BOUNDARY INTEGRAL EQUATION METHOD FOR STEADY STATE HEAT TRANSFER PROBLEM IN A PLANE

By

Mohammed Dedefo

A Thesis Submitted to
the program of Applied Mathematics
School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics(Specialization in Differential Equations)

Office of Graduate Studies
Adama Science and Technology University

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September, 2017

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Tamirat Temesgen(Ph.D)

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Declaration

I declare that this thesis is my work and has not been presented for a degree in any other university, and all sources of material used for this thesis have been duly acknowledged.

(Name)	(Signature)	(Date)

This Thesis has been submitted for examination with my approval as thesis advisor.

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APPROVAL PAGE

This is to certify that the thesis prepared by Mohammed Dedefo entitled ”Boundary Integral equation method for stead state heat transfer problem in a plane” submited in partial fulfilment of the requirement for the degree of Master of science in Applied Mathematics with the regulation of the University and meets the accepted standers with respect to originality.

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Abstract

In this thesis Dirichlet,Neumann and Mixed boundary value problems for poisson's equations for the steady state heat transfer in 2D are considered, by the method of Greens theorem and Green's identities. Green's theorem is an equation on $\partial\Omega$ reducing the dimension by one to solve boundary value problems using boundary integral equation method. Using those concepts Dirichlet,Neumann and Mixed boundary value problems of Poisson's equations are transformed to boundary integral equations and we have also show the equivalence of boundary value problem with boundary integral equation.

Acknowledgment

First and foremost I give my thanks to ALLHA(God), Next- I express my thanks to my Advisor Dr. Tamirat Temesgen for his constructive comments, suggestions and valuable advisees.Moreover I thanks all my colleagues and friends. Finally, I express my appreciation to my families for their encouragement.

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Notations and Abbriviation

Δ-Laplacian operator

Γ-Smooth boundary of Ω

$C^2(\Omega)$-Twice continuously differentiable

$C^1(\bar{\Omega})$-Continuously differentiable on the closest of the domain

δ-Dirac delta function

∇-Gradient operator

BVP-Boundary value problem

BIEM-Boundary integral equation method

Chapter 1

Introduction and Background of the study

1.1 Background of the study

The boundary integral equation (BIE) method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility of reducing boundary value problem (BVP) for partial differential equation in a domain to an integral equation on the boundary of the domain. The main ingredient necessary for reduction of BVP to a boundary integral equation is a fundamental solution to the original partial differential equation which is available in an analytic form. After the fundamental solution is used in the Green formula, one can reduce the boundary value problem to a boundary integral equation.
Boundary Integral Equation expresses the solution to the boundary value problem in terms of an integral equation. This method was first used to solve the Laplace partial differential equations, which govern steady-state heat transfer problems.In these cases, solution is expressed in terms of boundary integrals only, [9, 4, 2]. The application of Greens function is that it can be used to find the solution of Poissons equation.The significance of a boundary integral formulation is to drop the dimensionality of the problem by one, i.e.to transform the boundary value problem in to boundary integrals. The simplest approach is to use Greens second identity. In this theorem we assume the existence of two continuous functions u and v that have continuous first and second derivatives, [1].

The Boundary Integral Equation method becomes an appealing area of research twenty-five years ago. The method has been applied to a variety of heat transfer problems in the last thirty years, initial applications of the method were for steady state heat transfer problems. Boundary integral equation method has evolved as a powerful alternative for finite elements. The most severe restriction of the boundary integral equation approach is that it is applicable only to such problems where the fundamental solution is available, [7, 2]
The main advantage of the boundary integral equation method is that the ability to provide a complete solution in terms of boundary integrals, with substantial savings in modeling effort. As a method for solving problems of mathematical physics its origin in the work of G. Green. The method was employed as a mathematical tool to determine the necessary boundary conditions of mathematical physics, and not as a method to solve the problem. Because it was still

not possible to find the analytic solution of the derived singular integral equations. In the fore mentioned methods, the unknown boundary quantities have a direct physical or geometrical meaning and for this reason they are referred to as direct boundary integral equation method. The advantage of boundary integral equation method over finite element and Finite Difference method that requires discretization of the whole domain is the reduction in the degrees by one of freedom needed to model a given physical system. Such reduction is allowed by the underlying boundary integral formulation which requires, for its existence of solution, only the discretization of the boundary of the solution domain. This results not only in a reduction in size of the system matrix, but also in faster data preparation. The benefit of this approach is that it takes a 2D or3D partial differential equation and reduces it to an algorithm that involves 1D or 2D computation only, respectively is the great advantage of the method rather than other methods to solve boundary value problems. We also reformulate the partial differential equation as boundary integral equations;[6, 7, 13].
The Boundary integral equation has its own advantage and disadvantages, the advantage of boundary integral equation method are: discretization is only over the boundary of the body and reducing the numbers of unknown by one order. This method is particularly effective in computing the derivatives of the field function. It can easily handle concentrated forces and moments either inside the domain or on the boundary, it also allows evaluation of the solution and its derivatives at any point of the domain of the problem and at any instant in time. It uses an integral representation of the solution as a continuous mathematical expression. The method requires less data to run a program efficiently and less time is required for the solution of a problem due to a small system of equation. In such technique, unwanted information is much less than other techniques. This method is not costly, since the discretization is only on the surface in boundary integral equation method, thus the amount of data is small. It is also more accurate-due to the semi-analytical nature and use of integrals, more efficient in modeling due to the reduction of dimension and good for stress concentration and infinite domain problems. [7, 1, 8].

1.2 Statement of the problem

The study of boundary integral equation method for solving boundary value problem was discussed in many literatures. Boundary integral equation method for steady state heat transfer problem in a plane is aimed to find the solution of boundary value problem using Greens formula. The solution of boundary value problem can be found by applying Greens formula and Greens identities. In this thesis,we will try to develop, how to find the solution of boundary value problem for steady state heat transfer problem in a plane. Therefore, this research work address the following basic problems.

1. Transform boundary value problem (BVP) to boundary integral equation (BIE)
2. Investigating equivalence of boundary value problem and boundary integral equation.
3. Solve steady state heat transfer problem.

1.3 Objective of the Study

1.3.1 General Objective

The general objective of the study is to solve steady state heat transfer problem in a plane using boundary integral equation method.

Specific Objective

The specific objectives of this study are: -

1. To transform boundary value problem for steady state heat transfer problem in to boundary integral equation.

2. To show equivalence of boundary integral equation with the original boundary value problems.

3. To solve steady state heat transfer problem using boundary integral equation method.

1.4 Significance of the study

The study will contribute to improve our understanding of solving steady state heat transfer problem and related problems. It also devlops how to apply Green's theorem to solve steady state problems. Furthermore, the study serves as a base for the researchers who have an interest to conduct further research on the topic.

Chapter 2

Review Literature

In this chapter some related literatures from different sources such as published journals and reference book which are cited are reviewed.The review of related literature presented in this chapter focus on Boundary integral equation method for solving boundary value problems.
Boundary Integral Equation Method is a method for solving problems of mathematical physics.It has its orgion in the work of G.Green. He formulated in 1828,the integral representation of the solution for the Dirichlet and Neumann problem of Laplace equation and Poisson's equation by introducing the so called Green's function for those problems. In 1872,Betti presented a general method for integrating the equation of elasticity and deriving their solution in integral form.Basically,this may be regarded as a direct extension of Green's approach to the Navier equations of elasticity. In 1885,Somigliana used Betti's[7] reciprocal theorem to derive the integral representation of the solution for the elasticity problems. Sherman,Mikhlin and Muskhelishvili used complex functions to develop boundary integral equation methods for the solution of the plane elasticity problems.
Boundary integral equation method is now firmly established as an important technique to solve boundary-value problem and consists the transformation of the partial differential equation describing the behavior of the unknown inside and on the boundary of the domain in to an integral equation relating only boundary domain in[7] . The integral equation has been extensively employed to formulate boundary-value problems of potential theory of analytical solution to such equations are limited to very simple geometries and are obtained using the Greens function. The Greens function method of solving boundary-value problems is applicable to find the solution of partial differential equations. The Gauss-Green theorem is a fundamental identity which relates the integral of the derivatives of a function over a domain Ω to the integral of that function on its the boundary Γ [7, 12].
The most important work related to the boundary integral equation for solving boundary value problems comes from George Green. Green in (1828) presented the three Greens identities. Greens identities are a set of three vector derivative and integral identities which can be derived with the vector derivatives identities on its domain.
Boundary integral equation method was neglected until the end of fifties,then,with the advent of computers,the method come back to the spotlight as an appealing numerical method for solving engineering problems. Numerical methods were developed for the solution of boundary integral equations and difficult physical problems of complex boundary geometry,which could not be tackled by other methods, were solved for the first time by the boundary integral equation method. The first works that laid the function of boundary integral equation as a

computational technique appeared in the early sixties. Jaswon and Symm used Fredholm's equations to solve some two dimensional problems of potential theory.
All the fore-mentioned problems are governed by second order partial differential equations. Another group of problems are those described by the harmonic equation. In this case,the integral representation of the solution was derived from the Rayleigh-Green identity and the approach was applied to plate bending and plane elasticity,with the letter being formulated in terms of Airy's stress function.[7]
The formulation consists of two boundary integral equations,one for each of the unknown boundary quantities. The first one arises from the boundary character of the integral representation of the field function,while the second is obtained from the integral representation either of the Laplacian of the field function or its derivative along the normal to the boundary.
The essential re-formulation of the partial differential equation that underlies the BIEM consists of an integral equation that is defined on the boundary of the domain and an integral that relates the boundary solution to the solution of a point in the domain. The formulation of boundary integral equation method is based on an integral statement of elasticity, and this can be cast in to a relation involving unknowns only over the boundary of the domain under study. This originally lead to the boundary integral equation and are early work in the field was reported by Rizzo(1967)and Cruse(1969), [7].
In the late eighties, one could find numerous publications in the literature,where the boundary integral equation method was applied to a wide variety of engineering problems,among them are static and dynamic linear or non-linear problems of electricity/ problems of elastic dynamics,fluid dynamics,electricity and electromagnetism heat conduction etc.It could be said that today the boundary integral equation method has matured and become a power full methods for the analysis of engineering problems,and alternative to the domain methods.[7, 13]
This study will contribute to improve our understanding how to transform boundary value problem to boundary integral equation and showing equivalence of boundary value problem with boundary integral equation.

Chapter 3

Preliminary Concepts

Some preliminary mathematical concepts that are necessary for developing the boundary integral equation methods are the divergence theorem of Gauss, Greens reciprocal identity (Greens second theorem) and the fundamental solution. These concepts are used particularly for transformation of the differential equations which govern the response of physical systems within a domain in to integral equations on the boundary for two-dimensional domain.

Definition 3.1. A partial differential equation(PDE) is a relation that involves partial derivatives of an unknown function. Let the unknown function be u,and $x, y, z, ...$ be independent variables, i.e., $u = u(x, y, z, ...)$.Often,one of these variables represents the time. Thus,a partial(DE) is an equation of the form

$$F(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, ..., u_{xxx}, ...) = 0 \quad (*)$$

In (*) we have used the subscript notation for the partial differentiation, i.e.,

$$u_x = \frac{\partial u}{\partial x}, u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, and\ so\ on.$$

We will always assume that the unknown function u is sufficiently formulated. So that all necessary partial derivatives exist and corresponding mixed partial derivatives are equal, e.g.,

$$u_{xy} = u_{yx}, u_{xzx} = u_{xxz}, and\ so\ on.$$

As in the case of ordinary differential equations,we define the order of partial differential equation (*) to be the highest order partial derivatives appearing in the equation.
Furthermore,we say that the partial differential equation (*) is linear, if F is linear as a function of the variables $u, u_x, u_y, u_z, u_{xx}, ...$, i.e., F is linear combination of the unknown function and its derivatives,Equation(*)is said to be quasilinear if F is linear as a function of the highest-order derivatives. The following are examples of partial differential equations:

$$u_x + u_y = 3u_z - 2x^2 - 5z \ , (first - order\ linear)$$

$$u_{xx} + u = 4x^2 \ , \qquad (second - order\ linear)$$

$$5xyu_{xy} - 3zu_y + 2u = 0, \qquad (second - order\ linear)$$

$$u_{xz} + 2uu_y - 4z = 0 \qquad (second - order\ quasilinear)$$

The key defining property of a partial differential equation is that there are more than, one independent variables and there is a dependent variable that is an unknown function of these variables $u(x, y)$,we will often denotes its derivatives by subscripts, thus $\frac{\partial u}{\partial x} = u_x$ and so on. A partial differential equation is an identity that relates the independent variables, the dependent variable u, and the partial derivative of u . The most important partial differential equations are Laplace equation and Poisson equation,because both are elliptic partial equation.For this thesis work we tray to consider the non homogeneous Poisson equation,because for homogeneous Laplace equation there is no heat transfer in a plane.The solution of boundary value problem can be found by applying Green's identities.
For homogeneous Poisson equation

$$\Delta u = 0 \quad in \quad \Omega, \tag{3.1}$$

and $\frac{\partial u}{\partial n}$ is on the boundary Γ, when Ω is a bounded domain with boundary Γ, there is no heat transfer in a plane. We will consider the steady state heat flow problem described by the Poissons equation in two independent variables has the form

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), x, y \in \Omega. \tag{3.2}$$

The solution u of the equation (3.2) represents the amount of heat at a point (x, y) in the domain Ω due to the source $f(x, y)$ distributed over Ω. Poissons equation when f is a given function is an interesting and relevant equation to study many application [11].

3.1 The Gauss' theorem(divergence theorem)

Theorem 3.1. *[7] Let Ω be domain in space bounded by a closed boundary Γ; u and v be vector field acting on this domain. The divergence theorem establishes that the total flux of the vector field u and v across the closed boundary Γ must be equal to the domain integral of the divergence of this vector.*

$$\int_\Omega \nabla.u d\Omega = \int_\Gamma u.n ds$$

Proof: let u and v be a vector field for which the integration is carried out first with respect to x and then with respect to y; we can write

$$\int_\Omega \frac{\partial u}{\partial x} d\Omega = \int_{y_1}^{y_2} (\int_{x_1}^{x_2} \frac{\partial u}{\partial x} dx) dy = \int_{y_1}^{y_2} (u(x_2, y) - u(x_1, y)) dy \tag{3.3}$$

where

$$x_1 = x_1(y) \quad and \quad x_2 = x_2(y) \tag{3.4}$$

From the detail of figure 3.1, we have

$$\frac{dy}{ds} = cos\alpha, n_x \Rightarrow dy = n_x ds \tag{3.5}$$

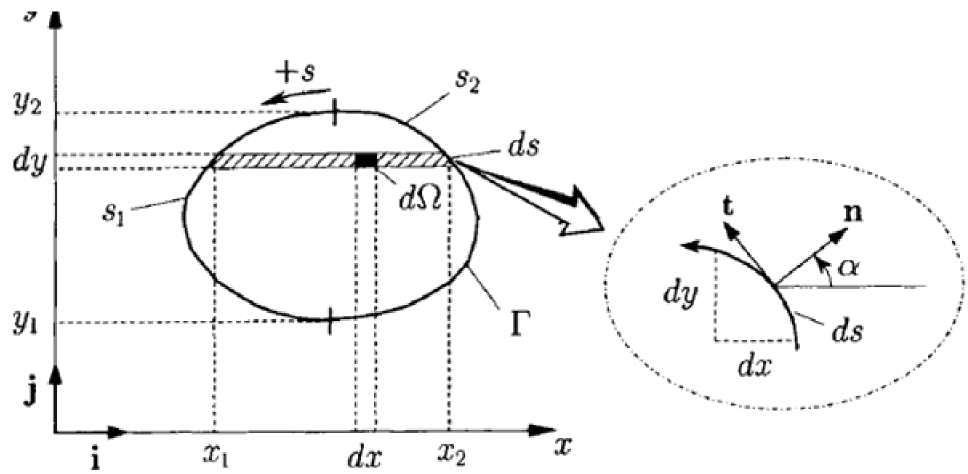

Figure 3.1: Integral over a plane domain Ω bounded by curve Γ

$$-\frac{dx}{ds} = sin\alpha, n_y \Rightarrow dx = -n_y ds \tag{3.6}$$

Where n_x and n_y are the components of the unit vector n, which is normal to the boundary Γ. The negative sign in equation(3.6) is due to the fact that the dx and sinα have opposite signs when the angle α is measured in the counter-clock wise sense with respect to the positive x -direction. Consequently equation (3.3) becomes

$$\int_{y_1}^{y_2} u(x_2, y)dy = \int_{s_2} u(x_2, y)n_x ds - \int_{s_1} u(x_1, y)n_x ds \tag{3.7}$$

In the previous expression the integration on s_1 is performed in the negative direction (clock-wise) when y varies from y_1 to y_2. Using uniform direction for the integration over s, both terms in equation (3.7) can be combined in a single expression

$$\int_\Omega \frac{\partial u}{\partial x} d\Omega = \int_\Gamma u n_x ds \tag{3.8}$$

Interchanging x and y in equation (3.7) we obtain

$$\int_\Omega \frac{\partial u}{\partial y} d\Omega = \int_\Gamma u n_y ds \tag{3.9}$$

If v is another function of x and y then equation (3.8) and (3.9) result in

$$\begin{aligned}
\int_{\Omega} \frac{\partial(uv)}{\partial x} d\Omega = \int_{\Gamma} uvn_x ds = \int_{\Omega} v\frac{\partial u}{\partial x} d\Omega + \int_{\Omega} u\frac{\partial v}{\partial x} d\Omega \\
\Rightarrow \int_{\Omega} v\frac{\partial u}{\partial x} d\Omega = -\int_{\Omega} u\frac{\partial v}{\partial x} + \int_{\Gamma} uvn_x ds \qquad (3.10) \\
\int_{\Omega} \frac{\partial(uv)}{\partial y} d\Omega = \int_{\Gamma} uvn_y ds = \int_{\Omega} v\frac{\partial u}{\partial y} d\Omega + \int_{\Omega} u\frac{\partial v}{\partial y} d\Omega \\
\Rightarrow \int_{\Omega} v\frac{\partial u}{\partial y} d\Omega = -\int_{\Omega} u\frac{\partial v}{\partial y} + \int_{\Gamma} uvn_y ds \qquad (3.11)
\end{aligned}$$

Equation (3.10) and equation (3.11) state the integration by parts in two dimensions and are known as the Gauss-Green theorem. Using vector notation equation (3.10) and equation (3.11) can also written as

$$\int_{\Omega} \nabla.u d\Omega = \int_{\Gamma} u.n ds \qquad (3.12)$$

Therefore, equation (3.12) is Gauss-Greens theorem(divergence theorem)

3.2 Green's identites

In solving Dirichlet,Neumann and Mixed boundary value problems we use Green's theorem. Green's theorem gives information about values of the function inside domain and values on the boundary. The main objective of this section is using Greens theorem and show how we can solve those boundary value problems using boundary integral equation method. To find the solution of boundary value problems we use Greens first and Greens second identities.
Green's first identity
This identity is derived from the divergence theorem applied to vector field

Theorem 3.2. *[10] Let Ω be a bounded region in the plane whose boundary Γ is closed and piecewise smooth carve. If $u, v \in C^1(\bar{\Omega}) \bigcap C^2(\Omega)$ then*

$$\int_{\Gamma} u\frac{\partial v}{\partial n} ds = \int_{\Omega} u\Delta v dx + \int_{\Omega} \nabla u.\nabla v dx \qquad (3.13)$$

Proof:-Using divergence theorem we verify this as follows

$$\int_{\Gamma} F.n ds = \int_{\Omega} \nabla.F dx \qquad (3.14)$$

Consider u and v be functions of two variables and let $F = u\nabla v$ in equation(3.14)then from the product rule we have

$$(vu_x)_x = v_x u_x + vu_{xx} \qquad (3.15)$$

$$(vu_y)_y = v_y u_y + vu_{yy} \tag{3.16}$$

and similarly for the other partial derivatives,adding up equation(3.15)and equation(3.16)which gives

$$\nabla.(v\nabla u) = \nabla u.\nabla v + v\Delta u \tag{3.17}$$

Integrating both side of equation (3.17)over the region Ω, and using equation(3.14)for the integral on the left hand side, we get

$$\begin{aligned}\int_\Gamma u\nabla v.nds &= \int_\Omega \nabla.(u\nabla v)dx \\ &= \int_\Omega (u\nabla.\nabla v + \nabla u.\nabla v)dx \\ &= \int_\Omega u\Delta v dx + \int_\Omega \nabla u.\nabla v dx\end{aligned}$$

Since $\nabla v.n = \frac{\partial v}{\partial n}$ is the directional derivative in the out ward normal direction and ∇ is gradient of scalar quantity, then the Greens first identity will be written in the form of

$$\int_\Gamma u\frac{\partial v}{\partial n}ds = \int_\Omega \nabla u.\nabla v dx + \int_\Omega u\Delta v dx \tag{3.18}$$

We can think of this identity as the generalization of integration by parts, in the sense that the one derivative is transformed from the function u to the function v under the integral, which results in the switched sign and a boundary term, much integration by part for the functions of single variable.

Green's second identity

Theorem 3.3. *[7] Let $u, v \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$ be function which are twice continuously differentiable in Ω and once continuously differentiable on Γ . Then*

$$\int_\Omega (u\Delta v - v\Delta u)dx = \int_\Gamma (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}).ds$$

where $\frac{\partial v}{\partial n}$ is the directional derivative of u in the direction of the outward pointing normaln to the surface elements ds ,$\frac{\partial u}{\partial n} = \nabla u.n = \nabla nu$.

Proof:Applying Gauss theorem to the function $F = u\nabla v$ gives

$$\int_\Omega \nabla.(u\nabla v)dx = \int_\Gamma (u\nabla v).ds \tag{3.19}$$

The left hand side expanded as

$$\int_\Omega \nabla.(u\nabla v)dx = \int_\Omega (\nabla u.\nabla v + u\Delta v)dx \tag{3.20}$$

Therefore

$$\int_\Omega (\nabla u.\nabla v + u\Delta v)dx = \int_\Gamma (u\nabla v).ds. \tag{3.21}$$

Exchanging u and v

$$\int_\Omega (\nabla v.\nabla u + v\Delta u)dx = \int_\Gamma (v\nabla u).ds. \tag{3.22}$$

Subtracting Equation (3.22) from Equation (3.21) yields

$$\int_\Omega (u\Delta v - v\Delta u)dx = \int_\Gamma (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}).ds. \tag{3.23}$$

Therefore equation (3.23) is Greens second identity.it is valid for the pair of function u and v .

Definition 3.2. [12] A continuous function satisfying Laplace's equation in an open region Ω, with continuous first and second order derivatives, is called an harmonic function.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Laplace equation arises as a steady state problem for heat equation which does not vary with time.This equation has no dependence on time, just on the special variable $x\ and\ y$,this means that the Laplace equation described steady state siuatation on the temperature distribution.

3.3 Fundamental Solution

Distributions are objects that generalize the classical notation of functions in mathematical analysis.Distribution make it possible to differentiate functions whose derivatives do not exist in the classical sense.
[15]A fundamental solution $K(x)$ for the Laplace equation is a distribution satisfying

$$\Delta K(x) = \delta(x), \tag{3.24}$$

where δ is the delta distribution supported at $x = 0$.In order to solve (3.24) we should first observe that Δ is symmetric in the variables $x_1, ..., x_n$ and $\delta(x)$ is also radially symmetric (i.e, its value only depends on $r = |x|$). Thus we try to solve (3.24) with a radially symmetric function $K(x)$.
Since $\delta(x) = 0$ for $x \neq 0$, we see that (3.24) requires K to be harmonic for $r \geq 0$. For the radially symmetric function K, Laplace equation becomes $(K = K(r))$:

$$\frac{\partial^2 K}{\partial r^2} + \frac{n-1}{r}\frac{\partial K}{\partial r} = 0 \tag{3.25}$$

The general solution to (3.25) is

$$K(r) = \begin{cases} \mathrm{c}_1 + c_2 \mathrm{log} r & if n = 2 \\ \mathrm{c}_1 + c_2 r^{2-n} & if n \geq 3 \end{cases}. \tag{3.26}$$

After we determine c_2, we find the fundamental solution for the Laplace operator:

$$K(x) = \begin{cases} \frac{1}{2\pi}\log r & if n = 2 \\ \frac{1}{(2-n)w_n r^{2-n}} & if n \geq 3 \end{cases}.$$

We can derive, (3.26) for any given n.
The fundamental solution of the Laplace operator for $n = 2$ derived as:
Starting with the Laplcian in $\mathbb{R}^2$ for a radially symmetric function K,

$$K" + \frac{2}{r}K' = 0,$$

and letting $K = \frac{1}{r}w(r)$, we obtained the equation: $w = c_1 r + c_2$ which is implied $K = c_1 + \frac{c_2}{r}$
We now find the constant c_2 that ensures that for $v \in C_0^\infty(\mathbb{R}^2)$we have

$$\int_{\mathbb{R}^2} K(|x|)\Delta v(x)dx = v(0)$$

Suppose $v(x) = 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small ϵ>0 , let $\Omega_\epsilon = \Omega - B_\epsilon(0)$ $K(|x|)$ *is harmonic* $(\Delta K(|x|) = 0)$ *in* Ω_ϵ.
Consider Green's identity $(\partial\Omega_\epsilon = \partial\Omega \bigcup \partial B_\epsilon(0))$

$$\int_{\Omega_\epsilon} K(|x|)\Delta v dx = \int_{\partial\Omega} (K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n})ds + \int_{\partial B_\epsilon(0)} (K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n})ds$$

$$\lim_{\epsilon \to 0}[\int_{\Omega_\epsilon} K(|x|)\Delta v dx] = \int_\Omega K(|x|)\Delta v dx$$

Since

$$K(x) = c_1 + \frac{c_2}{r} \ integrable\ at\ x = 0. On\ \partial B\epsilon(0), K(|x|) = K(\epsilon).\ Thus$$

$$|\int_{\partial B\epsilon(x)} K(|x|)\frac{\partial v}{\partial n}ds| = |K(\epsilon)|\int_{\partial B\epsilon(0)} |\frac{\partial v}{\partial n}|ds \leq |c_1 + \frac{c_2}{r}|4\pi\epsilon^2 \max|\nabla v| \to 0\ as\ \epsilon \to 0 \quad (3.27)$$

$$\int_{\partial B\epsilon(0)} v(x)\frac{\partial K(|x|)}{\partial n}dS = \int_{\partial B\epsilon(0)} \frac{c_2}{\epsilon^2}v(x)dS$$

$$= \int_{\partial B\epsilon(0)} \frac{c_2}{\epsilon^2}v(0)dS + \int_{\partial B\epsilon(0)} \frac{c_2}{\epsilon^2}[v(x) - v(0)]dS$$

$$= \frac{c_2}{\epsilon^2}v(0)4\pi\epsilon^2 + 4\pi c_2 \max_{x \in \partial B\epsilon(0)} |v(x) - v(0)|$$

since

$$4\pi c_2 \max_{x \in \partial B\epsilon(0)} |v(x) - v(0)| \to 0, v\ is\ continuous$$

$$= 4\pi c_2 v(0) \to -v(0).$$

Thus, taking $4\pi c_2 = -1,\ i.e,\ c_2 = -\frac{1}{4\pi}$ we obtain

$$\int_\Omega K(|x|)\Delta v dx = \lim_{\epsilon \to 0}\int_{\Omega\epsilon} K(|x|)\Delta v dx = v(0),$$

that is

$$K(r) = -\frac{1}{4\pi r} \tag{3.28}$$

There fore equation(3.28) is the fundamental solution of Laplace operator.[15].

Theorem 3.4. *[15]Representation formula*
Let Ω be bounded domain in $\mathbb{R}^2$ and let n be the unit exterior normal to Γ, Let $u \in C^2(\bar{\Omega})$ then the value of u at any point $x \in \Omega$ is given by the formula

$$u(x) = \frac{1}{2\pi}\int_{\Omega} \Delta u(y)\log|x-y|dy + \frac{1}{2\pi}\int_{\Gamma}[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}]ds \tag{3.29}$$

Proof:Consider the Green's identity.

$$\int_{\Omega}(u\Delta v - v\Delta u)dy = \int_{\Gamma}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds$$

where v is harmonic function $v(y) = \log|x-y|$, which is singular at $x \in \Omega$
In order to be able to apply Green's identity, we consider a new domain Ω_ϵ
$\Omega_\epsilon = \Omega - B_\epsilon(x)$
Since $u, v \in C_2(\bar{\Omega}_\epsilon)$, Green's identity can be applied. Since v is harmonic ($\Delta v = 0$) in Ω_ϵ and since $\partial\Omega\epsilon = \partial\Omega\bigcup\partial B\epsilon(x)$ we have

$$-\int_{\Omega_\epsilon}\Delta u(y)\log|x-y|dy \tag{3.30}$$

$$= \int_{\Gamma}[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}]ds + \int_{\partial B\epsilon(x)}[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}]ds$$

We will show that the formula (3.29) is obtained by letting $\epsilon \to 0$

$$\lim_{\epsilon\to 0}[-\int_{\Omega_\epsilon}\Delta u(y)\log|x-y|dy] = -\int_{\Omega}\Delta u(y)\log|x-y|dy$$

Since

$$\log|x-y| \ \textit{is integrable at}\ x = y$$

The first integral on the right of (3.30) does not depend on ϵ. Hence the limit as $\epsilon \to 0$ of the second integral on the right of (3.30) exists, and in order to obtain (3.29), need

$$\lim_{\epsilon\to 0}\int_{\partial B\epsilon(x)}[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}]ds = 2\pi u(x)$$

$$\int_{\partial B\epsilon(x)}[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}]ds = \int_{\partial B\epsilon(x)}[\frac{1}{\epsilon}u(y) - \log\epsilon\frac{\partial u(y)}{\partial n}]ds$$

$$= \int_{\partial B\epsilon(x)}\frac{1}{\epsilon}u(x)ds + \int_{\partial B\epsilon(x)}[\frac{1}{\epsilon}[u(y) - u(x)] - \log\epsilon\frac{\partial u(y)}{\partial n}]ds$$

$$= 2\pi u(x) + \int_{\partial B\epsilon(x)} [\frac{1}{\epsilon}[u(y) - u(x)] - \log\epsilon \frac{\partial u(y)}{\partial n}]ds$$

the last integral tends to 0 as $\epsilon \to 0$

$$|\int_{\partial B\epsilon(x)} [\frac{1}{\epsilon}[u(y) - u(x)] - \log\epsilon \frac{\partial u(y)}{\partial n}]ds| \leq \frac{1}{\epsilon}\int_{\partial B\epsilon(x)} |u(y) - u(x)| + \log\epsilon \int_{\partial B\epsilon(x)} |\frac{\partial u(y)}{\partial n}|ds$$

$$\leq 2\pi \max_{y\in\partial B\epsilon(x)} |u(y) - u(x)| + 2\pi\epsilon \;\log\epsilon \; \max_{y\in\bar{\Omega}} |\nabla u(y)|$$

Since

$$2\pi \max_{y\in\partial B\epsilon(x)} |u(y) - u(x)| \to 0, \; (u \; is \; continous \; in \; \bar{\Omega})$$

and

$$2\pi\epsilon \;\log\epsilon \; \max_{y\in\bar{\Omega}} |\nabla u(y)| \to 0, (|\nabla u| \; isfinite)$$

Theorem 3.5. *[11]Suppose that u is harmonic inside Ω and continuous on the boundary Γ, let (x_0, y_0) be points inside Ω, then*

$$u(x_0, y_0) = \frac{1}{2\pi}\int_\Gamma [u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}]ds = \frac{1}{2}\int_\Omega (u\Delta v - v\Delta u)dy$$

where $v(x, y) = \frac{1}{2}\ln[(x - x_0)^2 + (y - y_0)^2]$.

Chapter 4

Boundary Integral Equation Method

A steady state heat transfer problem in a plane with heat sources acting inside a domain Ω, with a boundary Γ of Poisson's equation is given by

$$\Delta u = f(x, y) \quad x, y \in \Omega, \tag{4.1}$$

where u is temperature,f is a known function,the governing equation for $f = 0$ is known as the Laplace equation, whereas for $f \neq 0$ is known as the Poissons equation. A boundary value problem is finding function which satisfies a given partial differential equation and particular boundary conditions, the boundary value problems for steady state heat transfer problem for Poisson's equation can be classified as Dirichlet, Neumann and Mixed boundary value problems,since Poisson equation is a Laplace equation.[7, 5].

4.1 Dirichlet Boundary Value Problem

Definition 4.1. Dirichlet boundary value is finding a function which solves a specified partial differential equation in the interior of a given region that takes prescribed values on the boundary of the region.

Assume $\Omega \subset \mathbb{R}^2$ is a connected domain of the Dirichlet boundary value problem (first boundary value problem) is to find a solution $u \in C^2(\Omega) \bigcap C(\bar{\Omega})$ of

$$\Delta u \ = f \ \ in \ \ \ \Omega, \tag{4.2}$$

$$u \ = g \ \ on \ \ \ \Gamma, \tag{4.3}$$

where u is specified along the whole boundary and g is given and continuous on Γ. f is a prescribed (ordered) continuous function on the boundary $\bar{\Omega}$ of the domain Ω. $\bar{\Omega}$ is closure of domain Ω. We may interpret the solution u of the Dirichlet problem as the steady state temperature distribution in a body, with the temperature prescribed at all points on the boundary.

4.1.1 Transformation of Dirichlet Boundary Value Problem to Integral Equation

Using representation formula we can obtain a solution of the Dirichlet boundary value problem for the Poissons equation

$$\begin{aligned} \Delta u = &\ f \ \ in \ \ \Omega, \\ u = &\ g \ \ on \ \ \Gamma, \end{aligned}$$

by the representation formula theorem (3.4)

$$u(x) = \int_\Omega G(x,y)f(y)dy \ + \ \int_\Gamma g\frac{\partial G(x,y)}{\partial n}ds. \tag{4.4}$$

If u is harmonic then

$$\begin{aligned} \Delta u = &\ 0 \ \ in \ \ \Omega, \\ u = &\ g \ \ on \ \ \Gamma, \end{aligned}$$

the solution of homogeneous Dirichlet boundary value problem is given by

$$u(x) = \int_\Gamma g\frac{\partial G(x,y)}{\partial n}ds. \tag{4.5}$$

Hence (4.5) is the integral representation for the solution of the Dirichlet boundary value problem for homogeneous equation.

Theorem 4.1. *Solutions of Poissons equation with zero boundary data.*
Let Ω be a region with boundary Γ as in theorem (3.5) and let $f(x,y)$ be a function on Ω and suppose that u is a solution of Poissons equation on Ω, then

$$\Delta u(x,y) \ = \ f(x,y), \tag{4.6}$$

when $u = 0$ on the boundary, then for all (x_0, y_0) in Ω

$$u(x_0,y_0) \ = \frac{1}{2\pi}\int_\Omega f(x,y)G(x,y,x_0,y_0)dx, \tag{4.7}$$

where G is Green's function for Ω.

Proof:-Since $\Delta u(x,y) = f(x,y)$ in Ω and so

$$\int_\Omega f(x,y)G(x,y,x_0,y_0)dx = \int_\Omega \Delta u(x,y)G(x,y,x_0,y_0)dx.$$

Since $\Delta u(x,y) = f(x,y)$,when $u = 0$ on the boundary.
Let Cr be a negatively oriented circle around (x_0, y_0) and let Ω_r be Ω minus Dr. The closed disk of radius r centered at (x_0, y_0), the boundary of Ωr consists of Γ plus the negatively oriented circle Cr. Applying Greens second identity in Ωr, when $u = 0$ on Γ. G is harmonic in Ωr and $G = 0$ on Γ and get

$$-\int_\Omega \Delta u(x,y)G(x,y,x_0,y_0)dx = \int_{Cr} u(x,y)\frac{\partial G}{\partial n}ds - \int_{Cr} G\frac{\partial u(x,y)}{\partial n}ds. \tag{4.8}$$

Let $r \to 0$ in (4.8) and establish the following three limits

$$\lim_{r\to 0}(-\int_{\Omega r} \Delta u(x,y)G(x,y,x_0,y_0)dx) = -\int_{\Omega r} \Delta u(x,y)G(x,y,x_0,y_0)dx \tag{4.9}$$

$$\lim_{r\to 0}\int_{Cr} u(x,y)\frac{\partial G}{\partial n}ds = 2\pi u(x_0,y_0) \tag{4.10}$$

$$\lim_{r\to 0} -\int_{Cr} G(x,y,x_0,y_0)\frac{\partial u(x,y)}{\partial n}ds = 0 \tag{4.11}$$

From equation (4.11) Green's function written as

$$G(x,y,x_0,y_0) = \frac{1}{2}ln[(x-x_0)^2 + (y-y_0)^2] + h(x,y,x_0,y_0)$$

Where h is harmonic in Ω and $\frac{\partial u}{\partial n}$ is continuous and bounded on Dr.
From equation (4.9)

$$\int_{\Omega} \Delta u(x,y)G(x,y,x_0,y_0)dx = -2\pi U(x_0,y_0) - 0$$

Hence

$$u(x_0,y_0) = \frac{1}{2\pi}\int_{\Omega} \Delta u(x,y)G(x,y,x_0,y_0)dx$$

therefore

$$u(x_0,y_0) = \frac{1}{2\pi}\int_{\Omega} f(x,y)G(x,y,x_0,y_0)dx \tag{4.12}$$

To solve the general Poissons equation with arbitrary boundary value, we combine equation (4.5) and equation (4.12).
Therefore the general solution of Poissons equation with the notation of theorem (3.5), of Poissons equation (4.6) with boundary condition $u(x,y) = g(x,y)$ for all (x,y) on Γ, then for all (x_0,y_0) in Ω

$$u(x_0,y_0) = \frac{1}{2\pi}\int_{\Omega} f(x,y)G(x,y,x_0,y_0)dx + \frac{1}{2\pi}\int_{\Gamma} g(x,y)\frac{\partial G}{\partial n}(x,y,x_0,y_0)ds \tag{4.13}$$

Therefore equation (4.13) is the integral representation of Dirichlet boundary value problem.
Thus equation (4.2 and 4.3) implies equation (4.13).
Therefore, the two formulations Dirichlet boundary value problem and equation (4.13) are equivalent.
Therefore Dirichlet boundary value problem is equivalent to boundary integral equation (4.13)

4.2 Neumann Boundary Value Problem

Definition 4.2. Neumann boundary value problem is a condition that specifies normal derivatives of unknown function on the boundary.

The Neumann boundary value problem (second boundary value problem) is to find the solution $u \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$ of

$$\Delta u = f \quad in \quad \Omega \tag{4.14}$$

$$\frac{\partial u}{\partial n} = g \quad in \quad \Gamma \tag{4.15}$$

when $\frac{\partial u}{\partial n}$ is directional derivative in the direction normal to boundary Γ specified along the whole boundary and g and f are given and continuous on boundary Γ and domain Ω respectively. Neumann boundary value problem is to find value of normal derivative of unknown function. The solution u may be interpreted as the steady state temperature distribution on the whole boundary. It requires additional condition for the existence of a solution than Dirichlet boundary value problem, that condition is known as compatibility condition.
The difference between Dirichlet boundary value problem and Neumann boundary value problem is that Dirichlet boundary value problem is to find values of unknown function given at each point of the boundary,while Neumann boundary value problem is to find values of normal derivatives of unknown function given at each points of the boundary.
Compatibility condition:- is a condition in which two first order partial differential equation f and g have common solution.
The Neumann boundary value problem will have no solution unless we assume that the average value of the function g on Γ is zero. This assumption is known as the compatibility condition. For instance, the compatibility condition for the Neumann boundary value problem:

$$\Delta u = 0 \quad in \quad \Omega.$$

$$\frac{\partial u}{\partial n} = g \quad on \quad \Gamma.$$

is

$$\int_\Gamma \frac{\partial u}{\partial n} = \int_\Gamma g ds = 0.$$

which follows is immediately from

$$\int_\Omega \Delta u d\Omega = \int_\Gamma \frac{\partial u}{\partial n} ds.$$

Let Ω be bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. Recall the following Gauss divergence theorem for $u, v \in C^1(\Omega)$,

$$\int_\Omega v \frac{\partial u}{\partial n} d\Omega = \int_\Gamma uv.nds - \int_\Omega u \frac{\partial v}{\partial n} d\Omega,$$

where n is the outward unit normal to the boundary Γ and ds is the element of arc length. As a consequence of Gauss divergence theorem, the following identity known as Greens identity hold true.

$$\int_\Omega v \Delta u d\Omega = \int_\Gamma v \frac{\partial u}{\partial n} ds - \int_\Omega \nabla u . \nabla v d\Omega.$$

Integrating the second term of the right hand side once more by parts we obtain

$$\int_{\Omega} v\Delta u d\Omega = \int_{\Omega} u\Delta v d\Omega + \int_{\Gamma} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds$$

Here, $\frac{\partial}{\partial n}$ indicates differentiation in the direction of the exterior normal to Γ.
From the identity (3.12), the special case when $v = 1$ yields

$$\int_{\Omega} \Delta u ds = \int_{\Gamma} \frac{\partial u}{\partial n} ds.$$

Another special case of interest by choosing $v = u$, in this case, equation (3.23) yields the identity

$$\int_{\Omega} |\Delta u|^2 d\Omega + \int_{\Omega} u\Delta u d\Omega = \int_{\Gamma} u\frac{\partial u}{\partial n} ds.$$

If $\Delta u = 0 \ \ in \ \ \Omega$ then for u is continuous.

$$\int_{\Omega} |\Delta u|^2 d\Omega = 0.$$

this implies $\nabla v = 0 \Rightarrow u = constant$
This observation leads to uniqueness theorem for the Neumann boundary value problem up to constant. In other cases, the constants must be zero, hence $u = v$.

4.2.1 Transformation of Neumann boundary value problem to integral equation

For the interior Neumann boundary value problem of equation (4.14 and 4.15) The boundary condition is $\frac{\partial u}{\partial n} = g$ on Γ and using it, we can written equation (3.29) as

$$u(x) = \frac{1}{\pi}\int_{\Gamma} u(y)\frac{\partial}{\partial n}[\log|x-y|]ds$$

$$u(x) = \frac{1}{\pi}\int_{\Gamma} f(y)\log|x-y|ds, x \in \Gamma \tag{4.16}$$

Equation (4.16) is an integral representation of the Neumann boundary value problem. Thus equation (4.14 and 4.15) implies equation (4.16). The two formulations equation (4.14 and 4.15) and equation (4.16) are equivalent.
Therefore, equation (4.16) is the integral solution of Neumann boundary value problem, where Ω be a smooth domain and u be a smooth solution of equations (4.14 and 4.15).
Solving Neumann boundary value problem seems like solving integral equation, therefore the Neumann boundary value problem equations (4.14 and 4.15) are equivalent to equation (4.16).

4.3 Mixed Boundary Value Problem

Mixed boundary value problems are practical situations that met in most potential and others mathematical physics problems. In this case the boundaries can have values of the functions

specified on them as a Dirichlet boundary condition, and derivatives as Neumann boundary conditions. Mixed boundary value problem is finding value of unknown function on Γ_1 and value of normal derivative of unknown function on Γ_2.
The mixed boundary value problem (third boundary value problem) is to find a solution $u \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$ of

$$\Delta u = f \ in \ \Omega \tag{4.17}$$

$$u = \bar{u} \ on \ \Gamma_1 \tag{4.18}$$

$$\frac{\partial u}{\partial n} = \bar{u}_n \ on \ \Gamma_2 \tag{4.19}$$

Where $\bar{u}$ and $\bar{u}_n$ are given and continuous function on Γ_1 and Γ_2 respectively is a prescribed continuous function. To find value of unknown and value of normal derivative of unknown functions. $\Gamma_1 \bigcup \Gamma_2 = \Gamma$ and $\Gamma_1 \bigcap \Gamma_2 = \emptyset$, when Γ is the boundary Γ_1 and Γ_2 are some prates of the boundary. The solution u may be interoperated as the steady state heat transfer in a body from the boundary at which the heat radiates (spread out) freely in to surrounding medium of prescribed temperature.

4.3.1 Transformation of Mixed Boundary Value Problem to integral equation

Solving boundary value problem is the same to solving integral equation, the solution of the boundary value problem which is governed by the Poisson equation

$$\Delta u = f \ in \ \Omega$$

And has mixed boundary conditions

$$u = \bar{u} \ on \ \Gamma_1$$

$$\frac{\partial u}{\partial n} = \bar{u}_n \ on \ \Gamma_2$$

The solution can be obtained in the following ways.
Application of Greens identity equation (3.23) for the function u and v that satisfy equation (4.17) and equation (4.18) respectively

$$\varepsilon(p)u = \int_\Omega uf d\Omega - \int_\Gamma (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds$$

The corresponding boundary integral equation for the smooth boundary is

$$\frac{1}{2}u = \int_\Omega vf d\Omega - \int_\Gamma (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds \tag{4.20}$$

there for equation(4.20)is the integral representation of Mixed boundary value problem.

Mixed boundary value problem can be transformed to boundary integral equation, therefore, mixed boundary value problem is equivalent to (4.20) boundary integral equation.

Chapter 5

Conclusion and Recommendation

5.1 Conclution

The result of the work presented in this thesis is how boundary value problems for Poisson's equation is transformed to boundary integral equations. To find the solution of boundary value problem we transform boundary value problem to boundary integral equation by applying Green's identities. The divergence theorem, Greens first identity, Greens second identity and fundamental solution are the concepts used to transform boundary value problems to boundary integral equation. Since the main objective of this thesis is transforming boundary value problems to boundary integral equation and showing the equivalence of boundary value problems with boundary integral equation,each boundary value problems of Poisson's equation transformed to equivalent boundary integral equation. Therefore boundary integral equation is successfully applied for modeling different types of bodies.

5.2 Recommendation

In this thesis we have introduce the boundary integral equation methods used to solve boundary value problems, it is advisable for riders and other researchers to start work sited in the reference for further understanding and for further investigating of the concept and application of the method in the same or related area.

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