EXPONENTIALLY FITTED FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED FREDHOLM INTEGRO DIFFERENTIAL EQUATIONS

Mohammed Sumebo Hogeme

A Thesis Submitted to the Department of Applied Mathematics
School of Applied Natural Science
Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Mathematics (Numerical Analysis)

Office of Graduate Studies
Adama Science and Technology University

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Adama, Ethiopia

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Mohammed Sumebo Hogeme

Advisor: Dr. Mesfin Mekuria Woldaregay

Co-Advisor: Dr. Tekle Gemechu Dinka

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Declaration

I hereby declare that this master thesis entitled ***"Exponentially fitted finite difference method for singularly perturbed Fredholm integero-differential equations"*** is my own work and has not been submitted to any university for similar purpose. The references used in this proposal are duly recognized by proper citations.

Name of Student	Signature	Date

Recommendation

We, the advisor(s) of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"Exponentially fitted finite difference method for singularly perturbed Fredholm integero-differential equations"*** prepared under our guidance by Mohammed Sumebo Hogeme submitted in partial fulfillment of the requirements for the degree of Mater's of Science in Applied Mathematics. Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures..

Major Advisor	Signature	Date
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Approval Page

We, the advisors of the thesis entitled **Exponentially fitted finite difference method for second order singularly perturbed Fredholm integro-differential equations** and developed by Mohammed Sumebo Hogeme, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

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We, the undersigned, members of the Board of Reviewers of the thesis open defense by Mohammed Sumebo Hogeme have read and evaluated the thesis entitled ***"Exponentially fitted finite difference method for singularly perturbed Fredholm integro-differential equations"*** and the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics (Numerical Analysis).

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List of Acronyms

SPFIDEs Singularly perturbed Fredholm integro differential equations

BVP Boundary Value Problem

SPP Singularly Perturbed Problems

SPDEs Singularly Perturbed Differential Equations

ROC Rate of Convergence

SPVIDEs Singularly perturbed Voltera integro differential equations

VFIDEs Voltera Fredholm integro differntial equations

Abstract

In this thesis, we developed an ε *uniform convergent scheme for solving singularly perturbed linear second order Fredholm integro differential equation. A parameter-uniform numerical method was constructed using exponentially fitted finite difference method to approximate the differential part and the composite Simpson's 1/3 rule for the integral part. The schemes stability and convergence analysis has been carried out . The maximum absolute errors and rate of convergence is tabulated for different values of perturbation parameter* ε *and mesh sizes using different numerical test examples.*

Chapter One
Introduction

1.1 Background of the study

Integro-differential equation (IDE) is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. In recent years, there has been a growing interest in the integro-differential equations (IDEs) which are a combination of differential and integral equations. This is an important branch of modern mathematics and arise frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc (Ren et al., 1999). Based on the integral boundaries it can be classified in to Volterra integral equations and Fredholm integral equations. Voltera-Fredholm integro differential equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type. In addition to this if the unknown function appears outside the integral sign , then it is called an integral equation of the second kind. On the other hand, if the unknown function appears only under the integral sign, is called Volterra integral equation of the first kind (Sastry, 2012). The form of Voltera integro differential equation is given by

$$u^{(n)}(x) = f(x) + \int_a^x k(x,t)u(t)dt, u^{(r)}(0) = 0 \leq r \leq n-1 \tag{1.1}$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$. The Volterra integro-differential equation appeared in so many physical applications like glass forming process, nano hydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert.

The integral equations which contains both Fredholm and Voltera integro differential equations either in a split integral or a mixed form are called Voltera-Fredholm integro differential equations. Those integral equations can arise from different places such as parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic, from various physical biological models and etc. They have led to many scientific computing's and they play important roles in different branches of science involving aero- dynamics, the economy, electricity and electronics, industrial networks, hydrodynamics, oceanography and chemistry see the detail in (Kashkaria & Syam, 2017). Mainly, VFIDEs have been used widely for population growth, medicine processes and pandemic research. For instance , the dissipation of tumor cells and the response of immune system were modeled in (Bellomo et al., 1999).

Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$ inside and outside the integral sign respectively. The limits of integration in this case are fixed. The form of the equation is considered below:

$$u^{(n)}(x) = f(x) + \int_a^b k(x,t)u(t)dt, u^{(r)}(0) = b_r, 0 \leq r \leq n-1 \tag{1.2}$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$.
Fredholm was a Swedish mathematician who established the theory of integral equations. In 1903, his paper in Acta-mathematica played a major role in the establishment of operator theory. Fredholm integro-differential equations (FIDEs) have in large quantities applications in every branches of science. FIDEs arise from the mathematical modeling of many scientific phenomena, such as the study of fluid flow, physics, chemistry, biology, mechanics, astronomy, potential theory, electrostatics, control theory of industrial mathematics and chemical kinetics (Amiraliyev et al., 2021). In general, integro-differential equations have gained a lot of interest in multitude of uses, specifically in sciences related to nature and engineering. Major usages of the integro-differential equations are visible in the mathematical modeling on spatio-temporal development of epidemics (Aziz & Fayyaz, 2013). It is impossible to get an analytical answer for such equations. Because of that, various numerical methods have been devoted to finding the approximate solutions to such equations.

Singularly perturbed differential equations are typically characterized by a small parameter ε called perturbation parameter multiplying some or all of the highest order terms in the differential equation. This problem undergo rapid changes within very thin layers near the boundary or inside the problem domain, so most of the conventional methods fail when this small parameter approaches to zero. These singularly perturbed differential equations arise in the modeling of various modern complicated processes, such as reaction - diffusion processes, epidemic dynamics, high Reynold's number flow in the fluid dynamics, heat transport problem. For more details on singular perturbation, one can refer the book(Bellman & Cooke, 1963).
In general, the solutions of such equations exhibit multi scale phenomena. Within certain thin sub-regions of the domain, the scale of some derivatives is significantly larger than other derivatives. These thin regions of rapid change are called, boundary or interior layers, as appropriate. Where ever the perturbation parameter occur the material quantities usually

change rapidly in the region of the associated boundary layers while the solution behaves regularly and varies slowly in other parts. For this reason it can be expected that numerical difficulties may arise, if accurate approximations to the true solution are required inside, or in a neighborhood of a boundary layer (Miller et al., 1996). Standard numerical methods, such as finite difference method, finite element methods and etc are not appropriate for practical applications when the perturbation parameter ε is sufficiently small. It is necessary to construct suitable numerical methods that are uniformly convergent with respect to ε. For the solution of singularly perturbed problems there are two approaches, fitted operator method that are reflect the nature of the solution in the boundary layers and fitted mesh methods and beside this the maximum norm is an appropriate to compute error analysis.

1.2 Statement of the problem

It is a well known fact that, for small values of ε, standard numerical methods for solving such types of problems are unstable and do not give accurate results. Therefore, it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the parameter value ε, i.e., methods that are convergent ε-uniformly (Amiraliyev et al., 2018) .

The accuracy and convergence of the methods need attention because the numerical treatment of singular perturbation problems is not easy and the solution depends on the perturbation parameter and mesh size (Turuna et al., 2020). The presence of the singular perturbation parameter ε leads to occurrences of oscillations or divergence in the computed solutions while using classical numerical methods (Woldaregay & Duressa, 2019). To avoid these oscillations or divergence, a large number of mesh points are required when ε is very small. It is difficult to handle such cases due to round-off error and computer processing ability. Therefore, to fill this gap associated with classical or standard numerical methods, this study aims at developing exponentially fitted finite difference method together with a composite Simpson's $\frac{1}{3}$ rule on uniform mesh to solve singularly perturbed Fredholm integro-differential equation of the form :

$$Lu := -\varepsilon u'' + a(x)u' + p(x)u = f(x) + \lambda \int_0^l k(x,s)u(s)ds \quad x \in (0,l) \tag{1.3}$$
$$u(0) = A \quad u(l) = B$$

where $\varepsilon \in$(0,1] is a perturbation parameter, λ is real parameter .
We assume that $a(x) \geq \alpha \geq 0, f(x), k(x)$ and $p(x)$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified.

1.3 Objectives

1.3.1 General Objective

The general objective of this study is to develop uniformly convergent numerical method for solving singularly perturbed Fredholm integro-differential equations.

1.3.2 Specific Objectives

The specific objectives of this study are:

- to construct exponentially fitted finite difference method together with a composite Simpson's 1/3 rule on uniform mesh for the numerical solution of linear second-order singularly perturbed Fredholm integro-differential equations,
- to establish stability analysis of the method,
- to establish the convergence of the method.

1.4 Significance of the Study

The result that obtained from this study may:

- used as a reference material for scholars who works on this area.
- provide a numerical method for solving singularly perturbed Fredholm integro-differential equations.

Chapter Two
Review of related literature

For the numerical solution of initial value problem of singularly perturbed Fredholm integro-differential equation, Amiraliyev et al. (2021) proposed a fitted finite difference scheme on a uniform mesh. The difference scheme is constructed by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. It is shown that the method exhibits uniform convergent in terms of perturbation parameter.

Loh and Phang (2019) derived the Bernoulli polynomials operational matrix method to solve Fredholm fractional integro-differential equation with right-sided Caputo's derivative with multi-fractional orders . They also derived analytical expressions of the expansion coefficient by Bernoulli polynomials approximation for both approximation of single- and double variable function. They used the double-variable expansion coefficient to approximate the Fredholm integral kernel by the kernel matrix and they compared the scheme with others methods using numerical examples. The computed numerical results have shown that the operational matrix based on Bernoulli polynomial performs better than Shifted Legendre polynomial in terms of accuracy.

Hosseini and Shahmorad (2003) developed a scheme based on the Tau method with arbitrary bases to find the numerical solution of Fredholm integro-differential equations. In that work, the differential part appearing in the equation is replaced by its operational Tau representation. Performance of the the method is demonstrated by different numerical examples.

Amiraliyev et al. (2018) proposed a fitted finite difference scheme on a uniform mesh for the solution of singularly perturbed Fredholm integro-differential equations. The difference scheme is constructed by the method of integral identities with the use of exponential ba- sis functions and interpolating quadrature rules with the weight and remainder terms in integral form. Their numerical result shows that the method exhibits uniform convergent in terms of parturbation parameter and first order accurate.

Tahernezhad and Jalilian (2020) Proposed a new scheme based on the exponential spline function for solving linear second-order Fredholm integro-differential equations.

Their approach consists of reducing the problem to a set of linear equations. They proved the convergence analysis of the method applied to the solution of integro-differential equations. The method is described and illustrated with numerical examples. The results reveal that the proposed method is accurate and easy to apply.

Cakir and Gunes (2022) suggested a fitted operator finite difference approximation for solving Singularly Perturbed Volterra–Fredholm integro-differential equations. They established the finite difference scheme on non uniform mesh by using interpolating quadrature rules and the linear basis functions. The stability and convergence of the method were examined in the discrete maximum norm. They also demonstrated the effectiveness and reliability of the suggested scheme using different numerical examples. The computed results reveal that the scheme is first order accurate.

Aigo (2013) constructed numerical approximation of Volterra integral equations of second kind using repeated simpson's and trapezoidal quadrature rule. In this paper a recurrence relation was derived and an approximation solution is obtained with a good convergence.

Yüzbaşi (2018) introduced an exponential collocation method for the solutions of the Fredholm–Volterra integro-differential equations under initial and boundary conditions. In addition, an error estimation technique was given for the approximate solutions and also the approximate solutions were improved by means of the residual method.

Zhao and Corless(2006) constructed sixth order compact finite difference method to solve second order integro-differential equations (IDE) with different boundary conditions, and in their work both of error estimates and numerical experiments confirm their compact finite difference method can get fifth order of accuracy. They also adjusted compact finite difference method for first order IDE and a system of IDE and gave numerical experiments for them. They also proposed algorithm which can solve nonlinear IDE and unsplit kernel of IDE.

Iragi and Munyakazi (2020) proposed a finite difference scheme to solve a family of singularly perturbed Volterra integro-differential equations (SPVIDE). The scheme was constructed based on the method of integral identities with appropriate quadrature rules with weights and remainder terms in the integral form and an euler finite difference operator was used on a piecewise uniform mesh of Shishkin type. In their paper they showed that the proposed scheme is first order uniformly convergent with respect to the perturbation and the mesh parameters.

Yapman et al. (2019) analyzed the convergence of the fitted mesh method applied to singularly perturbed Volterra delay integro-differential equation. Their mesh comprises a special nonuniform mesh on the first sub-interval and uniform mesh on another part. In their work, the difference scheme was constructed based on the method of integral identity by using appropriate interpolating quadrature rules with remainder terms in integral form. Error estimates are obtained using difference analogue of Gronwall's inequality with delay. In order to test convergence of the method, they implemented the present method on two particular problems the numerical results show that the proposed method is first - order uniformly accurate.

Kashkaria and Syam (2017) Presented a stochastic computational intelligence technique for solving a class of nonlinear Volterra–Fredholm integro-differential equations with mixed conditions. They used strength of feed forward artificial neural networks to accurately model the integrodifferential equation equation. They also compared their result with other numerical technique to show the efficiency of the proposed method.

Dawood et al.(2020) used Laplace discrete Adomian Decomposition Method (LDADM) to solve non-homogeneous nonlinear Volterra-Fredholm integro-differential equations. Their method is based upon the Laplace Adomian decomposition method coupled with some quadrature rules of numerical integration. They analyzed performance of the proposed method through absolute error measures between the approximated solutions and exact solutions. In that work series of experimental numerical results show that the proposed method performs in high accuracy and efficiency and their study highlights that the proposed method could be used to overcome the analytical approaches in solving nonlinear Volterra-Fredholm integro-differential equations.

Erdogan and Amiraliyev (2012) Studied singularly perturbed initial value problem for linear second-order delay differential equation . In order to solve the considered problem they used exponentially fitted difference scheme in a uniform mesh. Finally, the proposed scheme showed the uniform convergence with respect to perturbation parameter and mesh parameter.

Panda et al. (2021) constructed a post-processing technique employed to solve a SPVIDE using a piecewise uniform Shishkin mesh and also a fitted mesh finite difference approach is applied using a composite trapezoidal rule in the case of integral component and a finite difference operator for the derivative component. The proposed technique acquires a uniform convergence in accordance with the perturbation parameter.

Swaidan and Ali (2021) Proposed a computational method to solve non linear FIDE Using haar wavelet collocation Points. In that work they transformed the higher order differential equations to a set of algebraic equation using the Haar wavelet collocation points and its operational matrix. Their technique is mathematically simple and easy to utilize, then the required less computational complexity and provide more quantitatively reliable results. From the numerical problems, the result obtained from the proposed method almost coincides with the exact solutions of the problems.

Kurt and Sezer (2008) Formulated a practical matrix methods to solve the linear Fredholm integro- differential equation with constant coefficients. In their work, comparison of the results obtained by the present method with that obtained by other methods reveals that the present method is very effective and convenient. In addition to this the error analysis and illustrative examples are discussed to demonstrate the validity and applicability of the scheme.

Chapter Three
Methodology

3.1 Study design

In this study we applied both computational work and numerical experimentation.

3.2 Mathematical Procedure

We followed the following steps to attain the objectives of the study.

- Domain discretiziation of the given problem was performed using uniform mesh.
- Exponential fitting factor was computed.
- The quadrature method was constructed using composite Simpson's 1/3 rule .
- Exponential fitting factor was applied to the perturbation part of the problem.
- The scheme was constructed using central finite difference method to the differential part and Simpson's 1/3 rule to the integral part and system of equations ware generated.
- Stability bound and uniform convergence analysis of the scheme was performed.
- We wrote a code to solve the problem using matlab software
- The scheme was validated using different numerical examples.

Chapter Four
Description of the method

4.1 The continuous problem and asymptotic estimations

Consider the following boundary value problem

$$-\varepsilon u'' + a(x)u' + p(x)u = f(x) + \lambda \int_0^l k(x,t)u(t)dt \quad x \in (0,l) \tag{4.1}$$

$$u(0) = A \quad u(l) = B \tag{4.2}$$

Lemma 4.1 *If a,p, $f \in C^2[0,l]$, $\frac{\partial^s k}{\partial x^s} \in C[0,l]^2$, (s = 0,1,2) and*

$$|\lambda| < \frac{\alpha}{\max \int_0^l |k(x,t)|dt} \tag{4.3}$$

then for the solution u(x) of the problem 4.1 and 4.2 hold the following estimates

$$\|u\|_\infty \le C \tag{4.4}$$

$$|u'| \le C\left\{1 + \varepsilon^{-1}\exp\left(\frac{-\alpha(l-x)}{\varepsilon}\right)\right\}, \qquad x \in [0,l] \tag{4.5}$$

Proof. Using the maximum principle for the operator $L_0 u = -\varepsilon u'' + a'(x)u + p(x)u(x)$ we obtain the estimate

$$|u''| \le |A| + |B| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}|\lambda| \max_{0\le x\le l} \int_0^l |k(x,t)||u(t)|dt$$

Then, it follows that

$$\|u\|_\infty \le |A| + |B| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}|\lambda| \max_{0\le x\le l} \int_0^l |k(x,t)dt\|u\|_\infty$$

Finally we have

$$\|u\|_\infty(1 - \alpha^{-1}|\lambda| \max_{0\le x\le l} \int_0^l |k(x,t)|dt) \le |A| + |B| + \alpha^{-1}\|f\|_\infty$$

This leads to

$$\|u\|_\infty \leq C$$

Next to prove the estimate 4.5 we use the lemma stated blow

Lemma 4.2 *Let $F \in C^2[0,l]$. Then the solution u of 4.1 satisfies*
$|u_{(l)}^{(i)}| \leq c\varepsilon^{-i} \quad \forall i \geq 1$

where $F = f - p(x)u(x) + \lambda \int_0^l k(x,s)u(s)ds$
Proof From 4.1, $-\varepsilon u'' + au' = F$
Let $a(x)$ be an indefinite integral of a, we have

$$u(x) = u_{a(x)} + k_1 + k_2 \int_x^l \exp\left(-\varepsilon^{-1}(a(l) - a(x))\right) dt \tag{4.6}$$

where $u_{a(x)} = \int_x^l z(t)dt$ and

$$z(x) = \int_x^l \varepsilon^{-1} F(t) \exp\left(-\varepsilon^{-1}(a(t) - a(x))\right) dt$$

Using the inequality

$$\exp\left(-\varepsilon(a(t) - a(x))\right) \leq \exp\left(-\alpha\varepsilon(t - x)\right), \;\; x \leq t \tag{4.7}$$

and 4.5 we obtain

$$|z(x)| \leq c\varepsilon^{-1} \int_x^l \left[\exp\left(-\varepsilon^{-1}\alpha(l-x)\right) + k\varepsilon^{-1} \exp\left(-\varepsilon^{-1}\alpha(l-x)\right)\right] dt$$

$$\leq c\left(1 + \varepsilon^{-2}(l-x)\exp\left(-\varepsilon^{-1}\alpha(l-x)\right)\right)$$

Since $u_{a(x)} = \int_x^l z(x)dx$, hence $u_{a(x)} \leq c$.
From the computations the constants k_1 and k_2 must satisfy

$$k_1 + k_2 \int_0^l exp\left[-\varepsilon^{-1}(a(l) - a(t))\right] dt = A - u_{p(0)}, \quad k_1 = B$$

$$u_{p(0)} + k_2 \int_0^l exp\left[-\varepsilon^{-1}(a(l) - a(t))\right] dt = A - B$$

Since a(x) is bounded $(0,l),(a(l)-a(t))\leq c(l-t)$,

$$\int_0^l \exp\left(-\varepsilon^{-1}(a(l)-a(t))\right)dt \geq c\varepsilon$$

Integrating the above and simplifying we get $k_2 \leq c\varepsilon^{-1}$ finally $|u'(1)| = |k_2| \leq c\varepsilon^{-1}$ which is proved for $i=1$. For $i \geq 1$, we obtain by induction repeated differentiation of 4.1 see the detail in Kellogg and Tsan(1978) $\square$.

To prove 4.5, setting $z = u'$ then the equation 4.1 becomes

$$-\varepsilon z' + az = F \tag{4.8}$$

Using 4.5 and the inductive hypothesis

$$F(x) \leq c\left\{1+\varepsilon^{-1}\exp\left(-\alpha\varepsilon^{-1}(l-x)\right)\right\} \tag{4.9}$$

Let a(x) be the indefinite integral of a. Then we have

$$z(x) = B\exp\left(-\varepsilon^{-1}(a(l)-a(x))\right)+\varepsilon^{-1}\int_x^l F(t)\exp(-\varepsilon^{-1}(a(t)-a(x)))dt$$

From 4.9, inequality 4.7 and lemma 4.2

$$|z(x)| \leq c\varepsilon^{-1}\exp\left(-\alpha\varepsilon^{-1}(l-x)\right)+\varepsilon^{-1}\int_x^l [c\left\{1+\varepsilon^{-1}\exp\left(-\alpha\varepsilon^{-1}(l-x)\right)\right\}$$
$$\left(\exp(-\varepsilon^{-1}\alpha(t-x))\right)]dt$$

$$\leq c\varepsilon^{-1}\exp\left(-\alpha\varepsilon^{-1}(l-x)\right)+c\varepsilon^{-1}\int_x^l [\exp\left(-\alpha\varepsilon^{-1}(t-x)\right)$$
$$+\varepsilon^{-1}\exp\left(-\alpha\varepsilon^{-1}(l-x)\right)]dt$$

Integrating the expression in the integral and taking the common terms we obtain

$$|z| \leq C\left\{1+\varepsilon^{-1}\exp\left(\frac{-\alpha(l-x)}{\varepsilon}\right)\right\}$$

This implies that

$$|u'| \leq C\left\{1+\varepsilon^{-1}\exp\left(\frac{-\alpha(l-x)}{\varepsilon}\right)\right\}$$

4.2 Formulation of the numerical scheme

In this section, exponentially fitted finite difference method together with composite Simpson's 1/3 rule is applied to discretize and construct approximation equations for singularly perturbed Fredholm integro-differential equations. The exponential fitting factor is used to hinder the influence of the perturbation parameter in the boundary layer region. The theory of the asymptotic method is used for developing the exponential fitting factor. On the domain $[0,l]$, we introduce the equidistant meshes such that

$$\Omega_N = \left\{ x_0 = 0, x_i = ih, i = 1(1)N-1, x_N = l, h = \frac{l}{N} \right\}$$

where h is step size and N is the number of mesh points. For domain discretization, we apply an exponentially fitted finite difference method which helps us to hinder the influence of the singular perturbation parameter. First, we compute the exponential fitting factor for the given BVP as follows.

4.2.1 Computation of an exponential fitting factor

For the problem 1.3 to compute exponential fitting factor we follow the procedure used in Woldaregay et al.(2021). We use the technique designed in the theory of asymptotic method for solving singularly perturbed BVPs. From the theory of singular perturbation the zero order asymptotic solution of the singularly perturbed boundary value problems of the form

$$-\varepsilon u''(x) + a(x)u'(x) + p(x)u(x) = f(x) \quad x \in \Omega_x = (0,l). \tag{4.10}$$

and boundary condition

$$u(0) = A, u(l) = B \tag{4.11}$$

is given by

$$u(x) = u_0(x) + \frac{a(l)}{a(x)}(B - u_0(l)) \exp\left(-\int_x^l \left(\frac{a(x)}{\varepsilon} - \frac{p(x)}{a(x)}\right)dx \right) + O(\varepsilon) \tag{4.12}$$

Using Taylor's series expansion for $a(x)$ and $p(x)$ about $x = l$ and restriction to their first terms and simplifying gives

$$u(x) = u_0(x) + (B - u_0(l)) \exp\left(\frac{-a(l)(l-x)}{\varepsilon} \right) + O(\varepsilon) \tag{4.13}$$

where u_0 is the solution of the reduced problem which does not satisfy the boundary conditions. Considering for small values of h and evaluating the result in 4.13 at x_i gives

$$u(ih) = u_0(0) + (B - u_0(l)) \exp\left(-a(l)(\frac{l}{\varepsilon} - i\rho)\right) + O(\varepsilon) \tag{4.14}$$

where $\rho = \frac{h}{\varepsilon}$.
Using 4.14 we have

$$u_{i-1} = u_0(0) + (B - u_0(l)) \exp\left(-a(l)\left(\frac{l}{\varepsilon} - (i-1)\rho\right)\right) + O(\varepsilon) \tag{4.15}$$

$$u_{i+1} = u_0(0) + (B - u_0(l)) \exp\left(-a(l)\left(\frac{l}{\varepsilon} - (i+1)\rho\right)\right) + O(\varepsilon) \tag{4.16}$$

Consider an eqi-distant mesh $\Omega_x^N = \{x_i\}_{i=0}^N$ and denote h = $x_{i+1} - x_i$. For any mesh function v_i, define the following difference operators:

$$\begin{aligned}
D^+ v_i &\approx \frac{v_{i+1} - v_i}{h} \\
D^- v_i &\approx \frac{v_i - v_{i-1}}{h} \\
D^0 v_i &\approx \frac{D^+ v_i + D^- v_i}{2} \approx \frac{v_{i+1} - v_{i-1}}{2h} \\
D^+ D^- v_i &\approx \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}
\end{aligned} \tag{4.17}$$

To hinder the influence of the perturbation parameter, we introduce the fitting factor σ on the perturbation parameter part of problem 4.10 and applying the central finite difference scheme gives

$$-\varepsilon\sigma D^+ D^- u(x_i) + a(x_i) D^0 u(x_i) + p(x_i) u(x_i) = f(x_i) \tag{4.18}$$

for convenience, we take $a(x_i) = a_i, f(x_i) = f_i, p(x_i) = p_i, U(x_i) = U_i$. From 4.18 we have

$$-\varepsilon\sigma\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) + \frac{a_i}{2h}(u_{i+1} - u_{i-1}) = f_i - p_i u_i \tag{4.19}$$

Multiplying 4.19 by h we have

$$\frac{-\sigma}{\rho}(u_{i+1} - 2u_i + u_{i-1}) + \frac{a_i}{2}(u_{i+1} - u_{i-1}) = h(f_i - p_i u_i) \tag{4.20}$$

where $\rho = \frac{h}{\varepsilon}$. We assume that the solution converges uniformly to the solution of 4.12.
As h→0 equation 4.20 becomes

$$-lim_{h\to 0}\left(\frac{\sigma}{\rho}(u_{i+1}-2u_i+u_{i-1})+\frac{a_i}{2}(u_{i+1}-u_{i-1})\right) = lim_{h\to 0}(hf_i - hp_iu_i) \tag{4.21}$$

After solving the limit value the above equation becomes

$$\frac{-\sigma}{\rho}(u_{i+1}-2u_i+u_{i-1})+\frac{a_i}{2}(u_{i+1}-u_{i-1}) = 0 \tag{4.22}$$

Substituting 4.16 and 4.15 in to 4.22 we get

$$\begin{aligned}
&\frac{-\sigma}{\rho}[u_0(0)+(B-u_0(l))\exp\left(-a(l)(\frac{l}{\varepsilon}-(i+1)\rho)\right)-2u_0(0)\\
&\quad -2(B-u_0(l))\exp\left(-a(l)(\frac{l}{\varepsilon}-i\rho)\right)+u_0(0) \qquad (4.23)\\
&\quad +(B-u_0(l))\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i-1)\rho\right)\right)]\\
&\quad +\frac{a_i}{2}\left((B-u_0(l))\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i+1)\rho\right)\right)-(B-u_0(l))\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i-1)\rho\right)\right)\right)=0
\end{aligned}$$

After some simplifications we have

$$\begin{aligned}
&\frac{-\sigma}{\rho}(B-u_0(l))\exp\left(-a(l)(\frac{l}{\varepsilon}-(i+1)\rho)\right)+\frac{2\sigma}{\rho}(B-u_0(l))\exp\left(-a(l)(\frac{l}{\varepsilon}-i\rho)\right)\\
&\quad +\frac{a_i}{2}\left((B-u_0(l))\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i+1)\rho\right)\right)-(B-u_0(l))\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i-1)\rho\right)\right)\right)\\
&\quad -\frac{\sigma}{\rho}(B-u_0(l))\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i-1)\rho\right)\right)=0 \qquad (4.24)
\end{aligned}$$

Simplifying the above we obtain

$$\begin{aligned}
&\frac{-\sigma}{\rho}\left[\exp\left(-a(l)(\frac{l}{\varepsilon}-(i+1)\rho)\right)-2\exp\left(-a(l)(\frac{l}{\varepsilon}-i\rho)\right)+\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i-1)\rho\right)\right)\right]\\
&+\frac{a_i}{2}\left[\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i+1)\rho\right)\right)-\exp\left(-a(l)\left(\frac{l}{\varepsilon}-(i-1)\rho\right)\right)\right]=0
\end{aligned}$$

Taking the common factor and further simplifying we obtain

$$\begin{aligned}
&\frac{-\sigma}{\rho}\left[\exp\left(-a(l)(\frac{l}{\varepsilon}-\rho)\right)-2+\exp\left(-a(l)\left(\frac{l}{\varepsilon}+\rho\right)\right)\right]\\
&+\frac{a(l)}{2}\left[\exp\left(-a(l)\left(\frac{l}{\varepsilon}-\rho\right)\right)-\exp\left(-a(l)\left(\frac{l}{\varepsilon}+\rho\right)\right)\right]=0
\end{aligned}$$

Finally we have

$$\frac{\sigma}{\rho}\left[\exp\left(-a(l)(\frac{l}{\varepsilon}-\rho)\right)-2+\exp\left(-a(l)\left(\frac{l}{\varepsilon}+\rho\right)\right)\right]=$$
$$\frac{a(l)}{2}\left[\exp\left(-a(l)\left(\frac{l}{\varepsilon}-\rho\right)\right)-\exp\left(-a(l)\left(\frac{l}{\varepsilon}+\rho\right)\right)\right]$$

From the above we obtain the fitting factor for the boundary layer problems as:

$$\sigma_r=\frac{\rho a(l)}{2}\coth(\frac{\rho a(l)}{2}) \tag{4.25}$$

Similarly the fitting factor for the left boundary layer after simplification is given as

$$\sigma_l=\frac{\rho a(0)}{2}\coth(\frac{\rho a(0)}{2}) \tag{4.26}$$

Finally the fitting factor for the boundary layer problems will be

$$\sigma=\frac{\rho a(x_i)}{2}\coth(\frac{\rho a(x_i)}{2}) \tag{4.27}$$

Using the central finite difference method for the discretization of 4.10 , 4.50 and applying the exponential fitting factor we obtain:

$$-\varepsilon\sigma\left(\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}\right)+\frac{a_i}{2h}(u_{i+1}-u_{i-1})+p_iu_i=f_i$$

Explicitly it becomes

$$\left(\frac{-\varepsilon\sigma}{h^2}+\frac{-a_i}{2h}\right)u_{i-1}+\left(p_i+\frac{2\varepsilon\rho}{h^2}\right)u_i+\left(\frac{-\varepsilon\sigma}{h^2}+\frac{a_i}{2h}\right)u_{i+1}=f_i \tag{4.28}$$

From the above equation the numerical scheme for the problem 1.3 is

$$\left(\frac{-\varepsilon\sigma}{h^2}+\frac{-a_i}{2h}\right)u_{i-1}+\left(p_i+\frac{2\varepsilon\rho}{h^2}\right)u_i+\left(\frac{-\varepsilon\sigma}{h^2}+\frac{a_i}{2h}\right)u_{i+1}$$
$$=f_i+\lambda\int_0^l k(x,s)u(s)ds \tag{4.29}$$

4.2.2 Derivation of the quadrature method.

In order to form the system of linear equations, quadrature method is applied to approximate the integral parts in second-order of SPLFIDEs. Therefore, we use composite

Simpson's 1/3 rule to approximate the integral part. According to Jalius and Abdul Majid (2017) it can be expressed as:

$$\int_a^b y(x)dx = \frac{h}{3}\left[y(a) + 4\sum_{j=1}^{\frac{n}{2}} y(x_{2j-1}) + 2\sum_{j=1}^{(\frac{n}{2})-1} y(x_{2j}) + y(b)\right] - \frac{b-a}{180}h^4 y^{(4)}(\mu) \quad (4.30)$$

where $x_j = a + jh$ (j=1,2....n) are the abscissas of the partition points of the integration in interval [a, b].
$R_1 = \frac{b-a}{180}h^4 y^{(4)}(x)$ is the error term of 4.20 where $a \leq \mu \leq b$. The constant step size, h in 4.30 can be defined as:

$$h = \frac{b-a}{n}$$

If we apply composite Simpson's 1/3 rule with n sub intervals to (1.3) then the integral part in second-order SPLFIDE is approximated by

$$\begin{aligned}\int_a^b k(x,t)u(s)ds \approx \frac{h}{3}[&k(x,s_0)u(s_0) + 4k(x,s_1)u(s_1) + 2k(x,s_2)u(s_2) + ...\\ &+ 4k(x,s_{n-3})u(s_{n-3}) + 2k(x,s_{n-2})u(s_{n-2})\\ &+ 4k(x,s_{n-1})u(s_{n-1}) + k(x,s_n)u(t_n)]\end{aligned} \quad (4.31)$$

Substituting 4.31 in to 4.29 we obtain

$$\begin{aligned}&\left(\frac{-\varepsilon\sigma}{h^2} + \frac{-a_i}{2h}\right)u_{i-1} + \left(p_i + \frac{2\varepsilon\rho}{h^2}\right)u_i + \left(\frac{-\varepsilon\sigma}{h^2} + \frac{a_i}{2h}\right)u_{i+1}\\ &= f_i + \frac{\lambda h}{3}[k(x,s_0)u(s_0) + 4k(x,s_1)u(s_1) + 2k(x,s_2)u(t_2) + ...\\ &+ 4k(x,s_{n-3})u(s_{n-3}) + 2k(x,s_{n-2})u(s_{n-2})\\ &+ 4k(x,s_{n-1})u(s_{n-1}) + k(x,s_n)u(s_n)]\end{aligned} \quad (4.32)$$

From 4.32 and the procedure in (Iragi & Munyakazi, 2020) we have

$$\begin{aligned}&\left(\frac{-\varepsilon\sigma}{h^2} + \frac{-a_i}{2h}\right)u_{i-1} + \left(p_i + \frac{2\varepsilon\rho}{h^2}\right)u_i + \left(\frac{-\varepsilon\sigma}{h^2} + \frac{a_i}{2h}\right)u_{i+1}\\ &= f_i + \lambda h\sum_{j=0}^{N}\eta_j k_{i,j}u_j\end{aligned} \quad (4.33)$$

where

$$\eta_j = \begin{cases} \frac{1}{3} & \text{for } j = 0, N, \\ \frac{4}{3} & \text{for } j = 1,3,5,...N-1 \;, \\ \frac{2}{3} & \text{for } j = 2,4,6,...N-2 \end{cases}$$

Lastly, from the above system of equations for $u_1, u_2, ..., u_{n-2}, u_{n-1}$ are generated. Therefore, the generated system of linear algebraic equations can be written in matrix form as

$$(Z+W)U = F \tag{4.34}$$

where Z and W are coefficient matrix, F is a given function and U is an unknown function to be determined. The entries of Z, W and F are given as:

$$Z = \begin{cases} z_{i,i} = p_i + \frac{2\varepsilon\rho}{h^2} & \text{for } i = 1,2,3...N-1, \\ z_{i,i+1} = \frac{-\varepsilon\sigma}{h^2} + \frac{a_i}{2h} & \text{for } i = 1,2,3,...N-2 \;, \\ z_{i,i-1} = \frac{-\varepsilon\sigma}{h^2} + \frac{-a_i}{2h} & \text{for } i = 1,2,3,...N-1 \end{cases}$$

$$W = \begin{cases} \frac{4h\lambda}{3} k_{i,2j-1} & \text{for } j = 1 : \frac{N}{2}, \\ \frac{2h\lambda}{3} k_{i,2j} & \text{for } j = 1 : \frac{N}{2} - 1 \;. \end{cases}$$

$$F = \begin{cases} f_1 + \left((\lambda h \eta_0 k_{1,0} + \frac{2\sigma\varepsilon + ha_i}{h^2}) A + \lambda h \eta_N k_{1,N} B \right) & \\ f_i + \lambda h \left(\eta_0 k_{i,0} A + \eta_N k_{i,N} B \right) & i = 2,3...n-2 \\ f_{N-1} + \left(\lambda h \eta_0 k_{N-1,0} A + (\lambda h \eta_N k_{N-1,N} + \frac{2\sigma\varepsilon - ha_i}{2h^2}) B \right) & . \end{cases}$$

4.2.3 Truncation error of the scheme

The truncation error of 4.32 is given by

$$L^N(u_i(x) - U_i) = f_i - L^N u_i$$

$$= (L - L^N) u_i$$

$$= \left(-\varepsilon u_i'' + a(x)u_i' + p(x)u_i - f(x) - \lambda \int_0^l k(x,s)u(s)ds\right)$$
$$- \left(-\varepsilon\sigma D^+D^- u_i + a_i D^0 + p_i u_i - f_i - \lambda \int_0^l k(x,s)u(s)ds\right)$$

$$= \left(-\varepsilon u_i'' + a_i u'\right) - \left(-\varepsilon\sigma D^+D^- + a_i D^0 u_i\right) \tag{4.35}$$

$$= -\varepsilon\left(\frac{d^2}{dx^2} - \sigma D^+D^-\right)u(x) + a_i\left(\frac{d}{dx} - D^0\right)u(x)$$

$$L^N(u_i - U_i) \leq \left|\varepsilon\left[\frac{a(x)\rho}{2}\coth(\frac{a(l)\rho}{2}) - 1\right]D^+D^-u(x)\right| + \left|\varepsilon\left(\frac{d^2}{dx^2} - D^+D^-u(x)\right)\right|$$
$$+ \left|a(x)\left(\frac{d}{dx} - D^0\right)u(x)\right| \tag{4.36}$$

where $\rho = \frac{h}{\varepsilon}$ and $\sigma = \frac{\rho a(x)}{2}\coth(\frac{\rho a(l)}{2})$. For two constants c_1 and c_2 , we have $|\rho\coth(\rho) - 1| \leq c\rho^2$, for $\rho \leq 1$. Since $\coth(\rho) \to 1$ as $\rho \to \infty$,$|\coth(\rho) - 1| \leq c\rho$ for $\rho \geq 0$. Hence we can write for all $\rho \geq 0$

$$c_1\frac{\rho^2}{\rho+1} \leq \rho\coth(\rho) - 1 \leq c_2\frac{\rho^2}{\rho+1} \tag{4.37}$$

Using 4.37 we can write

$$\varepsilon\left[\frac{a(x)\rho}{2}\coth(\frac{a(l)\rho}{2}) - 1\right] \leq \frac{\rho^2}{\rho+1} \leq \varepsilon\frac{(\frac{h}{\varepsilon})^2}{\frac{h}{\varepsilon}+1} = \frac{h^2}{h+\varepsilon} \tag{4.38}$$

Using the Taylor series expansion we have the bound

$$\begin{aligned} &\left|D^+D^-u(x)\right| \leq C\|u'(\xi)\| \\ &\left|\left(\frac{d}{dx} - D^0\right)u(x)\right| \leq Ch^2\|u'(\xi)\| \\ &\left|\left(\frac{d^2}{dx^2} - D^+D^-\right)u(x)\right| \leq Ch^2\|u^{(4)}(\xi)\| \end{aligned} \tag{4.39}$$

Using 4.38 and 4.39 the truncation error becomes

$$\left(L^N(u_i(x) - U_i)\right) \leq C\left(\frac{h^2}{h+\varepsilon}\right)\|u'(\xi)\| + \varepsilon Ch^2\|u^{(4)}(\xi) + Ch^2\|u'(\xi)\| \tag{4.40}$$
$$R_2 \leq \left(\frac{Ch^2}{h+\varepsilon}\right)\|u'(\xi)\| + \varepsilon Ch^2\|u^{(4)}(\xi) + Ch^2\|u'(\xi)\| \tag{4.41}$$

4.3 Stability bound and uniform convergence analysis

Lemma 4.3 ***Discrete Maximum Principle***. *Let Ω_0^N be any mesh functions on $\bar{\Omega}$. Assume that the mesh function Φ_i satisfies $\Phi_0 \geq 0$. Then $L_\varepsilon^N \Phi_i \geq 0$ for all $1 \leq i \leq N$ implies that $\Phi_i \geq 0$ for all $0 \leq i \leq N$*

Proof. Let k be such that $\Phi_k = \min_{0 \leq i \leq N}$ and suppose that $\Phi_k < 0$. Then clearly $\Phi_k - \Phi_{k-1} \leq 0$ and so

$$L_\varepsilon^N \Phi_k = -\varepsilon\sigma \frac{\Phi_{k+1} - 2\Phi_k + \Phi_{k-1}}{(x_i - x_{i-1})^2} + \frac{a_i}{2}(\frac{\Phi_{k+1} - \Phi_{k-1}}{x_i - x_{i-1}}) + p_i \Phi_k - \lambda h \left(\sum_{j=0}^{N} \eta_j k_{ij} \Phi_k \right)$$

$$L_\varepsilon^N \Phi_k = -\varepsilon\sigma \frac{(\Phi_{k+1} - \Phi_k) - (\Phi_k - \Phi_{k-1})}{(x_i - x_{i-1})^2} + \frac{a_i}{2}(\frac{\Phi_{k+1} - \Phi_{k-1}}{x_i - x_{i-1}}) + \Phi_k \left(p_i - \lambda h \left(\sum_{j=0}^{N} \eta_j k_{ij} \right) \right)$$

It follows that $L_\varepsilon^N \Phi_k < 0$, which is contradiction. There for, $\Phi_k \geq 0$ and $\Phi_i \geq 0$ for all $i, 0 \leq i \leq N$. □

The uniqueness of the solution is guaranteed by this discrete maximum principle. The existence follows easily since, as for linear problems, the existence of the solution is implied by its uniqueness(Iragi & Munyakazi, 2020). The following lemma shows boundedness of the solution.

Lemma 4.4 *(Discrete stability estimate)*
Let $L_\varepsilon^N y_i$ be given as

$$L_\varepsilon^N y_i = \frac{-\sigma}{\rho}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_i}{2}(y_{i+1} - y_{i-1}) + p_i y_i \tag{4.42}$$

For the solution of the difference boundary value problem

$$L^N y_i = f_i \qquad i \geq 1$$
$$y_0 = \mu_1, y_N = \mu_2$$

the following inequality holds

$$\|y\|_\infty \leq \alpha^{-1} \max_{0 \leq i \leq N} |L^N y_i| + \max_{0 \leq x \leq l} \{|\mu_1| + |\mu_2|\} \tag{4.43}$$

where $\rho = \frac{h}{\varepsilon}$

Proof Consider the barrier functions defined by

$$\Phi_i^{\pm} = p^* \pm y_i$$

where

$$p^* = \alpha^{-1} \max_{0 \le i \le N} |L^N y_i| + \max_{0 \le x \le l} \{|\mu_1| + |\mu_2|\}$$

On the boundary points, we have

$$\begin{aligned} \Phi_0^{\pm} &= p^* \pm y_0 \\ &= p^* \pm \mu_1 \ge 0 \end{aligned}$$

$$\Phi_N^{\pm} = p^* \pm y_N = p^* \pm \mu_2 \ge 0$$

On the differential operator,

$$\begin{aligned} L\Phi^{\pm} &= \frac{-\sigma}{\rho}\left(p^* \pm y_{i+1} - 2(p^* \pm y_i) + p^* \pm y_{i-1}\right) \\ &+ \frac{a_i}{2}\left(p^* \pm y_{i+1} - (p^* \pm y_{i-1})\right) + p_i(p^* \pm y_i) \\ &= p_i p^* \pm f_i \\ &= p\left(\alpha^{-1} \max_{0 \le i \le N} |L^N y_i| + \max_{0 \le x \le l} \{|\mu_1| + |\mu_2|\}\right) \\ &\ge 0, \end{aligned} \tag{4.44}$$

Since $p_i > \alpha > 0$ which implies that $L\Phi^{\pm} \ge 0$. Hence, using the maximum principle in 4.3, we obtain $\Phi_i^{\pm} \ge 0 \ \forall i \in \overline{\Omega}$

Theorem 4.1 *Let U_i be the computed solutions of 4.33 and the coefficients $a(x)$ and $p(x)$ be sufficiently smooth functions . Then the bound of the solution U_i holds:*

$$\|L^N (u(x) - U_i)\| \le \frac{h^2}{h+\varepsilon}\left(1 + \varepsilon^{-4} \max_{0 \le x_i \le N-1} \exp\left(\frac{-\alpha(l - x_i)}{\varepsilon}\right)\right)$$

Proof From 4.40 the truncation error is

$$\|(L^N(u_i(x) - U_i)\| \le C\left(\frac{h^2}{h+\varepsilon}\right)\|u'(\xi)\| + \varepsilon C h^2 \|u^{(4)}(\xi)\| + C h^2 \|u'(\xi)\| \tag{4.45}$$

Using the bound for derivative of the solution in lemma 4.1 we obtain

$$\|(L^N u_i(x) - U_i)\| \le \frac{h^2}{h+\varepsilon}\left(1+\varepsilon^{-1}\max_{0\le x_i\le N-1}\exp\left(\frac{-\alpha(l-x_i)}{\varepsilon}\right)\right)$$
$$+\varepsilon C h^2\left(1+\varepsilon^{-4}\max_{0\le x_i\le N-1}\exp\left(\frac{-\alpha(l-x_i)}{\varepsilon}\right)\right)$$
$$+C h^2\left(1+\varepsilon^{-1}\max_{0\le x_i\le N-1}\exp\left(\frac{-\alpha(l-x_i)}{\varepsilon}\right)\right)$$

Since $\varepsilon^{-4} \ge \varepsilon^{-1}$ we obtain the bound

$$\|(L^N u_i(x) - U_i)\| \le \frac{h^2}{h+\varepsilon}\left(1+\varepsilon^{-4}\max_{0\le x_i\le N-1}\exp\left(\frac{-\alpha(l-x_i)}{\varepsilon}\right)\right) \tag{4.46}$$

Lemma 4.5 *For the solution of the difference scheme 4.33 the following inequality holds*

$$|u_i| \le \left(\max\{|A|,|B|\}+\alpha^{-1}\|f\|_\infty\right)\left(\exp\left(C\alpha^{-1}x_i\right)\right), 1\le i\le N-1$$

Proof The difference scheme can be written as

$$-\sigma\varepsilon u_i'' + a_i u_i' + p_i u_i = F$$

where

$$F = f_i + \lambda h\left(\sum_{j=1}^{N}\eta_j k_{ij} u_j\right) \tag{4.47}$$

From 4.47 we obtain

$$|F| \le \|f\|_\infty + \lambda h\left(\sum_{j=1}^{N}\eta_j |k_{ij}||u_j|\right) \tag{4.48}$$

Moreover from the maximum principle we obtain

$$|u_i|_\infty \le \left(\max\{|A|,|B|\}+\alpha^{-1}\|f\|_\infty+\alpha^{-1}\overline{K}\sum_{j=1}^{N}\eta_j|u_j|\right)$$

$$\le \Phi + \alpha^{-1}\overline{K}\sum_{j=1}^{N}\eta_j|u_j| \tag{4.49}$$

where

$$\Phi = \max\{|A|,|B|\}+\alpha^{-1}\|f\|_\infty$$

Using the difference analogue of Gronwall's inequality to 4.49 we get

$$|u_i| \leq \Phi \exp(\alpha^{-1}\overline{K}x_i)$$

Thus we can write that.

$$|u_i| \leq \left(\max\{|A|,|B|\} + \alpha^{-1}\|f\|_\infty\right)\exp(C\alpha^{-1}x_i)$$

Lemma 4.6 *For all integers N in a fixed number of mesh as* $\varepsilon \to 0$,*it holds*

$$\lim_{\varepsilon\to 0}\max_{1\leq x_i\leq N-1}\frac{\exp\left(\frac{-\alpha x_i}{\varepsilon}\right)}{\varepsilon^m} = 0$$

and

$$\lim_{\varepsilon\to 0}\max_{1\leq x_i\leq N-1}\frac{\exp\left(\frac{-\alpha(l-x_i)}{\varepsilon}\right)}{\varepsilon^m} = 0$$

where $x_i = ih, \forall i = 1,2,.....N-1$ *and* $m = 1,2,3....$

Proof To proof the stated lemma we follow the way in Woldaregay et al.(2021). With in the discrete domain $\{x_i\}_0^N$ for the interior grid points we have

$$\max_{1\leq x_i\leq N-1}\frac{\exp\left(\frac{-\alpha x_i}{\varepsilon}\right)}{\varepsilon^m} \leq \frac{\exp\left(\frac{-\alpha x_1}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-\alpha h}{\varepsilon}\right)}{\varepsilon^m},$$
$$\max_{1\leq x_i\leq N-1}\frac{\exp\left(\frac{-\alpha(l-x_i)}{\varepsilon}\right)}{\varepsilon^m} \leq \frac{\exp\left(\frac{-\alpha(l-x_{N-1})}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-\alpha h}{\varepsilon}\right)}{\varepsilon^m} \tag{4.50}$$

Since $x_1 = h$ and $l - x_{N-1} = h$. If we denote $\frac{1}{\varepsilon} = \eta$, we have

$$\lim_{\varepsilon\to 0}\frac{\exp\left(\frac{-\alpha h}{\varepsilon}\right)}{\varepsilon^m} = \lim_{\frac{1}{\varepsilon}=\eta\to\infty}\frac{\eta^m}{\exp(\alpha\eta h)} \tag{4.51}$$

Repeated use of L'Hospital's rule to the expression 4.51 gives us

$$\lim_{\frac{1}{\varepsilon}=\eta\to\infty}\frac{\eta^m}{\exp(\alpha\eta h)} = \lim_{\frac{1}{\varepsilon}=\eta\to\infty}\frac{m!}{(\alpha h)^m\exp(\alpha\eta h)} = 0 \tag{4.52}$$

Hence this completes the proof. □

Next to analyze the uniform convergence of the scheme, for the approximate solution the error is given by $z_i = u_i - U_i$ where u_i is the exact solution and U_i is the approximate solution.

From 4.33

$$L^N_\varepsilon z_i := \left(\frac{-\varepsilon\sigma}{h^2} + \frac{-a_i}{2h}\right) z_{i-1} + \left(p_i + \frac{2\varepsilon\rho}{h^2}\right) z_i + \left(\frac{-\varepsilon\sigma}{h^2} + \frac{a_i}{2h}\right) z_{i+1}$$
$$-\lambda h\left(\textstyle\sum_{j=0}^{N} \eta_j k_{ij} z_j\right) = R \tag{4.53}$$
$$z_= 0, \quad z_N = 0$$

Theorem 4.2 *If a,p, $f \in C^2[0,l]$, $\frac{\partial^s k}{\partial x^s} \in C[0,l]^2$, (s = 0,1,2) and*

$$|\lambda| < \frac{\alpha}{\max_{1\le x_i\le N}\sum_{j=0}^{N}\eta_j h|k_i, j|} \tag{4.54}$$

the solution U_i of 4.33 converges uniformly to the solution of 1.3 then the following error bound holds

$$\|u_i - U_i\|_{\infty,\Omega_N} \le Ch \tag{4.55}$$

Proof From maximum principle for expression 4.53 we have

$$\|z\|_{\infty,\Omega_N} \le \alpha^{-1}\|R + \lambda h\sum_{j=0}^{N}\eta_j k_{ij} z_j\|$$
$$\le \alpha^{-1}\|R\|_{\infty,\Omega_N} + \alpha^{-1}|\lambda|\max_{1\le i\le N}\sum_{j=0}^{N} h\eta_j|k_{i,j}|\|z_j\|_{\infty,\Omega_N}$$
$$\|z\|_{\infty,\Omega_N}\left(1 - \alpha^{-1}|\lambda|\max_{1\le i\le N}\sum_{j=0}^{N} h\eta_j|k_{i,j}|\right) \le \alpha^{-1}\|R\|_{\infty,\Omega_N}$$

Simplifying the above expression we obtain

$$\|z\|_{\infty,\Omega_N} \le \frac{\alpha^{-1}\|R\|_{\infty,\Omega_N}}{\left(1 - \alpha^{-1}|\lambda|\max_{1\le i\le N}\sum_{j=0}^{N} h\eta_j|k_{i,j}|\right)}$$

This implies that

$$\|z\|_{\infty,\Omega_N} \le C\|R\|_{\infty,\Omega_N} \tag{4.56}$$

where $C = \frac{\alpha^{-1}\|R\|_{\infty,\Omega_N}}{1-\alpha^{-1}|\lambda|\max_{1\le i\le N}\sum_{j=0}^{N} h\eta_j|k_{i,j}|}$

From 4.30 and 4.40

$R_1 = \frac{l}{180} h^4 \|u^{(4)}(\xi)\|$ and $R_2 \leq \left(\frac{Ch^2}{h+\varepsilon}\right) \|u'(\xi)\| + \varepsilon Ch^2 \|u^{(4)}(\xi) + Ch^2 \|u'(\xi)\|$
Since the remainder term $R = R_1 + R_2$ we obtain

$$|R| = |R_1 + R_2|$$

$$|R| \leq \left(\frac{Ch^2}{h+\varepsilon}\right) \|u'(\xi)\| + \varepsilon Ch^2 \|u^{(4)}(\xi)\| + Ch^2 \|u'(\xi)\|| + \frac{l}{180} h^4 \|u^{(4)}(\xi)\|$$

Simplifying the above expression we get

$$\|R\|_{\infty,\Omega_N} \leq Ch^2 \left(\frac{1}{h+\varepsilon} + 1\right) \|u'(\xi)\| + h^2 \left(\varepsilon C + \frac{\xi h^2}{180}\right) \|u^{(4)}(\xi)\| \tag{4.57}$$

Since the maximum error obtained when the value of epsilon is very small and hence

$$\|R\|_{\infty,\Omega_N} \leq Ch \tag{4.58}$$

This complete the proof $\square$.

Chapter Five

Results and discussions

In this section, we test the numerical method using examples. For this, we use the double mesh principle to obtain maximum pointwise errors and convergence rates.

Example 1 Consider the following boundary value problem:

$$-\varepsilon u''(x) + (1+x)u'(x) + \pi^2 u(x) = f(x) + \int_0^1 k(x,s)u(s)ds \quad x \in [0,1]$$

Subject to the boundary conditions

$$u(0) = u(1) = 0$$

where $k(x,s) = x+s, f(x) = \pi x \cos(\pi x)$.

In this example $a(x) = 1+x, p(x) = \pi^2, \lambda = 1, l = 1, k(x,s) = x+s, f(x) = \pi x \cos(\pi x)$. The fitting parameter σ for example 1 becomes

$$\sigma = \frac{a(x(i))\rho}{2} \coth\left(\frac{\rho a(x(i))}{2}\right)$$

$$= \frac{h(1+x(i))}{2\varepsilon} \coth\left(\frac{h(1+x(i))}{2\varepsilon}\right)$$

Thus the difference scheme for the given example becomes

$$-\frac{h(1+x(i))}{2\varepsilon} \coth\left(\frac{h(1+x(i))}{2\varepsilon}\right)\left(\frac{u_{i-1}-2u_i+u_{i+1}}{h^2} + a(i)\right) + (1+x_i)\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)$$
$$+ p_i - \sum_{j=0}^{j=N} \eta_j (x_i + s_j) u_j = \pi x_i cos(\pi x_i).$$
$$1 \le i \le N-1$$

Example 2 Consider singularly perturbed problem:

$$-\varepsilon u'' + (2+e^{-x})u' + (x+3)u + \frac{1}{2}\int_0^1 \left(e^{xcos(\pi s)} - 1\right) u(s)ds = \frac{1}{1+x}$$
$$u(0) = 1, \quad u(1) = 0$$

In this Example $a(x) = 2 + e^{-x}, p(x) = x + 3, \lambda = \frac{1}{2}, k(x,s) = e^{xcos(\pi s)} - 1$ and $f(x) = \frac{1}{1+x}$. The fitting parameter σ for example 2 becomes

$$\sigma = \frac{a(x(i))\rho}{2} \coth\left(\frac{\rho a(x(i))}{2}\right)$$

$$\sigma = \left(\frac{2 + e^{-x_i} h}{2\varepsilon}\right) \coth\left(\frac{h2 + e^{-x_i}}{2\varepsilon}\right)$$

Thus the difference scheme for example 2 becomes

$$\begin{aligned}
&-\left(\frac{(2+e^{-x_i})h}{2} \coth\left(\frac{h(2+e^{-x_i})}{2\varepsilon}\right)\right)\left(\frac{u_{i-1-2u_i+u_{i+1}}}{2h^2}\right) + \frac{2+e^{-x_i}}{2h}\left(u_{i+1} - u_{i-1}\right) \\
&+ (x+3)p_i + \frac{1}{2}\sum_{j=0}^{N} \eta_j \left(e^{x_i \cos(\pi s_j)} - 1\right) = \frac{1}{1+x}
\end{aligned}$$

Example 3 Consider singularly perturbed problem:

$$\begin{aligned}
&-\varepsilon u'' - 2u' + (x^2 - 5)u + \frac{1}{2}\int_0^1 \left(e^{xcos(\pi s)} - 1\right) u(s)ds = \frac{1}{1+x} \\
&u(0) = 1, \quad u(1) = 0
\end{aligned}$$

In this Example $a(x) = -2, p(x) = x^2 - 5, \lambda = 1, k(x,s) = e^{xcos(\pi s)} - 1$ and $f(x) = \frac{1}{1+x}$. The fitting parameter σ for example 3 becomes

$$\sigma = \frac{a(x(i))\rho}{2} \coth\left(\frac{\rho a(x(i))}{2}\right)$$

$$\sigma = \left(\frac{-h}{\varepsilon}\right) \coth\left(\frac{-h}{\varepsilon}\right)$$

Thus the difference scheme for example 2 becomes

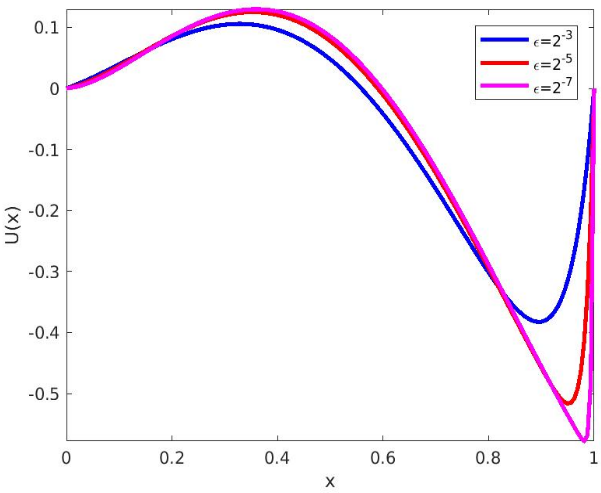
$$\begin{aligned}
&-\left(h \coth\left(\frac{-h}{\varepsilon}\right)\right)\left(\frac{u_{i-1-2u_i+u_{i+1}}}{2h^2}\right) + \frac{-1}{h}\left(u_{i+1} - u_{i-1}\right) \\
&+ (x^2 - 5)u_i + \sum_{j=0}^{N} \eta_j \left(e^{x_i \cos(\pi s_j)} - 1\right) = \frac{1}{1+x}
\end{aligned}$$

We use the formula in Doolan et al.(1980) to compute the maximum point wise error and rate of convergence.

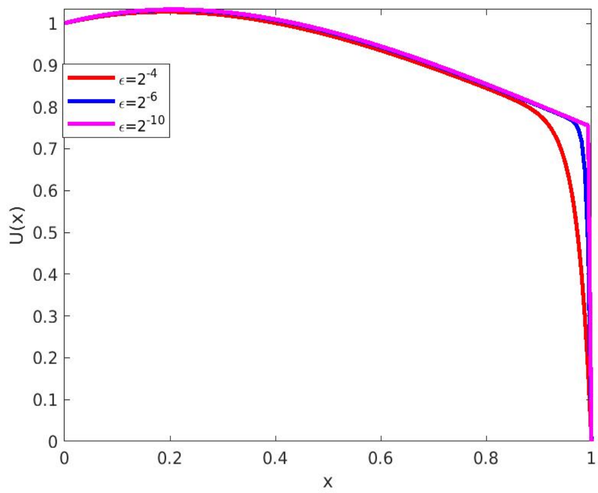
$$E_\varepsilon^N = \max_{0 \le i \le 2N} \left|U^N(x_i) - U^N(x_{2i})\right|$$

and

$$Roc^N = \log 2\left(\frac{E_r^N}{E_r^{2N}}\right)$$



(a) Solutions profile for Example 1 with boundary layer formation as ε goes small with mesh size N = 160.



(b) Solutions profile for Example 2 with boundary layer formation as ε goes small with mesh size N = 160.

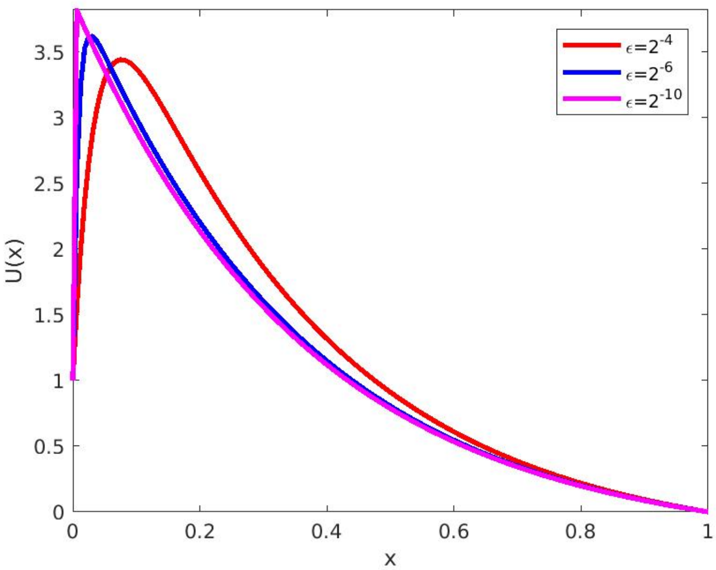
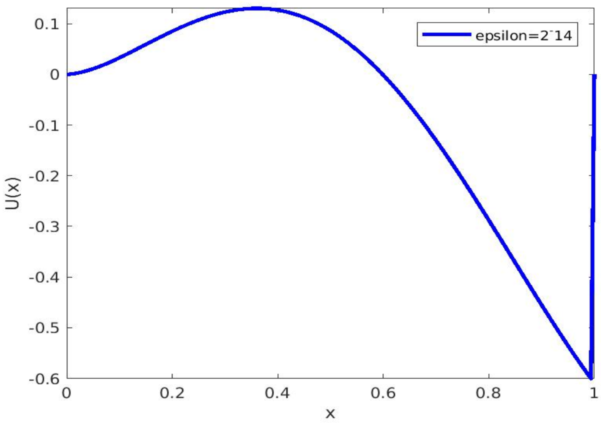
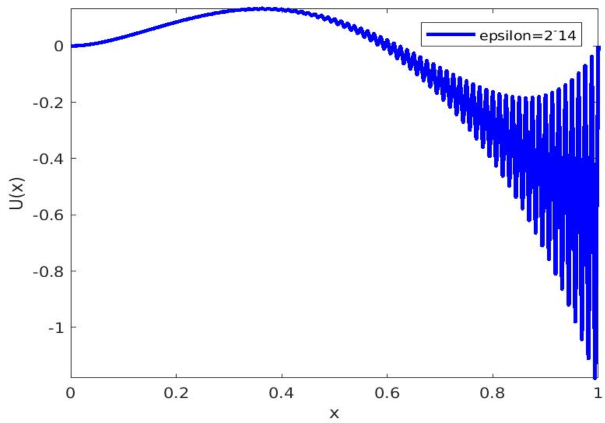


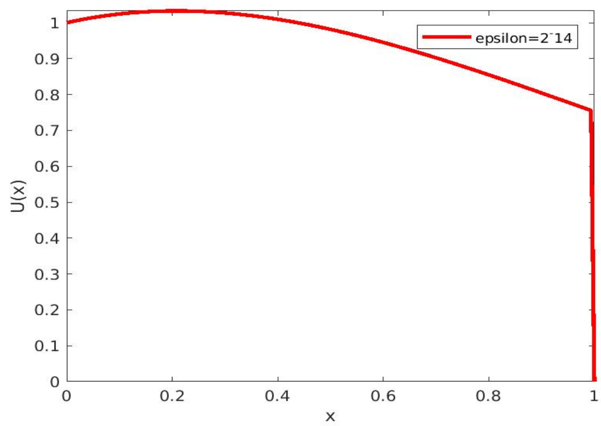
Figure 5.2: Solutions profile for Example 3 with boundary layer formation as ε goes small with mesh size N = 160.



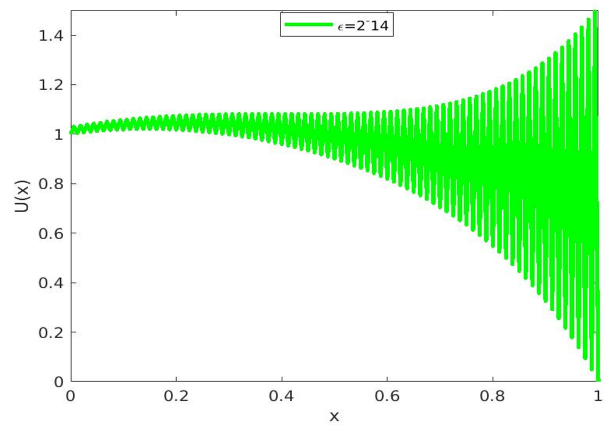
(a) Solutions profile for Example 1 with $\varepsilon = 2^{-14}$, N = 160.



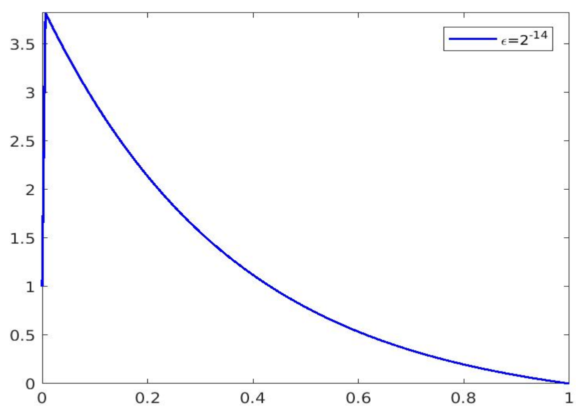
(b) Solution profile for example 1 with $\varepsilon = 2^{-14}, N = 160$ without fitting.



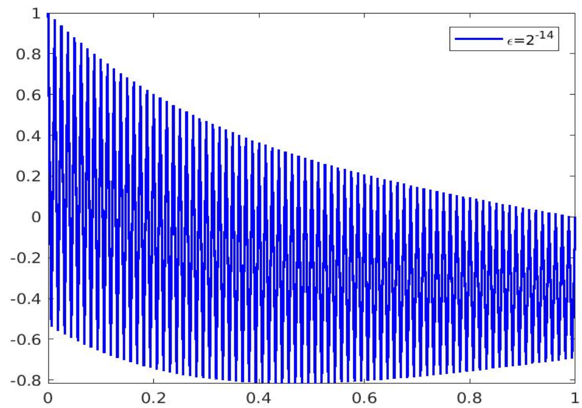
(a) Solutions profile for Example 2 with $\varepsilon = 2^{-14}$, N = 160.



(b) Solution profile for Example 2 with $\varepsilon = 2^{-14}, N = 160$ without fitting.



(a) Solutions profile for Example 3 with $\varepsilon = 2^{-14}$, N = 160.



(b) Solution profile for Example 3 with $\varepsilon = 2^{-14}, N = 160$ without fitting.

Table 5.1: Maximum absolute error and rate of convergence for Example 1

$\varepsilon \downarrow$	$N=20$	40	80	160	320
2^0	1.5885e-03	4.6295e-04	1.5990e-04	6.6492e-05	3.0550e-05
	1.77874	1.53368	1.26591	1.122008	
2^{-2}	3.1169e-03	9.8908e-04	4.3253e-04	2.0654e-04	1.0129e-04
	1.65595	1.19328	1.06638	1.02793	
2^{-4}	6.1666e-03	1.7468e-03	8.1235e-04	4.0353e-04	2.0162e-04
	1.81976	1.10454	1.00942	1.00104	
2^{-6}	1.2461e-02	4.0814e-03	1.1184e-03	5.3176e-04	2.6648e-04
	1.61029	1.86762	1.07259	0.99675	
2^{-8}	1.3813e-02	6.2991e-03	2.6950e-03	9.1256e-04	2.9722e-04
	1.13281	1.22486	1.56230	1.834	
2^{-10}	1.3814e-02	6.3601e-3	3.0410e-03	1.4703e-03	6.4743e-04
	1.11901	1.06503	1.04843	1.18331	
2^{-12}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3405e-04
	1.11901	1.06503	0.72047	1.33022	
2^{-14}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
	1.11901	1.06503	0.72047	1.3300	
2^{-16}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
	1.11901	1.06503	0.72047	1.3300	
2^{-18}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
	1.11901	1.06503	0.72047	1.3300	
2^{-20}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
	1.11901	1.06503	0.72047	1.3300	
$E^{N,M}$	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
$r^{N,M}$	1.11901	1.06503	0.72047	1.3300	

Table 5.2: Comparison of results for Example 1 maximum absolute error.

$\varepsilon \downarrow$	$N = 20$	40	80	160	320
Proposed scheme					
2^{0}	1.5885e-03	4.6295e-04	1.5990e-04	6.6492e-05	3.0550e-05
2^{-2}	3.1169e-03	9.8908e-04	4.3253e-04	2.0654e-04	1.0129e-04
2^{-4}	6.1666e-03	1.7468e-03	8.1235e-04	4.0353e-04	2.0162e-04
2^{-6}	1.2461e-02	4.0814e-03	1.1184e-03	5.3176e-04	2.6648e-04
2^{-8}	1.3813e-02	6.2991e-03	2.6950e-03	9.1256e-04	2.9722e-04
2^{-10}	1.3814e-02	6.3601e-3	3.0410e-03	1.4703e-03	6.4743e-04
2^{-12}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3405e-04
2^{-14}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
2^{-16}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
2^{-18}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
2^{-20}	1.3814e-02	6.3601e-03	3.0412e-03	1.8457e-03	7.3411e-04
Without fitting					
2^{0}	1.5803e-03	4.6329e-04	1.6153e-04	6.7124e-05	3.0731e-05
2^{-2}	3.2365e-03	1.2095e-03	5.0186e-04	2.2504e-04	1.0604e-04
2^{-4}	1.1999e-02	8.3482e-03	2.6138e-03	8.2057e-04	2.9417e-04
2^{-6}	1.0076e-01	2.3217e-02	1.2787e-02	8.4640e-03	2.3657e-03
2^{-8}	2.3518e-01	2.2135e-01	1.4627e-01	2.9765e-02	1.2969e-02
2^{-10}	4.6233e-01	5.1176e-01	3.9873e-01	2.6287e-01	1.6026e-01
2^{-12}	5.7052e-01	7.5415e-01	7.8034e-01	6.6149e-01	4.5284e-01
2^{-14}	6.0306e-01	8.3998e-01	9.5962e-01	9.7171e-01	8.8451e-01
2^{-16}	6.1160e-01	8.6352e-01	1.0133e+001	1.0811e+00	1.0866e+00
2^{-18}	6.1376e-01	8.6955e-01	1.0274e+00	1.1110e+00	1.1470e+00
2^{-20}	6.1430e-01	8.7107e-01	1.0309e+00	1.1187e00	1.1629e+00

Table 5.3: Maximum absolute error and rate of convergence for Example 2

$\varepsilon \downarrow$					
	$N=20$	40	80	160	320
2^{0}	1.9997e-04	9.7585e-05	4.9957e-05	2.5308e-05	1.2741e-05
	1.0350	0.9660	1.1.0999	0.8713	
2^{-2}	9.9284e-04	4.1426e-04	1.8494e-04	8.6493e-05	4.1718e-05
	1.2610	1.1634	1.0964	1.0519	
2^{-4}	2.7026e-03	1.0117e-03	4.0964e-04	1.8024e-04	8.3835e-05
	1.4176	13043	1.1844	1.1043	
2^{-6}	4.9748e-03	2.1478e-03	7.9740e-04	2.9961e-04	2.0936e-04
	1.2118	1.4295	1.4122	0.5171	
2^{-8}	5.1134e-03	2.6419e-03	1.3061e-03	5.5924e-04	3.2949e-04
	0.9527	1.0163	1.2237	0.7632	
2^{-10}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
	0.9527	0.9810	0.9904	0.9974	
2^{-12}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
	0.9527	0.9810	0.9904	0.9974	
2^{-14}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
	0.9527	0.9810	0.9904	0.9974	
2^{-16}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
	0.9527	0.9810	0.9904	0.9974	
2^{-18}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
	0.9527	0.9810	0.9904	0.9974	
2^{-20}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
$E^{N,M}$	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
$r^{N,M}$	0.9527	0.9810	0.9904	0.9974	

Table 5.4: Comparison of results for Example 2 maximum absolute error.

$\varepsilon \downarrow$	$N = 20$	40	80	160	320
Proposed scheme					
2^{0}	1.9997e-04	9.7585e-05	4.9957e-05	2.5308e-05	1.2741e-05
2^{-2}	9.9284e-04	4.1426e-04	1.8494e-04	8.6493e-05	4.1718e-05
2^{-4}	2.7026e-03	1.0117e-03	4.0964e-04	1.8024e-04	8.3835e-05
2^{-6}	4.9748e-03	2.1478e-03	7.9740e-04	2.9961e-04	2.0936e-04
2^{-8}	5.1134e-03	2.6419e-03	1.3061e-03	5.5924e-04	3.2949e-04
2^{-10}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
2^{-12}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
2^{-14}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
2^{-16}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
2^{-18}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
2^{-20}	5.1134e-03	2.6430e-03	1.3390e-03	6.7309e-04	3.3757e-04
Without fitting					
2^{0}	2.7731e-04	1.1939e-04	5.5374e-05	2.6659e-05	1.3079e-05
2^{-2}	2.6062e-03	5.0495e-04	1.4867e-04	7.3562e-05	3.6600e-05
2^{-4}	8.3083e-02	1.1205e-02	3.5362e-03	8.0619e-04	1.7388e-05
2^{-6}	2.0777e-01	6.6542e-01	1.0713e-02	1.2586e-02	3.8779e-03
2^{-8}	5.1220e-01	3.2443e-01	2.3770e-01	7.0572e-02	1.1463e-02
2^{-10}	9.5541e-01	8.4313e-01	6.1586e-01	3.5669e-01	2.4601e-01
2^{-12}	1.1500e+00	1.2153e+00	1.1180e+00	9.2646e-01	6.4904e-01
2^{-14}	1.2049e+00	1.3534e+00	1.3778e+00	1.2987e+00	1.1695e+00
2^{-16}	1.2189e+00	1.3913e+00	1.4715e+00	1.4688e+00	1.4029e+00
2^{-18}	1.2225e+00	1.4010e+00	1.4978e+00	1.5352e+00	1.5169e+00
2^{-20}	1.2223e+00	1.4034e+00	1.5045e+00	1.5547e+00	1.5683e+00

Table 5.5: Maximum absolute error and rate of convergence for Example 3

$\varepsilon \downarrow$	$N=20$	40	80	160	320
2^{0}	1.5956e-03	4.6601e-04	1.5760e-04	6.4359e-05	3.5072e-05
	1.77567	1.56409	1.29205	0.87582	
2^{-2}	4.4420e-02	1.0062e-02	2.1529e-03	5.0627e-04	3.2435e-04
	2.14229	2.22456	2.08830	0.64236	
2^{-4}	1.1969e-01	3.0296e-02	7.5168e-03	1.8476e-03	4.4538e-04
	1.98210	2.01093	2.02446	2.05254	
2^{-6}	3.1267e-01	1.0637e-01	2.9399e-02	7.5759e-03	1.9005e-03
	1.55554	1.85525	1.95628	1.99504	
2^{-8}	3.3850e-01	1.6456e-01	7.52230e-02	2.6644e-02	7.5299e-03
	1.04054	1.12967	1.49736	1.82310	
2^{-10}	3.3850e-01	1.6510e-01	8.1688e-02	4.0514e-02	1.8661e-02
	1.03581	1.01514	1.01170	1.11839	
2^{-12}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
	1.03581	1.01514	1.0069	1.0033	
2^{-14}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
	1.03581	1.01514	1.0069	1.0033	
2^{-16}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
	1.03581	1.01514	1.0069	1.0033	
2^{-18}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
	1.03581	1.01514	1.0069	1.0033	
2^{-20}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
$E^{N,M}$	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0778e-02
$r^{N,M}$	1.03581	1.01514	1.0069	1.0033	

Table 5.6: Comparison of results for Example 3 maximum absolute error.

$\varepsilon \downarrow$	$N = 20$	40	80	160	320
Proposed scheme					
2^{0}	1.5956e-03	4.6601e-04	1.5760e-04	6.4359e-05	3.5072e-05
2^{-2}	4.4420e-02	1.0062e-02	2.1529e-03	5.0627e-04	3.2435e-04
2^{-4}	1.1969e-01	3.0296e-02	7.5168e-03	1.8476e-03	4.4538e-04
2^{-6}	3.1267e-01	1.0637e-01	2.9399e-02	7.5759e-03	1.9005e-03
2^{-8}	3.3850e-01	1.6456e-01	7.52230e-02	2.6644e-02	7.5299e-03
2^{-10}	3.3850e-01	1.6510e-01	8.1688e-02	4.0514e-02	1.8661e-02
2^{-12}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
2^{-14}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
2^{-16}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
2^{-18}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
2^{-20}	3.3850e-01	1.6510e-01	8.1688e-02	4.0649e-02	2.0278e-02
Without fitting					
2^{0}	2.0524e-03	5.7858e-04	1.8488e-04	6.7776e-05	3.4274e-05
2^{-2}	4.9985e-02	1.1619e-02	2.6102e-03	6.7884e-04	3.6990e-04
2^{-4}	2.8831e-01	6.1046e-02	1.4509e-02	3.5996e-03	8.8899e-04
2^{-6}	1.8829e+00	7.6546e-01	2.2416e-01	4.8545e-02	1.1373e-02
2^{-8}	2.4750e+00	3.0050e+00	1.6924e+00	7.0571e-01	2.0953e-01
2^{-10}	5.2582e-01	2.4916e+00	4.0182e+00	2.8763e+00	1.6443e+00
2^{-12}	4.2037e-01	5.4338e-01	2.5178e+00	4.7300e+00	3.9406e+00
2^{-14}	3.9782e-01	4.8465e-01	5.5132e-01	2.5290e+00	5.1580e+00
2^{-16}	3.9240e-01	4.7115e-01	5.1856e-01	5.5508e-01	2.5342e+00
2^{-18}	3.9106e-01	4.6784e-01	5.1090e-01	5.3592e-01	5.5691e-01
2^{-20}	3.9073e-01	4.6702e-01	5.0901e-01	5.3153e-01	5.4470e-01

Discussion

The error analyses we performed in this work shows that the proposed method is first-order uniformly convergent. We computed the maximum absolute errors and rates of convergence for different values of mesh size N and perturbation parameter ε for the given examples in Tables 4.1, 4.2 and 4.5. In these Tables for each value of N as $\varepsilon \to 0$, the maximum absolute error becomes uniform and bounded this shows that the convergence of the scheme is independent of perturbation parameter. In Tables 4.2,4.4 and 4.6 we compared the results obtained from the proposed scheme for Example 1,2 and 3 with finite difference scheme without the fitting parameter. In Table 4.7 we also compared the result of the proposed scheme with the results in Chen et al. (2015) for Example 4. As one can observes from these tables the results obtained from the proposed scheme is stable accurate and an ε uniformly convergent.

Conclusion and Recommendation

Conclusion

In this study we developed exponentially fitted finite difference method to solve second order linear SPFIDE. We applied a central finite difference method for the differential part and Simpson's 1/3 for the integral part. An exponential fitting parameter is established based on the derivation from the asymptotic solution to hinder the influence of the singular perturbation on the discrete solution. Stability and convergence analysis of the proposed scheme is proved and the effect of perturbation parameter on the solution of the problem are shown by using figures. The method is shown to be uniformly convergent with order of convergence one. The results obtained from the proposed scheme are compered with standard finite difference scheme. It has been shown that the proposed method is stable and uniformly convergent.

Recommendation

In this thesis we focused only on exponentially fitted finite difference scheme for the solution of second order linear SPFIDEs. It is very interesting and it gives stable and accurate result. Hence we recommended that, further researchers can extend it in to finite element method with non linear problems.

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