difference schemes were used to approximate the first-order derivative terms, and a higher-order finite difference method is used to approximate the second-order derivative term in the spatial direction. An exponential fitting factor is calculated and introduced in the difference scheme to manage the effect of the small parameter. The stability of the scheme is examined and analyzed using the discrete maximum principle. It is proven that the developed scheme is parameter-uniform with the rate of convergence first order in both space and time. To validate the applicability of the proposed scheme, two numerical examples are considered, and a comparison was made with the results available in the literature. Additionally, the boundary layer behaviour of the solutions is given graphically.

4.1 Governing problem

Consider a class of singularly perturbed convection-diffusion equation of the form

$$\begin{cases} z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -\kappa(s, t)z(s, t - \tau) + \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & (s, t) \in \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), & t \in \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t), & t \in \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (4.1)$$

where $\mathfrak{L}_\varepsilon z(s, t) = -\varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) + \chi(s, t)z(s, t)$.

Here $\varepsilon \in (0, 1]$ and $\tau > 0$ are the perturbation parameter and the delay parameter, respectively. We pretended that the functions $\beta(s, t), \chi(s, t), \kappa(s, t), \gamma(s, t)$ on $\bar{\mathfrak{D}} = \bar{\Omega} \times \bar{\Lambda} = [0, 1] \times [0, T]$ and $\psi_b(s, t), \psi_l(t), \psi_r(t)$ on $\eta = \eta_l \cup \eta_r \cup \eta_b$ are smooth enough and bounded which satisfy $\chi(s, t) \geq \varpi > 0, \kappa(s, t) \geq \varphi > 0, \beta(s, t) \leq \mu < 0, \chi(s, t) + \kappa(s, t) \geq 0$. These conditions ensure that the problem represented by equation (4.1) has a boundary layer along $s = 0$. (Kumar et al., 2019).

4.2 Properties of the continuous solution

This section concerns with the analytical aspects of the solution and its derivatives. The existence and uniqueness of the solution (4.1) can be guaranteed by assuming that the data is Holder continuous, and imposing appropriate compatibility conditions at the corner points $(0, 0)$ and $(1, 0)$, and the delay terms are stated below Protter and Weinberger (1967)

$$\psi_l(0) = \psi_b(0, 0), \quad \psi_r(0) = \psi_b(1, 0), \quad (4.2)$$

$$\begin{aligned} \frac{d\psi_l(0)}{dt} - \varepsilon \frac{\partial^2 \psi_b(0, 0)}{\partial s^2} + \beta(0, 0) \frac{\partial \psi_b(0, 0)}{\partial s} + \chi(0, 0) \psi_b(0, 0) &= -\kappa(0, 0) \psi_b(0, -\tau) + \gamma(0, 0), \\ \frac{d\psi_r(0)}{dt} - \varepsilon \frac{\partial^2 \psi_b(1, 0)}{\partial s^2} + \beta(1, 0) \frac{\partial \psi_b(1, 0)}{\partial s} + \chi(1, 0) \psi_b(1, 0) &= -\kappa(1, 0) \psi_b(1, -\tau) + \gamma(1, 0). \end{aligned} \quad (4.3)$$

If these assumptions and conditions are satisfied, the problem (4.1) has a unique solution.

The reduced problem of (4.1) is taken as

$$\begin{cases} \frac{\partial z_0(s, t)}{\partial t} + \beta(s, t) \frac{\partial z_0(s, t)}{\partial s} + \chi(s, t) z_0(s, t) = -\kappa(s, t) z_0(s, t - \tau) + \gamma(s, t), (s, t) \in \mathfrak{D}, \\ z_0(s, t) = \psi_b(s, t), (s, t) \in \eta_b, \\ z_0(1, t) = \psi_r(t), t \in \Lambda. \end{cases} \quad (4.4)$$

As $\varepsilon \rightarrow 0$, the solution $z(s, t)$ of (4.1) approaches to the reduced solution $z_0(s, t)$. The operator $\mathfrak{L} = (\frac{\partial}{\partial t} + \mathfrak{L}_\varepsilon)$ of (4.1) satisfies the following maximum principle.

Lemma 4.2.1. *Protter and Weinberger (1967)(Maximum principle) Assume that $\nu(s, t) \in C^2(\mathfrak{D}) \cap C^0(\bar{\mathfrak{D}})$, satisfies $\nu(s, t) \geq 0, \forall (s, t) \in \eta$. If $\mathfrak{L}\nu(s, t) \geq 0, \forall (s, t) \in \mathfrak{D}$, then $\nu(s, t) \geq 0, \forall (s, t) \in \bar{\mathfrak{D}}$.*

Proof. The result is proved by contradiction. Suppose there is $(s^*, t^*) \in \bar{\mathfrak{D}}$, such that $\nu(s^*, t^*) = \min_{(s, t) \in \bar{\mathfrak{D}}} \nu(s, t) < 0$. From the assumption, note that $(s^*, t^*) \notin \{0, 1\}$ implies that $(s^*, t^*) \in (0, 1)$. Since $\nu(s^*, t^*) = \min_{(s, t) \in \bar{\mathfrak{D}}} \nu(s, t) < 0$, then we have $\nu_s(s^*, t^*) = \nu_t(s^*, t^*) = 0$ and $\nu_{ss}(s^*, t^*) \geq 0$. So, the operator gives that $\mathfrak{L}\nu(s^*, t^*) = \nu_t(s^*, t^*) - \varepsilon \nu_{ss}(s^*, t^*) + \beta(s^*, t^*) \nu_s(s^*, t^*) + \chi(s^*, t^*) \nu(s^*, t^*) < 0$. Which is contradiction to the assumption that made $\mathfrak{L}\nu(s^*, t^*) \geq 0, \forall (s, t) \in \mathfrak{D}$. Therefore, $\nu(s, t) \geq 0, \forall (s, t) \in \bar{\mathfrak{D}}$. ■

Lemma 4.2.2. *Miller et al. (1996)(Stability result) The solution $z(s, t)$ of (4.1) satisfies the following bound*

$$|z(s, t)| \leq \varpi^{-1} \|\gamma\| + \max\{|\psi_l(t)|, |\psi_b(s, t)|, |\psi_r(t)|\},$$

where ϖ is the lower bound of $\chi(s, t)$.

Proof. Define the two barrier functions

$\vartheta^\pm(s, t)$ as $\vartheta^\pm(s, t) = \varpi^{-1}\|\gamma\| + \max\{|\psi_l(t)|, |\psi_b(s, t)|, |\psi_r(t)|\} \pm z(s, t)$,
for the initial value case

$$\begin{aligned}\vartheta^\pm(s, 0) &= \varpi^{-1}\|\gamma\| + \max\{|\psi_l(t)|, |\psi_b(s, t)|, |\psi_r(t)|\} \pm z(s, 0), \\ &\geq \varpi^{-1}\|\gamma\|, \quad \text{since } \max\{|\psi_l(t)|, |\psi_b(s, t)|, |\psi_r(t)|\} \geq z(s, 0), \\ &\geq \varpi^{-1}\|\gamma\| \geq 0,\end{aligned}$$

for the boundary values cases

$$\begin{aligned}\vartheta^\pm(0, t) &= \varpi^{-1}\|\gamma\| + \max\{|\psi_l(t)|, |\psi_b(0, t)|, |\psi_r(t)|\} \pm z(0, t), \\ &\geq \varpi^{-1}\|\gamma\|, \quad \text{since } \max\{|\psi_l(t)|, |\psi_b(0, t)|, |\psi_r(t)|\} \geq z(0, t), \\ &\geq \varpi^{-1}\|\gamma\| \geq 0, \\ \vartheta^\pm(1, t) &= \varpi^{-1}\|\gamma\| + \max\{|\psi_l(t)|, |\psi_b(1, t)|, |\psi_r(t)|\} \pm z(1, t), \\ &\geq \varpi^{-1}\|\gamma\|, \quad \text{since } \max\{|\psi_l(t)|, |\psi_b(1, t)|, |\psi_r(t)|\} \geq z(1, t), \\ &\geq \varpi^{-1}\|\gamma\| \geq 0,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}\vartheta^\pm(s, t) &= \frac{\partial \vartheta^\pm(s, t)}{\partial t} - \varepsilon \frac{\partial^2 \vartheta^\pm(s, t)}{\partial s^2} + \beta(s, t) \frac{\partial \vartheta^\pm(s, t)}{\partial s} + \chi(s, t) \vartheta^\pm(s, t) \\ &= (\max\{\frac{\partial \psi_l(t)}{\partial t}, \frac{\partial \psi_b(s, t)}{\partial t}, \frac{\partial \psi_r(t)}{\partial t}\} \pm z_t(s, t)) \\ &\quad - \varepsilon (\max\{\frac{\partial^2 \psi_l(t)}{\partial s^2}, \frac{\partial^2 \psi_b(s, t)}{\partial s^2}, \frac{\partial^2 \psi_r(t)}{\partial s^2}\} \\ &\quad \pm z_{ss}(s, t)) + \beta(s, t) (\max\{\frac{\partial \psi_l(t)}{\partial s}, \frac{\partial \psi_b(s, t)}{\partial s}, \frac{\partial \psi_r(t)}{\partial s}\} \pm z_s(s, t)) \\ &\quad + \chi(s, t) (\varpi^{-1}\|\gamma\| + \max\{|\psi_l(t)|, |\psi_b(s, t)|, |\psi_r(t)|\} \pm z(s, t)) \geq 0.\end{aligned}$$

This implies that, $\mathfrak{L}\vartheta^\pm(s, t) \geq 0$. Hence, by maximum principle we have,

$\vartheta^\pm(s, t) \geq 0, \forall (s, t) \in \bar{\mathfrak{D}}$ which implies,

$$|z(s, t)| \leq \varpi^{-1}\|\gamma\| + \max\{|\psi_l(t)|, |\psi_b(s, t)|, |\psi_r(t)|\}.$$

Hence, the proof is completed. ■

Lemma 4.2.3. *The solution $z(s, t)$ of (4.1) satisfies $|z(s, t) - \psi_b(s, 0)| \leq Ct, (s, t) \in \bar{\mathfrak{D}}$, where $C > 0$ does not depend on ε .*

Proof. Define the function $m(s, t)$ as

$$m(s, t) = \begin{cases} |z(s, t) - \psi_b(s, 0)|, & 0 \leq t \leq T, \\ |z(s, t) - \psi_b(s, t)|, & -\tau \leq t \leq 0, \end{cases} \quad (4.5)$$

then (4.5) satisfies the PDE $\mathfrak{L}m(s, t) = g(s, t)$, where

$$\begin{aligned} g(s, t) &= \gamma(s, t) - \mathfrak{L}\psi_b(s, 0) - \kappa(s, t)\psi_b(s, -\tau) \\ &= \gamma(s, t) + \varepsilon(\psi_b)_{ss}(s, 0) - \beta(s, t)(\psi_b)_s(s, 0) - \chi(s, t)\psi_b(s, 0) - \kappa(s, t)\psi_b(s, -\tau), \end{aligned}$$

and on the boundary $\eta_b : m(s, t) = z(s, t) - \psi_b(s, t) = \psi_b(s, t) - \psi_b(s, t) = 0$, then $m(0, t) = \psi_l(t) - \psi_b(0, 0)$, and $m(1, t) = \psi_r(t) - \psi_b(1, 0)$, where $t \in [0, T]$. Now, taking the absolute values on both sides and imposing the compatibility conditions at corner points, we obtain $|m(0, t)| \leq C_1 t$, and $|m(1, t)| \leq C_2 t$, $t \in [0, T]$ for some constants C_1 and C_2 . On the other hand, by setting

$$r(s, t) = \begin{cases} Ct, & 0 \leq t \leq T, \\ 0, & -\tau \leq t \leq 0, \end{cases}$$

we have

$$\begin{cases} \mathfrak{L}r(s, t) = C(1 + t\chi(s, t)), \\ r(s, t) = 0, \quad (s, t) \in \eta_b, \\ r(0, t) = r(1, t) = Ct, \quad 0 \leq t \leq T. \end{cases}$$

Now, $|\mathfrak{L}m(s, t)| \leq \mathfrak{L}r(s, t)$ on $\mathfrak{D}$ and $|m(s, t)| \leq r(s, t)$ on η_b . Therefore, from the maximum principle, we obtain $|m(s, t)| \leq r(s, t)$ on $\bar{\mathfrak{D}}$, which implies that $|m(s, t)| \leq Ct$. $\blacksquare$

Lemma 4.2.4. *Let the solution of (4.1) be $z(s, t)$, then it satisfies the bound*

$$|z(s, t)| \leq C, \quad (s, t) \in \bar{\mathfrak{D}}.$$

Proof. Without loss of generality, assume the initial condition is zero. Lemma 4.2.3 yields

$$|z(s, t)| \leq Ct, \quad (s, t) \in \bar{\mathfrak{D}}.$$

Since $t \in (0, T]$, we obtain

$$|z(s, t)| \leq C, \quad (s, t) \in \bar{\mathfrak{D}}.$$

Hence, the proof is completed. $\blacksquare$

Lemma 4.2.5. *The derivative of the solution $z(s, t)$ of (4.1) with respect to t satisfies the following bound*

$$|z_t(s, t)| \leq C, \quad (s, t) \in \bar{\mathfrak{D}}.$$

Proof. On the sides $s = 0$ and $s = 1$ of $\bar{\mathcal{D}}$, we have $z = 0$, therefore $z_t = 0$. At $t = 0$, we have $z = 0, z_s = 0, z_{ss} = 0$. Substituting these values in (4.1), from equation it yields $z_t(s, 0) = \gamma(s, 0)$. If $t \in [-\tau, 0]$, then

$$z_t = \lim_{h \rightarrow 0} \frac{z(s, t - \tau + h) - z(s, t - \tau)}{h} = 0.$$

Thus for sufficiently large C on ψ_b , the following bound holds

$$|z_t(s, t)| \leq C.$$

Now, differentiating the operator $\mathfrak{L}z_t(s, t) = (\frac{\partial}{\partial t} + \mathfrak{L}_\varepsilon)z_t(s, t)$ with respect to t , then we obtain

$$\begin{aligned} \mathfrak{L}z_t(s, t) &= z_{tt} - \varepsilon z_{tss} + \beta(s, t)z_{ts} + \chi(s, t)z_t \\ &= (z_t - \varepsilon z_{ss} + \beta(s, t)z_s + \chi(s, t)z)_t - \beta_t(s, t)z_s - \chi_t(s, t)z. \end{aligned} \quad (4.6)$$

First we consider the case, where $z(s, t - \tau)$ is a known function, that is, in $t \in [0, \tau]$ of (4.6), if we use the given initial condition, we get $\mathfrak{L}z_t(s, t) = \gamma_t - \beta_t(s, t)z_s - \chi_t(s, t)z$. By using Lemma 4.2.4, we get $|\mathfrak{L}z_t(s, t)| \leq C$, for $(s, t) \in (0, 1) \times [0, \tau]$. Now applying the maximum principle, we obtain $|z_t(s, t)| \leq C$, for $(s, t) \in [0, 1] \times [0, \tau]$.

Next, we extend the domain of the defined PDE, i.e., consider $(s, t) \in [0, 1] \times [0, 2\tau]$. For $t \in [\tau, 2\tau]$, from (4.6) we get $\mathfrak{L}z_t(s, t) = \gamma_t - \beta_t(s, t)z_s - \chi_t(s, t)z + \kappa_t(s, t)z(s, t - \tau) - \kappa(s, t)z_t(s, t - \tau)$. From the fact that $|z_t(s, t)| \leq C$, also we have $|\mathfrak{L}z_t(s, t)| \leq C$ for $(0, 1) \times [\tau, 2\tau]$. Therefore, by maximum principle we get

$$|z_t(s, t)| \leq C.$$

C chosen to be sufficiently large. Since $\mathfrak{L}$ satisfies the maximum principle on $\bar{\mathcal{D}}$, we obtain the desired estimate

$$|z_t(s, t)| \leq C.$$

■

Lemma 4.2.6. *The derivatives of the exact solution $z(s, t)$ of (4.1) with respect to s satisfies*

$$|z_s(s, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu s/\varepsilon)), \quad (s, t) \in \bar{\mathcal{D}}.$$

Proof. In problem (4.1), the effect of delay occurs only in the time variable, and there is no effect of delay in the spatial variable, therefore, if we fix $t \in [0, T]$, the result can be obtained by using the argument of (Kellogg and Tsan, 1978). Thus, from (4.1), we have $-\varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) = g$, where $g = \gamma(s, t) - z_t(s, t) - \chi(s, t)z(s, t) - \kappa(s, t)z(s, t - \tau)$ and multiply

both sides by the integrating factor $\exp(-\varepsilon^{-1}B(s))$, where $B(s) = \int_0^s \beta(x, t)dx$ results in

$$\begin{aligned} \varepsilon^{-1} \exp(-\varepsilon^{-1}B(s))g(s, t) &= -\exp(-\varepsilon^{-1}B(s))z_{ss}(s, t) + \varepsilon^{-1}\beta(s, t) \exp(-\varepsilon^{-1}B(s))z_s, \\ -\frac{d}{ds} \left(\exp(-\varepsilon^{-1}B(s)) \frac{dz}{ds} \right) &= \varepsilon^{-1} \exp(-\varepsilon^{-1}B(s))g(s, t). \end{aligned}$$

Integrating the above expression from 0 to s gives

$$\exp(-\varepsilon^{-1}B(0))z_s(0, t) - \exp(-\varepsilon^{-1}B(s))z_s(s, t) = \varepsilon^{-1} \int_0^s g(x, t) \exp(-\varepsilon^{-1}B(x))dx.$$

This is further simplified to

$$z_s(s, t) = z_s(0, t) \exp(-\varepsilon^{-1}(B(0) - B(s))) + k_1(s, t), \quad (4.7)$$

where

$$k_1(s, t) = -\varepsilon^{-1} \int_0^s g(x, t) \exp(-\varepsilon^{-1}(B(x) - B(s)))dx. \quad (4.8)$$

Integrating the above equation from s to 1 to derive the bound of $z_s(0, t)$ results in

$$z(s, t) = z(1, t) + z_s(0, t) \int_s^1 \exp(-\varepsilon^{-1}(B(0) - B(x)))dx + \int_s^1 k_1(x, t)dx.$$

At $s = 0$, it becomes

$$z_s(0, t) = -\frac{\int_0^1 k_1(x, t)dx}{\int_0^1 \exp(-\varepsilon^{-1}(B(0) - B(x)))dx}$$

from the assumptions $z(0, t) = z(1, t) = 0$. Then substituting in (4.7), we get

$$z_s(s, t) = -\frac{\exp(-\varepsilon^{-1}(B(0) - B(s)))}{\int_0^1 \exp(-\varepsilon^{-1}(B(0) - B(x)))dx} \int_0^1 k_1(x, t)dx + k_1(s, t).$$

Now, taking the bound and yields

$$\begin{aligned} |z_s(s, t)| &\leq \frac{|\exp(-\varepsilon^{-1}(B(0) - B(s)))|}{\left| \int_0^1 \exp(-\varepsilon^{-1}(B(0) - B(x)))dx \right|} \left| \int_0^1 k_1(x, t)dx \right| + |k_1(s, t)| \\ &\leq \frac{|\exp(-\varepsilon^{-1}B(0))| |\exp(\varepsilon^{-1}B(s))|}{\left| \exp(-\varepsilon^{-1}B(0)) \right| \left| \int_0^1 \exp(\varepsilon^{-1}B(x))dx \right|} \left| \int_0^1 k_1(x, t)dx \right| + |k_1(s, t)| \\ &= \frac{|\exp(\varepsilon^{-1}B(s))|}{\left| \int_0^1 \exp(\varepsilon^{-1}B(x))dx \right|} \left| \int_0^1 k_1(x, t)dx \right| + |k_1(s, t)| \\ &\leq \hat{C} \left(1 + \frac{\exp(-\varepsilon^{-1}\mu s)}{\int_0^1 \exp(-\varepsilon^{-1}\bar{\mu}x)dx} \right), \text{ since } \beta(s, t) \leq \mu < 0, B(s) = \int_0^s \mu ds = \mu s, \end{aligned}$$

where $\bar{\mu}$ is an upper bound of $B(s)$ over $[1, 0]$ and $\hat{C}$ is also an upper bound of $k_1(s, t)$ over $\bar{\mathfrak{D}}$. Simplifying $\int_0^1 (\exp(-\varepsilon^{-1}\bar{\mu}x))dx$ gives $\frac{1}{\varepsilon^{-1}\bar{\mu}} - \frac{1}{\varepsilon^{-1}\bar{\mu}} \exp(-\varepsilon^{-1}\bar{\mu})$. From this result, we have

$$|z_s(s, t)| \leq \hat{C} \left(1 + \frac{\varepsilon^{-1}\bar{\mu} \exp(-\varepsilon^{-1}\mu s)}{(1 - \exp(-\varepsilon^{-1}\bar{\mu}))} \right).$$

Let $\gamma(\bar{\mu}, t) = \exp(-\varepsilon^{-1}\bar{\mu})$ be a real-valued function defined over $\bar{\mathfrak{D}}$ and differentiable on $\mathfrak{D}$. From Maclaurin series we have $\bar{\mu} \leq (1 - \exp(-\varepsilon^{-1}\mu))$, then we obtained

$$|z_s(s, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu s/\varepsilon)), \quad (s, t) \in \bar{\mathfrak{D}}$$

which is the desired bound. ■

Lemma 4.2.7. $|z_{tt}(s, t)| \leq C, (s, t) \in \mathfrak{D}$.

Proof. On the sides $s = 0$ and $s = 1$ of $\bar{\mathfrak{D}}$, we have $z = 0$ and therefore $z_{tt} = 0$. For $t \in [-\tau, 0]$, we have $z = 0$, $z_s = 0$, and $z_{ss} = 0$, therefore, from (4.6), we have

$$z_t(s, t) = \begin{cases} 0, & \text{for } t \in [-\tau, 0), \\ \gamma(s, 0), & \text{for } t = 0. \end{cases}$$

Also, we have

$$z_{ts}(s, t) = \begin{cases} 0, & \text{for } t \in [-\tau, 0), \\ \gamma_s(s, 0), & \text{for } t = 0, \end{cases}$$

and

$$z_{tss}(s, t) = \begin{cases} 0, & \text{for } t \in [-\tau, 0), \\ \gamma_{ss}(s, 0), & \text{for } t = 0, \end{cases}$$

$|z_{ts}(s, t)| \leq C$ and $|z_{tss}(s, t)| \leq C$ on ψ_b . Now, differentiating (4.1) with respect to t , we get

$$z_{tt}(s, t) - \varepsilon z_{tss}(s, t) + \beta_t(s, t)z(s, t) + \beta(s, t)z_t(s, t) = \gamma_t(s, t) - \kappa_t(s, t)z(s, t - \tau) - \kappa(s, t)z_t(s, t - \tau).$$

For $t \in [-\tau, 0]$, each term of the above equation, except the term z_{tt} , is now known to be bounded by C , which implies that $|z_{tt}| \leq C$, for $(s, t) \in \bar{\mathfrak{D}}$.

Consider the same operator defined in (4.1) and operating it on z_{tt} , we get

$$\begin{aligned} \mathfrak{L}z_{tt}(s, t) &= z_{ttt}(s, t) - \varepsilon z_{ttss}(s, t) + \beta(s, t)z_{tts}(s, t) + \chi(s, t)z_{tt} \\ &= (z_t(s, t) - \varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) + \chi(s, t))_{tt} - \chi_{tt}(s, t)z(s, t) - 2\chi_t(s, t)z_t(s, t). \end{aligned} \quad (4.9)$$

For $t \in [0, \tau]$, using the given initial condition, we get

$$\mathfrak{L}z_{tt}(s, t) = \gamma_{tt}(s, t) - \chi_{tt}(s, t)z(s, t) - 2\chi_t(s, t)z_t(s, t).$$

Since $\gamma_{tt}(s, t)$ is bounded function, which implies that $|\mathfrak{L}z_{tt}(s, t)| \leq C$. By the maximum principle, we have $|z_{tt}| \leq C$, $(s, t) \in [0, 1] \times [0, \tau]$.

For $t \in [\tau, 2\tau]$, from (4.9) we get

$$\mathfrak{L}z_{tt}(s, t) = \gamma_{tt}(s, t) - \chi_{tt}(s, t)z(s, t) - 2\chi_t(s, t)z_t(s, t) - (\kappa(s, t)z(s, t - \tau))_{tt}. \quad (4.10)$$

As $(s, t) \in [0, 1] \times [\tau, 2\tau]$, all the terms in the right-hand side of (4.10) are bounded, which implies that $|\mathfrak{L}z_{tt}(s, t)| \leq C$. By the maximum principle, we have $|z_{tt}| \leq C$, $(s, t) \in [0, 1] \times [0, 2\tau]$. Analogously, we obtained the required bound on the entire domain $\bar{\mathfrak{D}}$. ■

Lemma 4.2.8. *The derivative of $z_s(s, t)$ with respect to t satisfies*

$$|z_{st}(s, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu s/\varepsilon)), \quad (s, t) \in \bar{\mathfrak{D}}.$$

Proof. Differentiating (4.1) with respect to t , we get

$$\begin{aligned} -\varepsilon(z_{st})_s + \beta(s, t)z_{st} + \chi_t(s, t)z + \chi(s, t)z_t + z_{tt} \\ = -\kappa_t(s, t)z(s, t - \tau) - \kappa(s, t)z_t(s, t - \tau) + \gamma_t(s, t). \end{aligned}$$

Rewriting the above equation in the form $-\varepsilon(z_{st}(s, t))_s + \beta(s, t)z_{st}(s, t) = g_2$, where $g_2 = \gamma_t(s, t) - z_{tt}(s, t) - \chi(s, t)z(s, t) - \kappa(s, t)z(s, t - \tau)$ and multiply both sides by the integrating factor $\exp(-\varepsilon^{-1}B(s))$, where $B(s) = \int_0^s \beta(x, t)dx$ results in

$$\begin{aligned} \varepsilon^{-1} \exp(-\varepsilon^{-1}B(s))g_2(s, t) &= -\exp(-\varepsilon^{-1}B(s))z_{ss}(s, t) + \varepsilon^{-1}\beta(s) \exp(-\varepsilon^{-1}B(s))z_s, \\ -\frac{d}{ds} \left(\exp(-\varepsilon^{-1}B(s)) \frac{dz_t}{ds} \right) &= \varepsilon^{-1} \exp(-\varepsilon^{-1}B(s))g_2(s, t). \end{aligned}$$

Integrating the above expression from 0 to s gives

$$\exp(-\varepsilon^{-1}B(0))z_{st}(0, t) - \exp(-\varepsilon^{-1}B(s))z_{st}(s, t) = \varepsilon^{-1} \int_0^s g_2(x, t) \exp(-\varepsilon^{-1}B(x))dx.$$

This is further simplified to

$$z_{st}(s, t) = z_{ts}(0, t) \exp(-\varepsilon^{-1}(B(0) - B(s))) + k_2(s, t), \quad (4.11)$$

where

$$k_2(s, t) = -\varepsilon^{-1} \int_0^s g_2(x, t) \exp(-\varepsilon^{-1}(B(x) - B(s)))dx. \quad (4.12)$$

Integrating the above equation from s to 1 to derive the bound of $z_{st}(0, t)$ results in

$$z_t(s, t) = z_t(1, t) + z_{st}(0, t) \int_s^1 \exp(-\varepsilon^{-1}(B(0)) - B(x)) dx + \int_s^1 k_2(x, t) dx.$$

At $s = 0$, it becomes

$$z_{st}(0, t) = -\frac{\int_0^1 k_2(x, t) dx}{\int_0^1 (\exp(-\varepsilon^{-1}(B(0)) - B(x))) dx}$$

from the assumptions $z_t(0, t) = z_t(1, t) = 0$. Then substituting in (4.7), we get

$$z_{st}(s, t) = -\frac{\exp(-\varepsilon^{-1}(B(0) - B(s)))}{\int_0^1 (\exp(-\varepsilon^{-1}(B(0)) - B(x))) dx} \int_0^1 k_2(x, t) dx + k_2(s, t).$$

Now, taking the bound and yields

$$\begin{aligned} |z_{st}(s, t)| &\leq \frac{|\exp(-\varepsilon^{-1}(B(0) - B(s)))|}{\left| \int_0^1 (\exp(-\varepsilon^{-1}(B(0)) - B(x))) dx \right|} \left| \int_0^1 k_2(x, t) dx \right| + |k_2(s, t)| \\ &\leq \frac{|\exp(-\varepsilon^{-1}(B(0)))| |\exp(\varepsilon^{-1}B(s))|}{\left| \exp(-\varepsilon^{-1}(B(0))) \int_0^1 (\exp(\varepsilon^{-1}B(x))) dx \right|} \left| \int_0^1 k_2(x, t) dx \right| + |k_2(s, t)| \\ &= \frac{|\exp(\varepsilon^{-1}B(s))|}{\left| \int_0^1 (\exp(\varepsilon^{-1}B(x))) dx \right|} \left| \int_0^1 k_2(x, t) dx \right| + |k_2(s, t)| \\ &\leq \hat{C} \left(1 + \frac{\exp(-\varepsilon^{-1}\mu s)}{\int_0^1 (\exp(-\varepsilon^{-1}\bar{\mu}x)) dx} \right), \text{ since } \beta(s, t) \leq \mu < 0, B(s) = \int_0^s \mu ds = \mu s, \end{aligned}$$

where $\bar{\mu}$ is an upper bound of $B(s)$ over $[1, 0]$ and $\hat{C}$ is also an upper bound of $k_2(s, t)$ over $\bar{\mathfrak{D}}$. Simplifying $\int_0^1 (\exp(-\varepsilon^{-1}\bar{\mu}x)) dx$ gives $\frac{1}{\varepsilon^{-1}\bar{\mu}} - \frac{1}{\varepsilon^{-1}\bar{\mu}} \exp(-\varepsilon^{-1}\bar{\mu})$. From this result, we have

$$|z_{st}(s, t)| \leq \hat{C} \left(1 + \frac{-\varepsilon^{-1}\bar{\mu} \exp(-\varepsilon^{-1}\mu s)}{(1 - \exp(-\varepsilon^{-1}\bar{\mu}))} \right).$$

Let $\gamma(\bar{\mu}, t) = \exp(-\varepsilon^{-1}\bar{\mu})$ be a real-valued function defined over $\bar{\mathfrak{D}}$ and differentiable on $\mathfrak{D}$. Then we have

$$|z_{st}(s, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu s/\varepsilon)), (s, t) \in \bar{\mathfrak{D}}$$

which is the desired bound. The proof of this lemma can be also found in (Kellogg and Tsan, 1978; Das and Natesan, 2015). ■

Lemma 4.2.9. *The following bounds are satisfied for the derivatives of the solution $z(s, t)$ of*

(4.1) for s and t

$$\begin{aligned} \left| \frac{\partial^i z(s, t)}{\partial s^i} \right| &\leq C \left(1 + \varepsilon^{-i} \exp \left(-\frac{\mu}{\varepsilon} s \right) \right), \quad (s, t) \in \bar{\mathfrak{D}}, \quad i = 0, 1, 2, 3, 4, \\ \left| \frac{\partial^l z(s, t)}{\partial t^l} \right| &\leq C, \quad (s, t) \in \bar{\mathfrak{D}}, \quad l = 0, 1, 2, \end{aligned} \quad (4.13)$$

where $\beta(s, t)$ is lower bound of μ .

Proof. The proof follows from Lemma 4.2.4 - Lemma 4.2.8, it is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. ■

4.3 Numerical scheme

In this section, we discuss the numerical scheme used to solve the problem (4.1). The scheme involves a backward Euler method in the time direction, a higher-order finite difference method for the space derivatives, and Taylor series expansion for the delay term. As Tian (2000) stated, when τ is less than ε , treating the delay term using the Taylor series expansion works properly. Thus,

$$z(s, t - \tau) \approx z(s, t) - \tau z_t(s, t) + O(\tau^2). \quad (4.14)$$

Using (4.14) in (4.1) and simplifying, we get

$$\begin{cases} (1 - \tau \kappa(s, t)) z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -\kappa(s, t) z(s, t) + \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & (s, t) \in \eta_b, \\ z(0, t) = \psi_l(t), & t \in \eta_l, \\ z(1, t) = \psi_r(t), & t \in \eta_r. \end{cases} \quad (4.15)$$

4.3.1 Temporal semi-discretization

Using a uniform mesh in t -direction with step size $\Delta t = T/M$ given by $\Lambda^M = \{t_n = n\Delta t, 0 \leq n \leq M, t_M = T\}$, where M is the number of mesh points in $[0, T]$. Note that $T = r\tau$ for some positive integer r . Using the backward Euler scheme for the time derivative of (4.15), we get

$$\begin{cases} \mathfrak{L}^{\Delta t} Z^n(s) \equiv -\varepsilon \frac{d^2 Z^n(s)}{ds^2} + \beta(s, t_n) \frac{dZ^n(s)}{ds} + P(s, t_n) Z^n(s) = R(s, t_n), \\ Z^n(0) = \psi_l(t_n), \quad Z^n(1) = \psi_r(t_n), \quad 0 \leq n \leq M, \end{cases} \quad (4.16)$$

where $P(s, t_n) = \frac{1-\tau\kappa(s, t_n)}{\Delta t} + \kappa(s, t_n) + \chi(s, t_n)$ and $R(s, t_n) = \gamma(s, t_n) + \left(\frac{1-\tau\kappa(s, t_n)}{\Delta t}\right) Z^{n-1}(s)$. Now, we rewrite (4.16) as

$$-\varepsilon \frac{d^2 Z(s)}{ds^2} + \beta(s, t_n) \frac{dZ(s)}{ds} + P(s, t_n) Z(s) = R(s, t_n), \quad (4.17)$$

where $Z(s) = Z^n(s) = Z(s, t_n)$.

At each time step the local truncation error is defined as $e_n(s) := z(s, t_n) - Z^n(s)$, $0 \leq n \leq M$.

Lemma 4.3.10. *The local error estimate in the temporal direction satisfies the bound*

$$\|e_n\| \leq C_1(\Delta t)^2.$$

Proof. From (4.16), we have

$$\begin{aligned} (1 - \tau\kappa(s, t_n))Z^n(s) - Z^{n-1}(s) + \Delta t \mathfrak{L}Z^n(s) &= \Delta t R(s, t_n) \\ Z(t_{n-1}) &= (1 - \tau\kappa(t_n))Z(t_n) + \Delta t \mathfrak{L}Z(t_n) - \Delta t R(t_n) + \int_{t_{n-1}}^{t_n} (t_{n-1} - s) \frac{\partial^2 Z(s)}{\partial t^2} ds \\ &= (1 - \tau\kappa(t_n))Z(t_n) + \Delta t \mathfrak{L}Z(t_n) - \Delta t R(t_n) + O(\Delta t^2). \end{aligned}$$

Then $e_n(s)$ is the solution for the boundary value problem type

$$\begin{aligned} (1 - \tau\kappa(t_n))e(t_n) + \Delta t \mathfrak{L}e(t_n) &= O(\Delta t^2) \\ e_n(0) = 0 &= e_n(1) \end{aligned}$$

by maximum principle on the above operator $(1 - \tau\kappa(t_n))e(t_n) + \Delta t \mathfrak{L}e(t_n)$ gives

$$\|e_n\| \leq C_1(\Delta t)^2.$$

■

Lemma 4.3.11. *The global error E_n at n th time level satisfies the bound*

$$\|E_n\| \leq C(\Delta t), \quad 1 \leq n \leq M - 1.$$

Proof. Using Lemma 4.3.10, the global error E_n bound at n th time step is given as

$$\begin{aligned} \|E_n\| &= \left\| \sum_{l=1}^n e_l \right\| \leq \|e_1\| + \|e_2\| + \|e_3\| + \dots + \|e_n\| \\ &\leq C_1 T(\Delta t), \quad \text{since } (n)\Delta t \leq T \\ &= C(\Delta t), \quad \text{where } C_1 T = C, \end{aligned}$$

■

Lemma 4.3.12. For $0 \leq n \leq M - 1$, the solution of (4.16), satisfies the bound

$$\left| \frac{d^i Z^n(s)}{ds^i} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(-\frac{\mu}{\varepsilon} s \right) \right), \quad s \in \bar{\Omega}, \quad 0 \leq i \leq 4.$$

Proof. The proof follows from Lemma 4.2.6, it is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. ■

4.3.2 Spatial discretization

The spatial domain $[0, 1]$ into N equal number of sub-intervals with the length of h is given by $0 = s_0, s_1, \dots, s_N = 1$ and $s_m = mh, m = 0, 1, 2, \dots, N$. Assume smooth function $Z(s) = Z^n(s) = Z(s, t_n)$ in the interval $[0, 1]$. From Taylor's series expansion, we have

$$\begin{aligned} Z_{m+1} &\approx Z_m + hZ'_m + \frac{h^2}{2!}Z''_m + \frac{h^3}{3!}Z'''_m + \frac{h^4}{4!}Z^{(4)}_m + \frac{h^5}{5!}Z^{(5)}_m + \frac{h^6}{6!}Z^{(6)}_m + \frac{h^7}{7!}Z^{(7)}_m + \frac{h^8}{8!}Z^{(8)}_m + O(h^9), \\ Z_{m-1} &\approx Z_m - hZ'_m + \frac{h^2}{2!}Z''_m - \frac{h^3}{3!}Z'''_m + \frac{h^4}{4!}Z^{(4)}_m - \frac{h^5}{5!}Z^{(5)}_m + \frac{h^6}{6!}Z^{(6)}_m - \frac{h^7}{7!}Z^{(7)}_m + \frac{h^8}{8!}Z^{(8)}_m - O(h^9), \\ Z''_{m+1} &\approx Z''_m + hZ'''_m + \frac{h^2}{2!}Z^{(4)}_m + \frac{h^3}{3!}Z^{(5)}_m + \frac{h^4}{4!}Z^{(6)}_m + \frac{h^5}{5!}Z^{(7)}_m + \frac{h^6}{6!}Z^{(8)}_m + \frac{h^7}{7!}Z^{(9)}_m + \frac{h^8}{8!}Z^{(10)}_m + O(h^{11}), \\ Z''_{m-1} &\approx Z''_m - hZ'''_m + \frac{h^2}{2!}Z^{(4)}_m - \frac{h^3}{3!}Z^{(5)}_m + \frac{h^4}{4!}Z^{(6)}_m - \frac{h^5}{5!}Z^{(7)}_m + \frac{h^6}{6!}Z^{(8)}_m - \frac{h^7}{7!}Z^{(9)}_m + \frac{h^8}{8!}Z^{(10)}_m - O(h^{11}). \end{aligned} \quad (4.18)$$

Combining the corresponding results of (4.18), we arrive at

$$\begin{aligned} Z_{m-1} - 2Z_m + Z_{m+1} &= \frac{2h^2}{2!}Z''_m + \frac{2h^4}{4!}Z^{(4)}_m + \frac{2h^6}{6!}Z^{(6)}_m + \frac{2h^8}{8!}Z^{(8)}_m + O(h^{10}), \\ Z''_{m-1} - 2Z''_m + Z''_{m+1} &= \frac{2h^2}{2!}Z^{(4)}_m + \frac{2h^4}{4!}Z^{(6)}_m + \frac{2h^6}{6!}Z^{(8)}_m + \frac{2h^8}{8!}Z^{(10)}_m + O(h^{12}). \end{aligned} \quad (4.19)$$

From the second equation of (4.19), we get $\frac{h^4}{12}Z^{(6)}_m$, and we obtain

$$Z_{m-1} - 2Z_m + Z_{m+1} = \frac{h^2}{30} \left(Z''_{m-1} + 28Z''_m + Z''_{m+1} \right) + \zeta, \quad (4.20)$$

where $\zeta = \frac{h^4}{20}Z^{(4)}_m - \frac{13h^8}{302,400}Z^{(8)}_m + O(h^{10})$. Using (4.17), we get

$$\begin{aligned} -\varepsilon Z''_{m\pm 1} &= -\beta(s_{m\pm 1}, t_n)Z'_{m\pm 1} - P(s_{m\pm 1}, t_n)Z_{m\pm 1} + R(s_{m\pm 1}, t_n), \\ -\varepsilon Z''_m &= -\beta(s_m, t_n)Z'_m - P(s_m, t_n)Z_m + R(s_m, t_n). \end{aligned} \quad (4.21)$$

Next, using the non-symmetric FDM we approximate $Z'_{m\pm 1}$ and Z'_m as

$$\begin{aligned} Z'_m &= \frac{Z_{m+1} - Z_{m-1}}{2h} + O(h^2), \quad Z'_{m+1} = \frac{3Z_{m+1} - 4Z_m + Z_{m-1}}{2h} - hZ''_m + O(h^2), \\ Z'_{m-1} &= \frac{-Z_{m+1} + 4Z_m - 3Z_{m-1}}{2h} + hZ''_m + O(h^2). \end{aligned} \quad (4.22)$$

From (4.20), we have

$$-\varepsilon \left(\frac{Z_{m-1} - 2Z_m + Z_{m+1}}{h^2} \right) = \frac{1}{30} \left(-\varepsilon Z''_{m-1} - 28\varepsilon Z''_m - \varepsilon Z''_{m+1} \right). \quad (4.23)$$

Substituting (4.22) into (4.21) and the resulting equation into (4.23) after simplifying, we obtain

$$\begin{aligned} & - \left(\varepsilon - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) \left(\frac{Z_{m-1} - 2Z_m + Z_{m+1}}{h^2} \right) \\ & + \frac{\beta(s_{m-1}, t_n)}{60h} \left(-3Z_{m-1} + 4Z_m - Z_{m+1} \right) + \frac{28\beta(s_m, t_n)}{60h} \left(Z_{m+1} - Z_{m-1} \right) \\ & + \frac{\beta(s_{m+1}, t_n)}{60h} \left(Z_{m-1} - 4Z_m + 3Z_{m+1} \right) + \frac{P(s_{m-1}, t_n)}{30} Z_{m-1} \\ & + \frac{28P(s_m, t_n)}{30} Z_m + \frac{P(s_{m+1}, t_n)}{30} Z_{m+1} = \frac{1}{30} \left(R(s_{m-1}, t_n) + 28R(s_m, t_n) + R(s_{m+1}, t_n) \right). \end{aligned} \quad (4.24)$$

Computing the exponential fitting factor

In this part, we introduce the fitting factor σ and for the obtained scheme of (4.16) at $(m, n)th$ level. As the theory of singular perturbation given in O'Malley (1991) and as stated in Woldaregay et al. (2021), the zero-order asymptotic solution of the problem of the form

$$\begin{cases} -\varepsilon Z''(s) + \beta(s)Z' + \chi(s)Z(s) = \gamma(s), & s \in \Omega = (0, 1), \\ Z(0) = \psi_l, \quad Z(1) = \psi_r, \end{cases} \quad (4.25)$$

is given by

$$Z(s) \approx Z_0(s) + \frac{\beta(0)}{\beta(s)} \left(\psi_l - Z_0(0) \right) \exp \left(- \int_0^s \left(\frac{\beta(s)}{\varepsilon} - \frac{\chi(s)}{\beta(s)} \right) ds \right) + O(\varepsilon). \quad (4.26)$$

From Taylor's series expansion for $\beta(s)$ and $\chi(s)$ restricting to their first terms about $s = 0$ and the simplified form gives

$$Z(s) = Z_0(s) + (\psi_l - Z_0(0)) \exp \left(- \frac{\beta(0)}{\varepsilon} s \right), \quad (4.27)$$

where $Z_0(s)$ is the reduced problem solution. Considering h is fairly small and solving the result in (4.27) at s_m and $s_{m\pm 1}$ which gives

$$\begin{cases} Z(s_m) = Z(mh) \approx Z_0(0) + (\psi_l - Z_0(0)) \exp(-\beta(0)m\rho), \\ Z(s_{m\pm 1}) = Z((m \pm 1)h) \approx Z_0(0) + (\psi_l - Z_0(0)) \exp(-\beta(0)(m \pm 1)\rho), \end{cases} \quad (4.28)$$

where $\rho = \frac{h}{\varepsilon}$. We present an exponentially fitting factor σ of the scheme (4.24)

$$\begin{aligned} & - \left(\varepsilon\sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) \left(\frac{Z_{m-1} - 2Z_m + Z_{m+1}}{h^2} \right) \\ & + \frac{\beta(s_{m-1}, t_n)}{60h} \left(-3Z_{m-1} + 4Z_m - Z_{m+1} \right) + \frac{28\beta(s_m, t_n)}{60h} \left(Z_{m+1} - Z_{m-1} \right) \\ & + \frac{\beta(s_{m+1}, t_n)}{60h} \left(Z_{m-1} - 4Z_m + 3Z_{m+1} \right) + \frac{P(s_{m-1}, t_n)}{30} Z_{m-1} \\ & + \frac{28P(s_m, t_n)}{30} Z_m + \frac{P(s_{m+1}, t_n)}{30} Z_{m+1} = \frac{1}{30} \left(R(s_{m-1}, t_n) + 28R(s_m, t_n) + R(s_{m+1}, t_n) \right). \end{aligned} \quad (4.29)$$

Multiplying both side of (4.29) by h and taking the limit as $h \rightarrow 0$, it gives

$$\begin{aligned} & - \frac{\sigma}{\rho} \lim_{h \rightarrow 0} (Z_{m-1} - 2Z_m + Z_{m+1}) + \frac{\beta_0(t_n)}{60} \lim_{h \rightarrow 0} (-3Z_{m-1} + 4Z_m - Z_{m+1}) \\ & + \frac{28\beta_0(t_n)}{60} \lim_{h \rightarrow 0} (Z_{m+1} - Z_{m-1}) + \frac{\beta_0(t_n)}{60} \lim_{h \rightarrow 0} (Z_{m-1} - 4Z_m + 3Z_{m+1}) = 0. \end{aligned} \quad (4.30)$$

Using (4.28) and simplifying, we obtain

$$\left\{ \begin{aligned} & \lim_{h \rightarrow 0} (Z((m-1)h) - 2Z(mh) + Z((m+1)h)) \\ & = (\psi_l - Z_0(0)) e^{(-\beta(0)m\rho)} (e^{\beta(0)\rho} + e^{-\beta(0)\rho} - 2), \\ & \lim_{h \rightarrow 0} (-3Z((m-1)h) + 4Z(mh) - Z((m+1)h)) \\ & = (\psi_l - Z_0(0)) e^{(-\beta(0)m\rho)} (-3e^{\beta(0)\rho} - e^{-\beta(0)\rho} + 4), \\ & \lim_{h \rightarrow 0} (Z((m-1)h) - 4Z(mh) + 3Z((m+1)h)) \\ & = (\psi_l - Z_0(0)) e^{(-\beta(0)m\rho)} (e^{\beta(0)\rho} + 3e^{-\beta(0)\rho} - 4), \\ & \lim_{h \rightarrow 0} (Z((m+1)h) - Z((m-1)h)) \\ & = (\psi_l - Z_0(0)) e^{(-\beta(0)m\rho)} (e^{-\beta(0)\rho} - e^{\beta(0)\rho}). \end{aligned} \right. \quad (4.31)$$

Plugging back (4.31) into (4.30) after some simplification, we obtain

$$\frac{\sigma}{\rho} \left(e^{\beta_0(t_n)\rho} + e^{-\beta_0(t_n)\rho} - 2 \right) = \frac{\beta_0(t_n)}{60} \left(-30e^{\beta_0(t_n)\rho} + 30e^{-\beta_0(t_n)\rho} \right). \quad (4.32)$$

Solving (4.32) for σ , we get

$$\sigma = \frac{\rho\beta_0(t_n)}{2} \coth\left(\frac{\rho\beta_0(t_n)}{2}\right). \quad (4.33)$$

4.3.3 The discrete scheme

By utilizing the higher order finite difference scheme of (4.24) and applying the exponential fitting factor to (4.33), $1 \leq m \leq N-1$ and $0 \leq n \leq M-1$, we obtain the fully discrete scheme

$$\mathfrak{L}^{\Delta t, h} Z_m^n = \frac{1}{30}(R(s_{m-1}, t_n) + 28R(s_m, t_n) + R(s_{m+1}, t_n)), \quad (4.34)$$

where $\mathfrak{L}^{\Delta t, h} Z_m^n = -(\varepsilon\sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30})(\frac{Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n}{h^2}) + \frac{\beta(s_{m-1}, t_n)}{60h}(-3Z_{m-1}^n + 4Z_m^n - Z_{m+1}^n) + \frac{28\beta(s_m, t_n)}{60h}(Z_{m+1}^n - Z_{m-1}^n) + \frac{\beta(s_{m+1}, t_n)}{60h}(Z_{m-1}^n - 4Z_m^n + 3Z_{m+1}^n) + \frac{P(s_{m-1}, t_n)}{30}Z_{m-1}^n + \frac{28P(s_m, t_n)}{30}Z_m^n + \frac{P(s_{m+1}, t_n)}{30}Z_{m+1}^n$.

In the explicit form, we write

$$r_m^- Z_{m-1}^n + r_m^0 Z_m^n + r_m^+ Z_{m+1}^n = H_m^n, \quad (4.35)$$

where

$$\begin{aligned} r_m^- &= -\frac{1}{h^2} \left(\varepsilon\sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) - \frac{3\beta(s_{m-1}, t_n)}{60h} + \frac{P(s_{m-1}, t_n)}{30} \\ &\quad - \frac{28\beta(s_m, t_n)}{60h} + \frac{\beta(s_{m+1}, t_n)}{60h}, \\ r_m^0 &= \frac{2}{h^2} \left(\varepsilon\sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) + \frac{4\beta(s_{m-1}, t_n)}{60h} - \frac{4\beta(s_{m+1}, t_n)}{60h} + \frac{28P(s_m, t_n)}{30}, \\ r_m^+ &= -\frac{1}{h^2} \left(\varepsilon\sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) - \frac{\beta(s_{m-1}, t_n)}{60h} + \frac{28\beta(s_m, t_n)}{60h} \\ &\quad + \frac{3\beta(s_{m+1}, t_n)}{60h} + \frac{P(s_{m+1}, t_n)}{30}, \\ H_m^n &= \frac{1}{30} \left(R(s_{m-1}, t_n) + 28R(s_m, t_n) + R(s_{m+1}, t_n) \right). \end{aligned}$$

4.4 Convergence analysis

Lemma 4.4.13. *Das and Natesan (2015)(Discrete maximum principle) Assume that $Z_0^n \geq 0$, $Z_N^n \geq 0$ and $\mathfrak{L}^{\Delta t, h} Z_m^n \geq 0$, $\forall m = 1(1)N-1$, then $Z_m^n \geq 0$, $\forall m = 0(1)N$.*

Proof. Assume there is $k \in \{0, 1, 2, \dots, N\}$, such that $Z_k^n = \min_{0 \leq m \leq N} Z_m^n < 0$. Assume that $Z_k^n < 0$ and from the assumption, it is shown that $k \notin \{0, 1\}$. So that, we have $Z_{k+1}^n - Z_k^n > 0$ and $Z_k^n - Z_{k-1}^n < 0$. Then, we get $\mathfrak{L}^{\Delta t, h} Z_k^n < 0$ for $k = 1(1)N-1$. So, the assumption is wrong. Therefore, $Z_m^n \geq 0$, $\forall m = 0(1)N$. ■

Lemma 4.4.14. *Woldaregay et al. (2021)(Stability result) Let Z_m^n be the solution of (4.34). Then, it satisfies the following bound*

$$|Z_m^n| \leq \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\},$$

where $P(s_m, t_n) \geq \zeta > 0$.

Proof. let $Q = \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\}$ and define the barrier functions $\vartheta_{m,n}^\pm$ by $\vartheta_{m,n}^\pm = Q \pm Z_m^n$. On the boundaries, we get $\vartheta_{0,n}^\pm = Q \pm Z_0^n = \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \pm \psi_l(t_n) \geq 0$,
 $\vartheta_{N,n}^\pm = Q \pm Z_N^n = \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \pm \psi_r(t_n) \geq 0$. For $s_m, m = 1(1)N-1$, we obtain

$$\begin{aligned} \mathfrak{L}^{\Delta t, h} \vartheta_{m,n} &= - \left(\varepsilon \sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) \left(\frac{Q \pm Z_{m-1}^n - 2(Q \pm Z_m^n) + Q \pm Z_{m+1}^n}{h^2} \right) \\ &\quad + \frac{\beta(s_{m-1}, t_n)}{60h} \left(-3(Q \pm Z_{m-1}^n) + 4(Q \pm Z_m^n) - (Q \pm Z_{m+1}^n) \right) \\ &\quad + \frac{28\beta(s_m, t_n)}{60h} \left(Q \pm Z_{m+1}^n - (Q \pm Z_{m-1}^n) \right) \\ &\quad + \frac{\beta(s_{m+1}, t_n)}{60h} \left(R \pm Z_{m-1}^n - 4(Q \pm Z_m^n) + 3(Q \pm Z_{m+1}^n) \right) \\ &\quad + \frac{P(s_{m-1}, t_n)}{30} \left(Q \pm Z_{m-1}^n \right) + \frac{28P(s_m, t_n)}{30} \left(Q \pm Z_m^n \right) + \frac{P(s_{m+1}, t_n)}{30} \left(Q \pm Z_{m+1}^n \right) \\ &= \left(\frac{P(s_{m-1}, t_n)}{30} + \frac{28P(s_m, t_n)}{30} + \frac{P(s_{m+1}, t_n)}{30} \right) Q \pm \mathfrak{L}^{\Delta t, h} Z_m^n \\ &= \left(\frac{P(s_{m-1}, t_n)}{30} + \frac{28P(s_m, t_n)}{30} + \frac{P(s_{m+1}, t_n)}{30} \right) \left(\frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\zeta} \right. \\ &\quad \left. + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \right) \pm \frac{1}{30} \left(R(s_{m-1}, t_n) + 28R(s_m, t_n) + R(s_{m+1}, t_n) \right) \geq 0. \end{aligned}$$

By Lemma 4.4.13, we get $\vartheta_{m,n}^\pm \geq 0$, $m = 0(1)N$. Therefore, the needed bound is obtained. ■

Using Taylor's series approximation, we have the bounds

$$\begin{aligned} \left| - \left(\frac{d}{ds^2} - \delta_s^2 \right) Z^n(s_m) \right| &\leq Ch^2 \left\| \frac{d^4 Z^n(s_m)}{ds^4} \right\|, \\ \left| \frac{dZ^n(s_{m-1})}{ds} - \left(\frac{-Z_{m+1}^n + 4Z_m^n - 3Z_{m-1}^n}{2h} + h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right| &\leq Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\|, \\ \left| \frac{dZ^n(s_{m+1})}{ds} - \left(\frac{3Z_{m+1}^n - 4Z_m^n + Z_{m-1}^n}{2h} - h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right| &\leq Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\|, \\ \left| \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_m) \right| &\leq Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\|, \quad \left| \delta_s^2 Z^n(s_m) \right| \leq C \left\| \frac{d^2 Z^n(s_m)}{ds^2} \right\|, \end{aligned} \quad (4.36)$$

where $\|Z^{(k)}(s_m)\| = \max_{s_m \in (s_0, s_N)} |Z_n^{(k)}(s_m)|$, $k = 2, 3, 4$.

Now, we provide a theorem that presents an estimate for the error that arises due to the discretization of the space direction.

Theorem 4.4.1. *Consider the sufficiently smooth functions $\beta(s, t_n)$, $\chi(s, t_n)$ and $\kappa(s, t_n)$ of (4.16) so that $Z^n(s) \in C^4[0, 1]$. Then, the solution Z_m^n of (4.34) satisfies the bound*

$$|\mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n)| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3} \exp(-\frac{\mu}{\varepsilon}s_m) \right). \quad (4.37)$$

Proof. In the spatial direction, the local truncation error is given by

$$\begin{aligned} \left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| &= \left| -\varepsilon \left(\frac{d}{ds^2} - \sigma \delta_s^2 \right) Z^n(s_m) + \frac{\beta(s_{m-1}, t_n)}{60h} \right. \\ &\quad \left(\frac{dZ^n(s_{m-1})}{ds} - \left(\frac{-Z_{m+1}^n + 4Z_m^n - 3Z_{m-1}^n}{2h} + h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \\ &\quad + \frac{28\beta(s_m, t_n)}{60h} \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_m) + \frac{\beta(s_{m+1}, t_n)}{60h} \\ &\quad \left. \left(\frac{dZ^n(s_{m+1})}{ds} - \left(\frac{3Z_{m+1}^n - 4Z_m^n + Z_{m-1}^n}{2h} - h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \right| \\ &\leq \left| \left(\varepsilon(\beta(s_m, t_n) \frac{\rho}{2} \coth(\beta_0(t_n) \frac{\rho}{2}) - 1) \delta_s^2 Z^n(s_m) \right) \right| + \left| \varepsilon \left(\frac{d^2}{ds^2} - \delta_s^2 \right) Z^n(s_m) \right| \\ &\quad + \left| \frac{\beta(s_{m-1}, t_n)}{60h} \left(\frac{dZ^n(s_{m-1})}{ds} - \left(\frac{-Z_{m+1}^n + 4Z_m^n - 3Z_{m-1}^n}{2h} + h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \right| \\ &\quad + \left| \frac{28\beta(s_m, t_n)}{60h} \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_m) \right| + \left| \frac{\beta(s_{m+1}, t_n)}{60h} \right. \\ &\quad \left. \left(\frac{dZ^n(s_{m+1})}{ds} - \left(\frac{3Z_{m+1}^n - 4Z_m^n + Z_{m-1}^n}{2h} - h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \right|, \end{aligned} \quad (4.38)$$

where $\sigma = \beta(s_m, t_n) \frac{\rho}{2} \coth(\beta_0(t_n) \frac{\rho}{2})$ and $\rho = \frac{h}{\varepsilon}$.

For the constants C_1 and C_2 we have $|\rho \coth(\rho) - 1| \leq C_1 \rho^2$, for $\rho \leq 1$. For $\rho \rightarrow \infty$, since $\lim_{\rho \rightarrow \infty} \coth(\rho) = 1$ which gives $|\rho \coth(\rho) - 1| \leq C_1 \rho$.

Generally, $\forall \rho > 0$, we put as

$$C_1 \frac{\rho^2}{\rho + 1} \leq \rho \coth(\rho) - 1 \leq C_2 \frac{\rho^2}{\rho + 1}, \quad (4.39)$$

we obtain

$$\varepsilon \left[\beta(s_m, t_n) \frac{\rho}{2} \coth \left(\beta_0(t_n) \frac{\rho}{2} \right) - 1 \right] \leq \varepsilon \frac{(h/\varepsilon)^2}{h/\varepsilon + 1} = \frac{h^2}{h + \varepsilon}. \quad (4.40)$$

Using the bound for the difference of the derivatives in (4.36), (4.38) and (4.40), we obtain

$$\left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| \leq \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2 Z^n(s_m)}{ds^2} \right\| + Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\| + C\varepsilon h^2 \left\| \frac{d^4 Z^n(s_m)}{ds^4} \right\|.$$

From Lemma 4.3.12, we obtain the bound for the derivatives

$$\begin{aligned} \left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| &\leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-2} \exp\left(-\frac{\mu}{\varepsilon} s_m\right) \right) + Ch^2 \left[\left(1 + \varepsilon^{-3} \exp\left(-\frac{\mu}{\varepsilon} s_m\right) \right) \right. \\ &\quad \left. + \left(\varepsilon + \varepsilon^{-3} \exp\left(-\frac{\mu}{\varepsilon} s_m\right) \right) \right]. \end{aligned} \quad (4.41)$$

Evidently, $\varepsilon^{-2} \leq \varepsilon^{-3}$, then we obtain $\left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| \leq \frac{Ch^2}{h + \varepsilon} (1 + \varepsilon^{-3} \exp(-\frac{\mu}{\varepsilon} s_m))$ thus, it gives the desired bound. ■

Lemma 4.4.15. *Woldaregay et al. (2021)* For a fixed mesh and as $\varepsilon \rightarrow 0$, it gives

$$\lim_{\varepsilon \rightarrow 0} \max_m \frac{\exp(-\mu s_m / \varepsilon)}{\varepsilon^i} = 0, \quad i = 1, 2, 3, \dots,$$

where $s_m = mh, m = 1(1)N - 1$.

Proof. On the discrete domain $\{s_m\}_0^N$, for the interior grid points, we get

$$\max_{1 \leq m \leq N-1} \frac{\exp(-\mu s_m / \varepsilon)}{\varepsilon^i} \leq \frac{\exp(-\mu s_1 / \varepsilon)}{\varepsilon^i} = \frac{\exp(-\mu h / \varepsilon)}{\varepsilon^i},$$

since $s_1 = h = 1 - s_{N-1}$. Repeatedly using L'hospital's rule implies

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\exp(-\mu h / \varepsilon)}{\varepsilon^i} &= \lim_{v=1/\varepsilon \rightarrow \infty} \frac{v^i}{\exp(\mu h v)} \\ &= \lim_{v \rightarrow \infty} \frac{i!}{(\mu h)^i \exp(\mu h v)} = 0. \end{aligned}$$

This completes the proof. ■

Theorem 4.4.2. Let Z_m^n be the solution of (4.34), then we have the following uniform error bound

$$\sup_{\varepsilon \in (0,1]} \left\| Z^n(s_m) - Z_m^n \right\| \leq Ch, \quad m = 0(1)N.$$

Proof. Plugging Lemma 4.4.15 into (4.37), we arrive at

$$\left\| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right\| \leq \frac{Ch^2}{h + \varepsilon}.$$

Hence, the result leads $\|Z^n(s_m) - Z_m^n\| \leq \frac{Ch^2}{h+\varepsilon}$. Using the sup over all $\varepsilon \in (0, 1]$, we get

$$\sup_{\varepsilon \in (0,1]} \|Z^n(s_m) - Z_m^n\| \leq Ch.$$

■

Remark 4.4.1. When $\varepsilon > h$, the obtained scheme gives second-order uniform convergence. For $\varepsilon \ll h$, the scheme is first-order uniformly convergent in the spatial direction.

Theorem 4.4.3. Let z and Z are the solutions of (4.1) and (4.34) respectively, then we have the following uniform error bound

$$\sup_{\varepsilon \in (0,1]} \|z - Z\| \leq C(h + (\Delta t)). \quad (4.42)$$

Proof. The proof can be done by the combination of Lemma 4.3.11 and Theorem 4.4.2. ■

4.5 Numerical results

In this section some numerical experiments are carried out to validate the developed method is stable, more accurate and ε -uniformly convergent by considering two model examples.

Example 4.1. Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} - \frac{\partial z}{\partial s} + (\frac{1+s^2}{2})z = t^3 - z(s, t - \tau)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in (0, 2]$.

Example 4.2. Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} - (2 + s^2) \frac{\partial z}{\partial s} + sz(s, t) + z(s, t - \tau) = 10t^2 \exp(-t)s(1 - s)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in (0, 2]$.

Since the exact solutions of the considered examples are not known, the maximum pointwise absolute error and rate of convergence are calculated using the double mesh principle. So, we calculate the maximum pointwise absolute error by

$$E_\varepsilon^{N,M} = \max_{m,n} |Z_{m,n}^{N,M} - Z_{m,n}^{2N,2M}|, \quad (4.43)$$

and by using

$$E^{N,M} = \max_{m,n} (E_\varepsilon^{N,M}) \quad (4.44)$$

we estimate the ε -uniform error.

The rate of convergence of the proposed method is computed by

$$r_\varepsilon^{N,M} = \log_2(E_\varepsilon^{N,M} / E_\varepsilon^{2N,2M}). \quad (4.45)$$

The ε -uniform rate of convergence of the developed method is calculated by using

$$r^{N,M} = \log_2(E^{N,M}/E^{2N,2M}). \quad (4.46)$$

Tables 4.1 and 4.3 display the maximum pointwise absolute error, ε -uniform error, and ε -

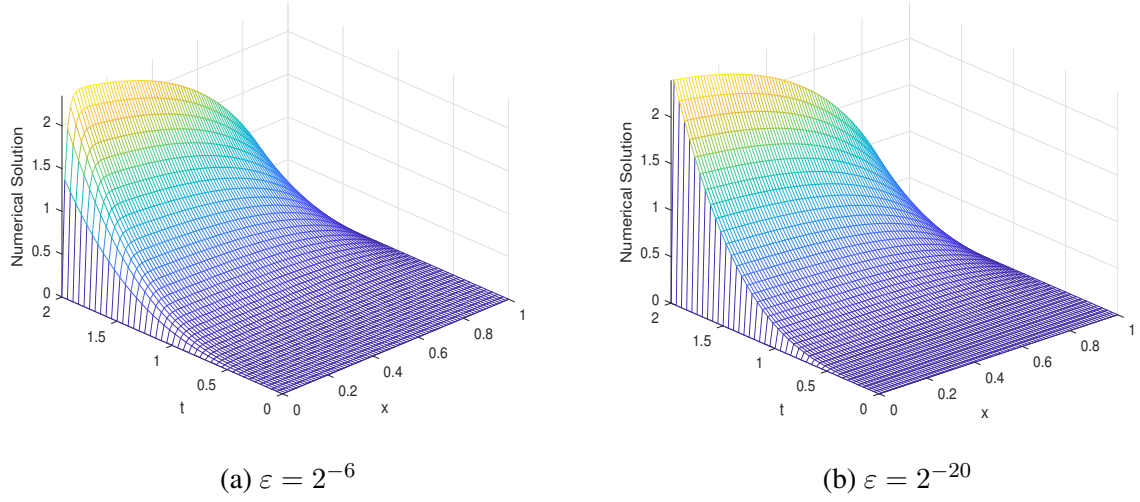


Figure 4.1: Numerical solution of the scheme for Example 4.1.

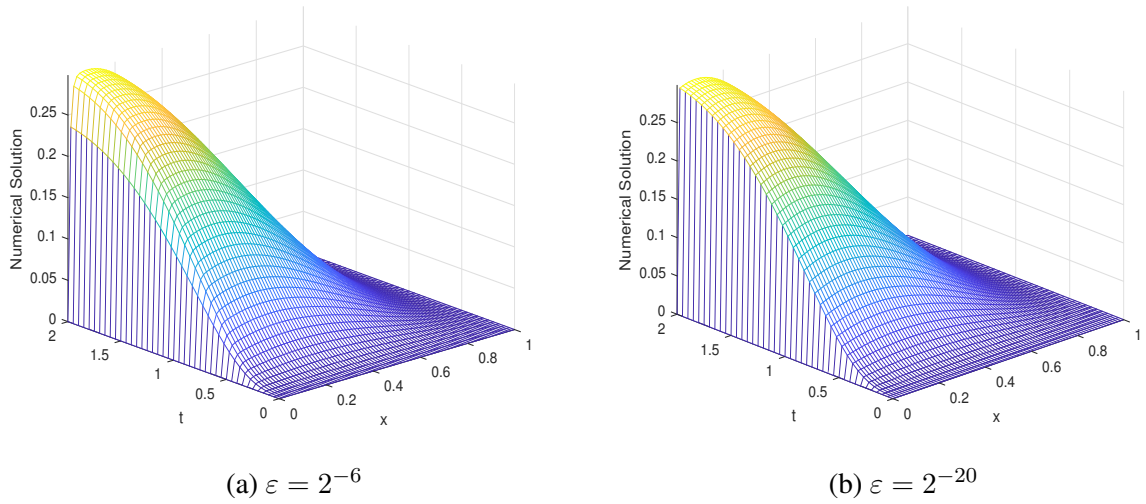


Figure 4.2: Numerical solution of the scheme for Example 4.2.

uniform rate of convergence of the proposed method for different values of ε and an equal number of mesh points. We observed that as the values of ε approach zero, the maximum pointwise absolute error remains uniform, indicating that the scheme is stable and uniformly convergent regardless of the values of ε . The numerical results agree with the theoretical result, which is confirmed by the ε -uniform error and ε -uniform rate of convergence of the method, shown in the last rows of each table.

Table 4.1: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 4.1 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
2^{-0}	1.3260e-03	7.4390e-04	3.9431e-04	2.0293e-04	1.0294e-04
2^{-2}	1.3097e-02	7.2784e-03	3.8340e-03	1.9676e-03	9.9661e-04
2^{-4}	2.7648e-02	1.5726e-02	8.3824e-03	4.3310e-03	2.2011e-03
2^{-6}	3.7674e-02	1.9780e-02	1.0209e-02	5.3769e-03	2.7671e-03
2^{-8}	3.7154e-02	2.2029e-02	1.2040e-02	5.6488e-03	2.8289e-03
2^{-10}	3.7206e-02	2.1942e-02	1.1903e-02	6.2248e-03	3.1986e-03
2^{-12}	3.7221e-02	2.1951e-02	1.1906e-02	6.1986e-03	3.1627e-03
2^{-14}	3.7224e-02	2.1953e-02	1.1907e-02	6.1992e-03	3.1627e-03
2^{-16}	3.7225e-02	2.1953e-02	1.1907e-02	6.1993e-03	3.1627e-03
2^{-18}	3.7225e-02	2.1953e-02	1.1908e-02	6.1994e-03	3.1627e-03
2^{-20}	3.7225e-02	2.1953e-02	1.1908e-02	6.1994e-03	3.1627e-03
$E^{N,M}$	3.7225e-02	2.1953e-02	1.1908e-02	6.1994e-03	3.1627e-03
$r^{N,M}$	0.76185	0.88249	0.94173	0.97097	–

Table 4.2: Comparison of $E^{N,M}$ and $r^{N,M}$ for Example 4.1.

Schemes $\downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
Proposed scheme					
$E^{N,M}$	3.7225e-02	2.1953e-02	1.1908e-02	6.1994e-03	3.1627e-03
$r^{N,M}$	0.76185	0.88249	0.94173	0.97097	–
Negero and Duressa (2022b)					
$E^{N,M}$	8.3951e-02	4.9224e-02	2.6666e-02	1.3880e-02	7.0816e-03
$r^{N,M}$	0.77019	0.88436	0.94199	0.97086	–

Table 4.3: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 4.2 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
2^{-0}	2.4094e-04	4.5191e-05	5.3745e-06	4.6776e-06	3.4511e-06
2^{-2}	5.8848e-04	1.7058e-04	1.1039e-04	6.2476e-05	3.3191e-05
2^{-4}	2.1117e-03	7.2963e-04	3.0344e-04	1.3611e-04	6.4377e-05
2^{-6}	4.6754e-03	2.2578e-03	8.2105e-04	2.8430e-04	1.0734e-04
2^{-8}	4.8645e-03	2.9411e-03	1.5511e-03	6.7433e-04	2.3656e-04
2^{-10}	4.8682e-03	2.9448e-03	1.6029e-03	8.4404e-04	4.2558e-04
2^{-12}	4.8691e-03	2.9452e-03	1.6032e-03	8.4463e-04	4.3880e-04
2^{-14}	4.8693e-03	2.9453e-03	1.6032e-03	8.4466e-04	4.3881e-04
2^{-16}	4.8693e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
2^{-18}	4.8694e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
2^{-20}	4.8694e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
$E^{N,M}$	4.8694e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
$r^{N,M}$	0.7253	0.8775	0.9245	0.9448	–

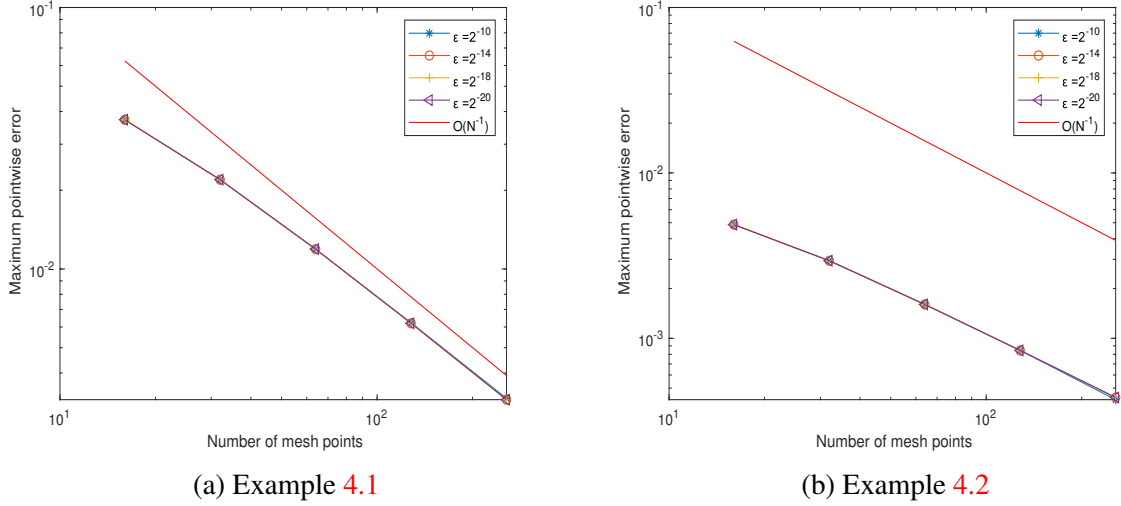


Figure 4.3: Maximum pointwise error in Log-Log scale plot.

Figures 4.1 and 4.2 reveal the numerical solution of the scheme for Example 4.1 and 4.2, respectively by considering different values of ε . These figures revealed that the boundary layer is formed on the left side of the spatial domain as ε approaches zero. Moreover, Figure 4.3 shows the maximum pointwise absolute error of the scheme by log-log scale plot. The result of the developed scheme are compared with some of the existing published work. Hence, in Table 4.2 displays the comparison of the proposed scheme with the results of the existing published work and provide better accurate results than the scheme in (Negero and Duressa, 2022b).

4.6 Summary

In this chapter, we presented a numerical scheme to solve a singularly perturbed parabolic convection-diffusion equation with a small time delay. The problem has a boundary layer on the left side of the domain, with a thickness of ε . The developed scheme combines the backward Euler method in the time direction and an exponentially fitted operator finite difference scheme in the spatial direction. The stability of the discrete scheme analyzed using the discrete maximum principle. To validate the theoretical findings, two model examples are considered, and the numerical results are presented using maximum pointwise absolute error, ε -uniform error, and ε -uniform rate of convergence in tables. The numerical results are in good agreement with the theoretical finding, and the developed method offers better accuracy, stability, and ε -uniform numerical results with a linear order of convergence. In the subsequent chapter, we consider the problem with a large time delay.

CHAPTER FIVE

FITTED OPERATOR METHOD FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATION WITH LARGE TIME DELAY

In this chapter, we extended the work done in the preceding chapter considering the problem with a large time delay. The solution of the problem has a boundary layer near the right side of the domain and the characteristics of the continuous solution are examined. The backward Euler scheme is used in the temporal direction on a uniform mesh. Non-symmetric finite difference schemes are used to approximate the first-order derivative terms, and a higher-order finite difference method is used for approximating the second-order derivative term. An exponential fitting factor is computed and incorporated into the difference scheme to handle the effect of the small parameter. The stability of the scheme is examined and analyzed using the discrete maximum principle. It is proven that the developed scheme is parameter-uniform and has first-order accuracy in both space and time. Two numerical examples are presented to demonstrate the applicability of the developed scheme. The results are compared with those available in the literature. In addition, the boundary layer behaviour of the solutions is illustrated graphically.

5.1 Governing problem

Let $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$ for $T > 0$, we consider SPDDE of the form

$$\begin{cases} z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -\kappa(s, t)z(s, t - \tau) + \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), & (s, t) \in \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), & t \in \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t), & t \in \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (5.1)$$

where $\mathfrak{L}_\varepsilon z(s, t) = -\varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) + \chi(s, t)z(s, t)$.

Here $\varepsilon \in (0, 1]$ and $\tau > 0$ are the perturbation parameter and the delay parameter, respectively. We pretended that the functions $\beta(s, t), \chi(s, t), \kappa(s, t), \gamma(s, t)$ on $\bar{\mathfrak{D}} = \bar{\Omega} \times \bar{\Lambda} = [0, 1] \times [0, T]$ and $\psi_b(s, t), \psi_l(t), \psi_r(t)$ on $\eta = \eta_l \cup \eta_r \cup \eta_b$ are smooth enough and bounded which meet $\chi(s, t) \geq \varpi > 0, \kappa(s, t) \geq \varphi > 0, \beta(s, t) \geq \mu > 0$. These conditions assure, problem (5.1) has a boundary layer near $s = 1$ (Das and Natesan, 2015).

5.2 Properties of the continuous solution

Under the premises, the data are Hölder continuous, if the compatibility conditions and assumption stated in (4.2) -(4.3) are satisfied, then the problem (5.1) has a unique solution.

Setting $\varepsilon = 0$, the reduced problem of (5.1) is given as

$$\begin{cases} \frac{\partial z_0(s,t)}{\partial t} + \beta(s,t) \frac{\partial z_0(s,t)}{\partial s} + \chi(s,t) z_0(s,t) = -\kappa(s,t) z_0(s,t-\tau) + \gamma(s,t), \\ z_0(s,t) = \psi_b(s,t), (s,t) \in \eta_b, \\ z_0(0,t) = \psi_l(t), t \in \bar{\Lambda}, \end{cases} \quad (5.2)$$

where $z_0(s,t)$ is the solution of the reduced problem.

The operator $\mathcal{L} = (\frac{\partial}{\partial t} + \mathcal{L}_\varepsilon)$ in (5.1) satisfies the maximum principle stated in Lemma 4.2.1 and the stability result of its solution $z(s,t)$ satisfies the bound given in Lemma 4.2.2.

Lemma 5.2.1. *The derivative of the solution $z(s,t)$ of (5.1) with respect of s satisfies*

$$|z_s(s,t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu(1-s)/\varepsilon)), (s,t) \in \bar{\mathcal{D}}.$$

Proof. In the problem (5.1), the effect of delay occurs only in the time variable t , and there is no effect of delay in the spatial variable s , therefore, if we fix $t \in [0, T]$, the result can be obtained by using the argument of (Kellogg and Tsan, 1978). Thus, from (5.1), we have $-\varepsilon z_{ss}(s,t) + \beta(s,t) z_s(s,t) = g$, where $g = \gamma(s,t) - z_t(s,t) - \chi(s,t) z(s,t) - \kappa(s,t) z(s,t-\tau)$ and multiply both sides of the above equation by the integrating factor $\exp(-\varepsilon^{-1} B(s))$, $B(s) = \int_s^1 \beta(x,t) dx$ result in

$$\begin{aligned} \varepsilon^{-1} \exp(-\varepsilon^{-1} B(x)) g(s,t) &= -\exp(-\varepsilon^{-1} B(s)) z_{ss}(s,t) + \varepsilon^{-1} \beta(s,t) \exp(-\varepsilon^{-1} B(s)) z_s, \\ -\frac{d}{ds} \left(\exp(-\varepsilon^{-1} B(s)) \frac{dz}{ds} \right) &= \varepsilon^{-1} \exp(-\varepsilon^{-1} B(s)) g(s,t). \end{aligned}$$

Integrating the above expression from s to 1 gives

$$\exp(-\varepsilon^{-1} B(1)) z_s(1,t) - \exp(-\varepsilon^{-1} B(s)) z_s(s,t) = -\varepsilon^{-1} \int_s^1 g(x,t) \exp(-\varepsilon^{-1} B(x)) dx.$$

This is further simplified to

$$z_s(s,t) = z_s(1,t) \exp(-\varepsilon^{-1} (B(1) - B(s))) + k_1(s,t), \quad (5.3)$$

where

$$k_1(s,t) = -\varepsilon^{-1} \int_s^1 g(x,t) \exp(-\varepsilon^{-1} (B(x) - B(s))) dx. \quad (5.4)$$

Integrating the above equation from s to 1 to derive the bound of $z_s(1, t)$ results in

$$z(s, t) = z(0, t) + z_s(1, t) \int_s^1 \exp(-\varepsilon^{-1}(B(1)) - B(x))dx + \int_s^1 k_1(x, t)dx.$$

At $s = 1$, it becomes

$$z_s(1, t) = -\frac{\int_0^1 k_1(x, t)dx}{\int_0^1 (\exp(-\varepsilon^{-1}(B(1)) - B(x)))dx}$$

from the assumptions $z(0, t) = z(1, t) = 0$. Then substituting in (5.3), we get

$$z_s(s, t) = -\frac{\exp(-\varepsilon^{-1}(B(1) - B(s)))}{\int_0^1 (\exp(-\varepsilon^{-1}(B(1)) - B(x)))dx} \int_0^1 k_1(x, t)dx + k_1(s, t).$$

Using inequality

$$\exp(-\varepsilon^{-1}(B(1) - B(s))) \leq \exp(-\varepsilon^{-1}\mu(1 - s)), \quad s \leq 1,$$

and the bound yields

$$\begin{aligned} |z_s(s, t)| &\leq \frac{|\exp(-\varepsilon^{-1}\mu(1 - s))|}{|\int_0^1 (\exp(-\varepsilon^{-1}\mu(1 - x)))dx|} \left| \int_0^1 k_1(x, t)dx \right| + |k_1(s, t)| \\ &\leq \hat{C} \left(1 + \frac{\exp(-\varepsilon^{-1}\mu(1 - s))}{\int_0^1 (\exp(-\varepsilon^{-1}\bar{\mu}(1 - x)))dx} \right), \end{aligned}$$

where $\bar{\mu}$ is an upper bound of $B(s)$ over $[1, 0]$ and $\hat{C}$ is also an upper bound of $C_1(s, t)$ over $\bar{\mathcal{D}}$. Simplifying $\int_0^1 (\exp(-\varepsilon^{-1}\bar{\mu}(1 - x)))dx$ gives $\frac{1}{\bar{\mu}\varepsilon^{-1}} - \frac{1}{\bar{\mu}\varepsilon^{-1}} \exp(-\bar{\mu}\varepsilon^{-1})$. From this result, we have

$$|z_s(s, t)| \leq \hat{C} \left(1 + \frac{\varepsilon^{-1}\bar{\mu} \exp(-\varepsilon^{-1}\mu(1 - s))}{(1 - \exp(-\varepsilon^{-1}\bar{\mu}))} \right).$$

Let $\gamma(\bar{\mu}, t) = \exp(-\varepsilon^{-1}\bar{\mu})$ be a real-valued function defined over $\bar{\mathcal{D}}$ and differentiable on $\mathcal{D}$. From Maclaurin series we have $\bar{\mu} \leq (1 - \exp(-\varepsilon^{-1}\mu))$, then we obtained

$$|z_s(s, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu(1 - s)/\varepsilon)), \quad (s, t) \in \bar{\mathcal{D}}$$

which is the desired bound. ■

Lemma 5.2.2. $|z_{tt}(s, t)| \leq C, (s, t) \in \mathcal{D}$.

Proof. On the sides $s = 0$ and $s = 1$ of $\bar{\mathcal{D}}$, we have $z = 0$ and therefore $z_{tt} = 0$. For

$t \in [-\tau, 0]$, we have $z = 0$, $z_s = 0$, and $z_{ss} = 0$, therefore, from (4.6), we have

$$z_t(s, t) = \begin{cases} 0, & \text{for } t \in [-\tau, 0), \\ \gamma(s, 0), & \text{for } t = 0. \end{cases}$$

Also, we have

$$z_{ts}(s, t) = \begin{cases} 0, & \text{for } t \in [-\tau, 0), \\ \gamma_s(s, 0), & \text{for } t = 0, \end{cases}$$

and

$$z_{tss}(s, t) = \begin{cases} 0, & \text{for } t \in [-\tau, 0), \\ \gamma_{ss}(s, 0), & \text{for } t = 0, \end{cases}$$

$|z_{ts}(s, t)| \leq C$ and $|z_{tss}(s, t)| \leq C$ on ψ_b . Now, differentiating (5.1) with respect to t , we get

$$z_{tt}(s, t) - \varepsilon z_{tss}(s, t) + \beta_t(s, t)z(s, t) + \beta(s, t)z_t(s, t) = \gamma_t(s, t) - \kappa_t(s, t)z(s, t - \tau) - \kappa(s, t)z_t(s, t - \tau).$$

For $t \in [-\tau, 0]$, each term of the above equation, except the term z_{tt} , is now known to be bounded by C , which implies that $|z_{tt}| \leq C$, for $(s, t) \in \bar{\mathcal{D}}$.

Consider the same operator defined in (5.1) and operating it on z_{tt} , we get

$$\begin{aligned} \mathfrak{L}z_{tt}(s, t) &= z_{ttt}(s, t) - \varepsilon z_{tss}(s, t) + \beta(s, t)z_{tts}(s, t) + \chi(s, t)z_{tt} \\ &= (z_t(s, t) - \varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) + \chi(s, t))_{tt} - \chi_{tt}(s, t)z(s, t) - 2\chi_t(s, t)z_t(s, t). \end{aligned} \quad (5.5)$$

For $t \in [0, \tau]$, using the given initial condition, we get

$$\mathfrak{L}z_{tt}(s, t) = \gamma_{ss}(s, t) - \chi_{tt}(s, t)z(s, t) - 2\chi_t(s, t)z_t(s, t). \quad (5.6)$$

Since $\gamma_{ss}(s, t)$ is bounded function, which implies that $|\mathfrak{L}z_{tt}(s, t)| \leq C$. By the maximum principle, we have $|z_{tt}| \leq C$, $(s, t) \in [0, 1] \times [0, \tau]$.

For $t \in [\tau, 2\tau]$, from (5.5) we get

$$\mathfrak{L}z_{tt}(s, t) = \gamma_{ss}(s, t) - \chi_{tt}(s, t)z(s, t) - 2\chi_t(s, t)z_t(s, t) - (\kappa(s, t)z(s, t - \tau))_{tt}. \quad (5.7)$$

As $(s, t) \in [0, 1] \times [\tau, 2\tau]$, all the terms in the right-hand side of (5.7) are bounded, which implies that $|\mathfrak{L}z_{tt}(s, t)| \leq C$. By the maximum principle, we have $|z_{tt}| \leq C$, $(s, t) \in [0, 1] \times [0, 2\tau]$. Analogously, we obtained the required bound on the entire domain $\bar{\mathcal{D}}$. ■

Lemma 5.2.3. *The derivative of $z_s(s, t)$ with respect to t satisfies*

$$|z_{st}(s, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu(1 - s)/\varepsilon)), \quad (s, t) \in \bar{\mathcal{D}}.$$

Proof. Differentiating (5.1) with respect to t , we get

$$\begin{aligned} & -\varepsilon(z_{st})_s + \beta(s, t)z_{st} + \chi_t(s, t)z + \chi(s, t)z_t + z_{tt} \\ & = -\kappa_t(s, t)z(s, t - \tau) - \kappa(s, t)z_t(s, t - \tau) + \gamma_s(s, t). \end{aligned}$$

Rewriting the above equation in the form $-\varepsilon(z_{ss}(s, t))_t + \beta(s, t)z_{st}(s, t) = g_2$, where $g_2 = \gamma_s(s, t) - z_{tt}(s, t) - \chi(s, t)z(s, t) - \kappa(s, t)z(s, t - \tau)$ and multiply both sides of the above equation by the integrating factor $\exp(-\varepsilon^{-1}B(s))$, $B(s) = \int_0^s \beta(x, t)dx$ result in

$$\begin{aligned} \varepsilon^{-1} \exp(-\varepsilon^{-1}B(s))g_2(s, t) &= -\exp(-\varepsilon^{-1}B(s))z_{ss}(s, t) + \varepsilon^{-1}\beta(s) \exp(\varepsilon^{-1}B(s))z_s, \\ & - \frac{d}{ds} \left(\exp(\varepsilon^{-1}B(s)) \frac{dz_t}{ds} \right) = \varepsilon^{-1} \exp(-\varepsilon^{-1}B(s))g_2(s, t). \end{aligned}$$

Integrating the above expression from s to 1 gives

$$\exp(-\varepsilon^{-1}B(1))z_{ts}(1, t) - \exp(-\varepsilon^{-1}B(s))z_{st}(s, t) = -\varepsilon^{-1} \int_s^1 g_2(x, t) \exp(-\varepsilon^{-1}B(x))dx.$$

This is further simplified to

$$z_{st}(s, t) = z_{ts}(1, t) \exp(\varepsilon^{-1}(B(1) - B(s))) + k_2(s, t), \quad (5.8)$$

where

$$k_2(s, t) = -\varepsilon^{-1} \int_s^1 g_2(x, t) \exp(\varepsilon^{-1}(B(x) - B(s)))dx. \quad (5.9)$$

Integrating the above equation from s to 1 to derive the bound of $z_{ts}(1, t)$ results in

$$z_t(s, t) = z_t(0, t) + z_{ts}(1, t) \int_s^1 \exp(-\varepsilon^{-1}(B(1) - B(x)))dx + \int_s^1 k_2(x, t)dx.$$

At $s = 1$, it becomes

$$z_{ts}(1, t) = -\frac{\int_0^1 k_2(x, t)dx}{\int_0^1 (\exp(-\varepsilon^{-1}(B(1) - B(x))))dx}$$

from the assumptions $z_t(0, t) = z_t(1, t) = 0$. Then substituting in (5.3), we get

$$z_{st}(s, t) = -\frac{\exp(\varepsilon^{-1}(B(1) - B(s)))}{\int_0^1 (\exp(\varepsilon^{-1}(B(1) - B(x))))dx} \int_0^1 k_2(x, t)dx + k_2(s, t).$$

Using inequality

$$\exp(-\varepsilon^{-1}(B(1) - B(s))) \leq \exp(-\varepsilon^{-1}\mu(1 - s)), s \leq 1,$$

and the bound yields

$$\begin{aligned} |z_{st}(s, t)| &\leq \frac{|\exp(-\varepsilon^{-1}\mu(1-s))|}{\left|\int_0^1 (\exp(-\varepsilon^{-1}\mu(1-x)))dx\right|} \left|\int_0^1 k_2(x, t)dx\right| + |k_2(s, t)| \\ &\leq \hat{C} \left(1 + \frac{\exp(-\varepsilon^{-1}\mu(1-s))}{\int_0^1 (\exp(-\varepsilon^{-1}\bar{\mu}(1-x)))dx}\right), \end{aligned}$$

where $\bar{\mu}$ is an upper bound of $B(s)$ over $[1, 0]$ and $\hat{C}$ is also an upper bound of $k_2(s, t)$ over $\bar{\mathfrak{D}}$. Simplifying $\int_0^1 (\exp(-\varepsilon^{-1}\bar{\mu}(1-x)))dx$ gives $\frac{1}{\bar{\mu}\varepsilon^{-1}} - \frac{1}{\varepsilon^{-1}\bar{\mu}(1-x)} \exp(-\bar{\mu}(1-s)\varepsilon^{-1})$. From this result, we have

$$|z_{st}(s, t)| \leq \hat{C} \left(1 + \frac{\varepsilon^{-1}\bar{\mu} \exp(-\varepsilon^{-1}\mu(1-s))}{(1 - \exp(-\varepsilon^{-1}\bar{\mu}(1-s)))}\right).$$

Let $\gamma(\bar{\mu}, t) = \exp(-\varepsilon^{-1}\mu(1-s))$ be a real-valued function defined over $\bar{\mathfrak{D}}$ and differentiable on $\mathfrak{D}$. Then we have

$$|z_{st}(s, t)| \leq C(1 + \varepsilon^{-1} \exp(-\mu(1-s)/\varepsilon)), (s, t) \in \bar{\mathfrak{D}}$$

which is the desired bound. The proof of this lemma can be also found in (Kellogg and Tsan, 1978; Das and Natesan, 2015). ■

Lemma 5.2.4. *The derivatives of the solution $z(s, t)$ of (5.1) with respect to s and t satisfy*

$$\begin{aligned} \left|\frac{\partial^i z(s, t)}{\partial s^i}\right| &\leq C \left(1 + \varepsilon^{-i} \exp\left(-\frac{\mu(1-s)}{\varepsilon}\right)\right), (s, t) \in \bar{\mathfrak{D}}, i = 0(1)4, \\ \left|\frac{\partial^l z(s, t)}{\partial t^l}\right| &\leq C, (s, t) \in \bar{\mathfrak{D}}, l = 0(1)2. \end{aligned} \quad (5.10)$$

Proof. The proof follows from Lemma 4.2.3- Lemma 4.2.5 and Lemma 5.2.3, which is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. ■

5.3 Numerical scheme

5.3.1 Temporal semi-discretization

The time domain $[0, T]$ is discretized uniformly with step size Δt as $\Lambda^M = \{t_n = n\Delta t, n = 0, 1, 2, \dots, M, t_M = T, \Delta t = T/M\}$ and $\Lambda^j = \{t_n = n\Delta t, n = 0, 1, 2, \dots, j, t_j = \tau, \Delta t = \tau/j\}$ with $M + 1$ mesh points in $[0, T]$ and $n + 1$ mesh points in $[-\tau, 0]$. We have $T = r\tau$ for some positive integer r .

Applying the backward Euler scheme for time derivative, we get

$$\begin{aligned} & \frac{Z^n(s) - Z^{n-1}(s)}{\Delta t} - \varepsilon \frac{d^2 Z^n(s)}{ds^2} + \beta(s, t_n) \frac{dZ^n(s)}{ds} + \chi(s, t_n) Z^n(s) \\ &= -\kappa(s, t_n) Z(s, t_{n-\tau}) + \gamma(s, t_n). \end{aligned} \quad (5.11)$$

Simplifying (5.11), we get

$$\mathfrak{L}^{\Delta t} Z^n(s) = \begin{cases} -\kappa(s, t_n) \psi_b(s) + \gamma(s, t_n) + \frac{Z^{n-1}(s)}{\Delta t}, \\ \text{for } n = 0, 1, 2, \dots, j, \quad s \in \Omega, \\ -\kappa(s, t_n) Z^{n-j}(s) + \gamma(s, t_n) + \frac{Z^{n-1}(s)}{\Delta t}, \\ \text{for } n = j+1, \dots, M-1, \quad s \in \Omega, \end{cases} \quad (5.12)$$

where $\mathfrak{L}^{\Delta t} Z^n(s) = -\varepsilon \frac{d^2 Z^n(s)}{ds^2} + \beta(s, t_n) \frac{dZ^n(s)}{ds} + P(s, t_n) Z^n(s)$ and $P(s, t_n) = (\frac{1}{\Delta t} + \chi(s, t_n))$ with the boundaries

$$Z^n(0) = \psi_l(t_n), \quad Z^n(1) = \psi_r(t_n), \quad n = 0(1)M. \quad (5.13)$$

Now, rewrite (5.12) as

$$\mathfrak{L}^{\Delta t} Z(s) = \begin{cases} -\kappa(s, t_n) \psi_b(s) + \gamma(s, t_n) + \frac{Z^{n-1}(s)}{\Delta t}, \\ \text{for } n = 0, 1, 2, \dots, j, \quad s \in \Omega, \\ -\kappa(s, t_n) Z^{n-j}(s) + \gamma(s, t_n) + \frac{Z^{n-1}(s)}{\Delta t}, \\ \text{for } n = j+1, \dots, M-1, \quad s \in \Omega, \end{cases} \quad (5.14)$$

where $\mathfrak{L}^{\Delta t} Z(s) = -\varepsilon \frac{d^2 Z(s)}{ds^2} + \beta(s, t_n) \frac{dZ(s)}{ds} + P(s, t_n) Z(s)$, $Z(s) = Z^n(s) \approx z(s, t_n)$ and $Q(s) = Q^n(s) = Q(s, t_n)$.

The local truncation error in the time direction is given as $e_n(s) := z(s, t_n) - Z^n(s)$, $n = 0(1)M$.

Lemma 5.3.5. *The local error e_n at t_n satisfies the bound*

$$\|e_n\| \leq C(\Delta t)^2.$$

Proof. From (5.12), we have

$$\begin{aligned} (1 + \Delta t \mathfrak{L}^{\Delta t}) Z^n(s) &= \Delta t Q(s, t_n) \\ Z(t_{n-1}) &= (1 + \Delta t \mathfrak{L}^{\Delta t}) Z(t_n) - \Delta t Q(t_n) + \int_{t_{n-1}}^{t_n} (t_{n-1} - s) \frac{\partial^2 Z(s)}{\partial t^2} ds \\ &= (1 + \Delta t \mathfrak{L}^{\Delta t}) Z^n(s) - \Delta t Q(t_n) + O(\Delta t^2). \end{aligned}$$

Then $e_n(s)$ is the solution of the boundary value problem of type

$$(1 + \Delta t \mathfrak{L}^{\Delta t})e(t_n) = O(\Delta t^2),$$

$$e_n(0) = 0 = e_n(1)$$

by maximum principle on the above operator $(1 + \Delta t \mathfrak{L}^{\Delta t})e(t_n)$ gives

$$\|e_n\| \leq C_1(\Delta t)^2.$$

■

Lemma 5.3.6. *The global error E_n at t_n satisfies the bound*

$$\|E_n\| \leq C(\Delta t), \quad n = 1(1)M - 1.$$

Proof. Using Lemma 5.3.5, the global error E_n bound at n th time step is given as

$$\begin{aligned} \|E_n\| &= \left\| \sum_{l=1}^n e_l \right\| \leq \|e_1\| + \|e_2\| + \|e_3\| + \dots + \|e_n\| \\ &\leq C_1 T(\Delta t), \quad \text{since } (n)\Delta t \leq T \\ &= C(\Delta t), \quad \text{where } C_1 T = C. \end{aligned}$$

■

Lemma 5.3.7. *For every $n = 0(1)M - 1$, the solution $Z^n(s)$ of (5.12)- (5.13) satisfies the bound*

$$\left| \frac{d^i Z^n(s)}{ds^i} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(\frac{-\mu}{\varepsilon} (1-s) \right) \right), \quad s \in \bar{\Omega}, \quad i = 0(1)4. \quad (5.15)$$

Proof. The proof follows from Lemma 5.2.1, it is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. ■

5.3.2 Spatial discretization

We discretize the spatial domain $[0, 1]$ into N equal number of sub-intervals with the length of h as $0 = s_0, s_1, \dots, s_N = 1$ and $s_m = mh, m = 0(1)N$. Consider a smooth function $Z(s)$ in the interval $[0, 1]$. Following similar procedures given in (4.18) -(4.19) of Chapter 4, we obtain

$$Z_{m-1} - 2Z_m + Z_{m+1} = \frac{h^2}{30} \left(Z''_{m-1} + 28Z''_m + Z''_{m+1} \right) + \zeta, \quad (5.16)$$

where $\zeta = \frac{h^4}{20}Z_m^{(4)} - \frac{13h^8}{302,400}Z_m^{(8)} + O(h^{10})$.

From (5.14), we draw

$$-\varepsilon \frac{d^2 Z(s)}{ds^2} = -\beta(s, t_n) \frac{dZ(s)}{ds} - P(s, t_n)Z(s) + Q(s), \quad (5.17)$$

where

$$Q(s) = \begin{cases} -\kappa(s, t_n)\psi_b(s) + \gamma(s, t_n) + \frac{Z^{n-1}(s)}{\Delta t}, \\ \text{for } n = 0, 1, 2, \dots, j, \quad s \in \Omega, \\ -\kappa(s, t_n)Z^{n-j}(s) + \gamma(s, t_n) + \frac{Z^{n-1}(s)}{\Delta t}, \\ \text{for } n = j+1, \dots, M-1, \quad s \in \Omega. \end{cases}$$

Using (5.17), we have

$$\begin{aligned} -\varepsilon Z''_{m\pm 1} &= -\beta(s_{m\pm 1}, t_n)Z'_{m\pm 1} - P(s_{m\pm 1}, t_n)Z_{m\pm 1} + Q_{m\pm 1}, \\ -\varepsilon Z''_m &= -\beta(s_m, t_n)Z'_m - P(s_m, t_n)Z_m + Q_m. \end{aligned} \quad (5.18)$$

Substituting the approximation given in (4.22) into (5.18), we have

$$\begin{aligned} -\varepsilon Z''_{m+1} &= -\beta(s_{m+1}, t_n) \left(\frac{3Z_{m+1} - 4Z_m + Z_{m-1}}{2h} - hZ''_m \right) - P(s_{m+1}, t_n)Z_{m+1} + Q_{m+1}, \\ -\varepsilon Z''_m &= -\beta(s_m, t_n) \frac{Z_{m+1} - Z_{m-1}}{2h} - P(s_m, t_n)Z_m + Q_m, \\ -\varepsilon Z''_{m-1} &= -\beta(s_{m-1}, t_n) \left(\frac{-Z_{m+1} + 4Z_m - 3Z_{m-1}}{2h} + hZ''_m \right) - P(s_{m-1}, t_n)Z_{m-1} + Q_{m-1}. \end{aligned} \quad (5.19)$$

From (5.16), we draw

$$-\varepsilon \left(\frac{Z_{m-1} - 2Z_m + Z_{m+1}}{h^2} \right) = \frac{1}{30} \left(-\varepsilon Z''_{m-1} - 28\varepsilon Z''_m - \varepsilon Z''_{m+1} \right) + \zeta. \quad (5.20)$$

Substituting (5.19) into (5.20), and rearranging, we obtain

$$\begin{aligned} & - \left(\varepsilon - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) \left(\frac{Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n}{h^2} \right) \\ & + \frac{\beta(s_{m-1}, t_n)}{60h} \left(-3Z_{m-1}^n + 4Z_m^n - Z_{m+1}^n \right) + \frac{28\beta(s_m, t_n)}{60h} \left(Z_{m+1}^n - Z_{m-1}^n \right) \\ & + \frac{\beta(s_{m+1}, t_n)}{60h} \left(Z_{m-1}^n - 4Z_m^n + 3Z_{m+1}^n \right) + \frac{P(s_{m-1}, t_n)}{30} Z_{m-1}^n + \frac{28P(s_m, t_n)}{30} Z_m^n \\ & + \frac{P(s_{m+1}, t_n)}{30} Z_{m+1}^n = \frac{1}{30} \left(Q_{m-1}^n + 28Q_m^n + Q_{m+1}^n \right), \end{aligned} \quad (5.21)$$

where

$$Q_m^n = \begin{cases} -\kappa(s_m, t_n)\psi_b(s_m) + \gamma(s_m, t_n) + \frac{Z_m^{n-1}}{\Delta t}, \\ \text{for } n = 0, 1, 2, \dots, j, \quad s \in \Omega, \\ -\kappa(s_n, t_m)Z_n^{m-j} + \gamma(s_m, t_n) + \frac{Z_m^{n-1}}{\Delta t}, \\ \text{for } n = j+1, \dots, M-1, \quad s \in \Omega. \end{cases} \quad (5.22)$$

Computing the exponential fitting factor

We introduce the exponential fitting factor σ to handle the effect of ε in the layer. From the singular perturbation theory stated in O'Malley (1991), the zero-order asymptotic solution of the problem of the form

$$\begin{cases} -\varepsilon Z''(s) + \beta(s)Z'(s) + \chi(s)Z(s) = \gamma(s), & s \in \Omega = (0, 1), \\ Z(0) = \psi_l, \quad Z(1) = \psi_r \end{cases} \quad (5.23)$$

is given as

$$Z(s) \approx Z_0(s) + \frac{\beta(0)}{\beta(s)} \left(\psi_r - Z_0(1) \right) \exp \left(- \int_s^1 \left(\frac{\beta(s)}{\varepsilon} - \frac{\chi(s)}{\beta(s)} \right) ds \right) + O(\varepsilon). \quad (5.24)$$

From Taylor's series approximation for $\beta(s)$ and $\chi(s)$ restricting to their first terms about $s = 1$ is given as

$$Z(s) \approx Z_0(s) + (\psi_r - Z_0(1)) \exp \left(\frac{-\beta(1)(1-s)}{\varepsilon} \right), \quad (5.25)$$

where $Z_0(s)$ is the solution of reduced problem. Taking $h \rightarrow 0$ and solving (5.25) at $s_{m\pm 1}$ and s_m , we get

$$\begin{aligned} Z(s_m) &\approx Z_0(0) + (\psi_r - Z_0(1)) \exp(-\beta(1)(\frac{1}{\varepsilon} - m\rho)), \\ Z(s_{m-1}) &\approx Z_0(0) + (\psi_r - Z_0(1)) \exp(-\beta(1)(\frac{1}{\varepsilon} - (m-1)\rho)), \\ Z(s_{m+1}) &\approx Z_0(0) + (\psi_r - Z_0(1)) \exp(-\beta(1)(\frac{1}{\varepsilon} - (m+1)\rho)), \end{aligned} \quad (5.26)$$

where $\rho = \frac{h}{\varepsilon}$. Multiplying (5.21) by h , and the term containing ε by σ , and evaluating the limit of the resulting equation as $h \rightarrow 0$, we get

$$\begin{aligned} & -\frac{\sigma}{\rho} \lim_{h \rightarrow 0} (Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n) + \frac{\beta(s_{m-1}, t_n)}{60} \lim_{h \rightarrow 0} (-3Z_{m-1}^n + 4Z_m^n - Z_{m+1}^n) \\ & + \frac{28\beta(s_m, t_n)}{60} \lim_{h \rightarrow 0} (Z_{m+1}^n - Z_{m-1}^n) + \frac{\beta(s_{m+1}, t_n)}{60} \lim_{h \rightarrow 0} (Z_{m-1}^n - 4Z_m^n + 3Z_{m+1}^n) = 0. \end{aligned} \quad (5.27)$$

From (5.26), we have

$$\left\{ \begin{array}{l} \lim_{h \rightarrow 0} (Z((m-1)h) - 2Z(mh) + Z((m+1)h)) \\ = (\psi_r - Z_0(1))e^{-\beta(1)(\frac{1-s_m}{\varepsilon})}(e^{-\beta(1)\rho} + e^{\beta(1)\rho} - 2), \\ \lim_{h \rightarrow 0} (-3Z((m-1)h) + 4Z(mh) - Z((m+1)h)) \\ = (\psi_r - Z_0(1))e^{-\beta(1)(\frac{1-s_m}{\varepsilon})}(-3e^{-\beta(1)\rho} - e^{\beta(1)\rho} + 4), \\ \lim_{h \rightarrow 0} (Z((m-1)h) - 4Z(mh) + 3Z((m+1)h)) \\ = (\psi_r - Z_0(1))e^{-\beta(1)(\frac{1-s_m}{\varepsilon})}(e^{-\beta(1)\rho} + 3e^{\beta(1)\rho} - 4), \\ \lim_{h \rightarrow 0} (Z((m+1)h) - Z((m-1)h)) \\ = (\psi_r - Z_0(1))e^{-\beta(1)(\frac{1-s_m}{\varepsilon})}(e^{\beta(1)\rho} - e^{-\beta(1)\rho}). \end{array} \right. \quad (5.28)$$

Substituting (5.28) into (5.27), and simplifying, yields

$$\frac{\sigma}{\rho} \left(e^{\beta(1)\rho} + e^{-\beta(1)\rho} - 2 \right) = \frac{\beta(s_m, t_n)}{60} \left(30e^{\beta(1)\rho} - 30e^{-\beta(1)\rho} \right) \quad (5.29)$$

Solving (5.29) for σ , we get

$$\sigma = \frac{\rho\beta(s_m, t_n)}{2} \coth\left(\frac{\rho\beta(1)}{2}\right). \quad (5.30)$$

5.3.3 The discrete scheme

Using the higher order finite difference scheme of (5.21) and fitting factor (5.30), we obtain the fully discrete scheme

$$\mathfrak{L}^{\Delta t, h} Z_m^n = \frac{1}{30} \left(Q_{m-1}^n + 28Q_m^n + Q_{m+1}^n \right), \quad (5.31)$$

where $\mathfrak{L}^{\Delta t, h} Z_m^n = - \left(\varepsilon\sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) \left(\frac{Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n}{h^2} \right) +$
 $\frac{\beta(s_{m-1}, t_n)}{60h} \left(-3Z_{m-1}^n + 4Z_m^n - Z_{m+1}^n \right) + \frac{28\beta(s_m, t_n)}{60h} \left(Z_{m+1}^n - Z_{m-1}^n \right) + \frac{\beta(s_{m+1}, t_n)}{60h} \left(Z_{m-1}^n - 4Z_m^n + \right.$
 $\left. 3Z_{m+1}^n \right) + \frac{P(s_{m-1}, t_n)}{30} Z_{m-1}^n + \frac{28P(s_m, t_n)}{30} Z_m^n + \frac{P(s_{m+1}, t_n)}{30} Z_{m+1}^n.$

In the explicit form, it becomes

$$R_m^- Z_{m-1}^n + R_m^0 Z_m^n + R_m^n Z_{m+1}^n = H_m^n, \quad (5.32)$$

where

$$\begin{aligned}
R_m^- &= -\frac{1}{h^2} \left(\varepsilon \sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) - \frac{3\beta(s_{m-1}, t_n)}{60h} + \frac{P(s_{m-1}, t_n)}{30} \\
&\quad - \frac{28\beta(s_m, t_n)}{60h} + \frac{\beta(s_{m+1}, t_n)}{60h}, \\
R_m^0 &= \frac{2}{h^2} \left(\varepsilon \sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) + \frac{4\beta(s_{m-1}, t_n)}{60h} - \frac{4\beta(s_{m+1}, t_n)}{60h} \\
&\quad + \frac{28P(s_m, t_n)}{30}, \\
R_m^+ &= -\frac{1}{h^2} \left(\varepsilon \sigma - \frac{h\beta(s_{m-1}, t_n)}{30} + \frac{h\beta(s_{m+1}, t_n)}{30} \right) - \frac{\beta(s_{m-1}, t_n)}{60h} + \frac{28\beta(s_m, t_n)}{60h} \\
&\quad + \frac{3\beta(s_{m+1}, t_n)}{60h} + \frac{P(s_{m+1}, t_n)}{30}, \\
H_m^n &= \frac{1}{30} \left(Q_{m-1}^n + 28Q_m^n + Q_{m+1}^n \right).
\end{aligned} \tag{5.33}$$

5.4 Convergence analysis

The resulting discrete scheme given in (5.31) satisfies the discrete maximum principle stated in Lemma 4.4.13 and the uniform stability result of its solution Z_m^n satisfies the bound given in Lemma 4.4.14.

Theorem 5.4.1. *Let the coefficients $\chi(s, t_n)$, $\beta(s, t_n)$ and $\kappa(s, t_n)$ of (5.12)-(5.13) be sufficiently smooth such that $Z^m(s) \in C^4[0, 1]$. Then, the solution Z_m^n of (5.31) satisfies the next bound*

$$|\mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n)| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3} \exp \left(\frac{-\mu(1 - s_m)}{\varepsilon} \right) \right). \tag{5.34}$$

Proof. The error estimate in the spatial direction is given as

$$\begin{aligned}
\left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| &= \left| -\varepsilon \left(\frac{d}{ds^2} - \sigma \delta_s^2 \right) Z^n(s_m) + \frac{\beta(s_{m-1}, t_n)}{30} \right. \\
&\quad \left(\frac{dZ^n(s_{m-1})}{ds} - \left(\frac{-Z_{m+1}^n + 4Z_m^n - 3Z_{m-1}^n}{2h} + h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \\
&\quad + \frac{28\beta(s_m, t_n)}{30} \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_m) + \frac{\beta(s_{m+1}, t_n)}{30} \\
&\quad \left(\frac{dZ^n(s_{m+1})}{ds} - \left(\frac{3Z_{m+1}^n - 4Z_m^n + Z_{m-1}^n}{2h} - h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \Bigg| \\
&\leq \left| \varepsilon(\beta(s_n, t_n) \frac{\rho}{2} \coth(\beta(1) \frac{\rho}{2}) - 1) \delta_s^2 Z^n(s_n) \right| \\
&\quad + \left| \varepsilon \left(\frac{d^2}{ds^2} - \delta_s^2 \right) Z^n(s_m) \right| + \left| \frac{\beta(s_{m-1}, t_n)}{30} \right. \\
&\quad \left(\frac{dZ^n(s_{m-1})}{ds} - \left(\frac{-Z_{m+1}^n + 4Z_m^n - 3Z_{m-1}^n}{2h} + h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \Bigg| \\
&\quad + \left| \frac{28\beta(s_m, t_n)}{30} \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_m) \right| + \left| \frac{\beta(s_{m+1}, t_n)}{30} \right. \\
&\quad \left(\frac{dZ^n(s_{m+1})}{ds} - \left(\frac{3Z_{m+1}^n - 4Z_m^n + Z_{m-1}^n}{2h} - h \frac{d^2 Z^n(s_m)}{ds^2} \right) \right) \Bigg|, \\
\end{aligned} \tag{5.35}$$

where $\sigma = \beta(s_m, t_n) \frac{\rho}{2} \coth(\beta(1) \frac{\rho}{2})$ and $\rho = \frac{h}{\varepsilon}$.

For the constants C_1 and C_2 we have $|\rho \coth(\rho) - 1| \leq C_1 \rho^2$, for $\rho \leq 1$. For $\rho \rightarrow \infty$, since $\lim_{\rho \rightarrow \infty} \coth(\rho) = 1$ which gives $|\rho \coth(\rho) - 1| \leq C_1 \rho$.

Generally, $\forall \rho > 0$, we express as

$$C_1 \frac{\rho^2}{\rho + 1} \leq \rho \coth(\rho) - 1 \leq C_2 \frac{\rho^2}{\rho + 1} \tag{5.36}$$

and we have

$$\varepsilon \left[\beta(s_m, t_n) \frac{\rho}{2} \coth(\beta(1) \frac{\rho}{2}) - 1 \right] \leq \varepsilon \frac{(h/\varepsilon)^2}{h/\varepsilon + 1} = \frac{h^2}{h + \varepsilon}. \tag{5.37}$$

From the bounds in (4.36), (5.35), and (5.37), we have

$$\left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| \leq \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2 Z^n(s_m)}{ds^2} \right\| + Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\| + C\varepsilon h^2 \left\| \frac{d^4 Z^n(s_m)}{ds^4} \right\|.$$

By Lemma 5.3.7, we have

$$\begin{aligned} \left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| &\leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-2} \exp \left(-\frac{\mu(1 - s_m)}{\varepsilon} \right) \right) \\ &\quad + Ch^2 \left[\left(1 + \varepsilon^{-3} \exp \left(-\frac{\mu(1 - s_m)}{\varepsilon} \right) \right) \right. \\ &\quad \left. + \left(\varepsilon + \varepsilon^{-3} \exp \left(-\frac{\mu(1 - s_m)}{\varepsilon} \right) \right) \right]. \end{aligned} \quad (5.38)$$

Obviously, $\varepsilon^{-3} \geq \varepsilon^{-2}$, then we draw

$$\left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3} \exp \left(\frac{-\mu(1 - s_m)}{\varepsilon} \right) \right). \quad (5.39)$$

Thus, it gives the wanted bound. ■

Lemma 5.4.8. *For a fixed mesh and as $\varepsilon \rightarrow 0$, it holds*

$$\lim_{\varepsilon \rightarrow 0} \max_m \frac{\exp(-\mu(1 - s_m)/\varepsilon)}{\varepsilon^i} = 0, \quad i = 1, 2, 3, \dots, \quad (5.40)$$

where $s_m = mh$, $1 \leq m \leq N - 1$.

Proof. On the discrete domain $\{s_m\}_0^N$, for the interior grid points, we get

$$\max_{1 \leq m \leq N-1} \frac{\exp(-\mu(1 - s_m)/\varepsilon)}{\varepsilon^i} \leq \frac{\exp(-\mu(1 - s_{N-1})/\varepsilon)}{\varepsilon^i} = \frac{\exp(-\mu h/\varepsilon)}{\varepsilon^i}, \quad (5.41)$$

since $h = 1 - s_{N-1}$. Repeatedly using L'hospital's rule implies

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\exp(-\mu h/\varepsilon)}{\varepsilon^i} &= \lim_{v=1/\varepsilon \rightarrow \infty} \frac{v^i}{\exp(\mu h v)} \\ &= \lim_{v \rightarrow \infty} \frac{i!}{(\mu h)^i \exp(\mu h v)} = 0. \end{aligned}$$

This completes the proof. ■

Theorem 5.4.2. *Let Z_m^n be the solution of (5.31), then we have the following uniform error bound*

$$\sup_{\varepsilon \in (0, 1]} \|Z^n(s_m) - Z_m^n\| \leq Ch, \quad m = 0(1)N. \quad (5.42)$$

Proof. Substituting Lemma 5.4.8 into (5.34), we arrive at

$$\left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| \leq \frac{Ch^2}{h + \varepsilon}. \quad (5.43)$$

Hence, the result leads

$$\left| Z^n(s_m) - Z_m^n \right| \leq \frac{Ch^2}{h + \varepsilon}. \quad (5.44)$$

Using the sup overall $\varepsilon \in (0, 1]$, we get

$$\sup_{\varepsilon \in (0, 1]} \|Z^n(s_m) - Z_m^n\| \leq Ch, m = 0(1)N. \quad (5.45)$$

■

Theorem 5.4.3. *Let z and Z are the solutions of (5.1) and (5.31) respectively, then we have the following uniform error bound*

$$\sup_{\varepsilon \in (0, 1]} \|z - Z\| \leq C(h + (\Delta t)). \quad (5.46)$$

Proof. The proof is considered by combining Lemma 5.3.5 and Theorem 5.4.2. ■

5.5 Numerical results

In this section, we carry out some numerical experiments to confirm the developed scheme is stable, more accurate and ε -uniformly convergent by considering two test examples.

Example 5.1. *Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} + (2 - s^2) \frac{\partial z}{\partial s} + sz(s, t) + z(s, t - \tau) = 10t^2 \exp(-t)s(1 - s)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-1, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in [0, 2]$.*

Example 5.2. *Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} + (2 - s^2) \frac{\partial z}{\partial s} + (s + 1)(t + 1)z(s, t) + z(s, t - \tau) = 10t^2 \exp(-t)s(1 - s)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-1, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in [0, 2]$.*

Since the exact solution of the examples is not known, we used a variant of the double mesh principle to perform the numerical experiment. The maximum pointwise absolute error is calculated using the formula given in (4.43), while the ε -uniform error is calculated with the formula given in (4.44). The convergence rate of the proposed method is computed by the formula given in (4.45), while the ε -uniform rate of convergence of the developed method is calculated with the formula given in (4.46).

The results for the model Examples 5.1 and 5.2, namely $E_\varepsilon^{N, M}$, $E^{N, M}$, $r_\varepsilon^{N, M}$, and $r^{N, M}$, were obtained for different values of ε and N and presented in Tables 5.1 and 5.2. As we observed that the maximum pointwise absolute error reduced as the step sizes decreased for each value of ε . As ε approaches zero, the maximum absolute pointwise error initially becomes large, and then it becomes constant. This indicates that the proposed scheme is ε -uniformly convergent, regardless of the value of ε . The calculated $E^{N, M}$ and $r^{N, M}$ using the proposed scheme are

Table 5.1: Computed $E_\varepsilon^{N,M}$, $E^{N,M}$, $r_\varepsilon^{N,M}$, and $r^{N,M}$ for Example 5.1.

$\varepsilon \downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
2^0	2.1285e-04	1.4601e-05	2.7352e-05	2.0267e-05	1.1789e-05
	3.8657	-0.9056	0.4325	0.7817	–
2^{-2}	9.8466e-04	8.9095e-05	1.1378e-04	8.8529e-05	5.2297e-05
	3.4662	-0.3528	0.3620	0.7594	–
2^{-4}	3.6424e-03	5.2203e-04	2.1648e-04	1.6404e-04	9.7451e-05
	2.8027	1.2699	0.4002	0.7513	–
2^{-6}	1.2902e-02	3.8264e-03	7.1066e-04	3.0405e-04	1.5008e-04
	1.7535	2.4288	1.2249	1.0186	–
2^{-8}	1.5384e-02	7.4554e-03	3.1045e-03	9.0706e-04	2.8724e-04
	1.0451	1.2639	1.7751	1.6589	–
2^{-10}	1.5389e-02	7.6241e-03	3.7907e-03	1.8490e-03	7.7045e-04
	1.0133	1.0081	1.0357	1.2630	–
2^{-12}	1.5389e-02	7.6241e-03	3.7923e-03	1.8943e-03	9.4730e-04
	1.0133	1.0075	1.0014	0.9998	–
2^{-14}	1.5389e-02	7.6241e-03	3.7923e-03	1.8943e-03	9.4771e-04
	1.0133	1.0075	1.0014	0.9992	–
2^{-16}	1.5389e-02	7.6241e-03	3.7923e-03	1.8943e-03	9.4771e-04
	1.0133	1.0075	1.0014	0.9992	–
2^{-18}	1.5389e-02	7.6241e-03	3.7923e-03	1.8943e-03	9.4771e-04
	1.0133	1.0075	1.0014	0.9992	–
2^{-20}	1.5389e-02	7.6241e-03	3.7923e-03	1.8943e-03	9.4771e-04
	1.0133	1.0075	1.0014	0.9992	–
$E^{N,M}$	1.5389e-02	7.6241e-03	3.7923e-03	1.8943e-03	9.4771e-04
$r^{N,M}$	1.0133	1.0075	1.0014	0.9992	–

Table 5.2: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, $r_{\varepsilon}^{N,M}$, and $r^{N,M}$ for Example 5.2.

$\varepsilon \downarrow$	Number of intervals N = M				
	16	32	64	128	256
2^0	1.3602e-04	1.4927e-05	1.4266e-05	1.0773e-05	6.3185e-06
	3.1878	0.0653	0.4052	0.7698	–
2^{-2}	3.8939e-04	6.6953e-05	6.2477e-05	3.9473e-05	2.1855e-05
	2.5400	0.0998	0.6625	0.8529	–
2^{-4}	9.8460e-04	2.5366e-04	1.3556e-04	6.9903e-05	3.5485e-05
	1.9566	0.9040	0.9555	0.9782	–
2^{-6}	3.5833e-03	1.2258e-03	4.0647e-04	1.4312e-04	5.5885e-05
	1.5476	1.5925	1.5059	1.3567	–
2^{-8}	4.8773e-03	2.9551e-03	1.2250e-03	4.0706e-04	1.2709e-04
	0.7229	1.2704	1.5895	1.6794	–
2^{-10}	4.8804e-03	3.0579e-03	1.6631e-03	8.3185e-04	3.3471e-04
	0.6745	0.8787	0.9995	1.3134	–
2^{-12}	4.8804e-03	3.0579e-03	1.6641e-03	8.6096e-04	4.3655e-04
	0.6745	0.8778	0.9507	0.9798	–
2^{-14}	4.8804e-03	3.0579e-03	1.6641e-03	8.6096e-04	4.3682e-04
	0.6745	0.8778	0.9507	0.9789	–
2^{-16}	4.8804e-03	3.0579e-03	1.6641e-03	8.6096e-04	4.3682e-04
	0.6745	0.8778	0.9507	0.9789	–
2^{-18}	4.8804e-03	3.0579e-03	1.6641e-03	8.6096e-04	4.3682e-04
	0.6745	0.8778	0.9507	0.9789	–
2^{-20}	4.8804e-03	3.0579e-03	1.6641e-03	8.6096e-04	4.3682e-04
	0.6745	0.8778	0.9507	0.9789	–
$E^{N,M}$	4.8804e-03	3.0579e-03	1.6641e-03	8.6096e-04	4.3682e-04
$r^{N,M}$	0.6745	0.8778	0.9507	0.9789	–

Table 5.3: Comparison of $E^{N,M}$ and $r^{N,M}$ for Example 5.1.

Schemes $\downarrow$	Number of intervals N = M				
	16	32	64	128	256
Present method					
$E^{N,M}$	1.5389e-02	7.6241e-03	3.7923e-03	1.8943e-03	9.4771e-04
$r^{N,M}$	1.0133	1.0075	1.0014	0.9992	–
Negero and Duressa (2022a)					
$E^{N,M}$	–	7.2307e-03	3.8523e-03	1.9892e-03	1.0107e-03
$r^{N,M}$	–	0.90842	1.0062	1.0155	0.98837
Kumar and Kumari (2019)					
$E^{N,M}$	3.41e-02	1.84e-02	9.38e-03	4.67e-03	2.31e-03
$r^{N,M}$	0.8901	0.9720	1.0062	1.0155	1.0063

Table 5.4: Comparison of $E^{N,M}$ and $r^{N,M}$ for Example 5.2.

Schemes ↓	Number of intervals $N = M$				
	16	32	64	128	256
Present method					
$E^{N,M}$	4.8804e-03	3.0579e-03	1.6641e-03	8.6096e-04	4.3682e-04
$r^{N,M}$	0.6745	0.8778	0.9507	0.9789	–
Negero and Duressa (2022b)					
$E^{N,M}$	6.40e-03	3.43e-03	1.75e-03	8.85e-04	4.44e-04
$r^{N,M}$	0.89986	0.97085	0.98361	0.99512	–
Kumar and Kumari (2020)					
$E^{N,M}$	1.86e-2	1.00e-2	5.48e-3	2.86e-3	1.46e-3
$r^{N,M}$	0.89	0.87	0.94	0.97	1.11
Kumar and Kumari (2019)					
$E^{N,M}$	3.06e-02	1.72e-02	9.00e-03	4.58e-03	2.30e-03
$r^{N,M}$	0.8311	0.9344	0.9746	0.9937	1.0000
Babu and Bansal (2021)					
$E^{N,M}$	4.9006e-02	2.8622e-02	1.5141e-02	7.7173e-03	3.8858e-03
$r^{N,M}$	0.775836	0.918608	0.972338	0.989898	0.995894

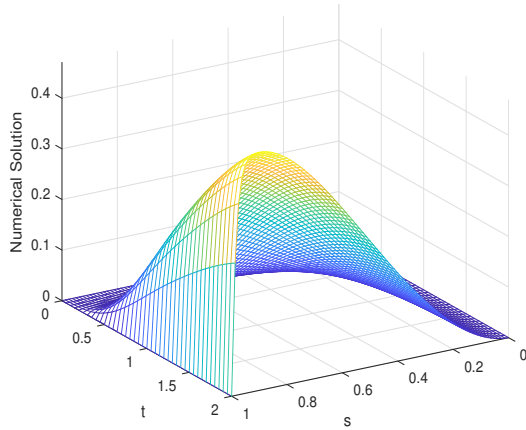
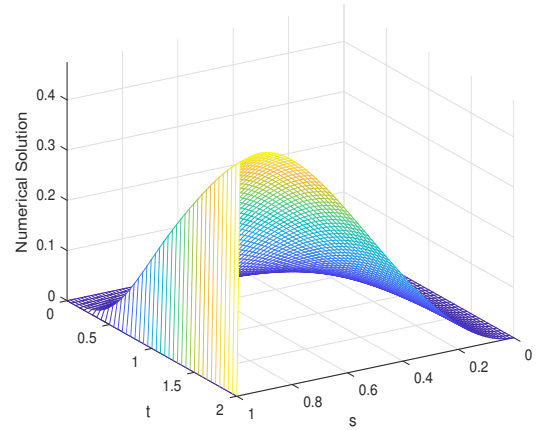
(a) $\varepsilon = 2^{-6}$ (b) $\varepsilon = 2^{-20}$

Figure 5.1: Numerical solution of the scheme for Example 5.1.

given in the last two rows and confirmed that the theoretical finding of the developed scheme is order one in both space and time direction.

Figures 5.1 and 5.2 reveal the numerical solutions of the method for Examples 5.1 and 5.2 for different values of ε , respectively for $N = 80$ and $M = 40$. Figure 5.3 displays the effect of ε on the solutions profile of the developed scheme for Examples 5.1 and 5.2. The figures show that a strong boundary layer is created on the right side of the spatial domain as ε approaches zero. Moreover, Figure 5.4 shows the maximum pointwise absolute errors of the scheme by the log-log scale plot. From these figures, it can be observed that the maximum

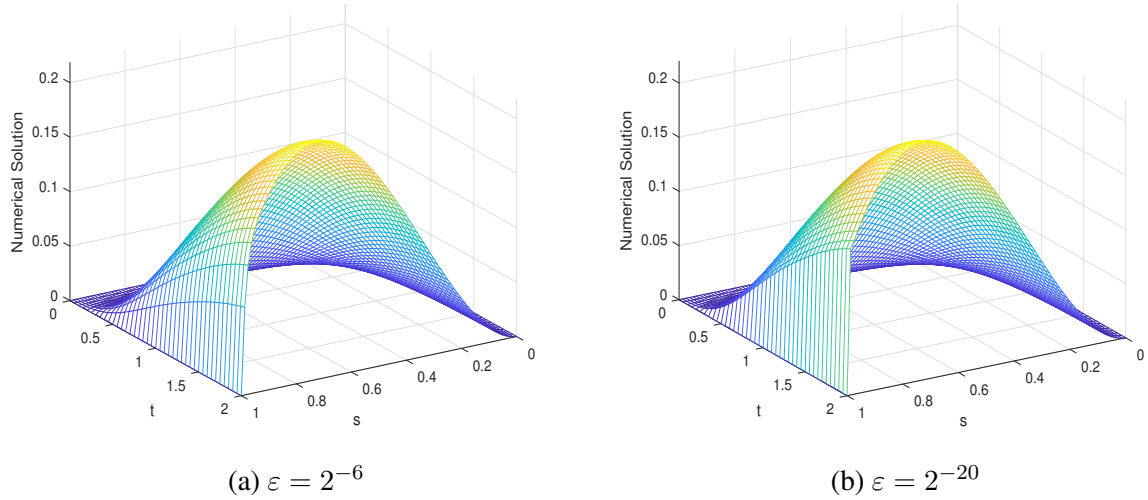
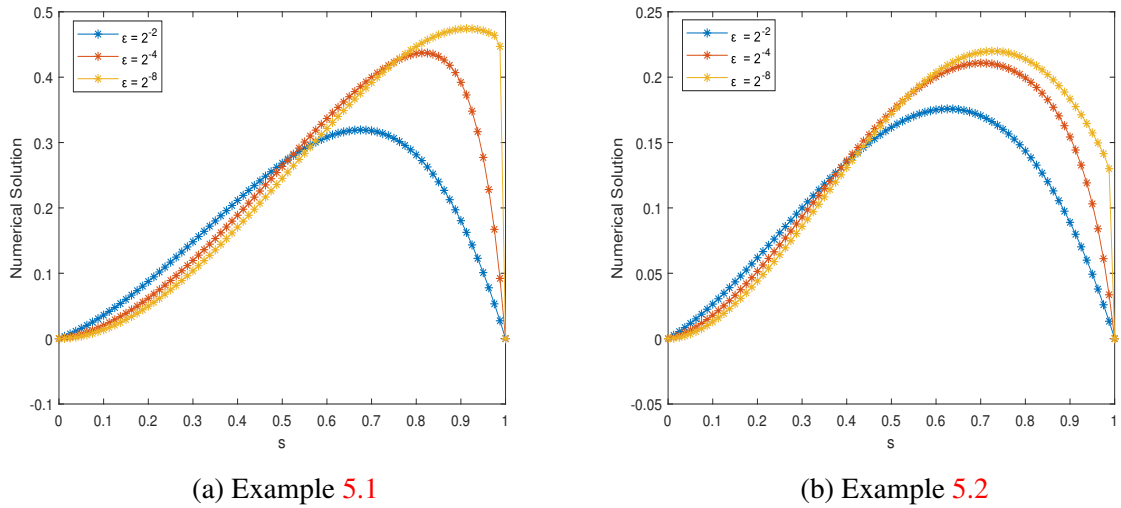


Figure 5.2: Numerical solution of the scheme for Example 5.2.

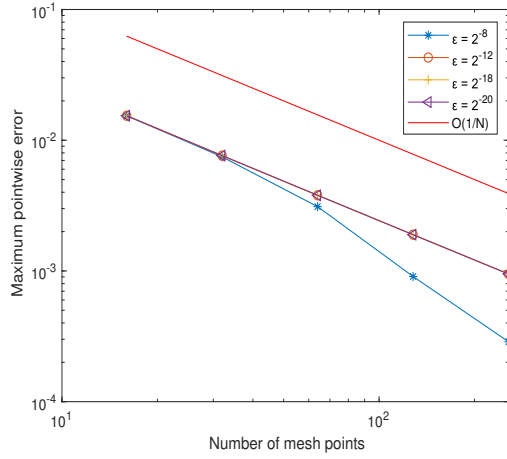
Figure 5.3: Effect of ε on the solution profiles.

pointwise absolute error decreases as the step sizes decrease for every value of ε and confirms the ε -uniform convergence of the proposed scheme.

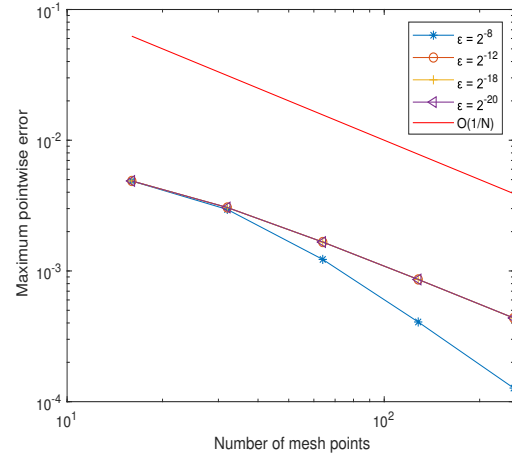
The results of the developed method are compared with the existing recently published works of Kumar and Kumari (2019); Negro and Duressa (2022a) for Example 5.1 in Table 5.3, while Table 5.4 presents the comparison of Example 5.2 with the existing number of recently published works of (Kumar and Kumari, 2020; Babu and Bansal, 2021; Negro and Duressa, 2022b; Kumar and Kumari, 2019). The results show that the proposed scheme outperforms existing methods in the literature in terms of accuracy.

5.6 Summary

We developed a numerical method to solve singularly perturbed parabolic convection-diffusion equations with large time delay. The solution to this problem shows a boundary layer on the



(a) Example 5.1



(b) Example 5.2

Figure 5.4: Maximum pointwise error in Log-Log scale plot.

right side of the domain. This solution has a steep gradient in the layer region due to the presence of ε . To handle this effect, we developed a method that uses the backward Euler scheme in the time direction and an exponentially fitted higher-order finite difference scheme in the spatial direction. Analyzed the stability of the discrete scheme using the discrete maximum principle. The numerical results are presented using maximum pointwise absolute error, ε -uniform error, rate of convergence and ε -uniform rate of convergence in tables. The results shows in good agreement with the theoretical findings. The effect of ε on the steepness of the boundary layer is revealed graphically. Further, the computational results are compared with some of the existing literature. The developed method contributes a better accuracy, stable, and ε -uniform solution with a linear order of convergence both in the spatial and the time directions. In the subsequent chapter, we consider the problem with a small time delay on piecewise uniform Shishkin mesh.

CHAPTER SIX

FITTED MESH METHOD FOR SINGULARLY PERTURBED CONVECTION DIFFUSION EQUATION WITH SMALL TIME DELAY

In this chapter, we extended the work of Chapter 4 using the fitted mesh finite difference method which has a boundary layer near the right side of the domain. The method discussed in the previous chapters did not have a boundary layer resolving behaviour. In this chapter we considered the boundary layer resolving method. As a result, we applied the backward Euler scheme in the temporal direction on a uniform mesh and the space variable is discretized on a piecewise uniform Shishkin mesh. The stability of the scheme is analyzed using the discrete maximum principle. It is proved that the developed scheme is parameter-uniform with an accuracy rate that is almost second-order in space and first-order in time. To validate the applicability of the proposed scheme, two numerical examples are considered and compared with the results available in the literature. Graphical information on the boundary layer behaviour of the solutions is provided.

6.1 Governing problem

Consider a class of singularly perturbed convection-diffusion equation of the form

$$\begin{cases} z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -\kappa(s, t)z(s, t - \tau) + \gamma(s, t), & (s, t) \in \mathfrak{D} \\ z(s, t) = \psi_b(s, t), & (s, t) \in \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), & t \in \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t), & t \in \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (6.1)$$

where $\mathfrak{L}_\varepsilon z(s, t) = -\varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) + \chi(s, t)z(s, t)$.

Here $\varepsilon \in (0, 1]$ and $\tau > 0$ are the perturbation parameter and the delay parameter, respectively. We pretended that the functions $\beta(s, t)$, $\chi(s, t)$, $\kappa(s, t)$, $\gamma(s, t)$ on $\bar{\mathfrak{D}} = \bar{\Omega} \times \bar{\Lambda} = [0, 1] \times [0, T]$ and $\psi_b(s, t)$, $\psi_l(t)$, $\psi_r(t)$ on $\eta = \eta_l \cup \eta_r \cup \eta_b$ are smooth enough and bounded which meet $0 < \varpi \leq \chi(s, t)$, $0 < \varphi \leq \kappa(s, t)$, $0 < \mu \leq \beta(s, t)$, $\chi(s, t) + \kappa(s, t) \geq 0$. These conditions assure, the problem (6.1) exhibits a boundary layer along $s = 1$ (Das and Natesan, 2015).

6.2 Properties of the continuous solution

Under the premises that the data are Hölder continuous, if the compatibility conditions and assumption stated in (4.2) -(4.3) are satisfied, then the problem (6.1) has a unique solution.

Putting $\varepsilon = 0$, the corresponding reduced problem (6.1) has the form of (5.2). The problem (6.1) satisfies the maximum principle stated in Lemma 4.2.1 and the stability result of its solution $z(s, t)$ satisfies the bound given in Lemma 4.2.2.

Lemma 6.2.1. *Let $z(s, t)$ be the solution of (6.1), then its derivative is estimated as*

$$\left| \frac{\partial^{i+l} z(s, t)}{\partial s^i \partial t^l} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(\frac{-\mu(1-s)}{\varepsilon} \right) \right), \quad (s, t) \in \mathfrak{D},$$

where i and l are non-negative integers satisfy $0 \leq i + l \leq 5$.

Proof. The proof follows from Lemma 4.2.3 - Lemma 4.2.5 and Lemma 5.2.3 which is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives can be obtained. ■

The estimation of the $z(s, t)$ derivative provided in Lemma 6.2.1 is insufficient to demonstrate uniform convergence of the proposed technique. As a result, it requires stronger bounds. These can be carried out by decomposing the smooth and singular components of the solution $z(s, t)$. At the start, this decomposition technique was considered by (Shishkin and Shishkina, 2011). Consequently, the solution $z(s, t)$ can be decomposed as

$$z(s, t) = \nu(s, t) + \omega(s, t), \quad (s, t) \in \bar{\mathfrak{D}},$$

where $\nu(s, t)$ is the smooth component and $\omega(s, t)$ is the singular component.

Given $\nu(s, t)$ as

$$\nu(s, t) = \sum_{i=0}^4 \varepsilon^i \nu_i(s, t), \quad (s, t) \in \bar{\mathfrak{D}},$$

where $\nu_i(s, t), i = 0, 1, 2, 3$, are the solutions of equation

$$\begin{cases} (\nu_0)_t(s, t) + \beta(s, t)(\nu_0)_s(s, t) + \chi(s, t)\nu_0(s, t) = -\kappa(s, t)\nu_0(s, t - \tau) + \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ \nu_0(s, t) = \psi_b(s, t), & (s, t) \in \eta_b, \\ \nu_0(0, t) = \psi_1(t), & t \in \bar{\Lambda}, \end{cases} \quad (6.2)$$

and

$$\begin{cases} (\nu_i)_t(s, t) + \beta(s, t)(\nu_i)_s(s, t) + \chi(s, t)(\nu_i)(s, t) = -\kappa(s, t)\nu_i(s, t - \tau) + (\nu_{i-1})_{ss}(s, t), \\ \hspace{15em} (s, t) \in \mathfrak{D}, \ i = 1, 2, 3, \\ \nu_i(s, t) = 0, \quad (s, t) \in \eta_b, \\ \nu_i(0, t) = 0, \quad t \in \bar{\Lambda}, \end{cases} \quad (6.3)$$

then, $\nu_4(s, t)$ fulfills

$$\begin{cases} (\nu_4)_t(s, t) + \mathfrak{L}_\varepsilon \nu_4(s, t) = -\kappa(s, t)\nu_4(s, t - \tau) + (\nu_3)_{ss}(s, t), \quad (s, t) \in \mathfrak{D}, \\ \nu_4(s, t) = 0, \quad (s, t) \in \eta_b, \\ \nu_4(0, t) = 0, \quad t \in \bar{\Lambda}, \\ \nu_4(1, t) = 0, \quad t \in \bar{\Lambda}. \end{cases} \quad (6.4)$$

Hence, $\nu(s, t)$ fulfills the next equation

$$\begin{cases} \nu_t(s, t) + \mathfrak{L}_\varepsilon \nu(s, t) = -\kappa(s, t)\nu(s, t - \tau) + \gamma(s, t), \quad (s, t) \in \mathfrak{D}, \\ \nu(s, t) = \psi_b(s, t), \quad (s, t) \in \eta_b, \\ \nu(0, t) = \psi_l(t), \quad t \in \bar{\Lambda}, \\ \nu(1, t) = \sum_{i=0}^4 \varepsilon^i \nu_i(1, t), \quad t \in \bar{\Lambda}, \end{cases} \quad (6.5)$$

and $\omega(s, t)$ fulfills the next equation

$$\begin{cases} \omega_t(s, t) + \mathfrak{L}_\varepsilon \omega(s, t) = -\kappa(s, t)\omega(s, t - \tau), \quad (s, t) \in \mathfrak{D}, \\ \omega(s, t) = 0, \quad (s, t) \in \eta_b, \\ \omega(0, t) = 0, \quad t \in \bar{\Lambda}, \\ \omega(1, t) = z(1, t) - \nu(1, t), \quad t \in \bar{\Lambda}. \end{cases} \quad (6.6)$$

The next lemma gives the estimates for the components of $\nu(s, t)$ and $\omega(s, t)$.

Lemma 6.2.2. *The derivatives of $\nu(s, t)$ and $\omega(s, t)$ stated in (6.5) and (6.6), respectively, are estimated as*

$$\begin{aligned} \left| \frac{\partial^{i+l} \nu}{\partial s^i \partial t^l} \right| &\leq C(1 + \varepsilon^{4-i}), \\ \left| \frac{\partial^{i+l} \omega}{\partial s^i \partial t^l} \right| &\leq C\varepsilon^{-i} \exp\left(-\frac{\mu(1-s)}{\varepsilon}\right), \quad (s, t) \in \mathfrak{D}, \end{aligned}$$

where i and l are non-negative integers satisfy $0 \leq i + l \leq 5$.

Proof. We can see that the PDEs (6.2) and (6.3) are independent of ε . Therefore, for the derivatives of ν_i defined in (6.2) and (6.3), $i = 0, 1, 2, 3$, can have the following ε -uniform

bound

$$\left| \frac{\partial^{i+l} \nu_i}{\partial s^i \partial t^l} \right| \leq C,$$

where i and l are non-negative integers satisfy $0 \leq i + l \leq 5$.

Since the solution $\nu_4(s, t)$ of (6.4) will have the same bound as stated in Lemma 6.2.1. Thus, by summing up the bounds for the derivatives of $\nu_i(s, t)$, $i = 0, 1, 2, 3, 4$, we get

$$\begin{aligned} \left| \frac{\partial^{i+l} \nu}{\partial s^i \partial t^l} \right| &\leq \left| \frac{\partial^{i+l} \nu_0}{\partial s^i \partial t^l} \right| + \varepsilon \left| \frac{\partial^{i+l} \nu_1}{\partial s^i \partial t^l} \right| + \varepsilon^2 \left| \frac{\partial^{i+l} \nu_2}{\partial s^i \partial t^l} \right| + \varepsilon^3 \left| \frac{\partial^{i+l} \nu_3}{\partial s^i \partial t^l} \right| + \varepsilon^4 \left| \frac{\partial^{i+l} \nu_4}{\partial s^i \partial t^l} \right| \\ &\leq C + C\varepsilon + C\varepsilon^2 + C\varepsilon^3 + C\varepsilon^4 \left(1 + \varepsilon^{-i} \exp \left(\frac{-\mu(1-s)}{\varepsilon} \right) \right) \\ &\leq C + C\varepsilon + C\varepsilon^2 + C\varepsilon^3 + C\varepsilon^4 + C\varepsilon^{4-i} \exp \left(\frac{-\mu(1-s)}{\varepsilon} \right) \\ &\leq C + C\varepsilon^{4-i} \exp \left(\frac{-\mu(1-s)}{\varepsilon} \right), \text{ since } \varepsilon \ll 1 \text{ and } \exp \left(\frac{-\mu(1-s)}{\varepsilon} \right) \leq 1, \\ &\leq C(1 + \varepsilon^{4-i}). \end{aligned} \quad (6.7)$$

Hence, the desired bound is obtained. ■

6.3 Numerical scheme

In this section, we discuss the numerical scheme used to solve the problem (6.1). The scheme involves a backward Euler method in the time direction, a higher-order finite difference method for the space derivatives.

6.3.1 Temporal semi-discretization

The interval $[0, T]$ is divided uniformly with M mesh points as $\Lambda^M = \{t_n = n\Delta t, n = 0, 1, 2, \dots, M, t_M = T, \Delta t = T/M\}$. The approximation of $z(s, t)$ in temporal discretization at the grid point n is denoted as $Z^n(s)$. Using the backward Euler scheme for the time derivative of (4.15) and simplifying, we get

$$\begin{cases} \mathfrak{L}^{\Delta t} Z^n(s) \equiv -\varepsilon \frac{d^2 Z^n(s)}{ds^2} + \beta(s, t_n) \frac{dZ^n(s)}{ds} + P(s, t_n) Z^n(s) = Q(s, t_n), \\ Z^n(0) = \psi_l(t_n), \quad Z^n(1) = \psi_r(t_n), \quad 0 \leq n \leq M, \end{cases} \quad (6.8)$$

where $P(s, t_n) = \left(\frac{1-\tau\kappa(s, t_n)}{\Delta t} + \kappa(s, t_n) + \chi(s, t_n) \right)$ and $Q(s, t_n) = \gamma(s, t_n) + \left(\frac{1-\tau\kappa(s, t_n)}{\Delta t} \right) Z^{n-1}(s)$.

In the time direction local error is donated as $e_n(s) := z(s, t_n) - Z^n(s)$, $0 \leq n \leq M$. Its local error bound is given in Lemma 4.3.10, whereas the global error bound associated with the method is given in Lemma 4.3.11.

Lemma 6.3.3. *For every $0 \leq n \leq M - 1$, the solution $Z^n(s)$ of (6.8) is bounded as*

$$\left| \frac{d^i Z^n(s)}{ds^i} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(\frac{-\mu(1-s)}{\varepsilon} \right) \right), \quad s \in \Omega, \quad 0 \leq i \leq 4.$$

Proof. The proof follows from Lemma 5.2.1, it is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. ■

To get the stronger bounds we decompose the solution of (6.8) as

$$Z^n(s) = V^n(s) + W^n(s), \quad s \in \bar{\Omega},$$

where the smooth component $V^n(s)$ fulfills

$$\begin{cases} \mathfrak{L}^{\Delta t} V^n(s) = Q(s, t_n), & s \in \bar{\Omega}, \\ Z^n(0) = V^n(0), \end{cases}$$

and the singular component $W^n(s)$ fulfills

$$\begin{cases} \mathfrak{L}^{\Delta t} W^n(s) = 0, & s \in \bar{\Omega}, \\ W^n(0) = 0, \quad W^n(1) = Z^n(1) - V^n(1). \end{cases}$$

The next lemma provides the estimate for the components $V^n(s)$ and $W^n(s)$.

Lemma 6.3.4. *The derivative of $V^n(s)$ and $W^n(s)$ have the following bound*

$$\begin{aligned} \left| \frac{d^i V^n(s)}{ds^i} \right| &\leq C(1 + \varepsilon^{4-i}), \quad i = 0, 1, 2, 3, 4, \\ \left| \frac{d^i W^n(s)}{ds^i} \right| &\leq C\varepsilon^{-i} \exp\left(\frac{-\mu(1-s)}{\varepsilon}\right), \quad i = 0, 1, 2, 3. \end{aligned}$$

Proof. The proof follows from Lemma 6.2.2. ■

6.3.2 Spatial discretization

In this section, we looked at the fully discrete scheme obtained after the space discretization of (6.8).

The piecewise-uniform Shishkin mesh

The piecewise uniform Shishkin mesh is shown for the space discretization of the domain $[0, 1]$. Since the boundary layer for the problem (6.1) is close to $s = 1$, the mesh ought to be condensing there. We split $[0, 1]$ into two sub-intervals, $[0, 1 - \sigma] \cup (1 - \sigma, 1]$ then split each sub-interval into $N/2$ equal intervals to characterize the piecewise-uniform mesh. The spatial grids are given as

$$\Omega^N = \{0 = s_0, s_1, \dots, s_{N/2} = 1 - \sigma, \dots, s_N = 1\},$$

where

$$s_m = \begin{cases} m \frac{2(1-\sigma)}{N}, & m = 0, 1, 2, \dots, N/2, \\ (1-\sigma) + (m - \frac{N}{2}) \frac{2\sigma}{N}, & m = (N/2) + 1, \dots, N, \end{cases} \quad (6.9)$$

and a positive integer $N \geq 4$. The transition point $1 - \sigma$, which separates the coarse and fine points of the mesh, is obtained by taking $\sigma = \min\{0.5, \sigma_0 \varepsilon \ln N\}$, where $\sigma_0 = 2/\mu$. The analysis has been done by assuming that $\sigma = \sigma_0 \varepsilon \ln N$, otherwise N is exponentially large compared to ε .

From the definition of s_m 's the piecewise-uniform mesh spacing is denoted as

$$H := h_m = \frac{2(1-\sigma)}{N}, \quad m = 1, 2, \dots, N/2, \\ h := h_m = \frac{2\sigma}{N}, \quad m = (N/2) + 1, \dots, N,$$

H and h are the mesh width in $[0, 1 - \sigma]$ and $[1 - \sigma, 1]$, respectively. From the preceding equation it is clear that $N^{-1} \leq H \leq 2N^{-1}$, $h = 2\sigma_0 N^{-1} \varepsilon \ln N$, and choosing $\sigma = 0.5$ gives uniform mesh.

Now, for the problem (6.1), we propose a higher-order finite difference scheme. Consider a smooth function $Z(s) = Z^n(s) = Z(s, t_n)$ in the domain $[0, 1]$.

From Taylor's series approximation, we get

$$\begin{aligned} Z_{m+1} &\approx Z_m + h_m Z'_m + \frac{h_m^2}{2!} Z''_m + \frac{h_m^3}{3!} Z'''_m + \frac{h_m^4}{4!} Z^{(4)}_m + \frac{h_m^5}{5!} Z^{(5)}_m + \frac{h_m^6}{6!} Z^{(6)}_m + \frac{h_m^7}{7!} Z^{(7)}_m \\ &\quad + \frac{h_m^8}{8!} Z^{(8)}_m + O(h^9), \\ Z_{m-1} &\approx Z_m - h_{m-1} Z'_m + \frac{h_{m-1}^2}{2!} Z''_m - \frac{h_{m-1}^3}{3!} Z'''_m + \frac{h_{m-1}^4}{4!} Z^{(4)}_m - \frac{h_{m-1}^5}{5!} Z^{(5)}_m + \frac{h_{m-1}^6}{6!} Z^{(6)}_m \\ &\quad - \frac{h_{m-1}^7}{7!} Z^{(7)}_m + \frac{h_{m-1}^8}{8!} Z^{(8)}_m - O(h^9). \end{aligned} \quad (6.10)$$

Combining the results in (6.10), we get

$$\begin{aligned} &\left(\frac{2h_{m-1}}{h_m + h_{m-1}} \right) Z_{m+1} - 2Z_m + \left(\frac{2h_m}{h_m + h_{m-1}} \right) Z_{m-1} \\ &= \left(\frac{h_m^2 + h_{m-1}^2}{2} \right) \left(Z''_m + \frac{1}{3} \left(h_m - h_{m-1} + \frac{h_m h_{m-1} (h_m - h_{m-1})}{h_m^2 + h_{m-1}^2} \right) Z'''_m \right. \\ &\quad \left. + \frac{1}{12} \left(h_m^2 + h_{m-1}^2 - \frac{2h_m^2 h_{m-1}^2}{h_m^2 + h_{m-1}^2} \right) Z^{(4)}_m \right) + \dots \end{aligned} \quad (6.11)$$

Equivalently,

$$\begin{aligned} Z''_{m+1} - 2Z''_m + Z''_{m-1} = & (h_m - h_{m-1})Z'''_m + \left(\frac{h_m^2 + h_{m-1}^2}{2!}\right)Z_m^{(4)} + \left(\frac{h_m^3 - h_{m-1}^3}{3!}\right)Z_m^{(5)} \\ & + \left(\frac{h_m^4 + h_{m-1}^4}{4!}\right)Z_m^{(6)} + \left(\frac{h_m^5 - h_{m-1}^5}{5!}\right)Z_m^{(7)} + \left(\frac{h_m^6 + h_{m-1}^6}{6!}\right)Z_m^{(8)} + \dots \end{aligned} \quad (6.12)$$

Using (6.12), we obtain

$$\begin{aligned} \left(\frac{h_m^2 + h_{m-1}^2}{2}\right)Z_m^{(4)} = & Z''_{m+1} - 2Z''_m + Z''_{m-1} - (h_m - h_{m-1})Z'''_m - \left(\frac{h_m^3 - h_{m-1}^3}{3!}\right)Z_m^{(5)} \\ & - \left(\frac{h_m^4 + h_{m-1}^4}{4!}\right)Z_m^{(6)} - \left(\frac{h_m^5 - h_{m-1}^5}{5!}\right)Z_m^{(7)} - \left(\frac{h_m^6 + h_{m-1}^6}{6!}\right)Z_m^{(8)} - \dots \end{aligned} \quad (6.13)$$

Substituting (6.13) into (6.11) and after some transcription, we get

$$-\frac{2\varepsilon}{h_m h_{m-1}} \left(\frac{h_{m-1}Z_{m+1} - \hat{h}_m Z_m + h_m Z_{m-1}}{\hat{h}_m} \right) = -\Delta_1 \varepsilon Z''_{m+1} - \Delta_2 \varepsilon Z''_m - \Delta_1 \varepsilon Z''_{m-1} + \zeta, \quad (6.14)$$

where

$$\begin{aligned} \Delta_1 = & \frac{1}{12} \left(\frac{h_m^2 + h_{m-1}^2}{h_m h_{m-1}} - \frac{2h_m h_{m-1}}{h_m^2 + h_{m-1}^2} \right), \quad \Delta_2 = \frac{h_m^2 + h_{m-1}^2}{2h_m h_{m-1}} - \frac{h_m^2 + h_{m-1}^2}{6h_m h_{m-1}} + \frac{h_m h_{m-1}}{3h_m^2 + 3h_{m-1}^2}, \quad \hat{h}_m = h_m + h_{m-1}, \\ \zeta = & -\frac{h_m - h_{m-1}}{12} \left(\frac{h_m^2 + h_{m-1}^2}{h_m h_{m-1}} - \frac{2h_m h_{m-1}}{h_m^2 + h_{m-1}^2} \right) Z'''_m - \frac{h_m^3 - h_{m-1}^3}{72} \left(\frac{2h_m^2 + h_{m-1}^2}{h_m h_{m-1}} - \frac{h_m h_{m-1}}{h_m^2 + h_{m-1}^2} \right) Z_m^{(5)} - O(h^6). \end{aligned}$$

Using (6.8), we draw

$$-\varepsilon \frac{d^2 Z(s)}{ds^2} = -\beta(s, t_n) \frac{dZ(s)}{ds} - P(s, t_n)Z(s) + Q(s, t_n), \quad (6.15)$$

Using (6.15) evaluating at s_m and $s_{m\pm 1}$, we have

$$\begin{aligned} -\varepsilon Z''_{m\pm 1} = & -\beta(s_{m\pm 1}, t_n)Z'_{m\pm 1} - P(s_{m\pm 1}, t_n)Z_{m\pm 1} + Q(s_{m\pm 1}, t_n), \\ -\varepsilon Z''_m = & -\beta(s_m, t_n)Z'_m - P(s_m, t_n)Z_m + Q(s_m, t_n). \end{aligned} \quad (6.16)$$

Next, on piecewise-uniform mesh finite difference approximations for Z'_{m+1} , Z'_m and Z'_{m-1} is

given as

$$\begin{aligned}
Z'_{m+1} &\approx \frac{1}{h_m h_{m-1} \hat{h}_m} \left((h_{m-1}^2 + 2h_m h_{m-1}) Z_{m+1} - \hat{h}_m^2 Z_m + h_m^2 Z_{m-1} \right) - \frac{\hat{h}_m}{2} Z''_m + O(h^2), \\
Z'_m &\approx \frac{1}{h_m h_{m-1} \hat{h}_m} \left(h_{m-1}^2 Z_{m+1} + (h_m^2 - h_{m-1}^2) Z_m - h_m^2 Z_{m-1} \right) + O(h^2), \\
Z'_{m-1} &\approx \frac{1}{h_m h_{m-1} \hat{h}_m} \left(-h_{m-1}^2 Z_{m+1} + \hat{h}_m^2 Z_m - (h_m^2 + 2h_m h_{m-1}) Z_{m-1} \right) + \frac{\hat{h}_m}{2} Z''_m + O(h^2).
\end{aligned} \tag{6.17}$$

Substituting (6.17) into (6.16) and plugging the resulting equation into (6.14) after simplifying, we have

$$\begin{aligned}
&\left(-\varepsilon + \frac{\beta(s_{m-1}, t_n) \hat{h}_m}{2} \Delta_1 - \frac{\beta(s_{m+1}, t_n) \hat{h}_m}{2} \Delta_1 \right) \left(\frac{2}{h_m h_{m-1} \hat{h}_m} \left(h_{m-1} Z_{m+1}^j - \hat{h}_m Z_m^n + h_m Z_{m-1}^n \right) \right) \\
&+ \Delta_1 \beta(s_{m-1}, t_n) \left(\frac{1}{h_m h_{m-1} \hat{h}_m} (-h_{m-1}^2 Z_{m+1}^n + \hat{h}_m^2 Z_m^n - (h_m^2 + 2h_m h_{m-1}) Z_{m-1}^n) \right) \\
&+ \Delta_2 \beta(s_m, t_n) \left(\frac{1}{h_m h_{m-1} \hat{h}_m} (h_{m-1}^2 Z_{m+1}^n + (h_m^2 - h_{m-1}^2) Z_m^n - h_m^2 Z_{m-1}^n) \right) \\
&+ \Delta_1 \beta(s_{m+1}, t_n) \left(\frac{1}{h_m h_{m-1} \hat{h}_m} ((h_{m-1}^2 + 2h_m h_{m-1}) Z_{m+1}^n - \hat{h}_m^2 Z_m^n + h_m^2 Z_{m-1}^n) \right) \\
&+ \Delta_1 P(s_{m+1}, t_n) Z_{m+1}^n + \Delta_2 P(s_m, t_n) Z_m^n + \Delta_1 P(s_{m-1}, t_n) Z_{m-1}^n \\
&= \Delta_1 Q(s_{m-1}, t_n) + \Delta_2 Q(s_m, t_n) + \Delta_1 Q(s_{m+1}, t_n).
\end{aligned} \tag{6.18}$$

6.3.3 The discrete scheme

For $0 \leq m \leq N$ and $0 \leq n \leq M$, the proposed fully discrete scheme is given as

$$\mathfrak{L}^{\Delta t, h} Z_m^n = \Delta_1 Q(s_{m-1}, t_n) + \Delta_2 Q(s_m, t_n) + \Delta_1 Q(s_{m+1}, t_n), \tag{6.19}$$

where

$$\begin{aligned} \mathfrak{L}^{\Delta t, h} Z_m^n = & \left(-\varepsilon + \frac{\beta(s_{m-1}, t_n) \hat{h}_m}{2} \Delta_1 - \frac{\beta(s_{m+1}, t_n) \hat{h}_m}{2} \Delta_1 \right) \\ & \left(\frac{2}{h_m h_{m-1}} \left(\frac{h_{m-1} Z_{m+1}^n - \hat{h}_m Z_m^n + h_m Z_{m-1}^n}{\hat{h}_m} \right) \right) \\ & + \Delta_1 \beta(s_{m-1}, t_n) \left(\frac{1}{h_m h_{m-1} \hat{h}_m} (-h_{m-1}^2 Z_{m+1}^n + \hat{h}_m^2 Z_m^n - (h_m^2 + 2h_m h_{m-1}) Z_{m-1}^n) \right) \\ & + \Delta_2 \beta(s_m, t_n) \left(\frac{1}{h_m h_{m-1} \hat{h}_m} (h_{m-1}^2 Z_{m+1}^n + (h_m^2 - h_{m-1}^2) Z_m^n - h_m^2 Z_{m-1}^n) \right) \\ & + \Delta_1 \beta(s_{m+1}, t_n) \left(\frac{1}{h_m h_{m-1} \hat{h}_m} ((h_{m-1}^2 + 2h_m h_{m-1}) Z_{m+1}^n - \hat{h}_m^2 Z_m^n + h_m^2 Z_{m-1}^n) \right) \\ & + \Delta_1 P(s_{m+1}, t_n) Z_{m+1}^n + \Delta_2 P(s_m, t_n) Z_m^n + \Delta_1 P(s_{m-1}, t_n) Z_{m-1}^n. \end{aligned}$$

After rearranging the term in (6.19), we get the leading system of equation

$$C_m^- Z_{m-1}^n + C_m^0 Z_m^n + C_m^+ Z_{m+1}^n = H_m^n, \quad (6.20)$$

where

$$\begin{aligned} C_m^+ = & -\frac{2\varepsilon}{h_m \hat{h}_m} + \Delta_1 \frac{\beta(s_{m-1}, t_n)}{h_m} - \Delta_1 \frac{\beta(s_{m+1}, t_n)}{h_m} - \Delta_1 \left(\frac{h_{m-1} \beta(s_{m-1}, t_n)}{h_m \hat{h}_m} \right) \\ & + \Delta_2 \left(\frac{h_{m-1} \beta(s_m, t_n)}{h_m \hat{h}_m} \right) + \Delta_1 \left(\frac{(h_{m-1} + 2h_m) \kappa(s_{m+1}, t_n)}{h_m \hat{h}_m} \right) + \Delta_1 P(s_{m+1}, t_n), \\ C_m^0 = & \frac{2\varepsilon}{h_m h_{m-1}} + \Delta_2 \left(\frac{h_m - h_{m-1}}{h_m h_{m-1}} \right) \beta(s_m, t_n) + \Delta_2 P(s_m, t_n), \\ C_m^- = & -\frac{2\varepsilon}{h_{m-1} \hat{h}_m} + \Delta_1 \frac{\beta(s_{m-1}, t_n)}{h_{m-1}} - \Delta_1 \frac{\beta(s_{m+1}, t_n)}{h_{m-1}} - \Delta_1 \left(\frac{(h_m + 2h_{m-1}) \beta(s_{m-1}, t_n)}{h_{m-1} \hat{h}_m} \right) \\ & - \Delta_2 \left(\frac{h_m \beta(s_m, t_n)}{h_{m-1} \hat{h}_m} \right) + \Delta_1 \left(\frac{h_m \beta(s_{m+1}, t_n)}{h_{m-1} \hat{h}_m} \right) + \Delta_1 P(s_{m-1}, t_n), \\ H_m^n = & \Delta_1 Q(s_{m-1}, t_n) + \Delta_2 Q(s_m, t_n) + \Delta_1 Q(s_{m+1}, t_n). \end{aligned}$$

6.4 Convergence analysis

The resulting discrete scheme given in (6.19) satisfies the discrete maximum principle stated in Lemma 4.4.13.

Lemma 6.4.5. *If φ_m^n be any mesh function such that $\varphi_0^n = \varphi_N^n = 0$. Then, we have*

$$|\varphi_m^n| \leq \frac{1}{\lambda} \max_i |\mathfrak{L}^{\Delta t, h} \varphi_i^n|, \quad i = 1, 2, \dots, N-1.$$

Proof. Take the barrier functions $\vartheta_{m,n}^\pm = q \pm \varphi_m^n$,

where $q = (1/\lambda) \max_i |\mathfrak{L}^{\Delta t, h} \varphi_i^n|$. Clearly $\vartheta_{0,n}^\pm \geq 0$, $\vartheta_{N,n}^\pm \geq 0$, next we show that $\mathfrak{L}^{\Delta t, h} \vartheta_{m,n}^\pm \geq 0$.

$$\begin{aligned} \mathfrak{L}^{\Delta t, h} \vartheta_{m,n}^\pm &= \mathfrak{L}^{\Delta t, h} (q \pm \varphi_m^n) = \mathfrak{L}^{\Delta t, h} ((1/\lambda) \max_i |\mathfrak{L}^{\Delta t, h} \varphi_i^n| \pm \varphi_m^n) \\ &= \mathfrak{L}^{\Delta t, h} ((1/\lambda) \max_i |\mathfrak{L}^{\Delta t, h} \varphi_i^n|) \pm \mathfrak{L}^{\Delta t, h} \varphi_m^n \\ &= \pm \mathfrak{L}^{\Delta t, h} \varphi_m^n + \frac{P(s_m, t_n)}{\lambda} (\max_i |\mathfrak{L}^{\Delta t, h} \varphi_i^n|) \geq 0. \end{aligned}$$

Hence, from the discrete comparison principle it gives

$$|\varphi_m^n| \leq \frac{1}{\lambda} \max_i |\mathfrak{L}^{\Delta t, h} \varphi_i^n|.$$

Theorem 6.4.1. (Error in the spatial direction) Let $Z^n(s)$ be the solution of (6.8) and Z_m^n be the solution of (6.19). Then, the error estimate is given as

$$\|\mathfrak{L}^{\Delta t, h} (Z^n(s_m) - Z_m^n)\| \leq C N^{-2} \ln^3 N.$$

Proof. Following similar way with that of the solution of (6.8), we decomposed the solution Z_m^n of (6.19) as

$$Z_m^n = V_m^n + W_m^n,$$

where the smooth component V_m^n fulfills

$$\begin{cases} \mathfrak{L}^{\Delta t, h} V_m^n = H_m^n, & s_m \in \bar{\Omega}^N, \\ V_0^n = \nu(0, t_n), & V_N^n = \nu(1, t_n), \end{cases}$$

and the singular component W_m^n fulfills

$$\begin{cases} \mathfrak{L}^{\Delta t, h} W_m^n = 0, & s_m \in \bar{\Omega}^N, \\ W^n(0) = 0, & W^n(1) = Z^n(1) - V^n(1). \end{cases}$$

The error at the node (s_m, t_n) can be written as

$$(Z - z)(s_m, t_n) = (V - \nu)(s_m, t_n) + (W - \omega)(s_m, t_n).$$

Applying the triangle inequality we have

$$|(Z - z)(s_m, t_n)| \leq |(V - \nu)(s_m, t_n)| + |(W - \omega)(s_m, t_n)|.$$

Now, the error of smooth and singular components can be estimated separately.

The truncation error for the smooth component is obtained classically.

Thus,

$$\begin{aligned}\Xi_m &= \mathfrak{L}^{\Delta t, h}(V - \nu)(s_m, t_n) = \mathfrak{L}^{\Delta t, h}\nu^n(s_m) - \mathfrak{L}^{\Delta t, h}V_m^n \\ &= \mathfrak{L}^{\Delta t, h}\nu^n(s_m) - H_m^n.\end{aligned}$$

Hence,

$$\Xi_m = C_m^- V_{m-1} + C_m^0 V_m + C_m^+ V_{m+1} - (\Delta_1 Q(s_{m-1}, t_n) + \Delta_2 Q(s_m, t_n) + \Delta_1 Q(s_{m+1}, t_n)). \quad (6.21)$$

Using (6.15) for $Q(s_{m-1}, t_n)$, $Q(s_m, t_n)$ and $Q(s_{m+1}, t_n)$ in (6.21), we get

$$\begin{aligned}\Xi_m &= C_m^- V_{m-1} + C_m^0 V_m + C_m^+ V_{m+1} \\ &\quad - (-\Delta_1 \varepsilon V_{m-1}'' + \Delta_1 \beta(s_{m-1}, t_n) V_{m-1}' + \Delta_1 P(s_{m-1}, t_n) V_{m-1}) \\ &\quad - (-\Delta_2 \varepsilon V_m'' + \Delta_2 \beta(s_m, t_n) V_m' + \Delta_2 P(s_m, t_n) V_m) \\ &\quad - (-\Delta_1 \varepsilon V_{m+1}'' + \Delta_1 \beta(s_{m+1}, t_n) V_{m+1}' + \Delta_1 P(s_{m+1}, t_n) V_{m+1}).\end{aligned} \quad (6.22)$$

Following similar relation obtained in (6.10), we approximate V_{m+1}'' , V_{m-1}'' , V_{m+1}' , V_{m-1}' , V_{m+1} , and V_{m-1} . Using these approximations in (6.22), we have

$$\Xi_m = \xi_{0,m} V_m + \xi_{1,m} V_m' + \xi_{2,m} V_m'' + \xi_{3,m} V_m''' + \xi_{4,m} V_m^{(iv)} + \dots, \quad (6.23)$$

where

$$\begin{aligned}\xi_{0,m} &= C_m^- + C_m^0 + C_m^+ - \Delta_1 P(s_{m-1}, t_n) - \Delta_2 P(s_m, t_n) - \Delta_1 P(s_{m+1}, t_n), \\ \xi_{1,m} &= -C_m^- h_{m-1} + C_m^+ h_m - \Delta_1 \beta(s_{m+1}, t_n) - \Delta_1 h_m P(s_{m+1}, t_n) - \Delta_2 \beta(s_m, t_n) \\ &\quad - \Delta_1 \beta(s_{m-1}, t_n) + \Delta_1 h_{m-1} P(s_{m-1}, t_n), \\ \xi_{2,m} &= C_m^- \frac{h_{m-1}^2}{2!} + C_m^+ \frac{h_m^2}{2!} + \Delta_1 \varepsilon - \Delta_1 h_m \beta(s_{m+1}, t_n) - \Delta_1 \frac{h_m^2}{2!} P(s_{m+1}, t_n) + \Delta_2 \varepsilon + \Delta_1 \varepsilon \\ &\quad + \Delta_1 h_{m-1} \beta(s_{m-1}, t_n) - \Delta_1 \frac{h_{m-1}^2}{2!} P(s_{m-1}, t_n), \\ \xi_{3,m} &= -C_m^- \frac{h_{m-1}^3}{3!} + C_m^+ \frac{h_m^3}{3!} + \Delta_1 \varepsilon h_m - \Delta_1 \frac{h_m^2}{2!} \beta(s_{m+1}, t_n) - \Delta_1 \frac{h_m^3}{3!} P(s_{m+1}, t_n) - \Delta_1 \varepsilon h_{m-1} \\ &\quad - \Delta_1 \frac{h_{m-1}^2}{2!} \beta(s_{m-1}, t_n) + \Delta_1 \frac{h_{m-1}^3}{3!} P(s_{m-1}, t_n), \\ \xi_{4,m} &= C_m^- \frac{h_{m-1}^4}{4!} + C_m^+ \frac{h_m^4}{4!} + \Delta_1 \varepsilon \frac{h_m^2}{2!} + \Delta_1 \varepsilon \frac{h_{m-1}^2}{2!} - \Delta_1 \frac{h_m^3}{3!} \beta(s_{m+1}, t_n) + \Delta_1 \frac{h_{m-1}^3}{3!} \beta(s_{m-1}, t_n) \\ &\quad - \Delta_1 \frac{h_m^4}{4!} P(s_{m+1}, t_n) - \Delta_1 \frac{h_{m-1}^4}{4!} P(s_{m-1}, t_n).\end{aligned}$$

After some calculation and using the estimate $h_m \leq 2N^{-1}$, $h_m + h_{m-1} \leq 4N^{-1}$, we have

$$\xi_{0,m} = 0, \quad \xi_{1,m} = 0, \quad \xi_{2,m} = 0, \quad \xi_{3,m} \leq C_1 N^{-2} \text{ and } \xi_{4,m} \leq C_2 N^{-2}(\varepsilon + N^{-1}).$$

Thus,

$$|\mathfrak{L}^{\Delta t, h}(V - \nu)| \leq CN^{-2}(\varepsilon + N^{-1}),$$

where $C = \max\{C_1, C_2\}$.

From the discrete maximum principle, we have

$$|V - \nu| \leq CN^{-2}(\varepsilon + N^{-1}). \quad (6.24)$$

The error of $(W - \omega)(s_m, t_n)$ is depend on the transition parameter value, i.e., $\sigma = 1/2$ or $\sigma = \sigma_0 \varepsilon \ln N$. So, we consider two cases:

Case 1 : When $\sigma = 1/2$ ($1/2 < \sigma_0 \varepsilon \ln N$). Here, the mesh is uniform with spacing of N^{-1} . The smooth component yields

$$|\mathfrak{L}^{\Delta t, h}(\omega^n(s_m) - W_m^n)| \leq CN^{-2} \left(\varepsilon \left| \frac{d^4 W^n}{ds^4} \right| + \left| \frac{d^3 W^n}{ds^3} \right| \right). \quad (6.25)$$

Now using Lemma 6.3.4 for the estimate $\left| \frac{d^4 W^n}{ds^4} \right|$ and $\left| \frac{d^3 W^n}{ds^3} \right|$ leads to

$$|\mathfrak{L}^{\Delta t, h}(\omega^n(s_m) - W_m^n)| \leq CN^{-2} \varepsilon^{-3}. \quad (6.26)$$

Since $1/2 < \sigma_0 \varepsilon \ln N$, we have $\varepsilon^{-1} < 2\sigma_0 \ln N$.

Thus,

$$|\mathfrak{L}^{\Delta t, h}(\omega^n(s_m) - W_m^n)| \leq CN^{-2} \ln^3 N. \quad (6.27)$$

From the discrete maximum principle, we have

$$|\omega^n(s_m) - W_m^n| \leq CN^{-2} \ln^3 N. \quad (6.28)$$

Case 2 : When $\sigma = \sigma_0 \varepsilon \ln N$ ($\sigma_0 \varepsilon \ln N < 1/2$). The mesh used is piecewise uniform with a uniform spacing of $2(1 - \sigma)/N$ in the subinterval $[0, 1 - \sigma)$ and $2\sigma/N$ in the subinterval $[1 - \sigma, 1]$. To find the errors in $[0, 1 - \sigma)$ and $[1 - \sigma, 1]$ separately.

In the first interval $[0, 1 - \sigma)$, Lemma 6.3.4 gives

$$|W^n(s)| \leq C \exp \left(\frac{-\mu(1-s)}{\varepsilon} \right) \leq C \exp \left(\frac{-\mu\sigma}{\varepsilon} \right), \quad 0 \leq m \leq N/2. \quad (6.29)$$

In this case $\sigma = \sigma_0 \varepsilon \ln N$ and so the above inequality gives

$$|\omega^n(s_m)| \leq CN^{-2}, \quad 0 \leq m \leq N/2. \quad (6.30)$$

The similar bound on W_m^n is given by

$$|W_m^n| \leq CN^{-2}, \quad 0 \leq m \leq N/2. \quad (6.31)$$

The triangle inequality gives

$$|\omega^n(s_m) - W_m^n| \leq CN^{-2}, \quad 0 \leq m \leq N/2. \quad (6.32)$$

Now by using the classical argument on the interval $[1 - \sigma, 1]$, we obtain

$$|\mathfrak{L}^{\Delta t, h}(\omega^n(s_m) - W_m^n)| \leq CN^{-2}\varepsilon^{-3}, \quad N/2 + 1 \leq m \leq N. \quad (6.33)$$

Since the mesh spacing is $2\sigma/N$ for the given interval and $\sigma = \sigma_0\varepsilon \ln N$, so

$$\begin{aligned} |\mathfrak{L}^{\Delta t, h}(\omega^n(s_m) - W_m^n)| &\leq CN^{-2}\varepsilon^{-3} \\ &\leq CN^{-2}\varepsilon^{-1} \ln^2 N \\ &\leq CN^{-2} \ln^3 N. \end{aligned}$$

From the discrete maximum principle, we have

$$|\omega^n(s_m) - W_m^n| \leq CN^{-2} \ln^3 N. \quad (6.34)$$

Combining the error estimates of the two sub-intervals for the singular components given in (6.27) and (6.34), we obtain

$$|\omega^n(s_m) - W_m^n| \leq CN^{-2} \ln^3 N, \quad 0 \leq m \leq N. \quad (6.35)$$

On combining the estimates for smooth and singular components given in (6.24) and (6.35), using the triangle inequality we get

$$\|\mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n)\| \leq CN^{-2} \ln^3 N. \quad (6.36)$$

■

Theorem 6.4.2. *Let z and Z are the solutions of (6.1) and (6.19) respectively, then we have the following uniform error bound*

$$\sup_{\varepsilon \in (0,1]} \|Z - z\| \leq C(N^{-2} \ln^3 N + (\Delta t)). \quad (6.37)$$

Proof. Lemma 4.3.11 and Theorem 6.4.1. ■

6.5 Numerical results

In this section some numerical experiments are conducted to verify the suggested scheme is stable, more accurate and ε -uniformly convergent by considering two test examples.

Example 6.1. Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} + (2 - s^2) \frac{\partial z}{\partial s} + sz(s, t) + z(s, t - \tau) = 10t^2 \exp(-t)s(1 - s)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in [0, 2]$.

Example 6.2. Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} + (2 - s^2) \frac{\partial z}{\partial s} + (s + 1)(t + 1)z(s, t) + z(s, t - \tau) = 10t^2 \exp(-t)s(1 - s)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in [0, 2]$.

Since the exact solutions to the examples are not known, the double mesh principle is used to calculate the maximum pointwise absolute error. Consequently, the maximum pointwise absolute errors are calculated using the formula given in (4.43). The ε -uniform errors are calculated by the formula given in (4.44). The rate of convergence of the proposed method is computed by the formula given in (4.45). The ε -uniform rate of convergence of the developed method is calculated by the formula given in (4.46).

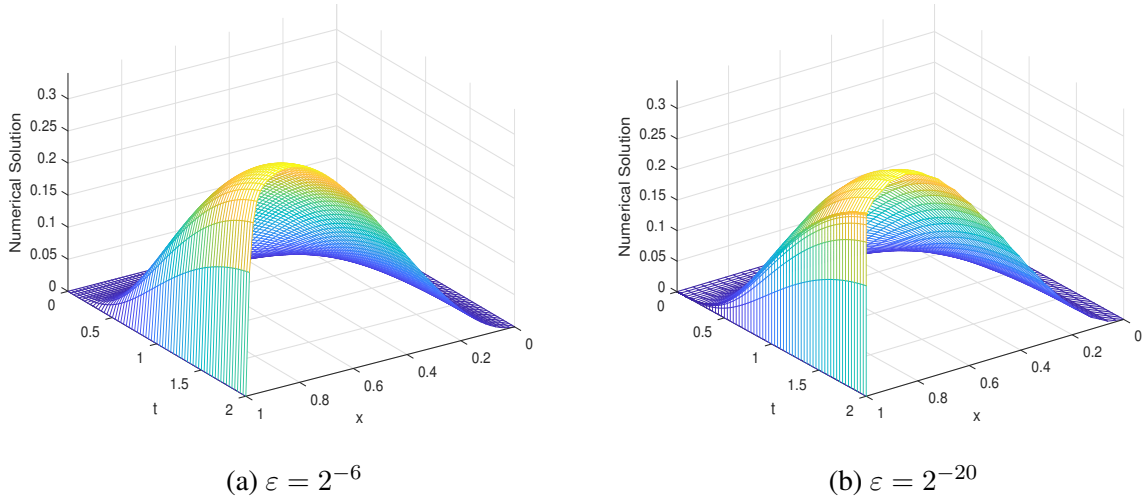


Figure 6.1: Numerical solution of the scheme for Example 6.1.

The numerical results delineated in Table 6.1 and Table 6.3 are obtained by taking $2N$ and $4M$ mesh size for different values of ε . From each table, we observe that the theoretical analysis of the scheme is confirmed. i.e. the developed scheme has ε -uniform with almost second order in space. Moreover, for different values of ε and an equal number of mesh points for Examples 6.1 and 6.2, respectively, $E_\varepsilon^{N,M}$, $E^{N,M}$, $r_\varepsilon^{N,M}$ and $r^{N,M}$ of the developed scheme are listed in Tables 6.2 and 6.4. In each table, the maximum pointwise absolute error and the corresponding rate of convergence calculated using a higher-order finite difference scheme on the piecewise uniform mesh by taking $\tau = 0.5\varepsilon$ are given in the last two rows.

Table 6.1: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, $r_{\varepsilon}^{N,M}$, and $r^{N,M}$ for Example 6.1 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	N=16 M= 10	32 40	64 120	128 480
2^{-8}	6.5777e-02 1.2961	2.6787e-02 1.4742	9.6414e-03 1.5539	3.2837e-03 –
2^{-10}	4.5519e-02 1.2343	1.9348e-02 1.2300	8.2483e-03 1.2115	3.6380e-03 –
2^{-12}	3.4227e-02 1.1504	1.5419e-02 1.6054	5.0674e-03 1.5612	1.7172e-03 –
2^{-14}	3.3335e-02 1.0516	1.6082e-02 1.5788	5.3835e-03 1.6723	1.6891e-03 –
2^{-16}	3.3663e-02 1.0501	1.6257e-02 1.5737	5.4615e-03 1.6467	1.7442e-03 –
2^{-18}	3.3713e-02 1.0483	1.6301e-02 1.5724	5.4812e-03 1.6414	1.7570e-03 –
2^{-20}	3.3595e-02 1.0423	1.6312e-02 1.5721	5.4861e-03 1.6400	1.7602e-03 –
2^{-22}	3.3043e-02 1.0182	1.6314e-02 1.5719	5.4873e-03 1.6398	1.7609e-03 –
2^{-24}	3.1891e-02 0.96730	1.6311e-02 1.5716	5.4876e-03 1.6397	1.7611e-03 –
2^{-26}	3.7873e-02 1.2167	1.6295e-02 1.5702	5.4877e-03 1.6396	1.7612e-03 –
$E^{N,M}$	6.5777e-02	2.6787e-02	9.6414e-03	3.6380e-03
$r^{N,M}$	1.2961	1.4742	1.4061	–

Table 6.2: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, $r_{\varepsilon}^{N,M}$, and $r^{N,M}$ for Example 6.1 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals N = M				
	32	64	128	256	512
2^{-6}	1.6741e-02 1.5255	5.8151e-03 1.7320	1.7506e-03 2.1738	3.8798e-04 2.2711	8.0381e-05 –
2^{-12}	1.5166e-02 1.6302	4.8992e-03 1.6684	1.5413e-03 1.2115	6.6556e-04 1.2112	2.8747e-04 –
2^{-18}	1.6050e-02 1.5948	5.3135e-03 1.7016	1.6336e-03 1.7169	4.9694e-04 1.8063	1.4209e-04 –
2^{-24}	1.6059e-02 1.5939	5.3200e-03 1.6998	1.6377e-03 1.7140	4.9921e-04 1.7976	1.4360e-04 –
2^{-30}	1.5733e-02 1.5645	5.3194e-03 1.6995	1.6378e-03 1.7140	4.9923e-04 1.7974	1.4362e-04 –
$E^{N,M}$	1.6741e-02	5.8151e-03	1.7506e-03	6.6556e-04	2.8747e-04
$r^{N,M}$	1.5255	1.7320	2.1738	1.2112	–

Table 6.3: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, $r_{\varepsilon}^{N,M}$, and $r^{N,M}$ for Example 6.2 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	N=16 M= 10	32 40	64 120	128 480
2^{-8}	5.1088e-02 1.0925	2.3957e-02 1.3656	9.2973e-03 1.5230	3.2350e-03 –
2^{-10}	3.4523e-02 1.0869	1.6252e-02 1.0502	7.8483e-03 1.1306	3.5845e-03 –
2^{-12}	2.5638e-02 1.4243	9.5527e-03 1.3055	3.8649e-03 1.2062	1.6751e-03 –
2^{-14}	2.3038e-02 1.6332	7.4267e-03 1.6106	2.4319e-03 1.5564	8.2684e-04 –
2^{-16}	2.2364e-02 1.7040	6.8642e-03 1.7495	2.0414e-03 1.8052	5.8414e-04 –
2^{-18}	2.2199e-02 1.7236	6.7215e-03 1.7915	1.9417e-03 1.8969	5.2139e-04 –
2^{-20}	2.2175e-02 1.7298	6.6856e-03 1.8025	1.9166e-03 1.9225	5.0558e-04 –
2^{-22}	2.2239e-02 1.7360	6.6763e-03 1.8052	1.9104e-03 1.9292	5.0161e-04 –
2^{-24}	2.2533e-02 1.7557	6.6727e-03 1.8056	1.9088e-03 1.9309	5.0062e-04 –
2^{-26}	2.3665e-02 1.8277	6.6666e-03 1.8047	1.9083e-03 1.9312	5.0037e-04 –
$E^{N,M}$	5.1088e-02	2.3957e-02	9.2973e-03	3.5845e-03
$r^{N,M}$	1.0925	1.3656	1.3750	–

Table 6.4: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, $r_{\varepsilon}^{N,M}$, and $r^{N,M}$ for Example 6.2 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals N = M				
	32	64	128	256	512
2^{-6}	6.0354e-03 1.4755	2.1704e-03 1.5697	7.3118e-04 1.7331	2.1995e-04 1.5026	7.7623e-05 –
2^{-12}	9.7572e-03 1.2515	4.0980e-03 1.1435	1.8550e-03 1.0871	8.7315e-04 1.1208	4.0151e-04 –
2^{-18}	6.9258e-03 1.6722	2.1732e-03 1.6363	6.9907e-04 1.5178	2.4413e-04 1.3632	9.4897e-05 –
2^{-24}	6.8770e-03 1.6840	2.1402e-03 1.6578	6.7827e-04 1.5492	2.3176e-04 1.3992	8.7871e-05 –
2^{-30}	6.7596e-03 1.6606	2.1381e-03 1.6571	6.7792e-04 1.5497	2.3156e-04 1.3996	8.7768e-05 –
$E^{N,M}$	9.7572e-03	4.0980e-03	1.8550e-03	8.7315e-04	4.0151e-04
$r^{N,M}$	1.2515	1.1435	1.0871	1.1208	–

Table 6.5: Comparison of $E_{\varepsilon}^{N,M}$ and $r_{\varepsilon}^{N,M}$ for Example 6.1 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals $N = M$			
	32	64	128	256
Presented Method				
2^{-16}	1.6005e-02	5.2937e-03	1.6208e-03	4.8908e-04
	1.5962	1.7076	1.7286	—
2^{-18}	1.6050e-02	5.3135e-03	1.6336e-03	4.9694e-04
	1.5948	1.7016	1.7169	—
2^{-20}	1.6060e-02	5.3185e-03	1.6367e-03	4.9868e-04
	1.5944	1.7002	1.7146	—
2^{-22}	1.6062e-02	5.3197e-03	1.6375e-03	4.9910e-04
	1.5942	1.6998	1.7141	—
Negero and Duressa (2022b)				
2^{-16}	6.3269e-03	3.3518e-03	1.7230e-03	8.7325e-04
	0.91656	0.96001	0.98046	—
2^{-18}	6.3269e-03	3.3518e-03	1.7230e-03	8.7325e-04
	0.91656	0.96001	0.98046	—
2^{-20}	6.3269e-03	3.3518e-03	1.7230e-03	8.7325e-04
	0.91656	0.96001	0.98046	—
2^{-22}	6.3269e-03	3.3518e-03	1.7230e-03	8.7325e-04
	0.91656	0.96001	0.98046	—

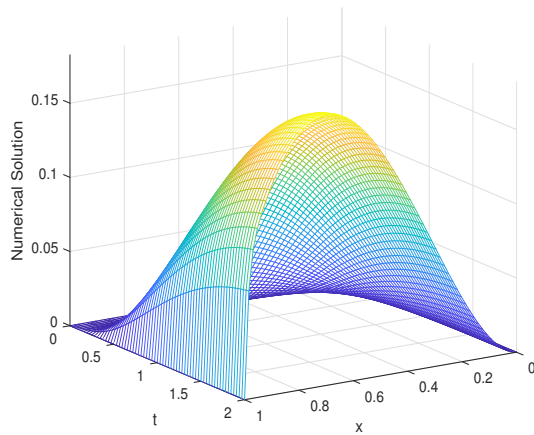
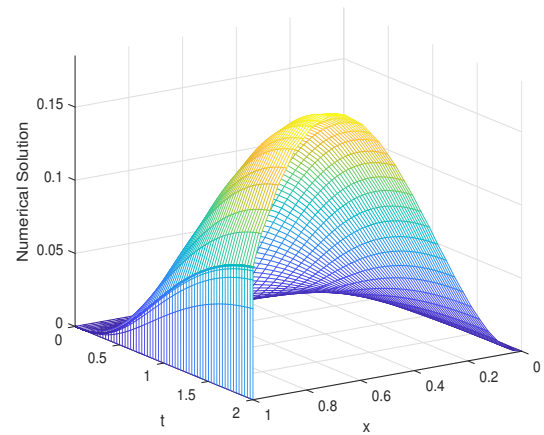
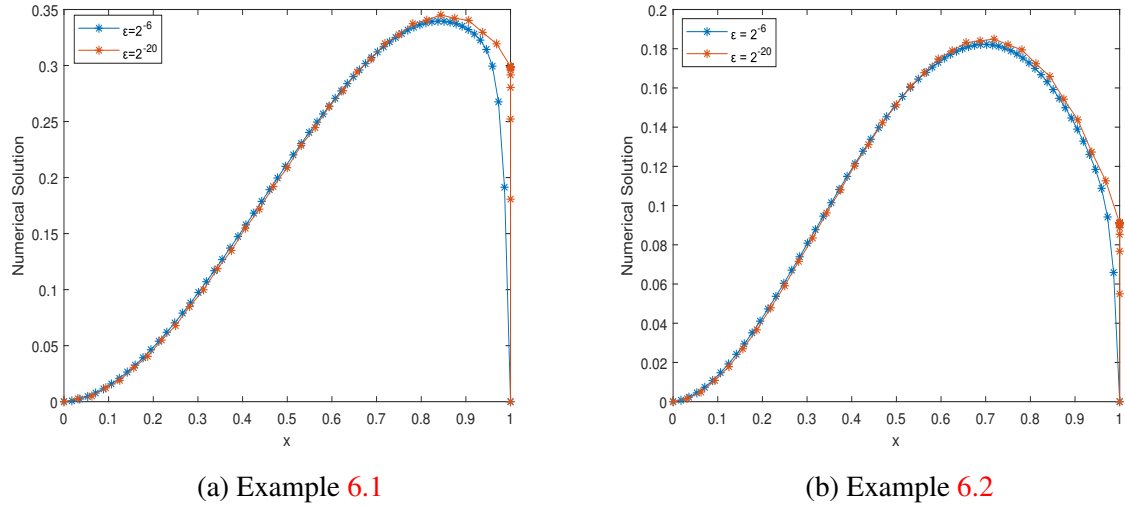
(a) $\varepsilon = 2^{-6}$ (b) $\varepsilon = 2^{-20}$

Figure 6.2: Numerical solution of the scheme for Example 6.2.

Figure 6.1 and Figure 6.2 display the surface plot of the solution of Examples 6.1 and 6.2 for distinct values of ε are given, respectively. In else way Figure 6.3 display the line plot numerical solutions of the developed scheme for Examples 6.1 and 6.2 for distinct values of ε . From the figures, we observe that a strong boundary layer is created on the right side of the spatial domain as ε becomes small starting from $\varepsilon = 2^{-6}$ to $\varepsilon = 2^{-20}$.

In Table 6.5, the comparison with the results of the proposed scheme with the existing

Figure 6.3: Effect of ε on the solution profiles.

recently published work of [Negero and Duressa \(2022b\)](#) is given for Example 6.1. As one follows from this table when the values of $\varepsilon \rightarrow 0$ and the number of mesh increased the developed scheme holds better accuracy than the existing published work.

6.6 Summary

A parameter uniform numerical scheme is constructed to solve the singularly perturbed convection-diffusion problem that involves a time delay term. The time derivative term of the problem is approximated by using the backward Euler scheme, and on a piecewise uniform Shishkin mesh, the higher order finite difference method is applied in the space direction. Using the comparison principle, the stability of the discrete scheme is examined and analyzed. The stability and uniform convergence of the method are discussed theoretically. Numerical results are delineated by applying maximum pointwise absolute error and ε -uniform rate of convergence in tables which are in acceptable agreement with the theoretical analysis. The proposed method contributes more accurate, stable, and ε -uniform with almost second order in space and first order in time. In the subsequent chapter, we discussed the spline method with the problem of having a small time delay on a uniform mesh.

CHAPTER SEVEN

FITTED TENSION SPLINE METHOD FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATION WITH SMALL TIME DELAY

In this chapter, we proposed a fitted tension spline numerical scheme to achieve more accurate results. To improve the accuracy of the obtained results in Chapter 4 and Chapter 6, we suggested the fitted tension spline method. In this chapter, we utilized a method which combines the backward Euler scheme in the temporal direction and the fitted spline method in the spatial direction on uniform meshes. It was proved that the proposed scheme is uniformly convergent with the rate of convergence of order one in both space and time. Three numerical examples were considered to validate the applicability of the proposed scheme. Comparison was made with the results available in the literature and preceding chapters. The chapter also provided graphical information on the boundary layer behaviour of the solutions.

7.1 Governing problem

On the domain $\mathfrak{D} = \Omega \times \Lambda = (0, 1) \times (0, T]$, we considered a SPP of the form

$$\begin{cases} z_t(s, t) + \mathfrak{L}_\varepsilon z(s, t) = -\kappa(s, t)z(s, t - \tau) + \gamma(s, t), (s, t) \in \mathfrak{D}, \\ z(s, t) = \psi_b(s, t), (s, t) \in \eta_b = [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), t \in \eta_l = \{(0, t) : 0 \leq t \leq T\}, \\ z(1, t) = \psi_r(t), t \in \eta_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (7.1)$$

where $\mathfrak{L}_\varepsilon z(s, t) = -\varepsilon z_{ss}(s, t) + \beta(s, t)z_s(s, t) + \chi(s, t)z(s, t)$.

$0 < \varepsilon \ll 1$ and $\tau > 0$ are given constants, the functions $\beta(s, t), \chi(s, t), \kappa(s, t), \gamma(s, t)$ on $\bar{\mathfrak{D}} = \bar{\Omega} \times \bar{\Lambda} = [0, 1] \times [0, T]$, and $\psi_b(s, t), \psi_l(t), \psi_r(t)$ on $\eta = \eta_l \cup \eta_r \cup \eta_b$ are sufficiently smooth and bounded that satisfy the assumptions: (i) $0 < \varpi \leq \chi(s, t), 0 < \varphi \leq \kappa(s, t), 0 < \mu \leq \beta(s, t), 0 \leq \chi(s, t) + \kappa(s, t), (s, t) \in \bar{\mathfrak{D}}$. (ii) $0 \leq \chi(s, t) + \kappa(s, t), \beta(s, t) \leq -\mu < 0, (s, t) \in \bar{\mathfrak{D}}$. Under the assumption (i), the solution of the problem (7.1) exhibits a boundary layer of width $O(\varepsilon)$ along $s = 1$ [Das and Natesan \(2015\)](#) and under the assumption (ii) the problem (7.1) exhibits a boundary layer of width $O(\varepsilon)$ along $s = 0$ [Kumar et al., 2019](#).

7.2 Properties of the continuous solution

The existence and uniqueness of the solution of (7.1) can be ensured by the compatibility conditions and assumption stated in (4.2) - (4.3). When the boundary layer exhibits in the right

end of the domain the corresponding reduced of problem (7.1) has the form (5.2). On the other hand, when the boundary layer exhibits in the left end of the domain the corresponding reduced problem of (7.1) has the form (4.4). The problem (7.1) satisfies the maximum principle stated in Lemma 4.2.1 and the stability result of its solution $z(s, t)$ satisfies the bound given in Lemma 4.2.2.

Lemma 7.2.1. *Let $z(s, t)$ be the solution of (7.1), then its derivative satisfy the estimate*

$$\left| \frac{\partial^{i+l} z(s, t)}{\partial s^i \partial t^l} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(\frac{-\mu}{\varepsilon} s \right) \right), \quad (s, t) \in \mathfrak{D}, \quad 0 \leq i + l \leq 5, \quad (7.2)$$

for the boundary layer along $s = 0$ and

$$\left| \frac{\partial^{i+l} z(s, t)}{\partial s^i \partial t^l} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(\frac{-\mu}{\varepsilon} (1 - s) \right) \right), \quad (s, t) \in \mathfrak{D}, \quad 0 \leq i + l \leq 5, \quad (7.3)$$

for the boundary layer along $s = 1$.

Proof. One can refer to Lemma 4.2.9 for the proof of (7.2) and Lemma 5.2.4 for the proof of (7.3). It is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives can be obtained. ■

7.3 Numerical scheme

In this section, we discuss the numerical method employed to solve the problem (7.1). The method involves a backward Euler method in the time direction, fitted tension spline method for the space derivatives.

7.3.1 Temporal semi-discretization

The time interval $[0, T]$ is divided uniformly with M mesh points as $\Lambda^M = \{t_n = n\Delta t, n = 0(1)M, t_M = T, \Delta t = T/M\}$. The approximation of $z(s, t)$ in temporal discretization at the grid point n is denoted as $Z^n(s)$. Using the backward Euler scheme for the time derivative of (4.15), and simplifying, we get

$$\begin{cases} \mathfrak{L}^{\Delta t} Z^n(s) \equiv -\varepsilon \frac{d^2 Z^n(s)}{ds^2} + \beta(s, t_n) \frac{dZ^n(s)}{ds} + P(s, t_n) Z^n(s) = R(s, t_n), \\ Z^n(0) = \psi_l(t_n), \quad Z^n(1) = \psi_r(t_n), \quad 0 \leq n \leq M, \end{cases} \quad (7.4)$$

where $P(s, t_n) = \left(\frac{1-\tau\kappa(s, t_n)}{\Delta t} + \kappa(s, t_n) + \chi(s, t_n) \right)$ and $R(s, t_n) = \gamma(s, t_n) + \left(\frac{1-\tau\kappa(s, t_n)}{\Delta t} \right) Z^{n-1}(s)$. Rewrite (7.4) as

$$\begin{cases} \mathfrak{L}^{\Delta t} Z(s) \equiv -\varepsilon \frac{d^2 Z(s)}{ds^2} + \beta(s, t_n) \frac{dZ(s)}{ds} + P(s, t_n) Z(s) = R(s, t_n), \\ Z(0) = \psi_l(t_n), \quad Z(1) = \psi_r(t_n), \quad 0 \leq n \leq M, \end{cases} \quad (7.5)$$

where $Z(s) = Z^n(s) \approx Z(s, t_n)$.

At each time step the local error is defined as $e_n(s) := z(s, t_n) - Z^n(s)$, $0 \leq n \leq M$. Its local error bound is given in Lemma 4.3.10, whereas the global error bound associated with the method is given in Lemma 4.3.11.

Lemma 7.3.2. *For $n = 0(1)M - 1$, $0 \leq i \leq 4$, then the derivative of the solution of (7.4) satisfy the following bounds*

$$\left| \frac{d^i Z^n(s)}{ds^i} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(-\frac{\mu}{\varepsilon} s \right) \right), \quad s \in \bar{\Omega}, \quad (7.6)$$

for the boundary layer along $s = 0$ and

$$\left| \frac{d^i Z^n(s)}{ds^i} \right| \leq C \left(1 + \varepsilon^{-i} \exp \left(-\frac{\mu}{\varepsilon} (1 - s) \right) \right), \quad s \in \bar{\Omega}, \quad (7.7)$$

for the boundary layer along $s = 1$.

Proof. One can refer Lemma 4.2.6 for the proof of (7.6) and Lemma 4.2.6 for the proof of (7.7). It is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives can be obtained. ■

7.3.2 Spatial discretization

The spatial domain $[0, 1]$ into N equal number of sub-intervals with the length of h is given by $0 = s_0, s_1, \dots, s_N = 1$ and $s_m = mh$, $m = 0(1)N$.

Description of the method

A function $\mathfrak{S}(s, \iota)$ in $C(\bar{\mathfrak{D}})$ interpolates $z(s)$ at the mesh points s_m that depends on the compression parameter ι and reduced to a cubic spline on $\bar{\Omega}$ as $\iota \rightarrow 0$ is referred as a parametric cubic spline function (Khan and Aziz, 2005). For each interval $[s_m, s_{m+1}]$, $m = 1(1)N - 1$ the spline function $\mathfrak{S}(s, \iota) = \mathfrak{S}(s)$ has the form

$$\begin{aligned} \mathfrak{S}_{ss}(s, t_n) - \iota \mathfrak{S}(s, t_n) = & [\mathfrak{S}_{ss}(s_m, t_n) - \iota \mathfrak{S}(s_m, t_n)] \frac{s_{m+1} - s}{h} \\ & + [\mathfrak{S}_{ss}(s_{m+1}, t_n) - \iota \mathfrak{S}(s_{m+1}, t_n)] \frac{s - s_m}{h}, \end{aligned} \quad (7.8)$$

where $\mathfrak{S}(s_m, t_n) = Z_m^n$ for $\iota > 0$ is called cubic spline in tension. Putting $\sqrt{\iota} = \frac{\alpha}{h}$ and from the homogeneous part of (7.8), we get

$$\mathfrak{S}_1(s, t_n) = \mathfrak{A} e^{\frac{\alpha}{h}(s-s_m)} + \mathfrak{B} e^{\frac{\alpha}{h}(s_{m+1}-s)}, \quad (7.9)$$

for arbitrary constant $\mathfrak{A}$ and $\mathfrak{B}$. Let the non-homogeneous part be given as

$$\mathfrak{S}_2(s, t_n) = -1/\iota \left([\mathfrak{S}_{ss}(s_m, t_n) - \iota \mathfrak{S}(s_m, t_n)] \frac{s_{m+1} - s}{h} + [\mathfrak{S}_{ss}(s_{m+1}, t_n) - \iota \mathfrak{S}(s_{m+1}, t_n)] \frac{s - s_m}{h} \right).$$

Thus, we have

$$\mathfrak{S}_2(s, t_n) = -\left(\frac{h}{\alpha}\right)^2 [\mathfrak{M}_m - \left(\frac{\alpha}{h}\right)^2 Z_m^n] \frac{s_{m+1} - s}{h} - \left(\frac{h}{\alpha}\right)^2 [\mathfrak{M}_{m+1} - \left(\frac{\alpha}{h}\right)^2 Z_{m+1}^n] \frac{s - s_m}{h}, \quad (7.10)$$

where $\mathfrak{M}_m = \mathfrak{S}_{ss}(s_m, t_n)$ and $\mathfrak{M}_{m+1} = \mathfrak{S}_{ss}(s_{m+1}, t_n)$. Using (7.9) and (7.10), we obtain

$$\begin{aligned} \mathfrak{S}(s, t_n) = & \mathfrak{A} e^{\frac{\alpha}{h}(s-s_m)} + \mathfrak{B} e^{\frac{\alpha}{h}(s_{m+1}-s)} - \left(\frac{h}{\alpha}\right)^2 [\mathfrak{M}_m - \left(\frac{\alpha}{h}\right)^2 Z_m^n] \frac{s_{m+1} - s}{h} \\ & - \left(\frac{h}{\alpha}\right)^2 [\mathfrak{M}_{m+1} - \left(\frac{\alpha}{h}\right)^2 Z_{m+1}^n] \frac{s - s_m}{h}. \end{aligned} \quad (7.11)$$

From (7.11), the arbitrary constants can be determined from the interpolation conditions $\mathfrak{S}(s_m, t_n)$ and $\mathfrak{S}(s_{m+1}, t_n)$, then

$$\begin{aligned} \mathfrak{S}(s, t_n) = & \frac{h^2}{\alpha^2 \sinh \alpha} \left[\mathfrak{M}_{m+1} \sinh \left(\frac{\alpha(s - s_m)}{h} \right) + \mathfrak{M}_m \sinh \left(\frac{\alpha(s_{m+1} - s)}{h} \right) \right] \\ & - \left[\frac{h}{\alpha^2} \mathfrak{M}_m - \frac{1}{h} Z_m^n \right] (s_{m+1} - s) - \left[\frac{h}{\alpha^2} \mathfrak{M}_{m+1} - \frac{1}{h} Z_{m+1}^n \right] (s - s_m). \end{aligned} \quad (7.12)$$

The derivative of (7.12) at (s_m, t_n) on the interval $[s_m, s_{m+1}]$ is

$$\mathfrak{S}(s_m^+, t_n) = \frac{Z_{m+1}^n - Z_m^n}{h} - \frac{h}{\alpha^2} [\mathfrak{M}_{m+1}(1 - \alpha \operatorname{csch} \alpha) + \mathfrak{M}_m(\alpha \coth \alpha - 1)], \quad (7.13)$$

and on the interval $[s_{m-1}, s_m]$ is

$$\mathfrak{S}(s_m^-, t_n) = \frac{Z_m^n - Z_{m-1}^n}{h} + \frac{h}{\alpha^2} [\mathfrak{M}_m(\alpha \coth \alpha - 1) + \mathfrak{M}_{m-1}(1 - \alpha \operatorname{csch} \alpha)]. \quad (7.14)$$

Equating (7.13) and (7.14) at the point s_m , we get

$$\frac{Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n}{h^2} = \frac{\mathfrak{M}_{m-1}(1 - \alpha \operatorname{csch} \alpha)}{\alpha^2} + \frac{2\mathfrak{M}_m(\alpha \coth \alpha - 1)}{\alpha^2} + \frac{\mathfrak{M}_{m+1}(1 - \alpha \operatorname{csch} \alpha)}{\alpha^2}. \quad (7.15)$$

Simplifying we obtain

$$\frac{Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n}{h^2} = \alpha_1 \mathfrak{M}_{m-1} + 2\alpha_2 \mathfrak{M}_m + \alpha_1 \mathfrak{M}_{m+1}, \quad m = 1(1)N - 1, \quad (7.16)$$

where $\alpha_1 = \alpha^{-2}(1 - \alpha \operatorname{csch} \alpha)$ and $\alpha_2 = \alpha^{-2}(\alpha \coth \alpha - 1)$. The continuity condition in (7.16) ensures the continuity of the first derivative of $\mathfrak{S}(s)$ at interior nodes. Using $Z_m'' = \mathfrak{M}_m =$

$\mathfrak{S}_{ss}(s_m, t_n)$ into (7.5), we obtain

$$\begin{cases} -\varepsilon \mathfrak{M}_m = -\beta(s_m, t_n) Z'_m - P(s_m, t_n) Z_m + R(s_m, t_n), \\ -\varepsilon \mathfrak{M}_{m\pm 1} = -\beta(s_{m\pm 1}, t_n) Z'_{m\pm 1} - P(s_{m\pm 1}, t_n) Z_{m\pm 1} + R(s_{m\pm 1}, t_n). \end{cases} \quad (7.17)$$

The local truncation error $\mathfrak{T}_1(h)$ obtained from (7.16) is

$$\mathfrak{T}_1(h) = \frac{h^4}{3}(-2\alpha_1 + \alpha_2)\beta(s_m, t_n)Z'''(s_m) + \frac{h^4}{12}(1 - 12\alpha_1)\beta(s_m, t_n)\varepsilon Z^{(4)}(s_m) + O(h^6), \quad (7.18)$$

for any choice of α_1 and α_2 whose sum is $1/2$. For the choice $\alpha_1 = 1/12$ and $\alpha_2 = 5/12$, we have

$$\mathfrak{T}_1(h) = \frac{\varepsilon h^6}{240} Z^{(6)}(s_m), \quad s_m \in [s_{m-1}, s_{m+1}]. \quad (7.19)$$

Using Taylor series expansions we approximate $Z'_{m\pm 1}$ and Z'_m as

$$\begin{cases} Z'_m \approx \frac{Z_{m+1} - Z_{m-1}}{2h}, \\ Z'_{m+1} \approx \frac{3Z_{m+1} - 4Z_m + Z_{m-1}}{2h}, \\ Z'_{m-1} \approx \frac{-Z_{m+1} + 4Z_m - 3Z_{m-1}}{2h}. \end{cases} \quad (7.20)$$

Substituting (7.20) into (7.17) and substituting the resulting equation into (7.16) and rearranging, we obtain

$$\begin{aligned} \mathfrak{L}^{\Delta t, h} Z_m^n &\equiv \frac{-\varepsilon}{h^2} (Z_{m-1} - 2Z_m + Z_{m+1}) + \frac{\alpha_1 \beta(s_{m-1}, t_n)}{2h} (-3Z_{m-1} + 4Z_m - Z_{m+1}) \\ &\quad + \frac{2\alpha_2 \beta(s_m, t_n)}{2h} (Z_{m+1} - Z_{m-1}) + \frac{\alpha_1 \beta(s_{m+1}, t_n)}{2h} (Z_{m-1} - 4Z_m + 3Z_{m+1}) \\ &\quad + \alpha_1 P(s_{m-1}, t_n) Z_{m-1} + 2\alpha_2 P(s_m, t_n) Z_m + \alpha_1 P(s_{m+1}, t_n) Z_{m+1} \\ &= \alpha_1 R(s_{m-1}, t_n) + 2\alpha_2 R(s_m, t_n) + \alpha_1 R(s_{m+1}, t_n) + \mathfrak{T}_1(h), \end{aligned} \quad (7.21)$$

where $Z_m^n \approx Z^n(s_m)$.

Computing the exponential fitting factor

In this part, we introduce the exponential fitting factor σ to handle the effect of ε in the layer. Multiplying (7.21) by σ with term containing ε , we get

$$\begin{aligned} & \frac{-\varepsilon\sigma}{h^2}(Z_{m-1} - 2Z_m + Z_{m+1}) + \frac{\alpha_1\beta(s_{m-1}, t_n)}{2h}(-3Z_{m-1} + 4Z_m - Z_{m+1}) \\ & + \frac{2\alpha_2\beta(s_m, t_n)}{2h}(Z_{m+1} - Z_{m-1}) + \frac{\alpha_1\beta(s_{m+1}, t_n)}{2h}(Z_{m-1} - 4Z_m + 3Z_{m+1}) \\ & + \alpha_1 P(s_{m-1}, t_n)Z_{m-1} + 2\alpha_2 P(s_m, t_n)Z_m + \alpha_1 P(s_{m+1}, t_n)Z_{m+1} \\ & = \alpha_1 R(s_{m-1}, t_n) + 2\alpha_2 R(s_m, t_n) + \alpha_1 R(s_{m+1}, t_n) + \mathfrak{T}_1(h). \end{aligned} \quad (7.22)$$

Thus, we consider two cases of the boundary layers.

Case I: Left-end boundary layer for $\beta(s, t) < 0$.

Multiplying (7.22) by h and taking a limit as $h \rightarrow 0$, which gives

$$\begin{aligned} & -\frac{\sigma}{\rho} \lim_{h \rightarrow 0} (Z_{m-1} - 2Z_m + Z_{m+1}) + \frac{\alpha_1\beta(0)}{2} \lim_{h \rightarrow 0} (-3Z_{m-1} + 4Z_m - Z_{m+1}) \\ & + \alpha_2\beta(0) \lim_{h \rightarrow 0} (Z_{m+1} - Z_{m-1}) + \frac{\alpha_1\beta(0)}{2} \lim_{h \rightarrow 0} (Z_{m-1} - 4Z_m + 3Z_{m+1}) = 0. \end{aligned} \quad (7.23)$$

Using the simplified zero-order asymptotic solution given in (4.31) into (7.23), we get

$$\sigma_0 = \rho\beta(0)(\alpha_1 + \alpha_2) \coth\left(\frac{\rho\beta(0)}{2}\right). \quad (7.24)$$

Case II: Right-end boundary layer for $\beta(s, t) > 0$.

Multiplying (7.22) by h and taking a limit as $h \rightarrow 0$, which gives

$$\begin{aligned} & -\frac{\sigma}{\rho} \lim_{h \rightarrow 0} (Z_{m-1} - 2Z_m + Z_{m+1}) + \frac{\alpha_1\beta(1)}{2} \lim_{h \rightarrow 0} (-3Z_{m-1} + 4Z_m - Z_{m+1}) \\ & + \alpha_2\beta(1) \lim_{h \rightarrow 0} (Z_{m+1} - Z_{m-1}) + \frac{\alpha_1\beta(1)}{2} \lim_{h \rightarrow 0} (Z_{m-1} - 4Z_m + 3Z_{m+1}) = 0. \end{aligned} \quad (7.25)$$

Using the simplified zero-order asymptotic solution given in (5.28) into (7.25), we obtain

$$\sigma_N = \rho\beta(1)(\alpha_1 + \alpha_2) \coth\left(\frac{\rho\beta(1)}{2}\right). \quad (7.26)$$

Therefore, we take the variable exponential fitting factor

$$\sigma_m = \rho\beta(s_m)(\alpha_1 + \alpha_2) \coth\left(\frac{\rho\beta(s_m)}{2}\right). \quad (7.27)$$

7.3.3 The discrete scheme

Inducing the obtained exponential fitting factor in (7.27) into (7.21) for $m = 1(1)N - 1$ and $n = 1(1)M - 1$, we obtain the fully discrete scheme

$$\mathfrak{L}^{\Delta t, h} Z_m^n = \alpha_1 R(s_{m-1}, t_n) + 2\alpha_2 R(s_m, t_n) + \alpha_1 R(s_{m+1}, t_n), \quad (7.28)$$

where $\mathfrak{L}^{\Delta t, h} Z_m^n = \frac{-\varepsilon\sigma_m}{h^2}(Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n) + \frac{\alpha_1\beta(s_{m-1}, t_n)}{2h}(-3Z_{m-1}^n + 4Z_m^n - Z_{m+1}^n) + \frac{2\alpha_2\beta(s_m, t_n)}{2h}(Z_{m+1}^n - Z_{m-1}^n) + \frac{\alpha_1\beta(s_{m+1}, t_n)}{2h}(Z_{m-1}^n - 4Z_m^n + 3Z_{m+1}^n) + \alpha_1 P(s_{m-1}, t_n)Z_{m-1}^n + 2\alpha_2 P(s_m, t_n)Z_m^n + \alpha_1 P(s_{m+1}, t_n)Z_{m+1}^n$.

In the explicit form, we write (7.28) as

$$\mathfrak{D}_m^- Z_{m-1}^n + \mathfrak{D}_m^0 Z_m^n + \mathfrak{D}_m^+ Z_{m+1}^n = \mathfrak{K}_m^n, \quad (7.29)$$

where

$$\begin{aligned} \mathfrak{D}_m^- &= -\frac{\varepsilon\sigma_m}{h^2} - \frac{3\alpha_1\beta(s_{m-1}, t_n)}{2h} - \frac{\alpha_2\beta(s_m, t_n)}{h} + \frac{\alpha_1\beta(s_{m+1}, t_n)}{2h} + \alpha_1 P(s_{m-1}, t_n), \\ \mathfrak{D}_m^0 &= \frac{2\varepsilon\sigma_m}{h^2} + \frac{2\alpha_1\beta(s_{m-1}, t_n)}{h} - \frac{2\alpha_1\beta(s_{m+1}, t_n)}{h} + 2\alpha_2 P(s_m, t_n), \\ \mathfrak{D}_m^+ &= -\frac{\varepsilon\sigma_m}{h^2} - \frac{\alpha_1\beta(s_{m-1}, t_n)}{2h} + \frac{\alpha_2\beta(s_m, t_n)}{h} + \frac{3\alpha_1\beta(s_{m+1}, t_n)}{2h} + \alpha_1 P(s_{m+1}, t_n), \\ H_m^n &= \alpha_1 R(s_{m-1}, t_n) + 2\alpha_2 R(s_m, t_n) + \alpha_1 R(s_{m+1}, t_n). \end{aligned}$$

7.4 Convergence analysis

The discrete scheme given in (7.28) satisfies the discrete maximum principle stated in Lemma 4.4.13 of Chapter 4.

Lemma 7.4.3. (Stability result) *If the solution of (7.28) be Z_m^n , then it satisfies*

$$|Z_m^n| \leq \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\lambda} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\},$$

where $P(s_m, t_n) \geq \lambda > 0$.

Proof. let $\Pi = \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\lambda} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\}$ and set the barrier functions $\vartheta_{m,n}^\pm$ by $\vartheta_{m,n}^\pm = \Pi \pm Z_m^n$.

On the boundaries, we obtain $\vartheta_{0,n}^\pm = \Pi \pm Z_0^n = \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\lambda} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \pm \psi_l(t_n) \geq 0$,

$\vartheta_{N,n}^\pm = \Pi \pm Z_N^n = \frac{\|\mathfrak{L}^{\Delta t, h} Z_m^n\|}{\lambda} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \pm \psi_r(t_n) \geq 0$. On the discretized

spatial domain s_m , $m = 1(1)N - 1$, we have

$$\begin{aligned}
\mathfrak{L}^{\Delta t, h} \vartheta_{m,n}^{\pm} &= -\frac{(\varepsilon \sigma_m)}{h} (\Pi \pm Z_{m-1}^n - 2(\Pi \pm Z_m^n) + \Pi \pm Z_{m+1}^n) \\
&\quad + \frac{\alpha_1 \beta(s_{m-1})}{2h} (-3(\Pi \pm Z_{m-1}^n) + 4(\Pi \pm Z_m^n) - (\Pi \pm Z_{m+1}^n)) \\
&\quad + \frac{\alpha_2 \beta(s_m)}{h} (\Pi \pm Z_{m+1}^n - (\Pi \pm Z_{m-1}^n)) \\
&\quad + \frac{\alpha_1 \beta(s_{m+1})}{2h} (\Pi \pm Z_{m-1}^n - 4(\Pi \pm Z_m^n) + 3(\Pi \pm Z_{m+1}^n)) \\
&\quad + \alpha_1 P(s_{m-1}, t_n) (\Pi \pm Z_{m-1}^n) + 2\alpha_2 P(s_m, t_n) (\Pi \pm Z_m^n) + \alpha_1 P(s_{m+1}, t_n) (\Pi \pm Z_{m+1}^n) \\
&= (\alpha_1 P(s_{m-1}, t_n) + 2\alpha_2 P(s_m, t_n) + \alpha_1 P(s_{m+1}, t_n)) \Pi \pm \mathcal{L}_\varepsilon^{\Delta t, h} Z_m^n \\
&= (\alpha_1 P(s_{m-1}, t_n) + 2\alpha_2 P(s_m, t_n) + \alpha_1 P(s_{m+1}, t_n)) \left(\frac{\|\mathcal{L}_\varepsilon^{\Delta t, h} Z_i^n\|}{\lambda} \right. \\
&\quad \left. + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \right) \pm \alpha_1 R(s_{m-1}, t_n) + 2\alpha_2 R(s_m, t_n) + \alpha_1 R(s_{m+1}, t_n) \geq 0.
\end{aligned} \tag{7.30}$$

From the discrete maximum principle, we get $\vartheta_{m,n}^{\pm} \geq 0$, $m = 0(1)N$. Hence, the necessary bound is satisfied. $\blacksquare$

Using Taylor's series approximation, we have the bounds

$$\begin{aligned}
&\left| -\left(\frac{d}{ds^2} - \delta_s^2\right) Z^n(s_m) \right| \leq Ch^2 \left\| \frac{d^4 Z^n(s_m)}{ds^4} \right\|, \\
&\left| \frac{dZ^n(s_{m-1})}{ds} - \left(\frac{-Z_{m+1}^n + 4Z_m^n - 3Z_{m-1}^n}{2h} \right) \right| \leq Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\|, \\
&\left| \frac{dZ^n(s_{m+1})}{ds} - \left(\frac{3Z_{m+1}^n - 4Z_m^n + Z_{m-1}^n}{2h} \right) \right| \leq Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\|, \\
&\left| \left(\frac{d}{ds} - \delta_s^0\right) Z^n(s_m) \right| \leq Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\|, \quad \left| \delta_s^2 Z^n(s_m) \right| \leq C \left\| \frac{d^2 Z^n(s_m)}{ds^2} \right\|,
\end{aligned} \tag{7.31}$$

where $\|Z^{(p)}(s_m)\| = \max_{s_0 \leq s_m \leq s_N} |Z_n^{(p)}(s_m)|$, $p = 2, 3, 4$.

For the constants $\rho > 0$, C_1 and C_2 , we have

$$C_1 \frac{\rho^2}{\rho + 1} \leq \rho \coth(\rho) - 1 \leq C_2 \frac{\rho^2}{\rho + 1}. \tag{7.32}$$

Following (7.32), we have

$$\begin{aligned}
|\varepsilon [\rho \beta(1)(\alpha_1 + \alpha_2) \coth(\frac{\rho \beta(1)}{2}) - 1] \delta_s^2 Z^n(s_m)| &\leq \varepsilon \frac{C(h/\varepsilon)^2}{h/\varepsilon + 1} \left\| \frac{d^2 Z^n(s_m)}{ds^2} \right\| \\
&= \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2 Z^n(s_m)}{ds^2} \right\|,
\end{aligned} \tag{7.33}$$

since $\alpha_1 + \alpha_2 \leq 1/2$.

The next theorem provides the error bound in the space direction for the boundary layer along $s = 1$.

Theorem 7.4.1. *Let $Z^n(s)$ be the solution of (7.4) and the solution Z_m^n of (7.28) satisfies the following error estimate*

$$|\mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n)| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3} \exp \left(-\frac{\mu}{\varepsilon}(1 - s_m) \right) \right). \quad (7.34)$$

Proof. In the spatial direction, the local truncation error is given by

$$\begin{aligned} \left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| &\leq \left| -\varepsilon \left(\frac{d}{ds^2} - \sigma_N \delta_s^2 \right) Z^n(s_m) + \alpha_1 \beta(s_{m-1}, t_n) \left(\frac{dZ^n(s_{m-1})}{ds} \right. \right. \\ &\quad \left. \left. - \left(\frac{-3Z_{m-1}^n + 4Z_m^n - Z_{m+1}^n}{2h} \right) \right) + 2\alpha_2 \beta(s_m, t_n) \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_m) \right. \\ &\quad \left. + \alpha_1 \beta(s_{m+1}, t_n) \left(\frac{dZ^n(s_{m+1})}{ds} - \left(\frac{Z_{m-1}^n - 4Z_m^n + 3Z_{m+1}^n}{2h} \right) \right) \right| + |\mathfrak{T}_1(h)| \\ &\leq \left| \varepsilon [\rho \beta(1)(\alpha_1 + \alpha_2) \coth(\beta(1)\frac{\rho}{2}) - 1] \delta_s^2 Z^n(s_m) \right| + \left| \varepsilon \left(\frac{d^2}{ds^2} - \delta_s^2 \right) Z^n(s_m) \right| \\ &\quad + \left| \alpha_1 \beta(s_{m-1}, t_n) \left(\frac{dZ^n(s_{m-1})}{ds} - \left(\frac{-3Z_{m-1}^n + 4Z_m^n - Z_{m+1}^n}{2h} \right) \right) \right| \\ &\quad + \left| 2\alpha_2 \beta(s_m, t_n) \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_m) \right| + \left| \alpha_1 \beta(s_{m+1}, t_n) \right. \\ &\quad \left. \left(\frac{dZ^n(s_{m+1})}{ds} - \left(\frac{Z_{m-1}^n - 4Z_m^n + 3Z_{m+1}^n}{2h} \right) \right) \right| + |\mathfrak{T}_1(h)|. \end{aligned}$$

Using the bounds in (7.31) and (7.33), we obtain

$$\begin{aligned} \left| \mathfrak{L}_\varepsilon^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| &\leq \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2 Z^n(s_m)}{ds^2} \right\| + Ch^2 \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\| + C\varepsilon h^2 \left\| \frac{d^4 Z^n(s_m)}{ds^4} \right\| \\ &\quad + Ch^4(-2\alpha_1 + \alpha_2) \left\| \frac{d^3 Z^n(s_m)}{ds^3} \right\| + C\varepsilon h^4(1 - 12\alpha_1) \left\| \frac{d^4 Z^n(s_m)}{ds^4} \right\|. \end{aligned}$$

From Lemma 7.3.2, we obtain

$$\begin{aligned} \left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| &\leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-2} \exp \left(-\frac{\mu}{\varepsilon}(1 - s_m) \right) \right) \\ &\quad + Ch^2 \left[1 + \varepsilon^{-3} \exp \left(-\frac{\mu}{\varepsilon}(1 - s_m) \right) + \varepsilon + \varepsilon^{-3} \exp \left(-\frac{\mu}{\varepsilon}(1 - s_m) \right) \right] \\ &\quad + Ch^4(-2\alpha_1 + \alpha_2) \left(1 + \varepsilon^{-3} \exp \left(-\frac{\mu}{\varepsilon}(1 - s_m) \right) \right) \\ &\quad + Ch^4(1 - 12\alpha_1) \left(\varepsilon + \varepsilon^{-3} \exp \left(-\frac{\mu}{\varepsilon}(1 - s_m) \right) \right) \\ &\leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3} \exp \left(-\frac{\mu}{\varepsilon}(1 - s_m) \right) \right), \end{aligned}$$

since $\varepsilon^{-2} \leq \varepsilon^{-3}$ and obtain the desired bound. ■

Theorem 7.4.2. *Let Z_m^n be the solution of (7.28), then we have the following uniform error bound*

$$\sup_{\varepsilon \in (0,1]} \|Z^n(s_m) - Z_m^n\| \leq Ch, \quad m = 0(1)N.$$

Proof. Substituting Lemma 5.4.8 into (7.34), we arrive at

$$\left| \mathfrak{L}^{\Delta t, h}(Z^n(s_m) - Z_m^n) \right| \leq \frac{Ch^2}{h + \varepsilon}.$$

Hence, the result leads $|Z^n(s_m) - Z_m^n| \leq \frac{Ch^2}{h + \varepsilon}$. Using the sup over all $\varepsilon \in (0, 1]$, we get

$$\sup_{\varepsilon \in (0,1]} \|Z^n(s_m) - Z_m^n\| \leq Ch.$$

■

Theorem 7.4.3. *Let z and Z are the solutions of (7.1) and (7.28) respectively, then we have the following uniform error bound*

$$\sup_{\varepsilon \in (0,1]} \|z - Z\| \leq C(h + (\Delta t)).$$

Proof. The proof can be done by the combination of Lemma 4.3.11 and Theorem 7.4.2. ■

7.5 Numerical results

In this section some numerical experiments are carried out to confirm the developed scheme is stable, more accurate and ε -uniformly convergent by considering three test examples.

Example 7.1. *Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} - \frac{\partial z}{\partial s} + (\frac{1+s^2}{2})z = t^3 - z(s, t - \tau)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in (0, 2]$.*

Example 7.2. *Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} + (2 - s^2) \frac{\partial z}{\partial s} + sz(s, t) + z(s, t - \tau) = 10t^2 \exp(-t)s(1 - s)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in (0, 2]$.*

Example 7.3. *Consider the problem $\frac{\partial z}{\partial t} - \varepsilon \frac{\partial^2 z}{\partial s^2} + (2 - s^2) \frac{\partial z}{\partial s} + (s + 1)(t + 1)z(s, t) + z(s, t - \tau) = 10t^2 \exp(-t)s(1 - s)$, $(s, t) \in (0, 1) \times (0, 2]$ with interval condition $z(s, t) = 0$, on $(s, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $z(0, t) = 0$ and $z(1, t) = 0$, $t \in [0, 2]$.*

The numerical values are given for $\alpha_1 = 1/50$ and $\alpha_2 = 24/50$. If we do not know the exact solutions for the examples we are considering, we can estimate the maximum pointwise

absolute error by using the double mesh principle. Therefore, we calculate, the maximum pointwise absolute error by the formula given in (4.43), the ε -uniform error by the formula given in (4.44), the rate of convergence by the formula given in (4.45), and the ε -uniform rate of convergence by the formula given in (4.46).

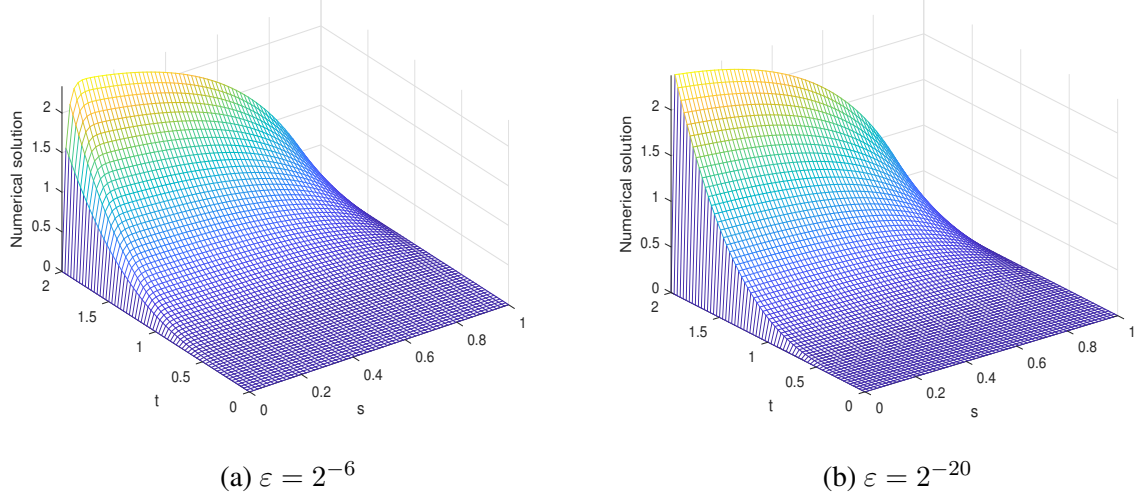


Figure 7.1: Numerical solution of the scheme for Example 7.1.

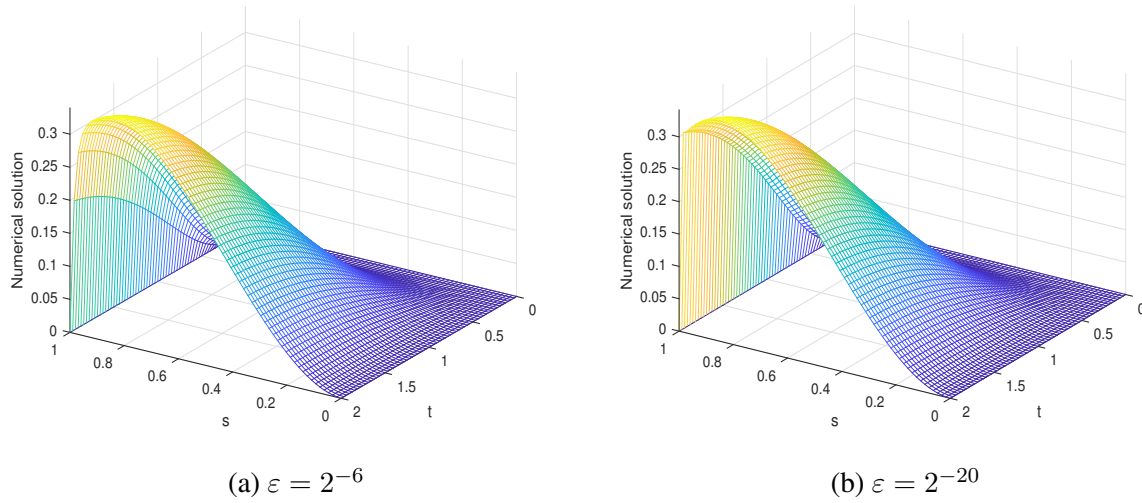


Figure 7.2: Numerical solution of the scheme for Example 7.2.

The $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and the corresponding $r^{N,M}$ of the proposed technique are revealed in Tables 7.1, 7.3, and 7.5 for each example, respectively, for different values of ε and N . These tables show that for every value of ε , the maximum pointwise absolute error monotonically decreases as the step sizes decrease, and as ε goes small, the maximum pointwise absolute error after getting large becomes constant, demonstrating ε -uniform convergence of the proposed scheme. On the other hand, the calculated $E^{N,M}$ and the corresponding $r^{N,M}$ using the proposed scheme are given in the last two rows, which confirms that the theoretical finding of the

Table 7.1: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 7.1 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
2^{-0}	1.5462e-02	7.3069e-04	3.9085e-04	2.0204e-04	2.0272e-04
2^{-2}	1.5462e-02	7.2071e-03	3.8153e-03	1.9627e-03	9.9539e-04
2^{-4}	2.6720e-02	1.5542e-02	8.3415e-03	4.3205e-03	2.1985e-03
2^{-6}	3.3768e-02	1.8165e-02	1.0080e-02	5.3565e-03	2.7620e-03
2^{-8}	3.2640e-02	1.9397e-02	1.0778e-02	5.2689e-03	2.8084e-03
2^{-10}	3.2691e-02	1.9261e-02	1.0440e-02	5.4740e-03	2.8614e-03
2^{-12}	3.2705e-02	1.9270e-02	1.0443e-02	5.4337e-03	2.7717e-03
2^{-14}	3.2709e-02	1.9272e-02	1.0444e-02	5.4343e-03	2.7716e-03
2^{-16}	3.2709e-02	1.9272e-02	1.0444e-02	5.4345e-03	2.7717e-03
2^{-18}	3.2710e-02	1.9273e-02	1.0444e-02	5.4345e-03	2.7717e-03
2^{-20}	3.2710e-02	1.9273e-02	1.0444e-02	5.4345e-03	2.7717e-03
$E^{N,M}$	3.3768e-02	1.9397e-02	1.0778e-02	5.4740e-03	2.8614e-03
$r^{N,M}$	0.7998	0.8477	0.9774	0.9359	–

Table 7.2: Comparison of $E_{\varepsilon}^{N,M}$ and $r_{\varepsilon}^{N,M}$ for Example 7.1.

Schemes $\downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
Proposed scheme					
$E^{N,M}$	3.3768e-02	1.9397e-02	1.0778e-02	5.4740e-03	2.8614e-03
$r^{N,M}$	0.7998	0.8477	0.9774	0.9359	–
Method in Chapter 4					
$E^{N,M}$	3.7225e-02	2.1953e-02	1.1908e-02	6.1994e-03	3.1627e-03
$r^{N,M}$	0.76185	0.88249	0.94173	0.97097	–
Negero and Duressa (2022b)					
$E^{N,M}$	8.3951e-02	4.9224e-02	2.6666e-02	1.3880e-02	7.0816e-03
$r^{N,M}$	0.77019	0.88436	0.94199	0.97086	–

Table 7.3: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 7.2 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals N = M				
	16	32	64	128	256
2^{-0}	2.3794e-04	4.2183e-05	3.7942e-06	4.8888e-06	3.6409e-06
2^{-2}	5.0365e-04	1.1446e-04	1.1509e-04	7.2877e-05	4.0330e-05
2^{-4}	1.3155e-03	3.3537e-04	2.4985e-04	1.4687e-04	7.9172e-05
2^{-6}	4.6137e-03	1.7222e-03	6.5369e-04	2.7011e-04	1.2039e-04
2^{-8}	6.3373e-03	4.5453e-03	2.0441e-03	7.1288e-04	2.4399e-04
2^{-10}	6.3448e-03	4.6914e-03	2.7609e-03	1.4423e-03	5.9485e-04
2^{-12}	6.3459e-03	4.6920e-03	2.7627e-03	1.4893e-03	7.7151e-04
2^{-14}	6.3461e-03	4.6921e-03	2.7628e-03	1.4893e-03	7.7196e-04
2^{-16}	6.3462e-03	4.6922e-03	2.7628e-03	1.4893e-03	7.7197e-04
2^{-18}	6.3462e-03	4.6922e-03	2.7628e-03	1.4893e-03	7.7197e-04
2^{-20}	6.3462e-03	4.6922e-03	2.7628e-03	1.4893e-03	7.7197e-04
$E^{N,M}$	6.3462e-03	4.6922e-03	2.7628e-03	1.4893e-03	7.7197e-04
$r^{N,M}$	0.4356	0.7641	0.8915	0.9480	–

Table 7.4: Comparison of $E_{\varepsilon}^{N,M}$ and $r_{\varepsilon}^{N,M}$ for Example 7.2.

Schemes $\downarrow$	Number of intervals N = M				
	16	32	64	128	256
Proposed scheme					
$E^{N,M}$	6.3462e-03	4.6922e-03	2.7628e-03	1.4893e-03	7.7197e-04
$r^{N,M}$	0.4356	0.7641	0.8915	0.9480	–
Method in Chapter 6					
$E^{N,M}$	–	1.6741e-02	5.8151e-03	1.7506e-03	6.6556e-04
$r^{N,M}$	–	1.5255	1.7320	2.1738	–
Negero and Duressa (2022b)					
$E^{N,M}$	1.1160e-02	6.3269e-03	3.3518e-03	1.7230e-03	8.7325e-04
$r^{N,M}$	0.81877	0.91656	0.96001	0.97086	–

Table 7.5: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 7.3 for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
2^{-0}	6.4188e-04	2.7416e-04	1.2553e-04	5.9884e-05	2.9227e-05
2^{-2}	1.5563e-03	7.0604e-04	3.3503e-04	1.6316e-04	8.0501e-05
2^{-4}	1.6730e-03	8.7110e-04	4.4811e-04	2.2739e-04	1.1454e-04
2^{-6}	2.1755e-03	1.0660e-03	3.9616e-04	2.0632e-04	1.1480e-04
2^{-8}	2.5911e-03	1.1849e-03	6.9552e-04	3.1188e-04	1.0684e-04
2^{-10}	2.5922e-03	1.2179e-03	6.8897e-04	3.8111e-04	1.9669e-04
2^{-12}	2.5921e-03	1.2179e-03	6.8880e-04	3.7968e-04	2.0016e-04
2^{-14}	2.5921e-03	1.2179e-03	6.8878e-04	3.7967e-04	2.0014e-04
2^{-16}	2.5921e-03	1.2179e-03	6.8878e-04	3.7967e-04	2.0014e-04
2^{-18}	2.5921e-03	1.2179e-03	6.8878e-04	3.7967e-04	2.0014e-04
2^{-20}	2.5921e-03	1.2179e-03	6.8878e-04	3.7967e-04	2.0014e-04
$E^{N,M}$	2.5922e-03	1.2179e-03	6.9552e-04	3.8111e-04	2.0016e-04
$r^{N,M}$	1.0898	0.8082	0.8679	0.9291	–

Table 7.6: Comparison of $E^{N,M}$ and $r^{N,M}$ for Example 7.3.

Schemes $\downarrow$	Number of intervals $N = M$				
	16	32	64	128	256
Proposed scheme					
$E^{N,M}$	2.5922e-03	1.2179e-03	6.9552e-04	3.8111e-04	2.0016e-04
$r^{N,M}$	1.0898	0.8082	0.8679	0.9291	–
Negero and Duressa (2022b)					
$E^{N,M}$	5.6185e-03	2.9471e-03	1.4861e-03	7.4255e-04	3.7060e-04
$r^{N,M}$	0.9308	0.98776	1.0010	1.0026	–
Method in Chapter 6					
$E^{N,M}$	–	9.7572e-03	4.0980e-03	1.8550e-03	8.7315e-04
$r^{N,M}$	–	1.2515	1.1435	1.0871	–

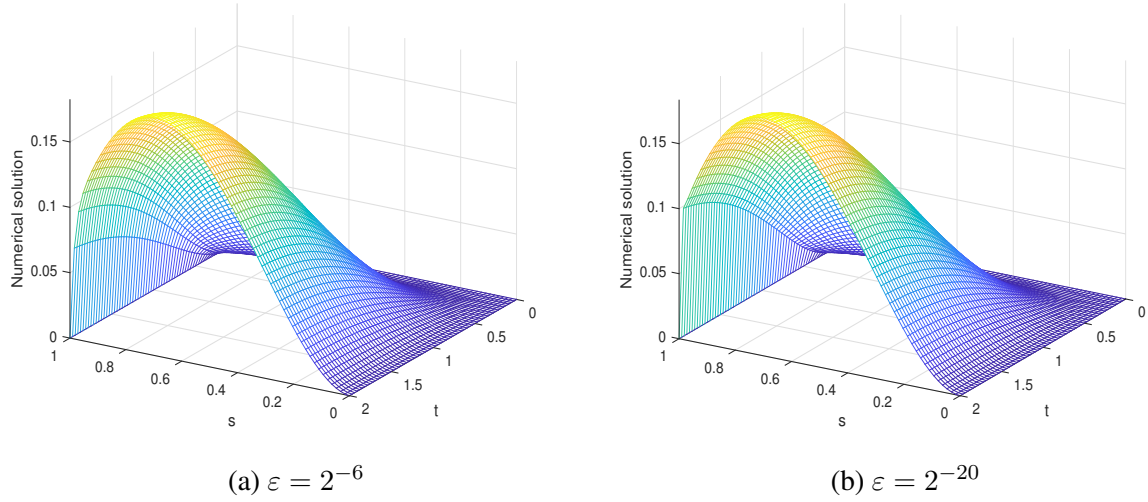


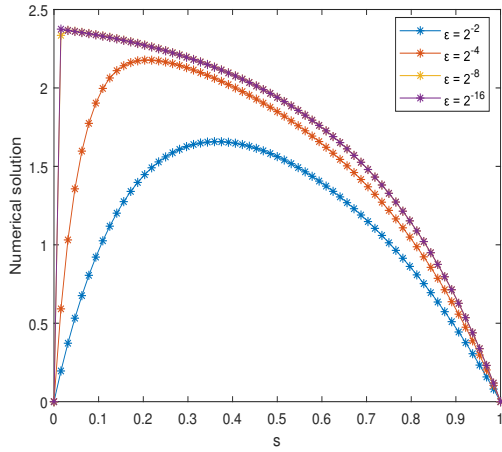
Figure 7.3: Numerical solution of the scheme for Example 7.3.

developed scheme is order one in the space.

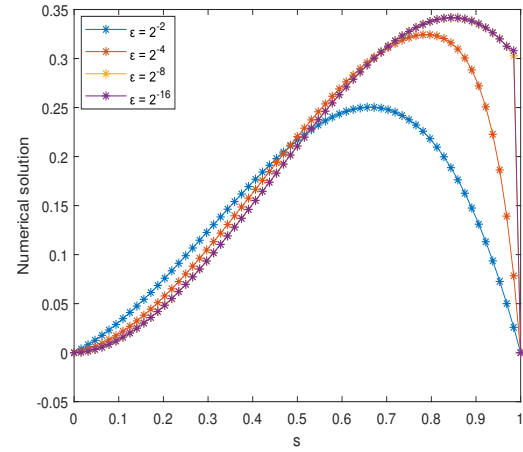
The numerical solutions of the proposed scheme for each example are revealed in Figures 7.1, 7.2, and 7.3. From Figure 7.1, it can be seen that a strong boundary layer is maintained along $s = 0$ as $\varepsilon \rightarrow 0$. Moreover, Figure 7.2 and 7.3 have revealed that a strong boundary layer is maintained along $s = 1$ as $\varepsilon \rightarrow 0$. Further, to show the effect of ε on the steepness of the boundary layer the solutions are depicted in Figure 7.4. In each figure, we observe that as $\varepsilon \rightarrow 0$ the width of the boundary layers decreases, which confirms the desired result, i.e., the boundary layer width of $O(\varepsilon)$. In addition, in Figure 7.5 the maximum pointwise absolute errors of the scheme are plotted by the log-log scale. From these figures, one can observe that maximum pointwise absolute error decreases as the step sizes decrease for every value of ε , which confirms the ε -uniform convergence of the proposed scheme. The comparison of results of the developed method with the existing recently published works is revealed in Table 7.2, 7.4 and 7.6. As we have seen in Table 7.2, the developed method improves results in Chapter 4 and in (Negero and Duressa, 2022b). In addition, in Table 7.4 and 7.6, it is shown that the obtained results are more accurate than that of results in Chapter 6 and in (Negero and Duressa, 2022b).

7.6 Summary

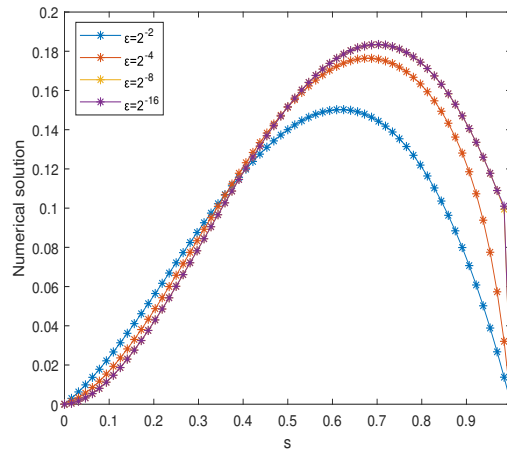
In this chapter, we initiated a fitted tension spline numerical scheme for a singularly perturbed parabolic problem with a small time delay. The solution to the problem exhibits a boundary layer on the left or right side of the domain, depending on the sign of the convection term. The proposed scheme comprises the backward Euler scheme in the time direction and an exponentially fitted tension spline scheme in the spatial direction. The applicability of the developed scheme is validated. The computational results are tabulated in terms of $E_\varepsilon^{N,M}$, $E^{N,M}$, and the corresponding $r^{N,M}$. The effect of ε on the steepness of the boundary layer is revealed graphi-



(a) Example 7.1



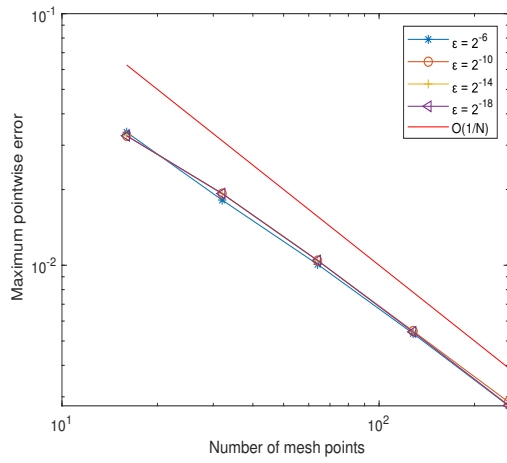
(b) Example 7.2



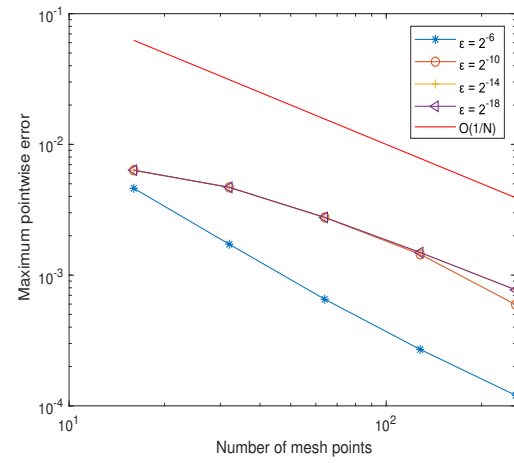
(c) Example 7.3

Figure 7.4: Effect of ε on the solution profiles.

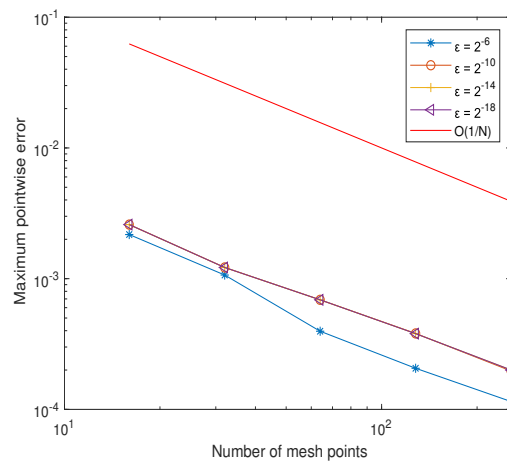
cally. Further, the computational results are compared with some of the existing literature and preceding chapters in terms of accuracy. The proposed method contributes a better accuracy, stable, and ε -uniform numerical result with a linear order of convergence. In the subsequent chapter we consider the degenerating convection-diffusion problem on uniform mesh.



(a) Example 7.1



(b) Example 7.2



(c) Example 7.3

Figure 7.5: Maximum pointwise error in Log-Log scale plot.

CHAPTER EIGHT

FITTED OPERATOR METHOD FOR SINGULARLY PERTURBED DEGENERATE CONVECTION-DIFFUSION EQUATION

In this chapter, an exponentially fitted operator finite difference method is developed to solve a singularly perturbed parabolic turning point (or degenerating coefficient) problem. The chapter discussed the priori bounds and properties of the continuous solution. We suggested method utilizes the Crank-Nicolson method in the time direction and an exponentially fitted finite difference method in the space direction on a uniform mesh. The stability of the scheme is analyzed using the comparison principle and discrete solution bounds. It is proved that the proposed scheme is uniformly convergent with the rate of convergence of order one in space and order two in time. Two numerical examples are considered to validate the applicability of the proposed method. Theoretical findings are supported by numerical results displayed in tables. The chapter also provided graphical information on the boundary layer behaviour of the solutions.

8.1 Governing problem

We considered a degenerate convection-diffusion equation of the form

$$\left\{ \begin{array}{l} \mathfrak{L}z(s, t) \equiv \varepsilon z_{ss}(s, t) + \beta(s)z_s(s, t) - b(s, t)z_t(s, t) - \chi(s, t)z(s, t) = \gamma(s, t), \\ (s, t) \in \mathfrak{D} = (0, 1) \times (0, T], \\ z(s, t) = \psi_b(s, t), (s, t) \in \eta_b := [0, 1] \times \{0\}, \\ z(0, t) = \psi_l(t), (s, t) \in \eta_l := \{0\} \times [0, T], \\ z(1, t) = \psi_r(t), (s, t) \in \eta_r := \{1\} \times [0, T], \end{array} \right. \quad (8.1)$$

where $\beta(s) = \beta_0(s)s^p$, $\beta_0(s) \geq \mu > 0$, $s \in [0, 1]$, $p \geq 1$, and the perturbation parameter $\varepsilon \in (0, 1]$. The coefficients $\beta(s)$, $b(s, t)$, $\chi(s, t)$, the source function $\gamma(s, t)$, and the boundary functions $\psi_b(s, t)$, $\psi_l(t)$, $\psi_r(t)$ on $\eta = \eta_b \cup \eta_l \cup \eta_r = \bar{\mathfrak{D}} - \mathfrak{D}$ are taken to be smooth and bounded that satisfies the conditions $0 \leq \beta(s)$, $0 < \alpha \leq b(s, t)$, $0 < \varpi \leq \chi(s, t)$, $(s, t) \in \bar{\mathfrak{D}} = [0, 1] \times [0, T]$ for some fixed $T > 0$. The coefficient of convection term $\beta(s)$ of (8.1) vanishes at $s = 0$, i.e., $\beta(0) = 0$. Such a problem is known as a turning point (or degenerating coefficient) problem. At the point where the convection coefficient vanishes, the problem has a boundary turning point or interior turning point. When the convection coefficient vanishes at the boundary, the problem is referred to as the boundary turning point problem, whereas the convection

coefficient vanishes inside the domain the problem is referred to as the interior turning point problem (Janani and Tamilselvan, 2020). For the case $p = 1$, the problem (8.1) becomes a simple turning point problem, and when $p > 1$ the problem (8.1) becomes a multiple turning point problem (Gelu and Duressa, 2023; Kumar and Kumari, 2020). In general, the solution of problem (8.1) possesses a parabolic boundary layer of width $O(\sqrt{\varepsilon})$ in the neighbourhood of the left boundary layer as ε closes zero.

8.2 Properties of the continuous solution

Taking $\varepsilon = 0$ in (8.1), we get the reduced problem

$$\begin{cases} \beta(s)z_s^0(s, t) - b(s, t)z_t^0(s, t) - \chi(s, t)z^0(s, t) = \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z^0(1, t) = \psi_r(t), & t \in (0, T], \\ z^0(s, t) = \psi_b(s, t), & (s, t) \in \eta_b, \end{cases} \quad (8.2)$$

which is a first-order hyperbolic PDE. For $\beta(0) = 0$ and $\chi(0, t) > 0$ the boundary $s = 0$ is a characteristic curve of (8.2), and the solution of (8.1) has a parabolic boundary layer closes $s = 0$.

The next lemma will help us to determine the bound of $z(s, t)$ and its derivatives. The solution of (8.1) fulfils the next lemma.

Lemma 8.2.1. *Dunne et al. (2003) (Minimum principle) Assume that $z(s, t) \in C^2(\mathfrak{D}) \cap C^0(\bar{\mathfrak{D}})$, satisfying $z(s, t) \geq 0$, $(s, t) \in \eta$. If $\mathfrak{L}z(s, t) \leq 0$, $(s, t) \in \mathfrak{D}$, then $z(s, t) \geq 0$, $(s, t) \in \bar{\mathfrak{D}}$.*

Proof. Let $\exists(s^*, t^*) \in \bar{\mathfrak{D}}$, such that $z(s^*, t^*) = \min_{(s, t) \in \bar{\mathfrak{D}}} z(s, t) < 0$. From the supposition, it is shown that $(s^*, t^*) \notin \{0, 1\} \times \{0, T\}$ implies that (s^*, t^*) in $(0, 1) \times (0, T)$. Since $z(s^*, t^*) = \min_{(s, t) \in \bar{\mathfrak{D}}} z(s, t) < 0$, then we have $z_s(s^*, t^*) = z_t(s^*, t^*) = 0$ and $z_{ss}(s^*, t^*) > 0$. So that, $\mathfrak{L}z(s^*, t^*) = \varepsilon z_{ss}(s^*, t^*) + \beta(s)z_s(s^*, t^*) - b(s, t)z_t(s^*, t^*) - \chi(s, t)z(s^*, t^*) > 0$. Which is a contradiction. Therefore, $z(s, t) \geq 0$, $\forall (s, t) \in \bar{\mathfrak{D}}$. ■

Lemma 8.2.2. *Dunne et al. (2003) (Stability estimate) The solution of (8.1) fulfills the estimate*

$$\|z\|_{\bar{\mathfrak{D}}} \leq \|z\|_{\eta} + \frac{T}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}}, \quad (8.3)$$

where $0 < \alpha \leq b(s, t)$.

Proof. Define the two barrier functions

$$\vartheta^{\pm}(s, t) \text{ as } \vartheta^{\pm}(s, t) = \|z\|_{\eta} + \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \pm z(s, t),$$

for the initial value case

$$\begin{aligned}\vartheta^\pm(s, 0) &= \|z\|_\eta + \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \pm z(s, 0), \\ &\geq \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}}, \quad \text{since } \|z\|_\eta \geq z(s, 0), \\ &\geq \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \geq 0,\end{aligned}$$

for the boundary values cases

$$\begin{aligned}\vartheta^\pm(0, t) &= \|z\|_\eta + \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \pm z(0, t), \\ &\geq \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}}, \quad \text{since } \|z\|_\eta \geq z(0, t), \\ &\geq \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \geq 0, \\ \vartheta^\pm(1, t) &= \|z\|_\eta + \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \pm z(1, t), \\ &\geq \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}}, \quad \text{since } \|z\|_\eta \geq z(1, t), \\ &\geq \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \geq 0,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}\vartheta^\pm(s, t) &= -b(s, t) \frac{\partial \vartheta^\pm(s, t)}{\partial t} + \varepsilon \frac{\partial^2 \vartheta^\pm(s, t)}{\partial s^2} + \beta(s) \frac{\partial \vartheta^\pm(s, t)}{\partial s} - \chi(s, t) \vartheta^\pm(s, t) \\ &= -b(s, t) (\max\{\frac{\partial \psi_l(t)}{\partial t}, \frac{\partial \psi_b(s, t)}{\partial t}, \frac{\partial \psi_r(t)}{\partial t}\} \pm z_t(s, t)) \\ &\quad + \varepsilon (\max\{\frac{\partial^2 \psi_l(t)}{\partial s^2}, \frac{\partial^2 \psi_b(s, t)}{\partial s^2}, \frac{\partial^2 \psi_r(t)}{\partial s^2}\} \\ &\quad \pm z_{ss}(s, t)) + \beta(s) (\max\{\frac{\partial \psi_l(t)}{\partial s}, \frac{\partial \psi_b(s, t)}{\partial s}, \frac{\partial \psi_r(t)}{\partial s}\} \pm z_s(s, t)) \\ &\quad - \chi(s, t) (\|z\|_\eta + \frac{t}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}} \pm z(s, t)) \leq 0.\end{aligned}$$

This implies that, $\mathfrak{L}\vartheta^\pm(s, t) \leq 0$. Hence, by minimum principle we have,

$\vartheta^\pm(s, t) \geq 0, \forall (s, t) \in \bar{\mathfrak{D}}$ which implies,

$$\|z\|_{\bar{\mathfrak{D}}} \leq \|z\|_\eta + \frac{T}{\alpha} \|\gamma\|_{\bar{\mathfrak{D}}}.$$

Hence, the proof is completed. ■

Lemma 8.2.3. *The derivatives of the solution $z(s, t)$ of (8.1) with respect to t satisfy the following estimate*

$$|z_t(s, t)| \leq C, (s, t) \in \bar{\mathfrak{D}}$$

Proof. On the sides $s = 0$ and $s = 1$ of $\bar{\mathfrak{D}}$, we have $z = 0$, therefore $z_t = 0$. At $t = 0$,

we have $z = 0, z_s = 0, z_{ss} = 0$. Substituting these values in (8.1), from equation it yields $z_t(s, 0) = \gamma(s, 0)$. Thus for sufficiently large C along the boundary, the following estimate holds

$$|z_t(s, t)| \leq C.$$

Now, differentiating (8.1) with respect to t , then we obtain

$$\begin{aligned} \mathfrak{L}z_t(s, t) &= -b(s, t)z_{tt} + \varepsilon z_{tss} + \beta(s)z_{ts} - \chi(s, t)z_t \\ &= (-b(s, t)z_t + \varepsilon z_{ss} + \beta(s)z_s - \chi(s, t)z)_t + b_t(s, t)z + \chi_t(s, t)z \\ &= \gamma_t + b_t(s, t)z + \chi_t(s, t)z. \end{aligned} \quad (8.4)$$

From the fact that γ is sufficiently smooth, $|\mathfrak{L}z_t(s, t)| \leq C$. Now applying the minimum principle, one can conclude that $|z_t(s, t)| \leq C$, for $(s, t) \in \bar{\mathfrak{D}}$. ■

Lemma 8.2.4. *The derivatives of the exact solution $z(s, t)$ of (8.1) with respect to s satisfies*

$$|z_s(s, t)| \leq C(1 + \varepsilon^{-1/2} \exp(-\delta s/\sqrt{\varepsilon})), \quad (s, t) \in \bar{\mathfrak{D}}, \quad 1 \leq j + 2l \leq 4, \text{ where } \delta = \sqrt{\varpi}.$$

Proof. The problem (8.1) can be written as $\varepsilon z_{ss}(s, t) + \beta(s)z_s(s, t) = g$, where $g = \gamma(s, t) + b(s, t)z_t(s, t) + \chi(s, t)z(s, t)$. Fixing $t \in [0, T]$ and by using the argument of [Vulanović and Farrell \(1993\)](#), considering a neighborhood of $I = (0, \sqrt{\varepsilon})$, $\forall s \in I$. For some $s^* \in I$ such that

$$z_s(s^*, t) = \frac{z(\sqrt{\varepsilon}, t) - z(0, t)}{\sqrt{\varepsilon}}. \quad (8.5)$$

From (8.5), we have

$$|z_s(s^*, t)| = \left| \frac{z(\sqrt{\varepsilon}, t) - z(0, t)}{\sqrt{\varepsilon}} \right| = \varepsilon^{-1/2} |z(\sqrt{\varepsilon}, t) - z(0, t)| \leq C\varepsilon^{-1/2} \|z\|. \quad (8.6)$$

For $s \in \bar{I}$,

$$\begin{aligned} z_s(s, t) &= z_s(s^*, t) + z_s(s, t) - z_s(s^*, t) \\ &= z_s(s^*, t) + \int_{s^*}^s z_{ss}(x, t) dx \\ &= z_s(s^*, t) + \varepsilon^{-1} \int_{s^*}^s (g(x, t) - \beta(x)) z_s(x, t) dx \\ |z_s(s, t)| &\leq |z_s(s^*, t)| + C\varepsilon^{-1} \int_{s^*}^s (\|z\| + \|\gamma\|) dx \\ &= |z_s(s^*, t)| + C\varepsilon^{-1} (\|z\| + \|\gamma\|) \varepsilon^{1/2}. \end{aligned} \quad (8.7)$$

Inserting (8.7) into the above equation results in

$$|z_s(s, t)| \leq C\varepsilon^{-1/2} \leq C(1 + \varepsilon^{-1/2} \exp(-\delta s/\sqrt{\varepsilon})).$$

■

Lemma 8.2.5. $|z_{tt}(s, t)| \leq C, (s, t) \in \bar{\mathfrak{D}}.$

Proof. On the sides $s = 0$ and $s = 1$ of $\bar{\mathfrak{D}}$, we have $z = 0$ and therefore $z_{tt} = 0$. and $|z_{ts}(s, t)| \leq C$ and $|z_{tss}(s, t)| \leq C$ along $t = 0$. Therefore, $|z_{tt}| \leq C$, along the boundary. Now, differentiating (8.4) with respect to t , we get

$$\begin{aligned} \mathfrak{L}z_{tt}(s, t) &= -b(s, t)z_{ttt} + \varepsilon z_{tts} + \beta(s)z_{tts} - \chi(s, t)z_{tt} \\ &= (-b(s, t)z_t + \varepsilon z_{ss} + \beta(s)z_s - \chi(s, t)z)_{tt} \\ &= \gamma_{tt}. \end{aligned}$$

Since $\gamma_{tt}(s, t)$ is bounded function, which implies that $|\mathfrak{L}z_{tt}(s, t)| \leq C$. By the minimum principle, we have $|z_{tt}| \leq C, (s, t) \in \bar{\mathfrak{D}}.$ ■

Lemma 8.2.6. *The derivatives of $z_s(s, t)$ with respect to t satisfies*

$$|z_{st}(s, t)| \leq C(1 + \varepsilon^{-1/2} \exp(-\delta s/\sqrt{\varepsilon})) \quad (s, t) \in \bar{\mathfrak{D}}.$$

Proof. Differentiating (8.1) with respect to t , we get

$$\varepsilon(z_{st})_s(s, t) + \beta(s)z_{st}(s, t) - \chi_t(s, t)z(s, t) - \chi(s, t)z_t(s, t) - b(s, t)z_{tt}(s, t) - b_t(s, t)z_t(s, t) = \gamma_t.$$

Rewriting the above equation in the form $\varepsilon(z_{st}(s, t))_s + \beta(s)z_{st}(s, t) = g_2$, where $g_2 = \gamma_t(s, t) + \chi_t(s, t)z + \chi(s, t)z_t(s, t) + b(s, t)z_{tt}(s, t) + b_t(s, t)z_t(s, t)$. Fixing $t \in [0, T]$ and by using the argument of [Vulanović and Farrell \(1993\)](#), considering a neighborhood of $I = (0, \sqrt{\varepsilon})$, $\forall s \in I$. For some $s^* \in I$ such that

$$z_{st}(s^*, t) = \frac{z_t(\sqrt{\varepsilon}, t) - z_t(0, t)}{\sqrt{\varepsilon}}. \quad (8.8)$$

From (8.8), we have

$$|z_{st}(s^*, t)| = \left| \frac{z_t(\sqrt{\varepsilon}, t) - z_t(0, t)}{\sqrt{\varepsilon}} \right| = \varepsilon^{-1/2} |z_t(\sqrt{\varepsilon}, t) - z_t(0, t)| \leq C\varepsilon^{-1/2} \|z_t\|. \quad (8.9)$$

For $s \in \bar{I}$,

$$\begin{aligned}
 z_{st}(s, t) &= z_{st}(s^*, t) + z_{st}(s, t) - z_{st}(s^*, t) \\
 &= z_{st}(s^*, t) + \int_{s^*}^s (z_{st})_s(x, t) dx \\
 &= z_{st}(s^*, t) + \varepsilon^{-1} \int_{s^*}^s (g_2(x, t) - \beta(x) z_{st}(x, t)) dx \\
 |z_{st}(s, t)| &\leq |z_{st}(s^*, t)| + C\varepsilon^{-1} \int_{s^*}^s (\|z_t\| + \|\gamma_t\|) dx \\
 &= |z_{st}(s^*, t)| + \varepsilon^{-1} (\|z_t\| + \|\gamma_t\|) \varepsilon^{1/2}.
 \end{aligned} \tag{8.10}$$

Inserting (8.8) into the above equation results in

$$|z_{st}(s, t)| \leq C\varepsilon^{-1/2} \leq C(1 + \varepsilon^{-1/2} \exp(-\delta s/\sqrt{\varepsilon})).$$

■

Lemma 8.2.7. *Majumdar and Natesan (2019); Mbroh et al. (2021)* Let the solution of (8.1) be $z(s, t)$, then the derivative of $z(s, t)$ satisfies the estimate

$$\left\| \frac{\partial^{l+i} z}{\partial s^i \partial t^l} \right\|_{\bar{\mathfrak{D}}} \leq C(1 + \varepsilon^{-i/2} \exp(-\delta s/\sqrt{\varepsilon})), \quad (s, t) \in \bar{\mathfrak{D}}, \quad 1 \leq i + 2l \leq 4, \tag{8.11}$$

Proof. The proof easily follows from Lemma 8.2.3 -Lemma 8.2.6 for the integers and it is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. ■

8.3 Numerical scheme

8.3.1 Temporal semi-discretization

We discretize the problem (8.1) via the Crank-Nicolson scheme in the temporal direction on a uniform mesh. A uniform partition of $[0, T]$ with the step length Δt is given by $\Lambda^M = \{t_n = n\Delta t, n = 0(1)M, t_M = T, \Delta t = T/M\}$ having M mesh points. Then the semi-discrete result of problem (8.1) yields

$$\begin{cases} (\frac{\Delta t}{2} \mathfrak{L}^{\Delta t} - 1) Z^{n+1}(s) = -(1 + \frac{\Delta t}{2} \mathfrak{L}^{\Delta t}) Z^n(s) + \frac{\Delta t}{2} (\gamma(s, t_n) + \gamma(s, t_{n+1})) \\ Z^{n+1}(0) = \psi_l(t_{n+1}), \quad Z^{n+1}(1) = \psi_r(t_{n+1}), \end{cases} \tag{8.12}$$

where $\mathfrak{L}^{\Delta t} Z^{n+1}(s) = \varepsilon \frac{d^2 Z^{n+1}(s)}{ds^2} + \beta(s) \frac{dZ^{n+1}(s)}{ds} - \chi(s, t_{n+1}) Z^{n+1}(s)$.

The approximation of $Z(s, t_{n+1})$ is denoted by $Z^{n+1}(s)$ at the $(n + 1)^{th}$ time level. The operator $(\frac{\Delta t}{2} \mathfrak{L}^{\Delta t} - 1) Z^{n+1}(s)$ satisfies the next minimum principle.

Lemma 8.3.8. *Majumdar and Natesan (2019) (Semi-discrete minimum principle) The function $Z^{n+1}(s) \in C^2(\bar{\mathfrak{D}})$ such that $Z^{n+1}(0) \geq 0$, $Z^{n+1}(1) \geq 0$, and $(\frac{\Delta t}{2}\mathfrak{L}^{\Delta t} - 1)Z^{n+1}(s) \leq 0$ on $\mathfrak{D}$. Then $Z^{n+1}(s) \geq 0$ on $\bar{\mathfrak{D}}$.*

Proof. Assume $\exists s^* \in \bar{\mathfrak{D}}$ such that $Z^{n+1}(s^*) = \min_{s \in \bar{\mathfrak{D}}} Z^{n+1}(s) < 0$. From the assumption, it is shown that $s^* \notin \{0, 1\}$ implies that $s^* \in (0, 1)$. Since $Z^{n+1}(s^*) = \min_{s \in \bar{\mathfrak{D}}} Z^{n+1}(s) < 0$, then we have $\frac{d}{ds}Z^{n+1}(s^*) = 0$ and $\frac{d^2}{ds^2}Z^{n+1}(s^*) > 0$. So, the operator gives that $(\frac{\Delta t}{2}\mathfrak{L}^{\Delta t} - 1)Z^{n+1}(s^*) > 0$. Which is a contradiction to the assumption. Therefore, $Z^{n+1}(s) \geq 0, \forall s \in \bar{\mathfrak{D}}$. ■

The local truncation error for the time discretization at each time step yields $e_{n+1}(s) := z(s, t_{n+1}) - Z^{n+1}(s)$, $n = 0(1)M$, where $Z^{n+1}(s)$ is the solution obtained after one step of the semi-discrete scheme by pickings the exact value $z(s, t_n)$, rather than $z^n(s)$ as the starting value.

Lemma 8.3.9. *If $\left| \frac{\partial^l z(s, t)}{\partial t^l} \right| \leq C$, $(s, t) \in \bar{\mathfrak{D}}$ and $0 \leq l \leq 2$, then the local truncation error of the scheme (8.12) satisfies*

$$\|e_{n+1}\| \leq C_1(\Delta t)^3. \quad (8.13)$$

Proof. From the Taylor's series approximation of $z(s, t_n)$ and $z(s, t_{n+1})$, centering at $z(s, t_{n+\frac{1}{2}})$, we get

$$\begin{aligned} z(s, t_n) &= z(s, t_{n+\frac{1}{2}}) - \frac{\Delta t}{2}z_t(s, t_{n+\frac{1}{2}}) + \frac{(\Delta t)^2}{8}z_{tt}(s, t_{n+\frac{1}{2}}) + O((\Delta t)^3), \\ z(s, t_{n+1}) &= z(s, t_{n+\frac{1}{2}}) + \frac{\Delta t}{2}z_t(s, t_{n+\frac{1}{2}}) + \frac{(\Delta t)^2}{8}z_{tt}(s, t_{n+\frac{1}{2}}) + O((\Delta t)^3) \end{aligned} \quad (8.14)$$

From (8.14), we get

$$\frac{z(s, t_{n+1}) - z(s, t_n)}{\Delta t} = z_t(s, t_{n+\frac{1}{2}}) + O((\Delta t)^2) \quad (8.15)$$

Using the approximation into (8.12), we get

$$\begin{aligned} \frac{z(s, t_{n+1}) - z(s, t_n)}{\Delta t} &= \varepsilon z_{ss}(s, t_{n+\frac{1}{2}}) + \beta(s)z_s(s, t_{n+\frac{1}{2}}) \\ &\quad - (\chi(s, t_{n+\frac{1}{2}}))z(s, t_{n+\frac{1}{2}}) - \gamma(s, t_{n+\frac{1}{2}}) + O((\Delta t)^2), \end{aligned}$$

where

$$\begin{aligned}\varepsilon z_{ss}(s, t_{n+\frac{1}{2}}) &= \frac{\varepsilon u_{ss}(s, t_{n+1}) + \varepsilon z_{ss}(s, t_n)}{2} + O((\Delta t)^2), \\ z_s(s, t_{n+\frac{1}{2}}) &= \frac{z_s(s, t_{n+1}) + z_s(s, t_n)}{2} + O((\Delta t)^2), \\ z(s, t_{n+\frac{1}{2}}) &= \frac{z(s, t_{n+1}) + z(s, t_n)}{2} + O((\Delta t)^2), \\ \gamma(s, t_{n+\frac{1}{2}}) &= \frac{\gamma(s, t_{n+1}) + \gamma(s, t_n)}{2} + O((\Delta t)^2)\end{aligned}$$

since the error $e_{n+1} =: z(s, t_{n+1}) - Z^{n+1}(s)$ and it satisfies the following semi-discrete differential operator

$$\begin{aligned}\left(\frac{\Delta t}{2} \mathfrak{L}^{\Delta t} - 1\right)e_{n+1}(s) &= O((\Delta t)^3), \\ e_{n+1}(0) &= 0 = e_{n+1}(1).\end{aligned}\tag{8.16}$$

Thus, by applying the minimum principle for the operator $(\frac{\Delta t}{2} \mathfrak{L}^{\Delta t} - 1)$, we get

$$\|e_{n+1}\| \leq C_1(\Delta t)^3\tag{8.17}$$

The proof of this lemma can be also found in (Woldaregay et al., 2021). ■

Lemma 8.3.10. *The global error at t_{n+1} satisfies the bound*

$$\|E_{n+1}\| \leq C(\Delta t)^2, \quad n = 1(1)M - 1.$$

Proof. Using (8.17) up to the $(n + 1)^{th}$ time step, we get the global error bound at the $(n + 1)^{th}$ time level

$$\begin{aligned}\|E_{n+1}\| &= \left\| \sum_{l=1}^{n+1} e_l \right\| \leq \|e_1\| + \|e_2\| + \|e_3\| + \dots + \|e_{n+1}\| \\ &\leq C_1 T (\Delta t)^2, \quad \text{since } (n + 1)\Delta t \leq T \\ &= C(\Delta t)^2, \quad \text{where } C_1 T = C,\end{aligned}$$

where C is not depends on ε and Δt (Mbroh et al., 2021). ■

Lemma 8.3.11. *For $n = 0(1)M - 1$, the derivatives of the solution (8.12) satisfies the estimate*

$$\left| \frac{d^i Z^{n+1}(s)}{ds^i} \right| \leq C(1 + \varepsilon^{-i/2} \exp(-\delta s / \sqrt{\varepsilon})), \quad s \in \bar{\mathfrak{D}}, \quad 0 \leq i \leq 4.\tag{8.18}$$

Proof. The proof follows from Lemma 8.2.4, it is done for $i = 1$. For $i > 1$ followed by

induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. The proof is also found in (Majumdar and Natesan, 2019; Mbroh et al., 2021). ■

8.3.2 Spatial discretization

We discretized the space domain $[0, 1]$ into N equal number of points $s_m = mh$, $0 \leq m \leq N$ with the mesh spacing h is given as $\Omega^N = \{0 = s_0, s_1, \dots, s_N = 1, s_m = mh\}$.

The divided differences operator is given as

$$\begin{aligned} \delta_s^+ Z_m^n &= \frac{Z_{m+1}^n - Z_m^n}{h}, \quad \delta_s^- Z_m^n = \frac{Z_m^n - Z_{m-1}^n}{h}, \quad \delta_s^0 Z_m^n = \frac{Z_{m+1}^n - Z_{m-1}^n}{2h}, \\ \delta_s^2 Z_m^n &= \frac{Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n}{h^2}. \end{aligned} \quad (8.19)$$

Applying the central finite difference scheme in (8.12), for $m = 1(1)N - 1$, we get

$$\begin{cases} (\frac{\Delta t}{2} \mathfrak{L}^{\Delta t, h} - 1) Z_m^{n+1} = -(1 + \frac{\Delta t}{2} \mathfrak{L}^{\Delta t, h}) Z_m^n + \frac{\Delta t}{2} (\gamma(s_m, t_n) + \gamma(s_m, t_{n+1})), \\ Z^{n+1}(0) = \psi_l(t_{n+1}), \quad Z^{n+1}(1) = \psi_r(t_{n+1}), \end{cases} \quad (8.20)$$

where $\mathfrak{L}^{\Delta t, h} Z_m^{n+1} = \varepsilon \delta_s^2 Z_m^{n+1} + \beta(s_m) \delta_s^0 Z_m^{n+1} - \chi(s_m, t_{n+1}) Z_m^{n+1}$.

Computing the exponential fitting factor

In this part, we introduce the fitting factor σ to handle the effect of ε in the layer region. Multiplying (8.20) by σ with the term containing ε , we obtain

$$\begin{cases} (\frac{\Delta t}{2} \mathfrak{L}^{\Delta t, h} - 1) Z_m^{n+1} = -(1 + \frac{\Delta t}{2} \mathfrak{L}^{\Delta t, h}) Z_m^n + \frac{\Delta t}{2} (\gamma(s_m, t_n) + \gamma(s_m, t_{n+1})), \\ Z^{n+1}(0) = \psi_l(t_{n+1}), \quad Z^{n+1}(1) = \psi_r(t_{n+1}), \end{cases} \quad (8.21)$$

where $\mathfrak{L}^{\Delta t, h} Z_m^{n+1} = \sigma \varepsilon \delta_s^2 Z_m^{n+1} + \beta(s_m) \delta_s^0 Z_m^{n+1} - \chi(s_m, t_{n+1}) Z_m^{n+1}$.

Multiplying (8.21) by h and taking a limit as $h \rightarrow 0$ and after some simplification we obtained

$$\lim_{h \rightarrow 0} \frac{\sigma}{\rho} (Z_{m-1}^{n+1} - 2Z_m^{n+1} + Z_{m+1}^{n+1}) + \beta(s_m) \frac{Z_{m+1}^{n+1} - Z_{m-1}^{n+1}}{2} = 0. \quad (8.22)$$

Consequently, (8.22) satisfied if and only if

$$\begin{cases} \lim_{h \rightarrow 0} \frac{\sigma}{\rho} (Z_{m-1}^{n+1} - 2Z_m^{n+1} + Z_{m+1}^{n+1}) + \beta(s_m) \frac{Z_{m+1}^{n+1} - Z_{m-1}^{n+1}}{2} = 0, \\ \lim_{h \rightarrow 0} \frac{\sigma}{\rho} (Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n) + \beta(s_m) \frac{Z_{m+1}^n - Z_{m-1}^n}{2} = 0. \end{cases} \quad (8.23)$$

At $(m, n + 1)^{th}$ levels, we obtain

$$\sigma = -\frac{\rho}{2} \frac{\lim_{h \rightarrow 0} \beta(s_m)(Z_{m+1}^{n+1} - Z_{m-1}^{n+1})}{\lim_{h \rightarrow 0} (Z_{m-1}^{n+1} - 2Z_m^{n+1} + Z_{m+1}^{n+1})}. \quad (8.24)$$

Using the simplified zero-order asymptotic solution given in (4.28), we have

$$\begin{aligned} \lim_{h \rightarrow 0} Z_{m+1}^{n+1} &= Z_0(0) + (\alpha_l - Z_0(0)) \exp(-\beta(0)m\rho) \exp(\beta(0)\rho), \\ \lim_{h \rightarrow 0} Z_{m-1}^{n+1} &= Z_0(0) + (\alpha_l - Z_0(0)) \exp(-\beta(0)m\rho) \exp(-\beta(0)\rho). \end{aligned} \quad (8.25)$$

Using (8.25) and (8.24) after some simplification the variable fitting factor is

$$\sigma_m = \frac{\rho\beta(s_m)}{2} \coth\left(\frac{\rho\beta(s_m)}{2}\right). \quad (8.26)$$

8.3.3 The discrete scheme

Applying the central finite difference method and the fitting factor σ_m , for $m = 1(1)N - 1$, and $n = 0(1)M - 1$, then we have the following fully discrete scheme

$$\begin{cases} (\frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} - 1)Z_m^{n+1} = -(1 + \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h})Z_m^n + \frac{\Delta t}{2}(\gamma(s_m, t_n) + \gamma(s_m, t_{n+1})), \\ Z^{n+1}(0) = \psi_l(t_{n+1}), \quad Z^{n+1}(1) = \psi_r(t_{n+1}), \end{cases} \quad (8.27)$$

where $\mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1} = \varepsilon \sigma_m \delta_s^2 Z_m^{n+1} + \beta(s_m) \delta_s^0 Z_m^{n+1} - \chi(s_m, t_{n+1}) Z_m^{n+1}$.

We rewrite in an explicit form as

$$Q_m^- Z_{m-1}^{n+1} + Q_m^0 Z_m^{n+1} + Q_m^+ Z_{m+1}^{n+1} = R_m^- Z_{m-1}^n + R_m^0 Z_m^n + R_m^+ Z_{m+1}^n + \Delta t \gamma(s_m, t_{n+\frac{1}{2}}), \quad (8.28)$$

where

$$\begin{aligned} Q_m^- &= \frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} - \frac{\beta(s_m)}{2h} \right), \\ Q_m^0 &= -\frac{\Delta t \varepsilon \sigma_m}{h^2} - \left(\frac{\Delta t}{2} \chi(s_m, t_{n+1}) + 1 \right), \\ Q_m^+ &= \frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} + \frac{\beta(s_m)}{2h} \right), \end{aligned} \quad (8.29)$$

and

$$\begin{aligned}
R_m^- &= -\frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} - \frac{\beta(s_m)}{2h} \right), \\
R_m^0 &= \frac{\Delta t \varepsilon \sigma_m}{h^2} + \left(\frac{\Delta t}{2} \chi(s_m, t_n) - 1 \right), \\
R_m^+ &= -\frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} - \frac{\beta(s_m)}{2h} \right), \\
H_m &= \frac{\Delta t}{2} \left(\gamma(s_m, t_n) + \gamma(s_m, t_{n+1}) \right).
\end{aligned} \tag{8.30}$$

8.4 Convergence analysis

In this part, we study the stability and uniform convergence for the developed technique of (8.27).

Lemma 8.4.12. *Dunne et al. (2003) (Discrete minimum principle) Assume a function $Z(s_m, t_{n+1})$ defined on the discretized domain $\bar{\mathfrak{D}}^{N,M} = \bar{\Omega}^N \times \bar{\Lambda}^M$. If $Z(s_m, t_{n+1}) \geq 0$, $(s_m, t_{n+1}) \in \eta^{N,M} = \bar{\mathfrak{D}}^{N,M} \cap \eta$ and $(\frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} - 1)Z(s_m, t_{n+1}) \leq 0$, $(s_m, t_{n+1}) \in \mathfrak{D}^{N,M}$, then $Z(s_m, t_{n+1}) \geq 0$, $(s_m, t_{n+1}) \in \bar{\mathfrak{D}}^{N,M}$.*

Proof. Assume $\exists (s_m^*, t_{n+1}^*) \in \bar{\mathfrak{D}}^{N,M}$, such that $Z(s_m^*, t_{n+1}^*) = \min_{(s_m, t_{n+1}) \in \bar{\mathfrak{D}}} Z(s_m, t_{n+1}) < 0$. From the assumption, it is shown that $(s_m^*, t_{n+1}^*) \notin \eta^{N,M}$ implies that $(s_m^*, t_{n+1}^*) \in (0, 1) \times (0, T)$. Since $Z(s_m^*, t_{n+1}^*) = \min_{(s_m, t_{n+1}) \in \bar{\mathfrak{D}}} Z(s_m, t_{n+1}) < 0$, then we have $Z_s(s_m^*, t_{n+1}^*) = 0$, $Z_t(s_m^*, t_{n+1}^*) = 0$ and $Z_{ss}(s_m^*, t_{n+1}^*) \geq 0$. So that, $(\frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} - 1)Z(s_m, t_{n+1}) > 0$. Which is a contradiction. Therefore, $Z(s_m, t_{n+1}) \geq 0$, $(s_m, t_{n+1}) \in \bar{\mathfrak{D}}^{N,M}$. ■

Lemma 8.4.13. *Mbroh et al. (2021) (Stability result) Let Z_m^{n+1} be the solution of (8.27), then we have*

$$|Z_m^{n+1}| \leq \frac{\|\mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1}\|}{\varpi} + \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\},$$

where $\chi(s, t) \geq \varpi > 0$.

Proof. Let $P = \frac{\|\mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1}\|}{\varpi} + \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\}$ and set the barrier functions $\theta_{m,n+1}^\pm$ given by $P \pm Z_m^{n+1}$. On the boundaries, we obtain $\theta_{0,n+1}^\pm = P \pm Z_0^{n+1} = \frac{\|\mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1}\|}{\varpi} + \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\} \pm \psi_l(t_{n+1}) \geq 0$, $\vartheta_{N,n+1}^\pm = P \pm Z_N^{n+1} = \frac{\|\mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1}\|}{\varpi} + \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\} \pm \psi_r(t_{n+1}) \geq 0$. For the

discretized domain $s_m, m = 1(1)N - 1$, then

$$\begin{aligned}
& \left(\frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} - 1 \right) \theta_{m, n+1}^\pm = \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} \theta_{m, n+1}^\pm - \theta_{m, n+1}^\pm \\
& = \frac{\Delta t}{2} \varepsilon \sigma_m \left(\frac{P \pm Z_{m-1}^{n+1} - 2(P \pm Z_m^{n+1}) + P \pm Z_{m+1}^{n+1}}{h^2} \right) + \frac{\Delta t}{2} \beta(s_m) \left(\frac{P \pm Z_{m+1}^{n+1} - (P \pm Z_{m-1}^{n+1})}{2h} \right) \\
& \quad - \frac{\Delta t}{2} \chi(s_m, t_{n+1}) \left(P \pm Z_m^{n+1} \right) - \left(P \pm Z_m^{n+1} \right) \\
& = \pm \frac{\Delta t}{2} \varepsilon \sigma_m \left(\frac{Z_{m-1}^{n+1} - 2Z_m^{n+1} + Z_{m+1}^{n+1}}{h^2} \right) \pm \frac{\Delta t}{2} \beta(s_m) \left(\frac{Z_{m+1}^{n+1} - Z_{m-1}^{n+1}}{2h} \right) \\
& \quad \pm \left(-\frac{\Delta t}{2} \chi(s_m, t_{n+1}) - 1 \right) Z_m^{n+1} - \left(\frac{\Delta t}{2} \chi(s_m, t_{n+1}) + 1 \right) P \\
& = \pm \left(\frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} - 1 \right) Z_m^{n+1} - \left(\frac{\Delta t}{2} \chi(s_m, t_{n+1}) + 1 \right) P \\
& = \pm \Delta t \gamma(s_m, t_{n+\frac{1}{2}}) - \left(\frac{\Delta t}{2} \chi(s_m, t_{n+1}) + 1 \right) \frac{\| \mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1} \|}{\varpi} + \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\} \leq 0.
\end{aligned}$$

Applying discrete minimum principle it implies $\theta_{m, n+1}^\pm \geq 0$, $m = 0(1)N$. Hence we obtained the desired bounds. $\blacksquare$

Next, we determine the truncation error in the spatial direction by considering the semi-discrete scheme of (8.12) and the fully discrete scheme of (8.27).

Theorem 8.4.1. *For the coefficients $\beta(s)$ and $\chi(s, t_{n+1})$ of (8.12) such that $Z^{n+1}(s) \in C^4[0, 1]$. Then, the solution Z_m^{n+1} of (8.27) fulfills the following truncation error estimate*

$$| \mathfrak{L}_\sigma^{\Delta t, h}(Z^{n+1}(s_m) - Z_m^{n+1}) | \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3/2} \exp(-\delta s_m / \sqrt{\varepsilon}) \right). \quad (8.31)$$

Proof. The estimate in the space direction is expressed as

$$\begin{aligned}
\left| \mathfrak{L}_\sigma^{\Delta t, h}(Z^{n+1}(s_m) - Z_m^{n+1}) \right| & = \left| \varepsilon \left(\frac{d^2}{ds^2} - \sigma_m \delta_s^2 \right) Z^{n+1}(s_m) + \beta(s_m) \left(\frac{d}{ds} - \delta_s^0 \right) Z^{n+1}(s_m) \right| \\
& \leq \left| \varepsilon \left(\beta(s_m) \frac{\rho}{2} \coth(\beta(s_m) \frac{\rho}{2}) - 1 \right) \delta_s^2 Z^{n+1}(s_m) \right| \\
& \quad + \left| \varepsilon \left(\frac{d^2}{ds^2} - \delta_s^2 \right) Z^{n+1}(s_m) \right| + \left| \beta(s_m) \left(\frac{d}{ds} - \delta_s^0 \right) Z^{n+1}(s_m) \right|,
\end{aligned} \quad (8.32)$$

where $\sigma_m = \beta(s_m) \frac{\rho}{2} \coth(\beta(s_m) \frac{\rho}{2})$ and $\rho = \frac{h}{\varepsilon}$.

Given C_1 and C_2 are constants, then $|\rho \coth(\rho) - 1| \leq C_1 \rho^2$, for $\rho = h/\varepsilon \leq 1$. For $h/\varepsilon \rightarrow \infty$, since $\lim_{h/\varepsilon \rightarrow \infty} \coth(h/\varepsilon) = 1$ which gives $|h/\varepsilon \coth(h/\varepsilon) - 1| \leq C_1 h/\varepsilon$. Generally, $\forall \rho > 0$ we express as

$$C_1 \frac{\rho^2}{\rho + 1} \leq \rho \coth(\rho) - 1 \leq C_2 \frac{\rho^2}{\rho + 1}, \quad (8.33)$$

we get

$$\varepsilon \left| \beta(s_m) \frac{\rho}{2} \coth(\beta(s_m) \frac{\rho}{2}) - 1 \right| \leq \varepsilon \frac{(h/\varepsilon)^2}{h/\varepsilon + 1} = \frac{h^2}{h + \varepsilon}. \quad (8.34)$$

From Taylor's series expansion, we get the bound

$$\begin{aligned} \left| \delta_s^2 Z^{n+1}(s_m) \right| &\leq C \left\| \frac{d^2}{ds^2} Z^{n+1}(s_m) \right\|, \\ \left| \left(\frac{d}{ds} - \delta_s^0 \right) Z^{n+1}(s_m) \right| &\leq Ch^2 \left\| \frac{d^3}{ds^3} Z^{n+1}(s_m) \right\|, \\ \left| \left(\frac{d^2}{ds^2} - \delta_s^2 \right) Z^{n+1}(s_m) \right| &\leq Ch^2 \left\| \frac{d^4}{ds^4} Z^{n+1}(s_m) \right\|. \end{aligned} \quad (8.35)$$

Substituting equations (8.34) and (8.35) into (8.32), we get

$$\left| \mathfrak{L}_\sigma^{\Delta t, h} (Z^{n+1}(s_m) - Z_m^{n+1}) \right| \leq \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2}{ds^2} Z^{n+1}(s_m) \right\| + Ch^2 \left(\left\| \frac{d^3}{ds^3} Z^{n+1}(s_m) \right\| + \varepsilon \left\| \frac{d^4}{ds^4} Z^{n+1}(s_m) \right\| \right). \quad (8.36)$$

Using (8.18), we have the bound for the derivatives

$$\begin{aligned} |\mathfrak{L}_\sigma^{\Delta t, h} (Z^{n+1}(s_m) - Z_m^{n+1})| &\leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-1} \exp(-\delta s_m / \sqrt{\varepsilon}) \right) \\ &\quad + Ch^2 \left((1 + \varepsilon^{-3/2} \exp(-\delta s_m / \sqrt{\varepsilon})) + (\varepsilon + \varepsilon^{-1} \exp(-\delta s_m / \sqrt{\varepsilon})) \right). \end{aligned} \quad (8.37)$$

Evidently, $\varepsilon^{-3/2} \geq \varepsilon^{-1}$, so

$$|\mathfrak{L}_\sigma^{\Delta t, h} (Z^{n+1}(s_m) - Z_m^{n+1})| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3/2} \exp(-\delta s_m / \sqrt{\varepsilon}) \right), \quad (8.38)$$

thus, we get the wanted bounds. ■

Lemma 8.4.14. *For a fixed N mesh numbers and as $\varepsilon \rightarrow 0$, we obtain*

$$\lim_{\varepsilon \rightarrow 0} \max_m \frac{\exp(-\delta s_m / \sqrt{\varepsilon})}{\varepsilon^{i/2}} = 0, \quad i = 1, 2, 3, \dots,$$

where $s_m = mh$, $m = 1(1)N - 1$.

Proof. On the discrete domain $\{s_m\}_0^N$, it is clear that the interior grid points satisfy the inequalities

$$\max_{1 \leq m \leq N-1} \frac{\exp(-\delta s_m / \sqrt{\varepsilon})}{\varepsilon^{i/2}} \leq \frac{\exp(-\delta s_1 / \sqrt{\varepsilon})}{\varepsilon^{i/2}} = \frac{\exp(-\delta h / \sqrt{\varepsilon})}{\varepsilon^{i/2}}, \quad (8.39)$$

since $s_1 = h = 1 - s_{1-N}$. Repeatedly using L'hospital's rule results in

$$\lim_{\varepsilon \rightarrow 0} \frac{\exp(-h\delta/\sqrt{\varepsilon})}{\varepsilon^i} = \lim_{v=1/\sqrt{\varepsilon} \rightarrow \infty} \frac{v^i}{\exp(\delta hv)} = \lim_{v \rightarrow \infty} \frac{i!}{(\delta h)^i \exp(\delta hv)} = 0.$$

This completes the proof. The proof is also considered in (Mbroh et al., 2020). ■

Theorem 8.4.2. *Let the solution of (8.27) be Z_m^{n+1} , then the uniform error is estimated by*

$$\sup_{\varepsilon \in (0,1]} \|Z^{n+1}(s_m) - Z_m^{n+1}\| \leq Ch, \quad (8.40)$$

where $m = 0(1)N$.

Proof. Plugging Lemma 8.4.14 into (8.31), we get

$$|\mathfrak{L}_\sigma^{\Delta t, h}(Z^{n+1}(s_m) - Z_m^{n+1})| \leq \frac{Ch^2}{h + \varepsilon}.$$

Hence, the result leads to $|Z^{n+1}(s_m) - Z_m^{n+1}| \leq \frac{Ch^2}{h + \varepsilon}$. Using the sup overall $\varepsilon \in (0, 1]$, we get

$$\sup_{\varepsilon \in (0,1]} \|Z^{n+1}(s_m) - Z_m^{n+1}\| \leq Ch. \quad \blacksquare$$

Theorem 8.4.3. *Let z and Z are the solutions of (8.1) and (8.27), respectively, then it satisfies a uniform error estimate*

$$\sup_{\varepsilon \in (0,1]} \|z - Z\| \leq C(h + (\Delta t)^2). \quad (8.41)$$

Proof. The proof is completed by combining Lemma 8.3.10 and Theorem 8.4.2. ■

8.5 Numerical results

In this section some numerical experiment are conducted to verify the proposed scheme is stable, more accurate and ε -uniformly convergent by considering two test examples.

Example 8.1. *Consider the problem*

$$\begin{cases} \varepsilon \frac{\partial^2 z}{\partial s^2} + s^p \frac{\partial z}{\partial s} - z - \frac{\partial z}{\partial t} = s^2 - 1, & (s, t) \in (0, 1) \times (0, 1], \\ z(s, 0) = (1 - s)^2, & s \in \bar{\mathfrak{D}}, \\ z(0, t) = 1 + t^2, & z(1, t) = 0, \quad t \in (0, 1]. \end{cases}$$

Example 8.2. Consider the problem

$$\begin{cases} \varepsilon \frac{\partial^2 z}{\partial s^2} + s^p \frac{\partial z}{\partial s} - (s + p)z - \frac{\partial z}{\partial t} = p(s^2 - 1)\exp(-t), & (s, t) \in (0, 1) \times (0, 1], \\ z(s, 0) = (1 - s)^2, & s \in \bar{\mathfrak{D}}, \\ z(0, t) = 1 + t^2, & z(1, t) = 0, \quad t \in (0, 1]. \end{cases}$$

If the exact solutions of the considered the examples are not known, the double mesh principle is used to estimate the maximum pointwise absolute errors. Therefore, we calculate the maximum pointwise absolute error by the formula given in (4.43), the ε -uniform error by the formula given in (4.44), the rate of convergence by the formula given in (4.45), and the ε -uniform rate of convergence by the formula given in (4.46).

Table 8.1: Computed $E_\varepsilon^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 8.1 for $p = 1$.

$\varepsilon \downarrow$	Number of intervals $N = M$			
	32	64	128	256
10^{-6}	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
10^{-8}	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
10^{-10}	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
10^{-12}	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
10^{-16}	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
10^{-18}	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
10^{-20}	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
$E^{N,M}$	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
$r^{N,M}$	0.8501	0.8906	0.9204	—

Table 8.2: Comparison of $E^{N,M}$ and $r^{N,M}$ for Example 8.1 for $p = 1$.

Schemes $\downarrow$	Number of intervals $N = M$			
	32	64	128	256
Proposed scheme				
$E^{N,M}$	8.0607e-03	4.4718e-03	2.4120e-03	1.2744e-03
$r^{N,M}$	0.8501	0.8906	0.9204	—
Kumar and Kumari (2020)				
$E^{N,M}$	3.0430e-02	1.1938e-02	6.1610e-03	3.2113e-03
$r^{N,M}$	1.3499	0.9543	0.9400	—

For distinguishable values of ε and N the obtained results for the model Examples 8.1 and 8.2, respectively, $E_\varepsilon^{N,M}$, $E^{N,M}$, and $r^{N,M}$ of the developed scheme (8.27) are delineated in Tables 8.1 and 8.2. From these tables, one can observe that the maximum pointwise absolute error decreases as the step sizes decrease for every value of ε , and as ε goes smaller, the maximum pointwise absolute error becomes constant, which displays ε -uniform convergence of the developed scheme regardless of ε . The calculated $E^{N,M}$ and the corresponding $r^{N,M}$ using the

Table 8.3: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ of Example 8.2 for $p = 1$.

$\varepsilon \downarrow$	Number of intervals $N = M$			
	32	64	128	256
10^{-6}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
10^{-8}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
10^{-10}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
10^{-12}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
10^{-16}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
10^{-18}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
10^{-20}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
$E^{N,M}$	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04
$r^{N,M}$	0.9128	0.9608	0.9846	–

Table 8.4: Comparison of $E_{\varepsilon}^{N,M}$ and $r_{\varepsilon}^{N,M}$ of Example 8.2 for $p = 1$.

$\varepsilon \downarrow$	Number of intervals $N = M$				
	32	64	128	256	512
Present method					
2^{-10}	5.7949e-03	3.2560e-03	2.5153e-03	1.5669e-03	8.7472e-04
	0.8317	0.3724	0.6828	0.8410	–
2^{-15}	5.8935e-03	3.1304e-03	3.8325e-03	1.7184e-03	7.1006e-04
	0.9128	-0.2919	1.1572	1.2751	–
2^{-20}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04	4.6537e-04
	0.9128	0.9608	0.9846	0.8045	–
2^{-25}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04	4.0825e-04
	0.9128	0.9608	0.9846	0.9934	–
2^{-30}	5.8935e-03	3.1304e-03	1.6083e-03	8.1279e-04	4.0825e-04
	0.9128	0.9608	0.9846	0.9934	–
Majumdar and Natesan (2017)					
2^{-10}	1.7481e-02	1.0302e-02	5.9462e-03	3.3853e-03	1.9118e-03
	0.7629	0.7929	0.8127	0.8244	–
2^{-15}	1.6983e-02	1.0060e-02	5.8145e-03	3.3187e-03	1.8729e-03
	0.7554	0.7901	0.8090	0.8254	–
2^{-20}	1.6951e-02	1.0051e-02	5.8101e-03	3.3167e-03	1.8717e-03
	0.7540	0.7907	0.8088	0.8254	–
2^{-25}	1.6948e-02	1.0050e-02	5.8101e-03	3.3167e-03	1.8717e-03
	0.7538	0.7906	0.8088	0.8254	–
2^{-30}	1.6947e-02	1.0050e-02	5.8101e-03	3.3167e-03	1.8717e-03
	0.7538	0.7906	0.8088	0.8254	–

Table 8.5: Computed $E_{\varepsilon}^{N,M}$ and $r_{\varepsilon}^{N,M}$ for $\varepsilon = 10^{-10}$.

$p \downarrow$	Example 8.1			Example 8.2		
	N=32	64	128	32	64	128
	M=32	64	128	32	64	128
1	8.0607e-03	4.4718e-03	2.4120e-03	5.8935e-03	3.1304e-03	1.6083e-03
	0.8501	0.8906	-	0.9128	0.9608	-
2	6.5919e-03	3.9087e-03	2.2569e-03	3.4578e-03	1.8498e-03	9.5601e-04
	0.75401	0.7924	-	0.9025	0.9523	-
3	5.6792e-03	3.4719e-03	2.0786e-03	5.0147e-03	2.7300e-03	1.4226e-03
	0.7100	0.7401	-	0.8773	0.9404	-
4	4.9951e-03	3.1524e-03	1.9183e-03	6.4596e-03	3.5805e-03	1.8817e-03
	0.6641	0.7166	-	0.8513	0.9281	-

Table 8.6: Computed $E_{\varepsilon}^{N,M}$ and $r_{\varepsilon}^{N,M}$.

$\varepsilon \downarrow$	Example 8.1			Example 8.2		
	N=16	64	256	16	64	256
	M=16	32	64	16	32	64
10^{-6}	1.3867e-02	4.4018e-03	1.2511e-03	1.0405e-02	3.0735e-03	7.9741e-04
	1.6555	1.8149	-	1.7593	1.9465	-
10^{-8}	1.3867e-02	4.4018e-03	1.2511e-03	1.0405e-02	3.0735e-03	7.9741e-04
	1.6555	1.8149	-	1.7593	1.9465	-
10^{-10}	1.3867e-02	4.4018e-03	1.2511e-03	1.0405e-02	3.0735e-03	7.9741e-04
	1.6555	1.8149	-	1.7593	1.9465	-
10^{-12}	1.3867e-02	4.4018e-03	1.2511e-03	1.0405e-02	3.0735e-03	7.9741e-04
	1.6555	1.8149	-	1.7593	1.9465	-

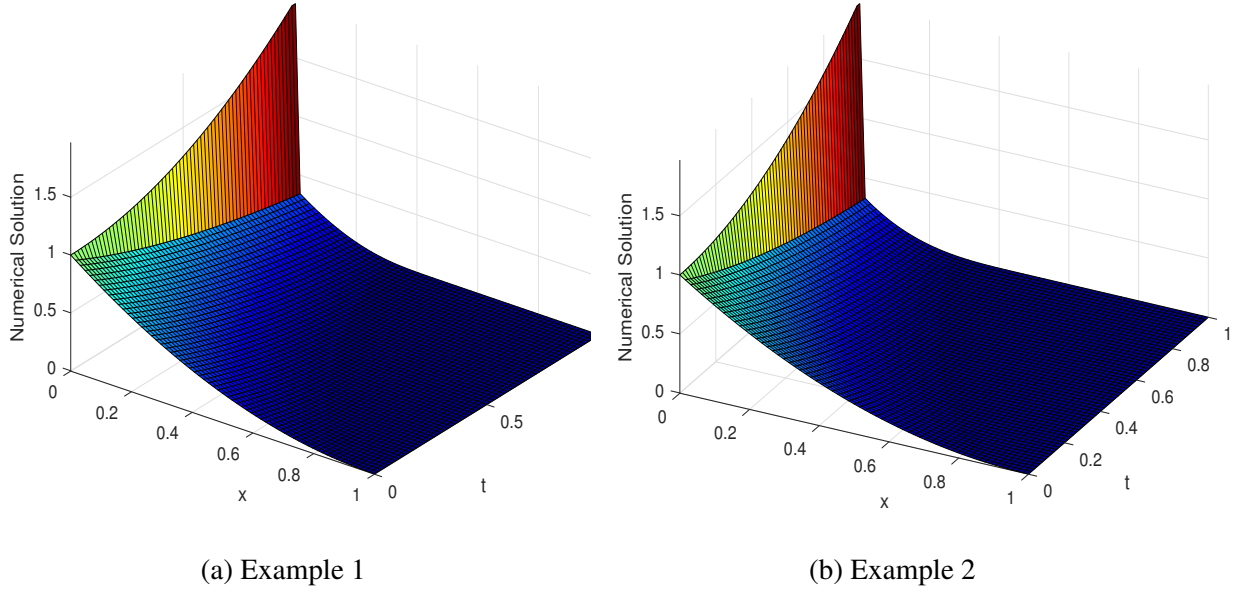


Figure 8.1: Numerical solution of the scheme with $N = M = 64$, $p = 1$, and $\varepsilon = 10^{-20}$.

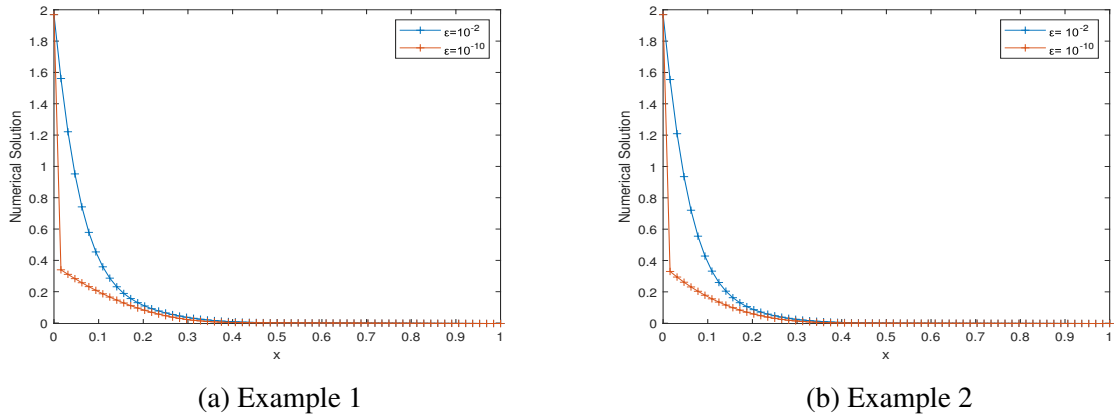


Figure 8.2: Effect of ε on the solution profiles with $N = M = 64$ and $p = 1$.

developed scheme are given in the last two rows of each table, which confirms that the theoretical finding of the developed scheme is ordered one in space. On the other hand, by applying $4N$ and $2M$ we confirmed that the proposed scheme has order two in time and delineated the results in Table 8.6. Furthermore, we look at the comparison of the developed scheme with some of the existing published works. However, as we have seen in Tables 8.2 and 8.4 the developed scheme has better accuracy than some of the existing published works. In Table 8.3, we provide ε -uniform pointwise error for distinct values of $p \geq 1$ for the two model examples. From the result, one can conclude that the error estimate does not depend on the values of p .

From the model examples and the bounds of the derivatives, one can expect that the problem has a boundary layer near $s = 0$. To show this boundary layer, the surface plot solutions of Example 8.1 and Example 8.2 are depicted in Figure 8.1, for the value of $\varepsilon = 10^{-20}$ with $N = M = 64$ and $p = 1$. In another way, Figure 8.2 visualizes the numerical solution of

Example 8.1 and Example 8.2. As one can see from the figures, the problem has a parabolic boundary layer, which exhibits a boundary layer of width $O(\sqrt{\varepsilon})$ in the neighborhood of $s = 0$ as $\varepsilon \rightarrow 0$.

8.6 Summary

A uniformly convergent numerical scheme for the singularly perturbed degenerate convection-diffusion equation is proposed. The scheme employed the Crank-Nicolson method in the time direction and an exponentially fitted central finite difference method in the space direction. Applying the comparison principle, the stability of the discrete scheme is analyzed. The stability and uniform convergence of the method are discussed theoretically. The numerical solutions are delineated by applying the maximum pointwise absolute error, ε -uniform error, and ε -uniform order of convergence in tables that are in acceptable agreement with the theoretical results by taking two test examples. The proposed method is stable and ε -uniform with first-order in space and order two in time. In the subsequent chapter, we extend to the time delay problem.

CHAPTER NINE

FITTED OPERATOR METHOD FOR SINGULARLY PERTURBED DEGENERATE CONVECTION-DIFFUSION EQUATION WITH SMALL TIME DELAY

In this chapter, we extended work of the preceding chapter by considering problem with small time delay. The considered problem has received little attention in the literature and motivated us to extend the study. In this chapter, we suggested method using the Crank-Nicolson method in the time direction and an exponentially fitted operator finite difference method in the space direction on a uniform mesh. The stability of the scheme is analyzed using the discrete maximum principle and discrete solution bounds. Two numerical examples are considered to validate the applicability of the developed method. It is proved that the considered method is stable and uniformly convergent with the rate of convergence of order one in space and order two in time. Theoretical findings are supported by numerical results displayed in tables. The chapter also provided graphical information on the boundary layer behavior of the solutions.

9.1 Governing problem

We consider a singularly perturbed parabolic delay differential equation with a degenerating convection term of the form

$$\begin{cases} \mathfrak{L}_\varepsilon z(s, t) - b(s, t)z_t(s, t) = \kappa(s, t)z(s, t - \tau) + \gamma(s, t), (s, t) \in \mathfrak{D} = (0, 1) \times (0, T], \\ z(s, t) = \psi_b(s, t), (s, t) \in \eta_b := [0, 1] \times [-\tau, 0], \\ z(0, t) = \psi_l(t), (s, t) \in \eta_l := \{0\} \times [0, T], \\ z(1, t) = \psi_r(t), (s, t) \in \eta_r := \{1\} \times [0, T], \end{cases} \quad (9.1)$$

where $\mathfrak{L}_\varepsilon z(s, t) = \varepsilon z_{ss}(s, t) + \beta(s)z_s(s, t) - \chi(s, t)z(s, t)$.

Here, $\beta(s) = \beta_0(s)s^p$, $\beta_0(s) \geq \mu > 0$, $s \in [0, 1]$, $p \geq 1$, the perturbation parameter $\varepsilon \in (0, 1]$, and the delay parameter τ . The coefficients $b(s, t)$, $\beta(s)$, $\chi(s, t)$, $\kappa(s, t)$, $\gamma(s, t)$, and the boundary functions $\psi_b(s, t)$, $\psi_l(t)$, $\psi_r(t)$ on $\eta = \eta_b \cup \eta_l \cup \eta_r = \bar{\mathfrak{D}} - \mathfrak{D}$ are taken to be smooth and bounded that fulfill the preconditions $0 \leq \beta(s)$, $0 < \alpha \leq b(s, t)$, and $0 < \varpi \leq \chi(s, t)$ $(s, t) \in \bar{\mathfrak{D}} = [0, 1] \times [0, T]$ for some fixed $T > 0$. For the case $p = 1$, the problem becomes a simple turning point problem and for the case $p > 1$ it becomes a multiple turning point problem (Kumar and Kumari, 2020).

9.2 Properties of the continuous solution

Taking $\varepsilon = 0$ in (9.1), we get the reduced problem

$$\begin{cases} \beta(s)z_s^0(s, t) - b(s, t)z_t^0(s, t) - \chi(s, t)z^0(s, t) = \kappa(s, t)z^0(s, t - \tau) + \gamma(s, t), & (s, t) \in \mathfrak{D}, \\ z^0(0, t) = \psi_l(t), z^0(1, t) = \psi_r(t), & t \in (0, 1], \\ z^0(s, t) = \psi_b(s, t), & (s, t) \in \eta_b, \end{cases} \quad (9.2)$$

which is a first-order hyperbolic PDE. For $\beta(0) = 0$ and $\chi(0, t) > 0$ the boundary $s = 0$ is a characteristic curve of (9.2), and the solution of (9.1) has a parabolic boundary layer closes to $s = 0$.

The problem (9.1) satisfies the minimum principle stated in Lemma 8.2.1 and the stability result of its solution $z(s, t)$ satisfies the bound given in Lemma 8.2.2.

Lemma 9.2.1. *Let the solution of (9.1) be $z(s, t)$, then the derivative of $z(s, t)$ satisfies the estimate*

$$\left\| \frac{\partial^{l+i} z}{\partial s^i \partial t^l} \right\|_{\bar{\mathfrak{D}}} \leq C(1 + \varepsilon^{-i/2} \exp(-\delta s \sqrt{\varepsilon})), \quad (s, t) \in \bar{\mathfrak{D}}, \quad 1 \leq i + 2l \leq 4, \quad (9.3)$$

where $\delta = \sqrt{\varpi}$.

Proof. The proof is done follows from Lemma 8.2.3 - Lemma 8.2.6 for the integers and it is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained. ■

9.3 Numerical scheme

In this section, we discuss the numerical scheme used to solve the problem (9.1). The scheme involves Crank-Nicolson scheme in the time direction and an exponentially fitted finite difference scheme in space direction. Taylor's expansion is used to estimate the delay term as given in (4.14) and substituting into (9.1), we get

$$\varepsilon z_{ss}(s, t) + (\tau \kappa(s, t) - 1)z_t(s, t) + \beta(s)z_s(s, t) - (\chi(s, t) + \kappa(s, t))z(s, t) = \gamma(s, t). \quad (9.4)$$

9.3.1 Temporal semi-discretization

Discretize the temporal direction using a uniform mesh with step size Δt given by $\Lambda^M = \{t_n = n\Delta t, n = 0, 1, 2, \dots, M, t_M = T, \Delta t = T/M\}$, where M is the number of mesh points in the domain of time $[0, T]$. For some positive integer r we have $T = r\tau$. The temporal derivative

of the problem (9.4) is discretized by using the Crank-Nicolson method, which gives as

$$\begin{cases} (\tau\kappa(s, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}^{\Delta t})Z^{n+1}(s) = (\tau\kappa(s, t_n) - 1 - \frac{\Delta t}{2}\mathfrak{L}^{\Delta t})Z^n(s) \\ + \frac{\Delta t}{2}(\gamma(s, t_n) + \gamma(s, t_{n+1})), \\ Z^{n+1}(0) = \psi_l(t_{n+1}), \quad Z^{n+1}(1) = \psi_r(t_{n+1}), \quad 0 \leq n \leq M, \end{cases} \quad (9.5)$$

where $\mathfrak{L}^{\Delta t}Z^{n+1}(s) = \varepsilon \frac{d^2 Z^{n+1}(s)}{ds^2} + \beta(s) \frac{dZ^{n+1}(s)}{ds} - (\chi(s, t_{n+1}) + \kappa(s, t_{n+1}))Z^{n+1}(s)$.

At the $(n+1)^{th}$ time level the approximation of $Z(s, t_{n+1})$ is denoted by $Z^{n+1}(s)$. The operator $(\tau\kappa(s, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}^{\Delta t})Z^{n+1}(s)$ satisfies the next minimum principle.

Lemma 9.3.2. *Consider the function $Z^{n+1}(s) \in C^2(\bar{\mathfrak{D}})$ such that $Z^{n+1}(0) \geq 0$, $Z^{n+1}(1) \geq 0$, and $(\tau\kappa(s, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}^{\Delta t})Z^{n+1}(s) \leq 0 \in \mathfrak{D}$, then $Z^{n+1}(s) \geq 0 \in \bar{\mathfrak{D}}$.*

Proof. Let $\exists s^* \in [0, 1]$ such that $Z^{n+1}(s^*) = \min_{s \in [0, 1]} Z^{n+1}(s) < 0$. From the assumption, it is clear that $s^* \notin \{0, 1\}$ implies that $s^* \in (0, 1)$. Since $Z^{n+1}(s^*) = \min_{s \in [0, 1]} Z^{n+1}(s) < 0$, then we have $\frac{d}{ds}Z^{n+1}(s^*) = 0$ and $\frac{d^2}{ds^2}Z^{n+1}(s^*) > 0$. So, the operator gives that $(\tau\kappa(s, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}^{\Delta t})Z^{n+1}(s) < 0$. Which is contradicted to $(\tau\kappa(s, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}^{\Delta t})Z^{n+1}(s) > 0, \forall s \in \mathfrak{D}$. Therefore, $Z^{n+1}(s) \geq 0, \forall s \in \bar{\mathfrak{D}}$. ■

Following, analyze the local truncation error for the temporal semi-discretization method described above. At each time step, the local truncation error is given by $e_{n+1}(s) := z(s, t_{n+1}) - Z^{n+1}(s), 0 \leq n \leq M$.

Lemma 9.3.3. *If $\left| \frac{\partial^l z(s, t)}{\partial t^l} \right| \leq C, (s, t) \in \bar{\mathfrak{D}}$ and $0 \leq l \leq 2$, then the local truncation error of the scheme (9.5) satisfies*

$$\|e_{n+1}\| \leq C_1(\Delta t)^3. \quad (9.6)$$

Proof. From the Taylor's series approximation of $z(s, t_n)$ and $z(s, t_{n+1})$, centering at $z(s, t_{n+\frac{1}{2}})$, we get

$$\begin{aligned} z(s, t_n) &= z(s, t_{n+\frac{1}{2}}) - \frac{\Delta t}{2}z_t(s, t_{n+\frac{1}{2}}) + \frac{(\Delta t)^2}{8}z_{tt}(s, t_{n+\frac{1}{2}}) + O(\Delta t^3), \\ z(s, t_{n+1}) &= z(s, t_{n+\frac{1}{2}}) + \frac{\Delta t}{2}z_t(s, t_{n+\frac{1}{2}}) + \frac{(\Delta t)^2}{8}z_{tt}(s, t_{n+\frac{1}{2}}) + O(\Delta t^3) \end{aligned} \quad (9.7)$$

From (9.7), we get

$$\frac{z(s, t_{n+1}) - z(s, t_n)}{\Delta t} = z_t(s, t_{n+\frac{1}{2}}) + O(\Delta t^2) \quad (9.8)$$

Substituting the approximation into (9.5), we get

$$\begin{aligned} (\tau\kappa(s, t) - 1) \frac{z(s, t_{n+1}) - z(s, t_n)}{\Delta t} &= -\varepsilon z_{ss}(s, t_{n+\frac{1}{2}}) - \beta(s)z_s(s, t_{n+\frac{1}{2}}) \\ &\quad + (\chi(s, t_{n+\frac{1}{2}}) + \kappa(s, t_{n+\frac{1}{2}}))z(s, t_{n+\frac{1}{2}}) + \gamma(s, t_{n+\frac{1}{2}}) + O(\Delta t^2), \end{aligned}$$

where

$$\begin{aligned}\varepsilon z_{ss}(s, t_{n+\frac{1}{2}}) &= \frac{\varepsilon u_{ss}(s, t_{n+1}) + \varepsilon z_{ss}(s, t_n)}{2} + O(\Delta t^2), \\ z_s(s, t_{n+\frac{1}{2}}) &= \frac{z_s(s, t_{n+1}) + z_s(s, t_n)}{2} + O(\Delta t^2), \\ z(s, t_{n+\frac{1}{2}}) &= \frac{z(s, t_{n+1}) + z(s, t_n)}{2} + O(\Delta t^2), \\ \gamma(s, t_{n+\frac{1}{2}}) &= \frac{\gamma(s, t_{n+1}) + \gamma(s, t_n)}{2} + O(\Delta t^2)\end{aligned}$$

since the error $e_{n+1} =: z(s, t_{n+1}) - Z^{n+1}(s)$ and it satisfies the following semi-discrete differential operator

$$\begin{aligned}(\tau \kappa(s, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}^{\Delta t}) e_{n+1}(s) &= O(\Delta t^3), \\ e_{j+1}(0) &= 0 = e_{j+1}(1).\end{aligned}\tag{9.9}$$

Thus, by applying the maximum principle, we get

$$\|e_{j+1}\| \leq C_1(\Delta t)^3.\tag{9.10}$$

■

Lemma 9.3.4. *Let the global error up to $(n+1)^{th}$ time step is denoted by E_{n+1} . At t_{n+1} it satisfies inequality*

$$\|E_{n+1}\| \leq C(\Delta t)^2, \quad 1 \leq n \leq M-1.$$

Proof. From Lemma 9.3.3, the global error at $(n+1)^{th}$ time step is given by

$$\begin{aligned}\|E_{n+1}\| &= \left\| \sum_{l=1}^{n+1} e_l \right\| \leq \|e_1\| + \|e_2\| + \|e_3\| + \dots + \|e_{n+1}\| \\ &\leq C_1 T (\Delta t)^2, \quad \text{since } (n+1)\Delta t \leq T \\ &= C(\Delta t)^2, \quad \text{where } C_1 T = C,\end{aligned}$$

where C is independent of ε and Δt .

■

Lemma 9.3.5. *For every $0 \leq n \leq M-1$, the solution $Z^{n+1}(s)$ of (9.5) satisfies the estimate*

$$\left| \frac{d^i Z^{n+1}(s)}{ds^i} \right| \leq C(1 + \varepsilon^{-i/2} \exp(-\delta s / \sqrt{\varepsilon})), \quad s \in \bar{\mathfrak{D}}, \quad 0 \leq j \leq 4.\tag{9.11}$$

Proof. The proof follows from Lemma 8.2.4, it is done for $i = 1$. For $i > 1$ followed by induction and repeated differentiation, the higher order derivatives up to the fourth order can be obtained.

■

9.3.2 Spatial discretization

We discretized the space domain $[0, 1]$ into N equal number of points $s_m = mh$, $0 \leq m \leq N$ with the mesh spacing h is given as $\Omega^N = \{0 = s_0, s_1, \dots, s_N = 1, s_m = mh\}$. Using the differences operators given in (8.19) and applying the central FDM in (9.5), for $m = 1, 2, \dots, N - 1$, we get

$$\begin{cases} \left(\tau\kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}^{\Delta t, h} \right) Z_m^{n+1} = \left(\tau\kappa(s_m, t_n) - 1 - \frac{\Delta t}{2} \mathfrak{L}^{\Delta t, h} \right) Z_m^n \\ + \frac{\Delta t}{2} (\gamma(s_m, t_n) + \gamma(s_m, t_{n+1})), \\ Z^{n+1}(0) = \psi_l(t_{n+1}), \quad Z^{n+1}(1) = \psi_r(t_{n+1}), \quad 0 \leq n \leq M, \end{cases} \quad (9.12)$$

where $\mathfrak{L}^{\Delta t, h} Z_m^{n+1} = \varepsilon \delta_s^2 Z_m^{n+1} + \beta(s_m) \delta_s^0 Z_m^{n+1} - (\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1})) Z_m^{n+1}$.

Next, we introduces an exponentially fitted operator FDM for spatial discretization that used to control the influence of ε to obtain an accurate and uniformly convergent numerical scheme.

Computing the exponential fitting factor

Here we introduce the fitting factor σ obtained from the semi-discrete scheme of (9.5) at the $(m, n + 1)^{th}$ level. To manage the influence of ε in the layer region, the exponentially fitting factor σ is introduced as

$$\mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1} = \varepsilon \sigma \delta_s^2 Z_m^{n+1} + \beta(s_m) \delta_s^0 Z_m^{n+1} - (\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1})) Z_m^{n+1}. \quad (9.13)$$

Consequently, following the results given in (8.22) - (8.25), we get the following variable fitting factor

$$\sigma_m = \frac{\rho \beta(s_m)}{2} \coth \left(\frac{\rho \beta(s_m)}{2} \right). \quad (9.14)$$

9.3.3 The discrete scheme

Applying the central finite difference method and the fitting factor σ_m , for $m = 1, 2, 3, \dots, N - 1$, and $n = 0, 1, 2, \dots, M - 1$, then we have the following fully discrete scheme

$$\begin{cases} \left(\tau\kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} \right) Z_m^{n+1} = \left(\tau\kappa(s_m, t_n) - 1 - \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} \right) Z_m^n \\ + \frac{\Delta t}{2} (\gamma(s_m, t_n) + \gamma(s_m, t_{n+1})), \\ Z^{n+1}(0) = \psi_l(t_{n+1}), \quad Z^{n+1}(1) = \psi_r(t_{n+1}), \quad 0 \leq n \leq M, \end{cases} \quad (9.15)$$

where $\mathfrak{L}_\sigma^{\Delta t, h} Z_m^{n+1} = \varepsilon \sigma_m \delta_s^2 Z_m^{n+1} + \beta(s_m) \delta_s^0 Z_m^{n+1} - (\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1})) Z_m^{n+1}$.

We rewrite in an explicit form as

$$Q_m^- Z_{m-1}^{n+1} + Q_m^0 Z_m^{n+1} + Q_m^+ Z_{m+1}^{n+1} = R_m^- Z_{m-1}^n + R_m^0 Z_m^n + R_m^+ Z_{m+1}^n + H_m^n, \quad (9.16)$$

where

$$\begin{aligned} Q_m^- &= \frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} - \frac{\beta(s_m)}{2h} \right), \\ Q_m^0 &= -\frac{\Delta t \varepsilon \sigma_m}{h^2} - \frac{\Delta t}{2} \left(\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1}) \right) + \tau \kappa(s_m, t_{n+1}) - 1, \\ Q_m^+ &= \frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} + \frac{\beta(s_m)}{2h} \right), \\ R_m^- &= -\frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} - \frac{\beta(s_m)}{2h} \right), \\ R_m^0 &= \frac{\Delta t \varepsilon \sigma_m}{h^2} + \frac{\Delta t}{2} \left(\chi(s_m, t_n) + \kappa(s_m, t_n) \right) + \tau \kappa(s_m, t_n) - 1, \\ R_m^+ &= -\frac{\Delta t}{2} \left(\frac{\varepsilon \sigma_m}{h^2} - \frac{\beta(s_m)}{2h} \right), \end{aligned} \quad (9.17)$$

and

$$\begin{aligned} H_1(t) &= \frac{\Delta t}{2} \left(\gamma(s_1, t_n) + \gamma(s_1, t_{n+1}) \right) - Q_1^- \psi_l(t_{n+1}), \\ H_m(t) &= \frac{\Delta t}{2} \left(\gamma(s_m, t_n) + \gamma(s_m, t_{n+1}) \right), \\ H_{N-1}(t) &= \frac{\Delta t}{2} \left(\gamma(s_{N-1}, t_n) + \gamma(s_{N-1}, t_{n+1}) \right) - Q_{N-1}^+ \psi_r(t_{n+1}), \quad m = 2, 3, \dots, N-2. \end{aligned} \quad (9.18)$$

9.4 Convergence analysis

The discrete scheme given in (9.15) satisfies the discrete minimum principle stated in Lemma 8.4.12.

Lemma 9.4.6. Assume $Z_m^{n+1} = 1 + s_m$, for $m = 0, 1, 2, \dots, N$. Then, a constant $C > 0$ such that $(\tau \kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h}) Z_m^{n+1} \geq C$, for $1 \leq m \leq N-1$.

Proof. We use a simple computation technique to prove, which enables to estimate a bound, i.e., ε -uniform for the norm of inverse of $(\tau\kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}_\sigma^{\Delta t, h})Z_m^{n+1}$.

$$\begin{aligned}
(\tau\kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}_\sigma^{\Delta t, h})Z_m^{n+1} &= (\tau\kappa(s_m, t_{n+1}) - 1)Z_m^{n+1} + \frac{\Delta t}{2}\sigma_m\varepsilon\delta_s^2 + \frac{\Delta t}{2}\beta(s_m)\delta_s^0 \\
&\quad - \frac{\Delta t}{2}(\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1}))Z_m^{n+1} \\
&= (\tau\kappa(s_m, t_{n+1}) - 1)(1 + s_m) + \frac{\Delta t}{2}\sigma_m\varepsilon\delta_s^2(1 + s_m) \\
&\quad + \frac{\Delta t}{2}\beta(s_m)\delta_s^0(1 + s_m) \\
&\quad - \frac{\Delta t}{2}(\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1}))(1 + s_m) \\
&= (\tau\kappa(s_m, t_{n+1}) - 1)(1 + s_m) + \frac{\Delta t}{2h^2}\sigma_m\varepsilon((1 + s_{m-1}) \\
&\quad - 2(1 + s_m) + (1 + s_{m+1})) \\
&\quad + \frac{\Delta t}{4h}\beta(s_m)((1 + s_{m+1}) - (1 + s_{m-1})) \\
&\quad - \frac{\Delta t}{2}(\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1}))(1 + s_m) \\
&= (\tau\kappa(s_m, t_{n+1}) - 1)(1 + s_m) + \frac{\Delta t}{2}\beta(s_m) \\
&\quad - \frac{\Delta t}{2}(\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1}))(1 + s_m) \geq C,
\end{aligned}$$

since $(\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1})) \geq \theta > 0$, $\beta(s_m) = \beta_0 s_m^p$,
 $\beta_0(s_m) \geq \mu > 0$.

■

Lemma 9.4.7. (Uniform stability result) Let Z_m^{n+1} be the solution of (9.15), then we have

$$|Z_m^{n+1}| \leq \frac{\|(\tau\kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}_\sigma^{\Delta t, h})Z_m^{n+1}\|}{(\tau\kappa(s_m, t_{n+1}) - 1) + \Delta\theta/2} + \max\{|\psi_l(t_{m+1})|, |\psi_r(t_{m+1})|\},$$

where $\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1}) \geq \theta > 0$.

Proof. Consider the two barrier functions $\vartheta_{m,n+1}^\pm$ as

$$\vartheta_{m,n+1}^\pm = \frac{\|(\tau\kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2}\mathfrak{L}_\sigma^{\Delta t, h})Z_m^{n+1}\|}{(\tau\kappa(s_m, t_{n+1}) - 1) + \Delta\theta/2} + C \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\} \pm Z_m^{n+1}.$$

Easily we can show that $\vartheta_{0,n+1}^\pm \geq 0$ and $\vartheta_{N,n+1}^\pm \geq 0$, then

$$\begin{aligned}
& \left(\tau \kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} \right) \vartheta_{m,n+1}^\pm \\
&= (\tau \kappa(s_m, t_{n+1}) - 1) \vartheta_{m,n+1}^\pm + \frac{\Delta t}{2} \varepsilon \sigma_m \delta_s^2 \vartheta_{m,n+1}^\pm + \frac{\Delta t}{2} \beta(s_m) \delta_s^0 \vartheta_{m,n+1}^\pm \\
&\quad - \frac{\Delta t}{2} \left(\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1}) \right) \vartheta_{m,n+1}^\pm \\
&= \left(\tau \kappa(s_m, t_{n+1}) - 1 - \frac{\Delta t}{2} (\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1})) \right) \left(\frac{\| (\tau \kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h}) Z_m^{n+1} \|}{(\tau \kappa(s_m, t_{n+1}) - 1) + \Delta t \theta / 2} \right. \\
&\quad \left. + C \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\} \right) \\
&\quad \pm Z_m^{n+1} \pm \frac{\Delta t}{2} \left(\varepsilon \sigma_m \delta_s^2 Z_m^{n+1} + \beta(s_m) \delta_s^0 Z_m^{n+1} - (\chi(s_m) + \kappa(s_m, t_{n+1})) \right) Z_m^{n+1} \\
&= \left(\tau \kappa(s_m, t_{n+1}) - 1 - \frac{\Delta t}{2} (\chi(s_m, t_{n+1}) + \kappa(s_m, t_{n+1})) \right) \left(\frac{\| (\tau \kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h}) Z_m^{n+1} \|}{(\tau \kappa(s_m, t_{n+1}) - 1) + \Delta t \theta / 2} \right. \\
&\quad \left. + C \max\{|\psi_l(t_{n+1})|, |\psi_r(t_{n+1})|\} \right) \pm \left(\tau \kappa(s_m, t_{n+1}) - 1 + \frac{\Delta t}{2} \mathfrak{L}_\sigma^{\Delta t, h} \right) Z_m^{n+1} \geq 0.
\end{aligned}$$

From the discrete maximum principle, we get $\vartheta_{m,n+1}^\pm \geq 0$, $m = 0, 1, \dots, N$. ■

In the next theorem, we need to determine the truncation error in the space direction in considering the semidiscrete problem in (9.5) and the fully discrete scheme in (9.15).

Theorem 9.4.1. *For the sufficiently smooth coefficient functions $\beta(s)$, $\chi(s, t_{n+1})$ and $\kappa(s, t_{n+1})$ of (9.5) so that $Z^{n+1}(s) \in C^4[0, 1]$. Then, the solution Z_m^{n+1} of the problem (9.15) satisfies*

$$\left| \mathfrak{L}_\sigma^{\Delta t, h}(Z^{n+1}(s_m) - Z_m^{n+1}) \right| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3/2} \exp(-\delta s_m / \sqrt{\varepsilon}) \right). \quad (9.19)$$

Proof. The local truncation error in the spatial direction is expressed by

$$\begin{aligned}
\left| \mathfrak{L}_\sigma^{\Delta t, h}(Z^{n+1}(s_m) - Z_m^{n+1}) \right| &= \left| \varepsilon \left(\frac{d^2}{ds^2} - \sigma_m \delta_s^2 \right) Z^{n+1}(s_m) + \beta(s_m) \left(\frac{d}{ds} - \delta_s^0 \right) Z^{n+1}(s_m) \right| \\
&\leq \left| \varepsilon \left(\beta(s_m) \frac{\rho}{2} \coth(\beta(s_m) \frac{\rho}{2}) - 1 \right) \delta_s^2 Z^{n+1}(s_m) \right| \\
&\quad + \left| \varepsilon \left(\frac{d^2}{ds^2} - \delta_s^2 \right) Z^{n+1}(s_m) \right| + \left| \beta(s_m) \left(\frac{d}{ds} - \delta_s^0 \right) Z^{n+1}(s_m) \right|,
\end{aligned} \quad (9.20)$$

where $\sigma_m = \beta(s_m) \frac{\rho}{2} \coth(\beta(s_m) \frac{\rho}{2})$ and $\rho = \frac{h}{\varepsilon}$.

For the constants C_1 and C_2 we have $|\rho \coth(\rho) - 1| \leq C_1 \rho^2$, for $\rho \leq 1$. For $\rho \rightarrow \infty$, since $\lim_{\rho \rightarrow \infty} \coth(\rho) = 1$ which gives $|\rho \coth(\rho) - 1| \leq C_1 \rho$.

Generally, $\forall \rho > 0$ we express as

$$C_1 \frac{\rho^2}{\rho + 1} \leq \rho \coth(\rho) - 1 \leq C_2 \frac{\rho^2}{\rho + 1}, \quad (9.21)$$

we get

$$\varepsilon \left| \beta(s_m) \frac{\rho}{2} \coth(\beta(s_m) \frac{\rho}{2}) - 1 \right| \leq \varepsilon \frac{(h/\varepsilon)^2}{h/\varepsilon + 1} = \frac{h^2}{h + \varepsilon}. \quad (9.22)$$

Substituting (8.34) and (9.22) into (9.20), we get

$$\begin{aligned} \left| \mathfrak{L}_\sigma^{\Delta t, h} (Z^{n+1}(s_m) - Z_m^{n+1}) \right| &\leq \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2}{ds^2} Z^{n+1}(s_m) \right\| + Ch^2 \left\| \frac{d^3}{ds^3} Z^{n+1}(s_m) \right\| \\ &\quad + C\varepsilon h^2 \left\| \frac{d^{(4)}}{ds^{(4)}} Z^{n+1}(s_m) \right\| \\ &\leq \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2}{ds^2} Z^{n+1}(s_m) \right\| + Ch^2 \left(\left\| \frac{d^3}{ds^3} Z^{n+1}(s_m) \right\| \right. \\ &\quad \left. + \varepsilon \left\| \frac{d^4}{ds^4} Z^{n+1}(s_m) \right\| \right). \end{aligned} \quad (9.23)$$

From Lemma 9.3.5, the bound for the derivatives of the solution gives

$$\begin{aligned} \left| \mathfrak{L}_\sigma^{\Delta t, h} (Z^{n+1}(s_m) - Z_m^{n+1}) \right| &\leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-1} \exp(-\delta s_m / \sqrt{\varepsilon}) \right) \\ &\quad + Ch^2 \left((1 + \varepsilon^{-3/2} \exp(-\delta s_m / \sqrt{\varepsilon})) \right. \\ &\quad \left. + (\varepsilon + \varepsilon^{-1} \exp(-\delta s_m \sqrt{\varepsilon})) \right). \end{aligned} \quad (9.24)$$

Obviously, $\varepsilon^{-1} \leq \varepsilon^{-3/2}$, then we arrive

$$\left| \mathfrak{L}_\sigma^{\Delta t, h} (Z^{n+1}(s_m) - Z_m^{n+1}) \right| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3/2} \exp(-\delta s_m \sqrt{\varepsilon}) \right). \quad (9.25)$$

Hence, it commits the sought bounds. ■

Theorem 9.4.2. *Let Z_m^{n+1} be the numerical solution of the problem (9.15). Then, it satisfies the following uniform error estimate*

$$\sup_{\varepsilon \in (0, 1]} \|Z^{n+1}(s_m) - Z_m^{n+1}\| \leq Ch,$$

where $m = 0, 1, 2, \dots, N$.

Proof. Substituting Lemma 8.4.14 into Theorem 9.4.1, we arrive at

$$\left| \mathfrak{L}_{\sigma}^{\Delta t, h}(Z^{n+1}(s_m) - Z_m^{n+1}) \right| \leq \frac{Ch^2}{h + \varepsilon}.$$

Hence, the result leads to

$$\left| Z^{n+1}(s_m) - Z_m^{n+1} \right| \leq \frac{Ch^2}{h + \varepsilon}.$$

Using the sup over all $\varepsilon \in (0, 1]$, we get

$$\sup_{\varepsilon \in (0, 1]} \|Z^{n+1}(s_m) - Z_m^{n+1}\| \leq Ch.$$

■

Theorem 9.4.3. *Let z and Z be the solutions of (9.1) and (9.15), respectively. Then, satisfies the following uniform error estimate*

$$\sup_{\varepsilon \in (0, 1]} \|z - Z\| \leq C(h + (\Delta t)^2).$$

Proof. This can be done by the combination of Lemma 9.3.3 and Theorem 9.4.2 holds the expected result. ■

9.5 Numerical results

In this section some numerical experiments are conducted to verify the developed scheme is stable, more accurate and ε -uniformly convergent by considering two model examples.

Example 9.1. *Consider the problem*

$$\begin{cases} \varepsilon \frac{\partial^2 z(s, t)}{\partial s^2} + s^p \frac{\partial z(s, t)}{\partial s} - z(s, t) - \frac{\partial z(s, t)}{\partial t} = s^2 - 1 + z(s, t - \tau)/2, & (s, t) \in \mathfrak{D} \\ z(s, t) = (1 - s)^2, & (s, t) \in [0, 1] \times [-\tau, 0] \\ z(0, t) = 1 + t^2, z(1, t) = 0, & t \in (0, 2]. \end{cases} \quad (9.26)$$

Example 9.2. *Consider the problem*

$$\begin{cases} \varepsilon \frac{\partial^2 z(s, t)}{\partial s^2} + s^p \frac{\partial z}{\partial s} - (\exp(s) + 1)z(s, t) - \frac{\partial z}{\partial t} = p(s^2 - 1)\exp(-t) + z(s, t - \tau), & (s, t) \in \mathfrak{D} \\ z(s, t) = (1 - s)^2, & (s, t) \in [0, 1] \times [-\tau, 0] \\ z(0, t) = 1 + t^2, z(1, t) = 0, & t \in (0, 2]. \end{cases} \quad (9.27)$$

Since the exact solutions of the considered examples are not available, the maximum pointwise absolute errors are estimated using the double mesh principle. So, we calculate, the maximum pointwise absolute error by the formula given in (4.43), the ε -uniform error by the formula

given in (4.44), the rate of convergence by the formula given in (4.45), and the ε -uniform rate of convergence by the formula given in (4.46).

Table 9.1: Computed $E_{\varepsilon}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ Example 9.1 with $p = 1$ for $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals $N = M$			
	16	32	64	128
10^{-6}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
10^{-8}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
10^{-10}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
10^{-12}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
10^{-14}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
10^{-16}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
10^{-18}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
10^{-20}	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
$E^{N,M}$	7.7886e-03	6.8154e-03	4.4277e-03	2.4436e-03
$r^{N,M}$	0.1926	0.6222	0.8576	–

Table 9.2: Computed $E^{N,M}$ and $r^{N,M}$ for Example 9.1 with $\varepsilon = 10^{-10}$ and $\tau = 0.5\varepsilon$.

p	Number of intervals $N = M$			
	16	32	64	128
2 $E^{N,M}$	1.9017e-02	1.0057e-02	5.1599e-03	2.6119e-03
$r^{N,M}$	0.9191	0.9628	0.9822	–
3 $E^{N,M}$	1.9116e-02	1.0072e-02	5.1619e-03	2.6122e-03
$r^{N,M}$	0.9244	0.9644	0.9826	–
4 $E^{N,M}$	1.9126e-02	1.0072e-02	5.1619e-03	2.6089e-03
$r^{N,M}$	0.9252	0.9644	0.9845	–
5 $E^{N,M}$	1.9127e-02	1.0073e-02	5.1610e-03	2.6084e-03
$r^{N,M}$	0.9251	0.9645	0.9845	–
6 $E^{N,M}$	1.9127e-02	1.0072e-02	5.1610e-03	2.6084e-03
$r^{N,M}$	0.9251	0.9645	0.9845	–

Using the proposed scheme the computed numerical results are presented in tabular and graphical ways. The computed numerical results for various values of ε are delineated in Tables 9.1 and 9.3. Each table includes information about the proposed method's maximum pointwise absolute error ($E_{\varepsilon}^{N,M}$), ε -uniform absolute error ($E^{N,M}$), and ε -uniform order of convergence ($r^{N,M}$) of Examples 9.1 and 9.2, respectively. These tables indicate that the maximum pointwise absolute error remains uniform as $\varepsilon \rightarrow 0$. The last rows of each table show $E^{N,M}$ and $r^{N,M}$ of the proposed scheme, and demonstrate that the numerical results agree with the theoretical finding. Such that, regardless of the values of ε , the developed scheme is stable and ε -uniform with an order of convergence one. In else way we delineate $E^{N,M}$ and $r^{N,M}$ for different values of p in Table 9.2 and Table 9.4 for Example 9.1 and Example 9.2, respectively. From these

Table 9.3: Computed $E_\varepsilon^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 9.2 with $p = 1$ and $\tau = 0.5\varepsilon$.

$\varepsilon \downarrow$	Number of intervals $N = M$			
	16	32	64	128
10^{-6}	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
10^{-8}	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
10^{-10}	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
10^{-12}	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
10^{-16}	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
10^{-18}	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
10^{-20}	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
$E^{N,M}$	4.6803e-04	2.9489e-04	1.8137e-04	1.0233e-04
$r^{N,M}$	0.6664	0.7012	0.82571	—

Table 9.4: Computed $E^{N,M}$ and $r^{N,M}$ for Example 9.2 with $\varepsilon = 10^{-10}$ and $\tau = 0.5\varepsilon$.

p	Number of intervals $N = M$			
	16	32	64	128
2 $E^{N,M}$	1.1527e-03	5.2670e-04	2.4861e-04	1.2034e-04
$r^{N,M}$	1.1300	1.0831	1.0468	—
3 $E^{N,M}$	1.2647e-03	5.5179e-04	2.5453e-04	1.2178e-04
$r^{N,M}$	1.1966	1.1163	1.0636	—
4 $E^{N,M}$	1.4717e-03	5.7471e-04	2.6019e-04	1.2138e-04
$r^{N,M}$	1.3566	1.1433	1.1000	—
5 $E^{N,M}$	1.6962e-03	5.9755e-04	2.6536e-04	1.2250e-04
$r^{N,M}$	1.5052	1.1711	1.1152	—
6 $E^{N,M}$	1.8605e-03	6.2028e-04	2.7098e-04	1.2390e-04
$r^{N,M}$	1.5847	1.1947	1.1290	—

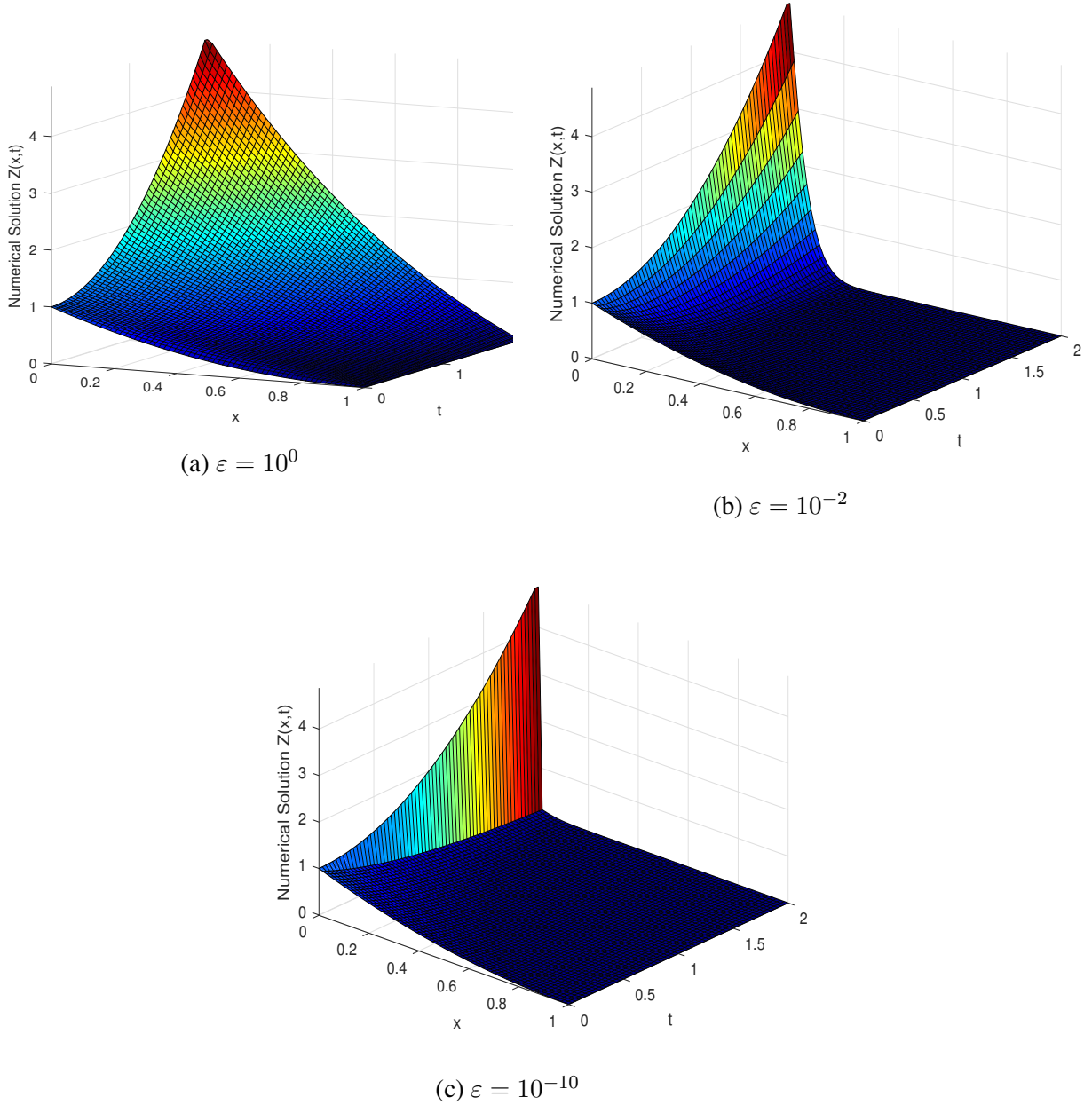


Figure 9.1: Numerical solution of Example 9.1 with $N = 64$ and $p = 1$.

tables, we observe the fact that the error bounds of the developed scheme do not depend on the values of p . The numerical results agree with the theoretical finding discussed in Theorem 9.4.3. The numerical solutions of Examples 9.1 and 9.2 are depicted in Figures 9.1 and 9.2, respectively. From these figures, it can be seen that the problem has a parabolic boundary layer in the neighbourhood of $s = 0$ as $\varepsilon \rightarrow 0$. These proved the fact that the solution of such a problem possesses a boundary turning point at the point where the convection coefficient vanishes.

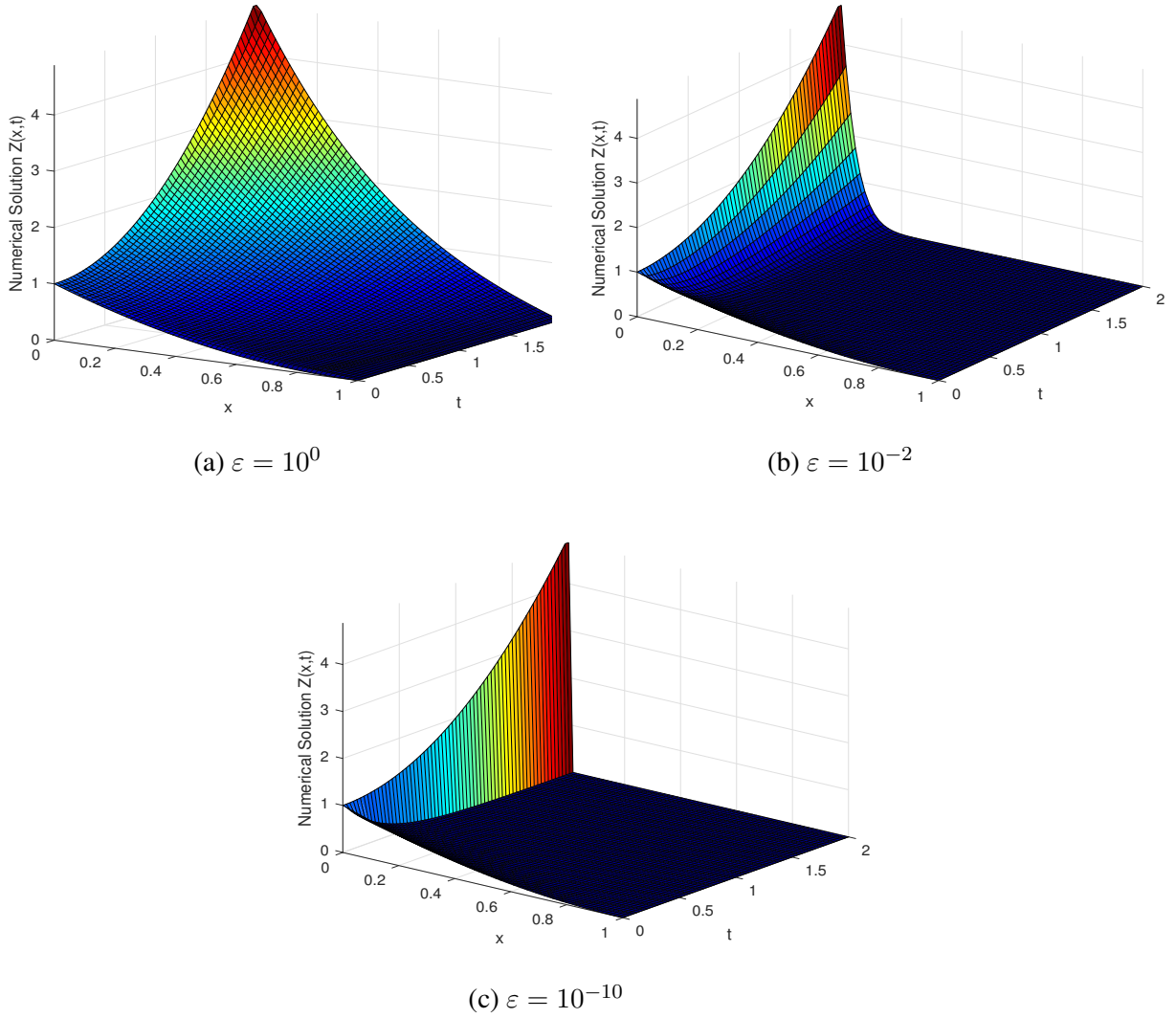


Figure 9.2: Numerical solution of Example 9.2 with $N = 64$ and $p = 1$.

9.6 Summary

In this chapter, we considered a singularly perturbed parabolic delay differential equation with a degenerating convection coefficient. Due to the presence of ε , the solution to the problem possesses a parabolic boundary layer near $s = 0$. To handle this, we used a scheme that employed the Crank-Nicolson method in the time direction and an exponentially fitted operator finite difference scheme in the spatial direction. Using the discrete minimum principle, the stability of the discrete scheme is examined and analyzed. Two numerical examples are conducted to validate the theoretical finding of the scheme and proved that the developed numerical scheme is stable and ε -uniformly convergent.

CONCLUSIONS AND RECOMMENDATIONS

This section presents summary of the results, highlights the contribution made in this dissertation, and provides the scope of a possible extension of the present work.

Conclusions

This dissertation develops parameter-uniform numerical schemes for solving a class of singularly perturbed parabolic partial differential equations. This dissertation has contributed six ε -uniformly convergent numerical schemes, which are summarized as follows. The preceding four schemes were developed using governing problem (1.3).

In Chapter 4, considered the problem with small delay on the left boundary layer. The delay term is tackled by using Taylor's approximation. The developed the scheme combines the backward Euler method in the time direction and an exponentially fitted operator finite difference scheme in the spatial direction. The stability estimate and convergence analysis of the formulated scheme are investigated and shown that the scheme is more accurate and uniformly convergent with the rate of first-order accuracy in both space and time.

In Chapter 5, employed the problem with large delay on the right boundary layer. The delay term is handled by choosing a special mesh. The developed the scheme combines the backward Euler method in the time direction and an exponentially fitted operator finite difference scheme in the spatial direction. The stability estimate and convergence analysis of the developed scheme are investigated and shown that the scheme has better accuracy and ε -uniformly convergent with the rate of first-order accuracy in both space and time.

In Chapter 6, used the problem with small delay on the right boundary layer. The delay term is tackled by using Taylor's approximation. The backward Euler method for time discretization on uniform mesh and the second-order central finite difference methods for space discretization on Shishkin mesh is employed. The obtained is considered sufficient number of mesh point inside the layer region. This reveals that the obtained method is a layer resolving method. This method improved error converges at a rate of almost second-order up to a logarithmic factor in space.

In Chapter 7, applied on the problem with a small time delay. The solution to the problem exhibits a boundary layer on the left or right side of the domain. The proposed scheme comprises the backward Euler scheme in the time direction and an exponentially fitted tension spline scheme in the spatial direction. The applicability of the developed scheme is validated. Further, the preceding chapters in terms of accuracy. The proposed method contributes a better accuracy over the preceding chapters, stable, and ε -uniform numerical result with a linear order of convergence.

However, the method which was developed in Chapter 7 produced more accurate results than the rest developed methods. On the other hand, the method developed in Chapter 6 considered sufficient numbers mesh points in the layer region and increase the rate of the order of convergence almost second-order up to a logarithmic factor.

In Chapter 8, the governing equation (1.4) is used to conduct an exponentially fitted finite difference method. The exponentially fitted finite difference method is employed in the space direction on a uniform mesh. The time derivative is discretized by the Crank–Nicolson scheme on a uniform mesh. Extensively the continuous solution behaviors, and the discrete analyses of the solution are established. A computational result reveals that the present method is ε -uniformly convergent, the error converges at the rate of first-order accuracy in space and second-order accuracy in time.

In Chapter 9, the governing equation (1.5) is used to conduct an exponentially fitted finite difference method. This problem is extended from Chapter 8 by considering delay term. Thus, we considered in the case when a delay parameter is smaller than the perturbation parameter. The delay parameter is tackled by using Taylor's approximation. The exponentially fitted finite difference method is employed in the space direction on a uniform mesh. The time derivative is discretized by the Crank–Nicolson scheme on a uniform mesh. Extensively the continuous solution behaviour, and the discrete analyses of the solution are established. The obtained result reveals that the present method is ε -uniformly convergent, the error converges at the rate of first-order accuracy in space and second-order accuracy in time. For each considered problem the theoretical findings of the results are confirmed and the boundary layer behaviour of the solutions is given graphically.

Recommendations

This dissertation attempts to analyze and develop some accurate and efficient numerical methods to solve a class of singularly perturbed parabolic problems that exhibit layer behaviors in the neighbourhood of the domain. Even if the developed methods are concerned with a class of singularly perturbed parabolic problems of convection-diffusion type with Dirichlet boundary conditions, the following areas are still our concerns.

- A class of singularly perturbed parabolic problems with a time delay with Robin boundary conditions.
- A class of singularly perturbed parabolic degenerating coefficient with time delay.
- A class of turning point problems involving interior layer.
- A class of singularly perturbed parabolic problems with a time delay with integral boundary conditions.

- One can extend the methods to solve singularly perturbed integro-differential equation.
- One can extend the methods to solve singularly perturbed non-linear differential equations and fractional-order derivative problems.

As far as the scope for further research is concerned, we intend to

- develop a higher-order ε -uniform numerical methods,
- apply equidistribution mesh, Bakhvalov's meshes, and Harmonic meshes.
- develops spline collocation methods to improve the accuracy and rate of convergence for the developed methods so far and the problems under consideration in this dissertation.

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APPENDIX A

LIST OF PUBLISHED AND SUBMITTED PAPERS

Based on the work in this dissertation, the following research articles are published or communicated

1. Tesfaye, S.K, Woldaregay, M.M, Dinka, T.G, and Duressa, G.F.(2022). Fitted computational method for solving singularly perturbed small time lag problem, BMC Research Notes, Springer, <https://doi.org/10.1186/s13104-022-06202-0>.
2. Tesfaye, S.K, Duressa, G.F., Woldaregay, M.M., and Dinka, T.G. (2023). Fitted computational method for singularly perturbed convection-diffusion equation with time delay. Frontiers in Applied Mathematics and Statistics. <https://doi.org/10.3389/fams.2023.1244490>.
3. Tesfaye, S.K., Dinka, T.G., Woldaregaya, M. M., and Duressa, G. F. (2024). Numerical analysis for a singularly perturbed parabolic differential equation with a time delay. Computational Mathematics and Mathematical Physics, 64(3):521–538. DOI: 10.1134/S096554252403014X.
4. Tesfaye, S.K., Duressa, G.F., Dinka, T.G., Woldaregay, M. M. (2024). Fitted tension spline scheme for a singularly perturbed parabolic problem with a time delay. Journal of Applied Mathematics. Vol. 2024. <https://doi.org/10.1155/2024/9458277>
5. Tesfaye, S.K., Duressa, G.F., Woldaregay, M.M., Dinka, T.G. (2024). A uniformly convergent numerical scheme for singularly perturbed parabolic turning point problem. Journal of Mathematical Modeling, 12(3):501–516. DOI: 10.22124/JMM.2024.26669.2349.
6. Numerical scheme for singularly perturbed parabolic delay differential equation with a degenerating coefficient. To be communicated.