THE ANALYSIS OF PREDATOR AND PREY SYSTEM BY CONSIDERING THE INTERFERENCE OF TOP PREDATOR IN A POLLUTED ENVIRONMENT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER'S IN MATHEMATICS

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By: Kiniso Abdi

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Dedication

To my Father.

STATEMENT OF THE AUTHOR

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Abstract

The modeling investigation in this research discusses the system level effects of a toxicant on a three species food chain system. The model has been studied to visualize the dynamical behaviour of prey and predator populations under toxicant stress. All the feasible equilibria of the systems are obtained and using stability theory the conditions are derived for the survival or extinction of species. In the model, we have assumed that the presence of top predator reduces the predatory ability of the intermediate predator. The stability analysis of the model is carried out and the sufficient conditions for the existence and extinction of the populations under the stress of toxicant are obtained. From the analysis of the model it is observed that the effect of toxicant on predator population increases the equilibrium density of prey population. Further, it is also found that the predation rate of the intermediate predator is a bifurcating parameter and Hopf-bifurcation occurs at some critical value of this parameter.

Finally, numerical simulation is carried out to support the analytical results. We support our analytical findings with numerical simulations.

Keywords: Stability, Bifurcation, Interference, Lyapunov function.

Chapter 1

Introduction

1.1 Background of the study

Species are regularly exposed to many natural and synthetic chemicals which are adversely affecting their growth rate directly or indirectly. The direct effects of toxicant on the species are alterations in their mortality and reproductive rates. The indirect effects are observed either through the food chain or through the reduction in the carrying capacity of the environment due to the degradation of the habitat. It is generally observed in nature that the toxicants decrease the growth rate of species and also their carrying capacity. The presence of toxicants in the environment affects not only the species but their resources also. These toxicants have very pronounced effects on the species if the availability of the resource is limited. There are many instances where the toxicants have been the main cause of extinction of many species and depletion of resources such as forestry, fertile crop and wild life.

Ecologists and mathematicians have often used food chain systems to describe the feeding relationships between species with in ecosystems and there has been considerable interest in the prey-predator model, especially for systems of three species Hastings and Powell(1991). However, the ecological communities in nature are observed to exhibit very complex dynamical behaviors and three species continuous time models are reported to have more complicated patterns.

In a food- chain system with prey-predator relationship, it is observed that the predator interference occur naturally in the presence of top predator on intermediate predator. Mathematical studies related to the effect of interference on the dynamics of prey-predator population have been carried out by several researchers. For examples see and references there in Lv and Zhao(2008).

The food-limited population model incorporated the concept of limited food and space. It was formulated by modifying the logistic growth equation considering the average growth rate to be a non-linear function of population density. The food-limited population models have been proposed by several researchers Sun and Loreau(2009) for the dynamics of a population where growth limitations were based on the proportion of available resources not utilized. Some studies about food-limited population models have been carried out by few authors Gomes et.al(2008), who have obtained an interesting results about the stability and Hopf bifurcation of positive solutions.

Three species food chain systems have received much attention from many applied mathematicians and ecologists in recent years D.M.Xiao(2001). In Zhao et.al(2011) studied and established an experimental marine food chain of three levels (microalgae $\rightarrow$ zooplankton $\rightarrow$ fish) to investigate the effect of feeding selectivity on the transfer of methyl mercury (MeHg) through the food chain system. Fish-eating birds in certain parts of the United States may ingest large amounts of methylmercury in their diet. The methylmercury-containing bacteria may be consumed by the next higher level in the food chain, or the bacteria may excrete the methylmercury to the water where it can quickly absorb to plankton, which are also consumed by the next level in the food chain, Dube and Hussain(2006). Because animals accumulate methylmercury faster than they eliminate it, animals consume higher concentrations of mercury at each successive level of the food chain. Small environmental concentrations of methylmercury can thus readily accumulate to potentially harmful concentrations in fish, fish-eating wildlife and people. Even at very low atmospheric deposition rates in locations remote from point sources, mercury biomagnification can result in toxic effects in consumers at the top of these aquatic food chains. Poisoning from pesticides can travel up the food chain Turner and Tester(1997); for example, birds can be harmed when they eat insects and worms that have consumed pesticides. A number of studies have shown that pesticides have had harmful effects on growth and reproduction of earthworms, which are in turn consumed by terrestrial vertebrates such as birds and small mammals.

Previously, some researches have been done on tri-trophic food-chain systems including toxicant effects on the survival or extinction of species in the system, Misra and Babu.A(2015). It has been observed that toxicants have very pronounced effects on the species if the availability of the resources is limited. Up to my knowledge, we have been conducted to investigate the effect of toxicant on top predator in food chain systems with "food-limited". Therefore, in this thesis, a mathematical model is proposed to study the effects of toxicants on a top predator in food-chain system with "food-limited". The present model may be suited for the food chain system comprising of Spider $\rightarrow$ Mouse $\rightarrow$ Snake and also for the food chain system consisting of Spider $\rightarrow$ Lizard $\rightarrow$ Hawk

In this thesis therefore we have studied the dynamical behaviour of a three species food-chain by the interference of top predator in a polluted environment and prey interference by intermediate predator using mathematical model.

1.2 Statement of the problem

Global dynamics of the prey-predator model exhibits the well known but highly controversial paradox of enrichment. The controversial part is, of course, not the mathematical analysis of the models; rather, it is the mathematical model itself which we used to describe the interaction between the prey and predator populations.

The problem of prey-predator mathematical model which describe the interactions between the prey and predator population under the interference of polluted environment has been studied by different authors. Mc Nair(1987) showed that several kinds of toxicant could exert a local destabilizing effort and create stable, large amplitude oscillations which would damp out if no toxicant was present and also obtained that polluted environment with legitimate entry existing dynamics was quite capable of amplifying rather than damping prey-predator oscillation.

Dubey et. al(2003) examined how flexible prey behaviour and imperfect knowledge of predation risk affect the stability of prey-predator population dynamics. Kuang and Beretta(1998) also considered a prey-predator system where individuals from a prey fish population could hide in holes where intermediate predators were unable to enter and top predators were attacked by toxicant. Sun and Loreau(2009) considered the dynamics of a prey-predator system with seasonal effects on additional food. Kuang and Beretta(1998) studied the global dynamics of a prey-predator model incorporating a prey refuge. The former researcher tried to show the positive equilibrium point is globally stable.

Hence in this research work, we tried to develop a mathematical model which describes the dynamics of the interaction between the prey and predator population using the relation of Hassel(1974) constant and the magnitude of interference among predators between the prey and predator population under the interference of polluted environment on the global dynamics of prey-predator model and which has actually a significant influencing factors on global dynamics of a prey-predator model. We again studied the positivity and boundedness of the solution and the stability

properties of positive interior equilibrium point and permanence of the basic mathematical model and studied the bifurcation analysis of the system.
This research address the following basic problems:

1. Formulation of a mathematical model that describes the dynamics of food-chain system in the case of top predator in a polluted environment.

2. Investigating the effect of toxicant on the dynamic interaction between the prey and predator population.

3. Interpretation of the mathematical model describing the dynamics of the interaction between the prey and predator in a polluted environment.

1.3 Objective of the Study

1.3.1 General objective

The general objective of this research work is to investigate the effect of polluted environment on the global dynamics of prey-predator interaction with top predator in the toxicant environment.

1.3.2 Specific objectives

The specific objectives of the study are to:

a. Develop a mathematical model which describes the dynamics of the interaction between the prey and predator population considering interference of top predator in a polluted environment.

b. Study local and global stability of the system.

c. Examine the ecological implication of the solution.

1.4 Significance of the study

1. To develop awareness on the analysis of prey-predator system by considering the interference of top predator in a polluted environment.

2. To suggest possible intervention mechanisms on how to sustain coexistence with top predator population.

3. Improve our understanding of the dynamics of the interaction between the prey and predator population considering interference of top predator in a polluted environment.

1.5 Expected outcomes

The expected outcomes of this research are:

a. Propose the mechanism that minimizes the effect of toxicant.

b. Strategies that protect the species in the polluted environment.

c. Indicate the effect of toxicant on top predator and intermediate consumer in the food chain system.

1.6 Limitation of the study

The following problems were limited me /researcher to do a research as needed.

- ⋆ Time constraint.
- ⋆ Shortage of material/references.
- ⋆ Lack of internet services in our working place.
- ⋆ Shortage of net-work to communicate my advisor to get a constructive idea from him with the difficulties.
- ⋆ Shortage on review of literature concepts.
- ⋆ Lack of skilled researchers, experience share.

1.7 Research Methodology

In this study we developed a mathematical model which describes the dynamics of food-chain system considering interference of top predator in a polluted environment using a system of non-linear differential equations is developed. We performed the qualitative analysis of the model. Both local and global stability analysis of the equilibrium points of the model equations would be done using variational matrix and Lyapunove functions respectively. The analytical solutions of the model equations was supplemented by numerical simulations by choosing appropriate system parameter using Matlab.

Chapter 2

LITERATURE REVIEW

In the early eighties a deterministic modelling approach to the problem of assessing the effects of a pollutant on an ecological system was proposed by Hallam and Luna(1984). Since then, such models have been the subject of many investigations and improvements. Population-toxicant coupling has been applied in several contexts, including Lotka-Volterra and chemostat-like environments, resulting in ordinary, integro-differential and stochastic models of Springer-Verlag and Heidelberg(2000). Usually a qualitative analysis was performed which focuses on the survival or extinction of populations. All these studies rely on the hypothesis of a complete spatially homogeneous environment.

A first attempt to consider a spatial structure has been carried out in Sherratt et. al(1995) where a reaction-diffusion model is proposed to describe the dynamics of a living population interacting with a toxicant present in the environment(external toxicant) through the amount of toxicant stored into the bodies of the living organisms (internal toxicant). However, as the authors pointed out, even if the resulting model presents many features which make stimulating its study, such a modelling approach is a rough approximation to the biological phenomena at hand. For the last one century, different people have developed many mathematical models for the dynamical study of food-chain system considering interference of top predator in a polluted environment. In this section, we review the ones that are directly related to the objectives of this study. Lotka in the mid 1920s for prey-predator interactions, mutualist and competitive mechanisms have been studied extensively in the recent years by researchers.

Simple prey predator food chain models often display rich nonlinear mathematical behavior, including varying numbers and stability of equilibrium states and limit cycles. A number of studies have been carried out in recent years to study the dynamics of three trophic-level food chain comprising of prey, specialist predator and generalist predators, using mathematical models based on Aziz-Alaoui(2002). It is observed in the marine ecosystem that the top predator is generalist predator in the food chain system consisting of phytoplankton$\rightarrow$ zooplankton$\rightarrow$ molluscs. Generally, the effects of toxicants are to decrease the growth rate of biological species as well as its carrying capacity. Several investigators have studied the effects of toxicants on biological species using mathematical models Misra and Raveendra Babu(2016).

Previously, some research work has been carried out on tri-trophic food-chain systems including toxicant effects on the survival or extinction of species using mathematical models. Misra and Raveendra Babu(2014) have studied the effects of toxicant in a three species food chain system with "food-limited" growth of prey population. In the model, the authors have assumed that the carrying capacity and growth rate of prey is affected by environmental toxicant which is being transferred to intermediate and top predator populations through food chain pathways. All the feasible equilibria of the system are obtained and the criteria for the survival or extinction of species under the effect of toxicant are derived. In Misra and Raveendra Babu(2014) the authors have considered a three species food–chain with Holling type-II functional responses and discussed the system level effects of a toxicant. They have assumed that the presence of top predator reduces the predatory ability of the intermediate predator and also the growth equation for the prey population in the absence of predator is assumed to be governed by a modified smith-type differential equation. The stability analyses of the models are carried out and the sufficient conditions for the existence and extinction of the populations under the stress of toxicant are obtained. Hallam and Luna(1984) have proposed a mathematical model of a toxicant-population interaction and investigated their behaviour.

This modelling investigation discusses system level effects of a toxicant on a population when exposure is via both environmental and food chain pathways. The main analytical results yield conditions sufficient for persistence and conditions sufficient for extinction of the toxicant stressed population. The model in the studied has demonstrated the increase in concentration of potentially toxic substances, such as cadmium, as one proceeds up the food chain. The food chain model also illustrated how interactive modelling between complex non-linear and linear compartment model can be accomplished. The authors have observed that when the toxicant (cadmium) is released into the water environment; the trophic level structure of the model indicates the general behaviour.

Thomann and Connolly(1984) have studied the age dependent food–chain species system under the exposure of PCB (poly clorinated biphenyl) concentration in the Lake Michigan, and their model was capable of reproducing the age–dependent trends and magnitude of PCB contamination observed in Lake Michigan alewife and lake trout in 1971. In the model, the underlying food chain system consists of a prey population, an intermediate predator population and a top predator population with Holling type-II functional responses. It is assumed in the model that the presence of the top predator reduces the predatory ability of the intermediate predator Aziz-Alaoui(2002). In the model, the growth equation for the prey population in the absence of predator is assumed to be governed by a modified Smith-type differential equation, Hallam and Luna(1984).

The state variables of the model are x(t), the density of the prey population; y(t), the density of the intermediate predator population; z(t), the density of the top predator population; $C_o(t)$, the organism toxicant concentration in the prey population; and $C_e(t)$, the environmental toxicant concentration.

In the model, it is assumed that the growth rate and the carrying capacity of prey are negatively affected by environmental toxicant Raveendra Babu(2016). Toxicant is transferred to intermediate predator and top predator populations through food chain pathways. For the prey population, a simple “food-limited” growth equation

Zhang et. al(2013), is considered. Lotka-Volterra type of prey-predator interaction is considered in the model. The model is formulated with the help of following system of ordinary differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= xr(U)\left(\frac{K(T)-x}{K(T)+r_o cx}\right) - a_1xy \\
\frac{dy}{dt} &= \beta_1 a_1 xy - a_2yz - \beta_{11}Vy - d_1y - b_1y^2 \\
\frac{dz}{dt} &= \beta_2 a_2 yz - \beta_{22}Wz - d_2z - c_3z^2 \\
\frac{dT}{dt} &= Q_0 - \delta_0 T - \alpha_1 xT \\
\frac{dU}{dt} &= \alpha_1 xT - \delta_1 U - \beta_3(U)a_1xy \\
\frac{dV}{dt} &= \beta_3(U)a_1xy - \delta_2 V - \beta_4(V)a_2yz \\
\frac{dW}{dt} &= \beta_4(V)a_2yz - \delta_3 W
\end{aligned} \tag{2.1}$$

The above systems of ordinary differential equations are associated with the following initial conditions:
$x(0) > 0, y(0) > 0, z(0) > 0, T(0) > 0, U(0) \geq 0, V(0) \geq 0, W(0) \geq 0.$
In the model, x is the density of prey population, y is the density of intermediate predator population, z is the density of top predator population, T is the concentration of toxicant in the environment, U is the concentration of toxicant in the prey population, V is the concentration of toxicant in the intermediate predator population and W is the concentration of toxicant in the top predator population.The first term in the prey equation describes "food-limited" growth rate function under the effect of toxicant; d_1 and d_2 are the death rates of intermediate predator and top predator populations respectively, b_1 and c_3 are the intraspecific competition rates due to crowding of intermediate and top predator populations respectively. K(T) represents the carrying capacity of prey which is negatively affected by T , the function r(U) denotes the specific growth rate of prey population which is negatively affected by U , β_1 and β_2 are conversion coefficients, β_3(U) and β_4(V) are toxicant transfer functions. Q_0 is the rate of introduction of toxicant into the environment. δ_0, δ_1, δ_2 and δ_3 are first order decay rates of toxicants in the environment as well as in the populations. β_{11} and β_{22} are the death rates of predators due to organismal toxicant concentration. k_0 is the natural carrying capacity. Food limited parameter

is denoted by c and r_0 is the intrinsic growth rate of prey. k_1 and r_1 are the constants which determine the rate of decrease of carrying capacity and growth rate of prey population respectively due to the presence of toxicant. α_1 is the depletion rate of toxicant in the environment due to its intake by the population. a_1 and a_2 are the loss of prey and intermediate predator due to intermediate and top predators. The constants c, a_1, a_2, a_3, a_4 and α_1 are positive.

While many of the mathematicians working in mathematical biology may regard these as important contributions that mathematics had for ecology, they are very controversial among ecologists up to this day.

The researcher have been conducted the effect of toxicant with three species in food chain system when prey population exposed to polluted environment .But did not conducted the effect of toxicant on prey population and the intermediate predator population if top predator population have affected with toxicant. Therefore, we have tried to conduct the effects of toxicant on prey population and the intermediate predator population in food chain system with food-limited.

Chapter 3

Model Formulation and Analysis

3.1 Mathematical Model

The model formulation has been carried out in light of the research papers by Misra and Raveendra Babu(2016). In the model, the underlying food chain system consists of a prey population, an intermediate predator population, and a top predator population, toxicant concentration in the top predator population and the environmental toxicant concentration with Holling type-II functional responses. It is assumed in the model that the absence of the top predator increases the predatory ability of the intermediate predator, Aziz-Alaoui(2002).In the model, the growth equation for the prey population in the absence of predator is assumed to be governed by a modified smith-type differential equation, Hallam and Luna(1984).The state variables of the model are x(t) is the density of the prey population; y(t) is the density of the intermediate predator population; z(t) is the density of the top predator population; c_o(t) is the organism toxicant concentration in the top-predator population; and c_e(t) is the environmental toxicant concentration.

Taking these as state variables, The mathematical model is formulated using a system of nonlinear ordinary differential equations, as under. **Model**

A

$$
\begin{aligned}
\frac{dx}{dt} &= r_0x(1-\frac{x}{k})-(\frac{P_1x}{Q_1+x})y & (3.1)\\
\frac{dy}{dt} &= \alpha_1(\frac{P_1x}{Q_1+x})y-(\frac{P_2y}{Q_2+y})z-d_1y-d_2y^2 & (3.2)\\
\frac{dz}{dt} &= \alpha_2(c_o)(\frac{P_2y}{Q_2+y})\frac{B(c+r-a)-acz}{B(c+r-a)+az}-d_3z & (3.3)\\
\frac{dc_0}{dt} &= mc_e+\frac{h}{m}\theta\beta-(l_1+l_2)c_o & (3.4)\\
\frac{dc_e}{dt} &= s_0+k_1l_1c_oz-k_1mc_ez-k_2c_e & (3.5)
\end{aligned}
$$

The initial conditions are: $x(0) = x_0 > 0; y(0) = y_0 > 0; z(0) = z_0 > 0; c_o(0) = 0; c_e(0) = c_e > 0$: In Model A; k and B are the population carrying capacity of prey and top populations respectively; r_0 and r are the intrinsic growth rate of the prey population top populations respectively; a is a measure of the population response to stress effects; d_1 and d_3 are the death rates of y and z respectively; d_2 is intra-specific competition because of over crowding of intermediate population. We assume that the toxicant concentration θ in the population is a constant; β is the average rate of food intake per unit organismal mass; l_1 and l_2 are egestion and depuration rates respectively; $k_1l_1c_o$z is the total toxicant ingested; k_1mc_ez is the total toxicant uptake from the environment; k_2c_e is the term which describes the loss due to detoxifying process such as hydrolysis, volatilization, etc.; The exogenous input to the body burden c_o is assumed to be from the environment at a rate proportional to the environmental concentration: mc_e, where m is the population rate of toxicant uptake per unit mass; s_0 represents exogenous input of toxicant into the environment; h is a constant numerically less than or equal to the numerical value of m; c is the rate of replacement of mass in the top population at saturation.

In the model$\frac{P_iu}{(Q_i+u)}$, $(i = 1, 2; u = x \text{ and } y)$, account for the interactions between two different species, representing the Holling type-II functional response. This functional response is parameterized by the constants P_i *and* $Q_i(i = 1, 2)$ and we verify that Q_i is the value of the prey population level when the predation rate per unit prey is half their maximum value Aziz-Alaoui(2002). Exposure to toxicant may lead to changes in fecundity and mortality rates of a population. This stress can be modeled by assuming that the growth rate of the population is a function of the body burden $\alpha_2(c_0) = r_1 - H(c_0)$. Here H is a non-decreasing function of c_0 with $H(0) = 0$ and r_1 is the constant which determine the rate of decrease of growth rate of top predator population due to the presence of toxicant. H (c_0) is a dose-response function, which is assumed to be linear and taken as $H(c_0) = r_1c_0$;α_1 and α_2 are the biomass conversion rate of predators population.

We can reduce the number of parameters in the above system by the following scaling transformations, but, for our analytical and numerical tests, we have continue to used the original system:

$x \to \frac{x}{Q_1}, y \to \frac{P_1y}{Q_1}, z \to \frac{P_1z}{P_2Q_1}, c_o \to \frac{r_1c_o}{k_1}, c_e \to \frac{r_1c_e}{m}, t \to d_1t$

Thus,the above system after re-scaling becomes as

Model B:

$$\frac{dx}{dt} = xe_1e_2(\frac{o_5 - x}{e_4 + x}) - \frac{u_1xy}{(1 + e_5z)(1 + x)} \tag{3.6}$$

$$\frac{dy}{dt} = \alpha_3\frac{o_6xy}{(1 + e_5z)(1 + x)} - \frac{u_2yz}{e_8 + y} - y \tag{3.7}$$

$$\frac{dz}{dt} = \alpha_4\frac{r_1o_7yz}{e_8 + y}(e_0(c_o)e_2(e_3 - z)) - u_3z \tag{3.8}$$

$$\frac{dc_o}{dt} = o_2c_e + o_3 - u_4c_o \tag{3.9}$$

$$\frac{dc_e}{dt} = u_0 + o_4c_oz - u_5c_ez - u_6c_e \tag{3.10}$$

The initial conditions are

$x(0) = x_0 > 0; y(0) = y_0 > 0; z(0) = z_0 > 0; c_o(0) = 0; c_e(0) = c_e0 > 0 :$

Here, $e_1 = \frac{c}{d_1}, e_2 = \frac{T_0r_0r_1}{ack_1Q_1}, e_3 = \frac{k_1}{r_1}, e_4 = \frac{T_0}{aQ_1}, e_5 = \frac{P_1}{P_2Q_1}, e_8 = \frac{P_1Q_2}{Q_1}, u_0 = \frac{s_0m}{r_0d_1},$

$u_1 = \frac{1}{d_1Q_1^2}, u_2 = \frac{1}{d_1}, u_3 = \frac{d_1}{d_2}, u_4 = \frac{(l_1+l_2)}{d_1}, u_5 = \frac{mk_1Q_1}{d_1}, u_6 = \frac{k_2}{d_1}, o_1 = \frac{\alpha_1k_1}{\alpha_2r_0}, o_2 = \frac{k_1}{d_1},$

$o_3 = \frac{k_1h\theta\beta}{r_0md_1}, o_4 = \frac{l_1mQ_1}{d_1}, o_6 = \frac{\alpha_1}{P_1d_1}, o_7 = \frac{P_2\alpha_2}{d_1},$

$T_0 = B(c + r - a); e_0(c_o) = e_3c_o; o_5 = e_2e_3; o_6 = e_6e_7; o_7 = e_9o_1$

All these parameters, of course, assume only positive values.

3.2 Boundedness of the Model:

To analyze Model B, in this section we need the bounds of dependent variables involved. First we see the boundedness of Model B. So here, we find the region of attraction for the Model in the following lemma.

Lemma 1. The set

$\Omega = \{(x, y, z, c_o, c_e)\} \in \Re^5_+ : 0 \leq x(t) \leq o_5, 0 \leq o_6 x(t) + u_1 y(t) + u_1 u_2 o_7 z(t) \leq \theta_1,$

$0 \leq c_o(t) + c_e(t) \leq \theta_2, o_6 x(t) + u_1 y(t) + \frac{u_1 u_2}{o_7} z(t) + c_e(t) \geq \theta_3\}$

is a region of attraction for all solutions initiating in the interior of the positive region (Referred from Lv and Zhao 2008), where

$\theta_1 = \frac{(1+e_1 o_5) o_5 o_6}{\Phi_1}, \theta_2 = \frac{u_0}{\Phi_2}, \theta_3 = \frac{u_0}{\Phi_3}, \Phi_1 = min\{1, 1, u_3\},$

$\Phi_2 = min\{u_4 - o_4 o_5, u_6 - o_2\}, \Phi_3 = max\{u_5 \theta_2, 1, u_3, u_6\}.$

Proof.

From equation (3.6) we obtain $\frac{dx}{dt} \leq x e_1 (o_5 - x)$:then by the usual comparison theorem, we get

$$x \leq o_5$$

Now, let us consider the following function

$\psi_1(t) = o_6 x(t) + u_1 y(t) + \frac{u_1 u_2}{o_7} z(t)$:

By using equations (3.6)-(3.10), we conclude

$\frac{d\psi_1(t)}{dt} + \Phi_1 \psi_1(t) \leq (1 + e_1 o_5) o_5 o_6$; Where $\Phi_1 = min\{1, 1, u_3\}$ and then by the usual comparison theorem, we get $\psi_1(t) \leq (1 + e_1 o_5) o_5 o_6$ as t$\rightarrow \infty$, and hence,

$o_6 x(t) + u_1 y(t) + \frac{u_1 u_2}{o_7} z(t) \leq \theta_1$;where $\theta_1 = (1 + e_1 o_5) \frac{o_5 o_6}{\Phi_1}$:

Now, let us consider the function:$\psi_2(t) = c_o(t) + c_e(t)$,by using equations (3.9) - (3.10), we get $\frac{d\psi_2(t)}{dt} + \Phi_2 \phi_2 \leq u_0$;

where $\Phi_2 = min\{u_4 - o_4 o_5, u_6 - o_2\}$ and then by the usual comparison theorem, we

get as $t \to \infty$,$\psi_2(t) \leq \frac{u_0}{\phi_2}$ and hence, $c_o(t) + c_e(t) \leq \theta_2$;
where $\theta_2 = \frac{u_0}{\Phi_2}$.Again let as consider the function $\psi_3(t) = o_6x(t) + u_1y(t) + \frac{u_1u_2}{o_7}z(t) + c_e(t)$:
By using the equation (3.2)-(3.6), we get $\frac{d\psi_3(t)}{dt} + \Phi_3\psi_3(t) \geq u_0$,where $\Phi_3 = max\{u_5\theta_2, 1, u_3, u_6\}$
and then by the usual comparison theorem, we get as $t \to \infty$,$\psi_3(t) \geq \frac{u_0}{\Phi_3}$ and hence

$o_6x(t) + u_1y(t) + \frac{u_1u_2}{o_7}z(t) + c_e(t) \geq \theta_3$; where $\theta_3 = \frac{u_0}{\Phi_3}$,
Hence the solution of the Models B is bounded in Ω.

3.3 Analysis of Model B

3.3.1 Equilibria of Model B:

Model B has following four non-negative equilibria in x, y, z, c_o and c_e space namely,$E_{10}(0,0,0,0,0)$, $\tilde{E_{11}}(\tilde{x}, 0, 0, \tilde{c_o}, \tilde{c_e})$,$\hat{E_{12}}(\hat{x}, \hat{y}, 0, \hat{c_o}, \hat{c_e})$ and $\bar{E_{13}}(\bar{x}, \bar{y}, \bar{z}, \bar{c_o}, \bar{c_e})$.
The existence of E_{10} is obvious. We prove the existence of $\tilde{E_{11}}$, $\hat{E_{12}}$ and $\bar{E_{13}}$ as follows:

1.Existence of $\tilde{E_{11}}(\tilde{x}, 0, 0, \tilde{c_o}, \tilde{c_e})$
from (3.6), $\tilde{x} = e_2e_3 > 0$,
from (3.9) and (3.10), $\tilde{c_e} = \frac{u_0u_1 + e_2e_3o_3o_4}{(e_2e_3u_5 + u_6)u_4 - o_2e_2e_3o_4}$,

$\tilde{c_e} > 0$ if $(e_2e_3u_5 + u_6)u_4 > o_2e_2e_3o_4$,

$\tilde{c_o} = \frac{o_2}{u_4}(\frac{u_0u_4 + e_2e_3o_4}{(e_2e_3u_5 + u_6)u_4 - o_2e_2e_3o_4}) + \frac{o_3}{u_4} > 0$.

2.Existence of $\hat{E_{12}}(\hat{x}, \hat{y}, 0, \hat{c_o}, \hat{c_e})$

From (3.7),$\hat{x} = \frac{1}{o_6 - 1}$,$\hat{x} > 0$ if $o_6 > 1$.

From (3.9) and (3.10), $\hat{c_e} = \frac{u_0u_4 + \hat{x}o_3o_4}{(\hat{x}u_5 + u_6)u_4 - o_2o_4\hat{x}}$,$\hat{c_e} > 0$ if $(\hat{x}u_5 + u_6)u_4 > o_2o_4\hat{x}$.

From (3.9), $\hat{c_o} = \frac{o_2}{u_4}(\frac{u_0u_4+\hat{x}o_3o_4}{(\hat{x}u_5+u_6)u_4-o_2o_4\hat{x}}) + \frac{o_3}{u_4} > 0$,$c_o > 0$ if $(\hat{x}u_5 + u_6)u_4 > o_2o_4\hat{x}$.

From (3.6), $\hat{y} = (1+\hat{x})\frac{e_1}{u_1}(\frac{e_2(e_3-\hat{c_o})-\hat{x}}{e_4+\hat{x}})$, $\hat{y} > 0$ if $e_3 > \hat{c_o}$ and $e_2(e_3-\hat{c_o}) > \hat{x}$.

3. Existence of $\bar{E_{13}}(\bar{x}, \bar{y}, \bar{z}, \bar{c_o}, \bar{c_e})$.

Now, we show the existence of $\bar{E_{13}}$ as follows:

Here $\bar{x}$,$\bar{y}$,$\bar{z}$,$\bar{c_o}$ and $\bar{c_e}$ are the positive solutions of the system of algebraic equations given below:

From (3.7), $\bar{y} = \frac{u_3e_8}{o_7-u_3}$,$\bar{y} > 0$ if $o_7 > u_3$.

From (3.6), $\bar{x} = \frac{(1+e_5z)(1+\frac{u_2z}{e_8+\bar{y}})}{o_6-(1+e_5z)(1+\frac{u_2z}{e_8+\bar{y}})} = h_1(z)$.

From (3.8), $\bar{c_o}e_3 - \frac{1}{e_2}(h_1(z) + \frac{u_1\bar{y}(e_4+h_1(z))}{e_1(1+h_1(z))(1+e_5z)}) = h_2(z)$.

From (3.9), $\bar{c_e} = \frac{1}{o_2}[u_4h_2(z) - o_3] = h_3(z)$.

Let

$$Q(z) = u_0 + o_4h_1(z)h_2(z) - (u_6 + u_5h_1(z))h_3(z). \tag{3.11}$$

To show the existence of $\bar{E_{13}}$,it suffices to show that equation (3.11) has a unique positive solution for this we may note that

$$Q(0) = u_0 + o_4h_1(0)h_2(0) - (u_6 + u_5h_1(0))h_3(0) > 0, \tag{3.12}$$
$$Q(k_0) = u_0 + o_4h_1(k_0)h_2(k_0) - (u_6 + u_5h_1(k_0))h_3(k_0) \tag{3.13}$$

This guarantees the existence of a root of for $Q(z) = 0$ for $0 < z < k_0$, say $\bar{z}$. Further, this root will be unique provided

$Q(z)' = o_4[h_1(z)h_2'(z) + h_1'(z)h_2(z)] - [(u_6 + u_5h_1(z))h_3'(z) + u_5h_1'(z)h_3(z)] < 0.$

Knowing the value of $\bar{z}$, the values of $\bar{x}$, $\bar{c_o}$ and $\bar{c_e}$ can be computed from above equations respectively.

3.3.2 Dynamical behaviour of Model B:

The general variational matrix corresponding to the Model B is as follows:

$$J(x,y,z,c_o,c_e)=\begin{bmatrix}-a_{11} & -a_{12} & a_{13} & -a_{14} & 0\\ a_{21} & a_{22} & -a_{23} & a_{24} & 0\\ 0 & a_{32} & a_{33} & a_{34} & 0\\ 0 & 0 & 0 & -a_{44} & a_{45}\\ -a_{51} & 0 & 0 & a_{54} & -a_{55}\end{bmatrix}$$

Where

$a_{11}=\frac{u_1 y}{(1+x)^2(1+e_5 z)}+\frac{e_1 x}{(e_4+x)^2}(e_2 e_0(c_o)+e_4)+e_1(\frac{r_1(c_o)e_2-x}{e_4+x})$,

$a_{12}=\frac{u_1 x}{(1+x)(1+e_5 z)}$,$a_{13}=\frac{u_1 e_5 xy}{(1+x)^2(1+e_5 z)^2}$,$a_{14}=\frac{xe_1 e_2}{e_4+x}$,

$a_{21}=\frac{o_6 y}{(1+x)^2(1+e_5 z)}$,$a_{22}=\frac{o_6 y}{(1+x)^2(1+e_5 z)}-\frac{e_8 u_2 z}{(e_8+y)^2}-1$,

$a_{23}=\frac{u_2 y}{e_8+y}$,$a_{32}=\frac{o_7 e_8 z}{(e_8+y)^2}$,$a_{33}=\frac{o_7 y}{e_8+y}-u_3$,$a_{44}=u_4$,$a_{45}=o_2$,

$a_{51}=(u_5 c_e-o_4 c_o)$,$a_{54}=o_4 x$,$a_{55}=u_5 x+u_6$.

1. At $\tilde{E_{10}}$,the eigenvalues of the characteristic equation are $e_1e_2e_3$,-1,$-u_3$,$-u_4$ and $-u_6$ showing the instability of $\tilde{E_{10}}$.

2. At $\tilde{E_{11}}$,two eigenvalues of the characteristic equation are $\frac{o_6\tilde{x}}{1+\tilde{x}}-1$,$-u_3$ and the other three eigenvalues are given by the roots of the following cubic equation

$$\lambda^3+\lambda^2 W_1+\lambda W_2+W_3=0, \tag{3.14}$$

Where,

$W_1=u_4+u_5\tilde{x}+u_6+\frac{\tilde{x}e_1(e_2e_0(\tilde{c_o})+e_4)}{(e_4+\tilde{x})^2}$,

$W_2=\frac{\tilde{x}e_1(e_2e_0(\tilde{c_o})+e_4)}{(e_4+\tilde{x})^2}(u_4+u_5\tilde{x}+u_6)+u_4(u_5\tilde{x}+u_6)-o_2o_4\tilde{x}$,

$W_3=u_4(u_5\tilde{x}+u_6)\frac{\tilde{x}e_1(e_2e_0(\tilde{c_o})+e_4)}{(e_4+\tilde{x})^2}-o_2((u_5\tilde{c_e}-o_4\tilde{c_o})\frac{\tilde{x}e_1e_2}{e_4+\tilde{x}}+o_2\tilde{x}^2\frac{e_1(e_2e_0(\tilde{c_o})+e_4)}{(e_4+\tilde{x}})^2)$.

According to Routh-Hurwitz criteria $\tilde{E_{11}}$ is locally asymptotically stable if $o_6e_2e_3<(1+e_2e_3)$,$o_2o_4<u_2u_5$, $W_1W_2-W_3>0$.

3.At $\hat{E_{12}}$,one of the eigenvalues of the characteristic equation is $\frac{o_7\hat{y}}{e_8+\hat{y}} - u_3$ and the other four eigenvalues are given by the roots of the following equation is

$$\lambda^4 + \lambda^3 G_1 + \lambda^2 G_2 + \lambda G_3 + G_4 = 0, \tag{3.15}$$

Where,

$G_1 = u_4 + u_5\hat{x} + u_6 + \frac{e_1\hat{x}}{(e_4+\hat{x})^2}(e_2e_0(\hat{c_o}) + e_4) - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2}$,

$G_2 = \frac{u_1o_6\hat{x}\hat{y}}{(1+\hat{x})^3}(u_4 + u_5\hat{x} + u_6)(\frac{e_1\hat{x}(e_2e_0(\hat{c_o})+e_4)}{(e_4+\hat{x})^2} - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2}) + u_4(u_5\hat{x} + u_6) - o_2o_4\hat{x}$,

$G_3 = (u_4 + u_5\hat{x} + u_6)\frac{u_1o_6\hat{x}\hat{y}}{(1+\hat{x})^3} + u_4(u_5\hat{x} + u_6)(\frac{e_1\hat{x}}{(e_4+\hat{x})^2}(e_2e_0(\hat{c_o}) + e_4) - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2})$

$-o_2[(u_5\hat{c_e} - o_4\hat{c_o})\frac{\hat{x}e_1e_2}{e_4+\hat{x}} + o_4\hat{x}(\frac{e_1\hat{x}}{(e_4+\hat{x})^2}(e_2e_0(\hat{c_o}) + e_4) - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2})]$,

$G_4 = (u_4(u_5\hat{x} + u_6) - o_2o_4\hat{x})\frac{u_1o_6\hat{x}\hat{y}}{(1+\hat{x})^3}$.

From the Routh-Hurwitz criteria it is found that $\hat{E_{12}}$ is locally asymptotically stable if the following conditions hold good.

$\hat{y} < \frac{u_3e_8}{o_7-u_3}$,$o_2o_4 < u_4u_5$,

$G_i > 0$,i=1,2,3,4, $G_1G_2 > G_3$ and $G_1G_2G_3 > (G_3^2 + G_1^2G_4)$.

4. The characteristic equation of $\bar{E_{13}}$ is as follows:

$$\lambda^5 + \lambda^4 N_1 + \lambda^3 N_2 + \lambda^2 N_3 + \lambda N_4 + N_5 = 0, \tag{3.16}$$

Where,

$N_1 = m_1 + m_9 + m_{13} - m_6$,

$N_2 = m_2m_5 - m_1m_6 + m_7m_8 + m_1m_9 - m_6m_9 - m_{10}m_{12} + m_1m_{13}$,

$N_3 = m_1m_7m_8 - m_3m_5m_8 + m_2m_5m_9 - m_1m_6m_9 + m_7m_8m_9$
$-m_4m_{10}m_{11} - m_1m_{10}m_{12} + m_6m_{10}m_{12} + m_2m_5m_{13} - m_1m_6m_{13}$
$+m_7m_8m_{13} + m_1m_9m_{13} - m_6m_9m_{13}$

$N_4 = m_1m_7m_8m_9 - m_3m_5m_8m_9 + m_4m_6m_{10}m_{11} - m_2m_5m_{10}m_{12} + m_1m_6m_{10}m_{12}$

$-m_7m_8m_{10}m_{12} - m_3m_5m_8m_{13} + m_1m_7m_8m_{13} + m_2m_5m_9m_{13} - m_1m_6m_9m_{13}$
$+m_7m_8m_9m_{13},$

$N_5 = m_3m_5m_8m_{10}m_{12} - m_4m_7m_8m_{10}m_{11} - m_1m_7m_8m_{10}m_{12}$
$-m_3m_5m_8m_9m_{13} + m_1m_7m_8m_9m_{13}.$

and

$m_1 = \frac{e_1\bar{x}}{(e_4+\bar{x})^2}(e_2e_0(\bar{c_o}) + e_4) - \frac{u_1\bar{x}\bar{y}}{(1+\bar{x})^2(1+e_5\bar{z})},$

$m_2 = \frac{u_1\bar{x}}{(1+\bar{x})(1+e_5\bar{z})}, m_3 = -\frac{u_1e_5\bar{x}\bar{y}}{(1+\bar{x})^2(1+e_5\bar{z})^2}, m_4 = \frac{\bar{x}e_1e_2}{e_4+\bar{x}},$

$m_5 = \frac{o_6\bar{y}}{(1+\bar{x})^2(1+e_5\bar{z})}, m_6 = \frac{u_2\bar{y}\bar{z}}{(e_8+\bar{y})^2}, m_7 = \frac{u_2\bar{y}}{e_8+\bar{y}},$

$m_8 = \frac{o_7e_8\bar{z}}{(e_8+\bar{y})^2}, m_9 = u_4, m_{10} = o_2, m_{11} = u_5\bar{c_e} - o_4\bar{c_o},$
$m_{12} = o_4\bar{x}, m_{13} = u_5\bar{x} + u_6.$

According to Routh-Hurwitz criteria, the equilibrium point $\bar{E_{13}}$ is locally asymptotically
stable if
$N_i > 0 (i = 1, 2, 3, 4, 5)$, $N_1N_2 > N_3$, $N_1N_2N_3 > (N_3^2 + N_1^2N_4)$ and
$(N_3N_4 - N_2N_5)(N_1N_2 - N_3) > (N_1N_4 - N_5)^2.$

It is difficult to interpret the results in ecological terms from these complicated expressions, however, numerical examples are taken and graphs are ploted to illustrate the dynamical behaviour of the system about equilibrium $\bar{E_{13}}$.

Again, in similar way the equilibrium population densities are functions of A_1 and the coefficients of the characteristic equation (3.16) are functions of the parameter A_1. Now we can use the notation $N_i = N_i(A_1)$ for i=1,2,3,4,5.Now noting that the quantities $N_i's$ are smooth functions of the parameter A_1.As we have explained the definition of Hopf-bifurcation in previous section. Without knowing eigenvalues, [13] proved that (referring the result to the current case): if $N_i(A_1)$,

$\Delta_1(A_1) = N_1(A_1)N_2(A_1) - N_3(A_1),$

$\Delta_2(A_1) = N_1(A_1)N_2(A_1)N_3(A_1) - (N_3^2(A_1) + N_1^2(A_1)N_4(A_1)),$

$\Delta_3(A_1) = [N_3(A_1)N_4(A_1) - N_2(A_1)N_5(A_1)][N_1(A_1)N_2(A_1) - N_3(A_1)]$
$-[N_1(A_1)N_4(A_1)N_5(A_1)]^2$ are smooth functions of the parameter $'A_1'$
in an open interval containing $\bar{A_1} \in \Re^+$ that following conditions hold:

(*i*) $N_1(\bar{A_1}) > 0$,$\Delta_1(\bar{A_1}) > 0$,$\Delta_2(\bar{A_1}) > 0$, and $\Delta_3(\bar{A_1}) = 0$;
(*ii*) $\frac{d\Delta_3(A_1)}{dA_1} |_(A_1 = \bar{A_1}) \neq 0$

then (*i*) and (*ii*) are mentioned in section 3.3.2, for the occurrence of a simple Hopf-bifurcation at $A_1 = \bar{A_1}$.

Hence, in the similar way, we can propose the following theorem:

Theorem 1. If a critical value $\bar{A_1}$ of parameter A_i be found such that $N_i(\bar{A_1}) > 0$,$\Delta_1(\bar{A_1}) > 0$,$\Delta_2(\bar{A_1}) > 0$,$\Delta_3(\bar{A_1}) = 0$ and further $\Delta_3' \neq 0$ (where primes denotes differentiation with respect to A_1) then system (1) undergoes Hopf-bifurcation around $\bar{E_{13}}$.

Theorem 2. Let the following inequalities hold in the region Ω:

$$\frac{u_1\bar{y}}{\sigma_{12}}(1 + e_5\theta_3) < \frac{e_1}{\sigma_{11}}(e_4 + e_2e_0(\bar{c_o})) \tag{3.17}$$

$$\frac{\theta_3 o_6}{\sigma_{12}}(1 + \bar{x})(1 + e_5\bar{z}) < (1 + \frac{u_2e_8\theta_1}{\sigma_{13}}) \tag{3.18}$$

$$o_7\theta_3(e_8 + \bar{y}) < u_2\sigma_{13} \tag{3.19}$$

$$6[\frac{u_1e_5}{\sigma_{11}}(1 + \bar{x})]^2 < p_{12}(u_2 - \frac{o_7\theta_3}{\sigma_{13}}(e_8 + \bar{y})) \tag{3.20}$$

$$6[\frac{e_1e_2}{\sigma_{11}}(e_8 + \bar{x})]^2 < p_{12}u_4 \tag{3.21}$$

$$6(u_5\bar{c_e} - o_4\theta_2)^2 < p_{12}(u_6 + \theta_1u_5) \tag{3.22}$$

$$2p_{11}^2 < p_{13}(u_2 - \frac{o_7\theta_3}{\sigma_{13}}(e_8 + \bar{y})) \tag{3.23}$$

$$4[o_2 + o_4\bar{x}]^2 < u_4(u_6 + \theta_1u_5) \tag{3.24}$$

$$Where,$$

$$K_1 > \frac{u_1(1 + \bar{x})}{o_6\bar{y}} \tag{3.25}$$

$p_{11} = K_1(\frac{e_5o_6\bar{x}\bar{y}}{\sigma_{12}}(1 + \theta_3) + \frac{u_2\bar{y}}{\sigma_{13}}(e_8 + \theta_3)) - \frac{o_7e_8\bar{z}}{\sigma_{13}}$,

$p_{12} = \frac{e_1}{\sigma_{11}}(e_4 + e_2e_0(\bar{c_o})) - \frac{u_1\bar{y}}{\sigma_{12}}(1 + e_5\theta_3)$,

$p_{13} = K_1((1 + \frac{u_1e_8\theta_1}{\sigma_{13}}) - \frac{\theta_3o_6}{\sigma_{12}}(1 + \bar{x})(1 + e_5\bar{z}))$,

$\sigma_{11} = (e_4 + \theta_1)(e_4\bar{x})$,$\sigma_{12} = (1 + \theta_1)(1 + \bar{x})(1 + e_5\theta_1)(1 + \bar{z})$,
$\sigma_{13} = (e_8 + \theta_1)(e_8 + \bar{y})$,

then the positive equilibrium $\bar{E_{13}}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive region Ω.

Proof. We consider the following positive definite function about $\bar{E_{13}}$:

$V_1 = (x - \bar{x} - \bar{x}ln(\frac{x}{\bar{x}})) + (\frac{K_1}{2})(y - \bar{y})^2 + (\frac{K_2}{2})(z - \bar{z})^2 + (\frac{K_3}{2})(c_o - \bar{c_o})^2 + (\frac{K_4}{2})(c_e - \bar{c_e})$.

$$\dot{V_1} = (x - \frac{x}{\bar{x}})(\frac{dx}{dt}) + K_1(y - \bar{y})(\frac{dy}{dt}) + K_2(z - \bar{z})(\frac{dz}{dt}) + K_3(c_o - \bar{c_o})(\frac{dc_o}{dt}) + K_4(c_e - \bar{c_e})(\frac{dc_e}{dt}).$$

Using system of equations (3.6)-(3.10), we get after some algebraic manipulations

$$\dot{V_1} = -(x - \bar{x})^2(\frac{e_1}{\sigma_{11}}(e_4 + e_2e_0(\bar{c_o})) - \frac{u_1\bar{y}}{\sigma_{12}}(1 + e_5z))$$

$$-(y - \bar{y})^2K_1((1 + \frac{u_2e_8z}{\sigma_{13}}) - \frac{xo_6}{\sigma_{12}}(1 + \bar{x})(1 + e_5\bar{z}))$$

$$-(z - \bar{z})^2K_2(u_2 - \frac{o_7y}{\sigma_{13}}(e_8 + \bar{y})) - (c_e - c_e)^2K_4(u_6 + xu_5)$$

$$-(c_o - \bar{c_o})^2K_3u_4 - (x - \bar{x})(y - \bar{y})\frac{1+e_5\bar{z}}{\sigma_{12}}(u_1(1 + \bar{x}) - K_1o_6\bar{y})$$

$$+(x - \bar{x})(z - \bar{z})\frac{u_1e_5\bar{y}}{\sigma_{11}}(1 + \bar{x}) - (x - \bar{x})(c_o - \bar{c_o})\frac{e_1e_2}{\sigma_{11}}(e_4 + \bar{x})$$

$$-(y - \bar{y})(z - \bar{z})(K_1(\frac{e_5o_6\bar{x}\bar{y}}{\sigma_{12}}(1 + x) + \frac{u_2\bar{y}}{\sigma_{13}}(e_8 + y)) - K_2\frac{o_7e_8\bar{z}}{\sigma_{13}})$$

$$-(x - \bar{x})(c_e - \bar{c_e})K_4(u_5\bar{c_e} - o_4c_o) + (c_o - \bar{c_o})(c_e - \bar{c_e})(o_2K_3 + K_4o_4\bar{x}).$$

Now, $\dot{V_1}$ can further be written as sum of the quadratic forms:

$$\dot{V_1} \leq -[((\frac{c_{11}}{2})(x - \bar{x})^2 + c_{12}(x - \bar{x})(z - \bar{z}) + (\frac{c_{33}}{2})(z - \bar{z})^2)$$

$$+((\frac{c_{11}}{2})(x - \bar{x})^2 + c_{14}(x - \bar{x})(c_o - \bar{c_o}) + (\frac{c_{44}}{2})(c_o - \bar{c_o})^2)$$

$$+((\frac{c_{11}}{2})(x - \bar{x})^2 + c_{15}(x - \bar{x})(c_e - \bar{c_e}) + (\frac{c_{55}}{2})(c_e - \bar{c_e})^2)$$

$$+((\frac{c_{22}}{2})(y - \bar{y})^2 + c_{23}(y - \bar{y})(z - \bar{z}) + (\frac{c_{33}}{2})(z - \bar{z})^2)$$

$$+((\frac{c_{44}}{2})(c_o - \bar{c_o})^2 + c_{45}(c_o - \bar{c_O})(c_e - \bar{c_e}) + (\frac{c_{55}}{2})(c_e - \bar{c_e})^2)].$$

Where,

$c_{11} = \frac{1}{3}(\frac{e_1}{\sigma_{11}}(e_4 + e_2e_0(\bar{c_o})) - \frac{u_1\bar{y}}{\sigma_{12}}(1 + e_5z))$,$c_{13} = \frac{u_1e_5\bar{y}}{\sigma_{11}}(1 + \bar{x})$,

$c_{14} = \frac{e_1 e_2}{\sigma_{11}}(e_4 + \bar{x})$,$c_{15} = (u_5 \bar{c_e} - o_4 c_o)$,

$c_{22} = K_1((1 + \frac{u_2 e_8 z}{\sigma_{13}}) - \frac{x o_6}{\sigma_{12}}(1 + \bar{x})(1 + e_5 \bar{z}))$,

$c_{23} = K_1(\frac{e_5 o_6 \bar{x} \bar{y}}{\sigma_{12}}(1 + x) + \frac{u_2 \bar{y}}{\sigma_{13}}(e_8 + y)) - \frac{o_7 e_8 \bar{z}}{\sigma_{13}}$,

$c_{33} = \frac{1}{2}(u_2 - \frac{o_7 y}{\sigma_{13}}(e_8 + \bar{y}))$, $c_{44} = \frac{u_4}{2}$,$c_{45} = o_2 + o_4 \bar{x}$,$c_{55} = \frac{1}{2}(u_6 + x u_5)$,

$\sigma_{11} = (e_4 + x)(e_4 + \bar{x})$,$\sigma_{12} = (1 + x)(1 + \bar{x})(1 + e_5 z)(1 + e_5 \bar{z})$,

$\sigma_{13} = (e_8 + y)(e_8 + \bar{y})$,$e_0(\bar{c_o}) = e_3 - \bar{c_o}$.
Now, by using Sylvesters criteria and by choosing $K_1 = \frac{u_1(1+\bar{x})}{o_6 \bar{y}} > 0$ and we get that $\dot{V_1}$ is negative definite under the following conditions:

$$c_{11} > 0 \tag{3.26}$$
$$c_{22} > 0 \tag{3.27}$$
$$c_{33} > 0 \tag{3.28}$$
$$c_{11}c_{22} > c_{12}^2 \tag{3.29}$$
$$c_{11}c_{33} > c_{13}^2 \tag{3.30}$$
$$c_{11}c_{44} > c_{14}^2 \tag{3.31}$$
$$c_{11}c_{55} > c_{15}^2 \tag{3.32}$$
$$c_{22}c_{33} > c_{23}^2 \tag{3.33}$$
$$c_{44}c_{55} > c_{45}^2 \tag{3.34}$$

We note that the fourth inequality, i.e., $c_{11}c_{22} > c_{12}^2$ is satisfied due to the proper choice of K_1
and other inequalities, (3.17) $\Rightarrow$ (3.26),(3.18) $\Rightarrow$ (3.27) , (3.19) $\Rightarrow$ (3.28), (3.20) $\Rightarrow$ (3.30), (3.21) $\Rightarrow$ (3.31),
(3.22) $\Rightarrow$ (3.32), (3.23) $\Rightarrow$ (3.33) and (3.24) $\Rightarrow$ (3.34).
Hence V_1 is a Lyapunov function with respect to $\bar{E_{13}}$,whose domain contains the region of attraction Ω, proving the theorem.

3.4 Numerical Example

In this section, we demonstrate the dynamical behavior of a three species food chain system with toxicant with the help of numerical examples.

3.4.1 Numerical Example for Model B

We choose the following values of parameters for $\tilde{E_{11}}$:

$r_1 = 0.081$, $m = 1.61$, $h = 0.01$, $\beta = 0.8$, $\alpha_1 = 0.6$, $\alpha_2 = 0.1$, $\alpha_3 = 0.31$, $l_1 = 1.2$, $l_2 = 0.599$, $s_0 = 0.2$, $k_1 = 0.1$, $k_2 = 0.11$, $t_0 = 0.41$, $B = 153.3$, $c = 0.1$, $a = 0.5$, $P_1 = 1.5$, $Q_1 = 43.1$, $d_1 = 0.9$, $P_2 = 0.5$, $Q_2 = 8.9$, $d_2 = 0.01$.

It is found that under the above set of parameters, the equilibrium point $\tilde{E_{11}}$= (11.9736, 0.0000, 0.0000, 0.2401, 0.2679) is locally asymptotically stable (see Fig.3.1).

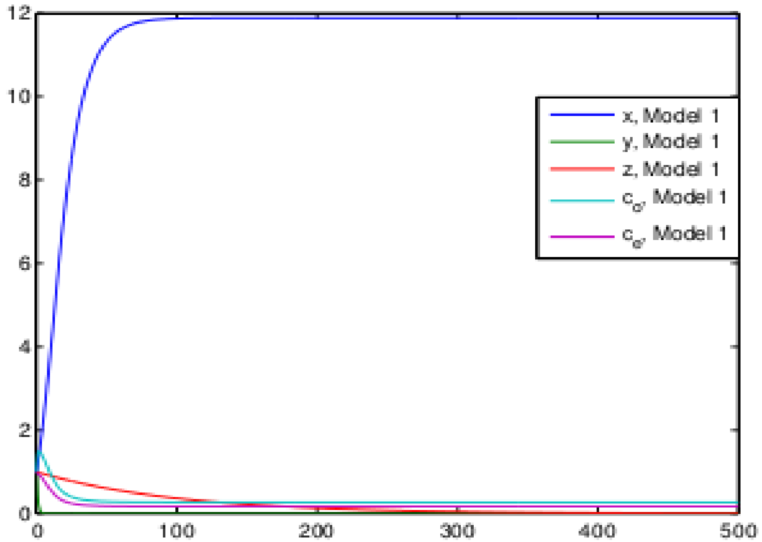

Figure 3.1: Time graph for the Model B, around the equilibrium point $\tilde{E_{11}}$, showing the stability behavior.

Now, we choose the following values of parameters for $\hat{E_{12}}$:

$r_1 = 0.02$, $\alpha_1 = 0.003$, $\alpha_2 = 0.02$, $m = 1.6$, B = 140.3, $P_1 = 1.3$, $Q_1 = 5.1$, $d_1 = 0.001$, $P_2 = 0.008$, $Q_2 = 6.5$.

With the above values of parameters and taking the remaining parameters to be the same as considered for $\tilde{E_{11}}$ of Model B, it is found that the equilibrium $\hat{E_{12}}$ = (1.7587, 1.0694, 0.0000, 0.8796, 0.9603) is locally asymptotically stable (see fig.3.2).

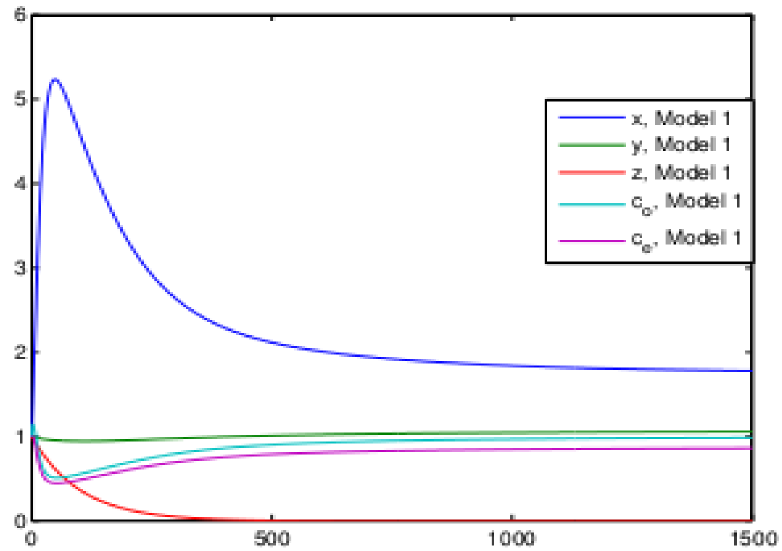

Figure 3.2: Time graph for the Model B, around the equilibrium point $\hat{E_{12}}$, showing the stability behavior.

Now, we choose the following values of parameters for $\bar{E_{13}}$:

$r_1 = 0.2$, $m = 1.00$, $P_1 = 1.1$, $d_1 = 0.001$, $Q_2 = 9.0$, $P_2 = 0.6$.

With the above values of parameters and taking the remaining parameters to be the same as considered for $\tilde{E_{11}}$ of Model B, it is found that the interior equilibrium $\bar{E_{13}} = (8.4842,\ 1.8039,\ 0.9731,\ 0.2989,\ 0.5211)$ is locally asymptotically stable (see Figs.3.3).

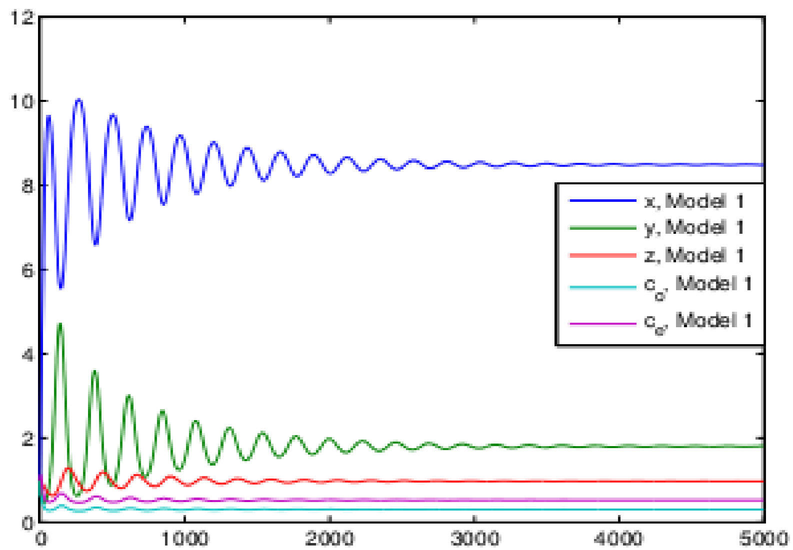

Figure 3.3: Time graph for the Model B, around the equilibrium point $\bar{E_{13}}$, showing the stability behavior.

Now, we study the Hopf-bifurcation of the Model B, taking A_1 as the bifurcating parameter. The transversality condition holds with the above set of parameters when $A_1 = \bar{A}_1 = 0.7015$. It is clear that the interior equilibrium point $\bar{E_{13}}$ of Model B is stable when $A_1 > \bar{A}_1$ and unstable when $A_1 \leq \bar{A}_1$ for which Hopf-bifurcation occurs(see Fig.3.4)

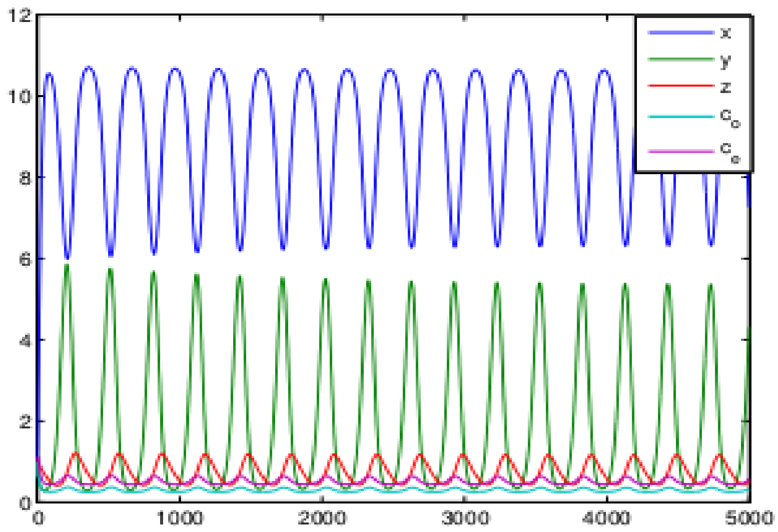

Figure 3.4: Time graph for the Model B, around the equilibrium point $\bar{E_{13}}$, showing the bifurcation behavior from Raveendra Babu and O.Misra(2015).

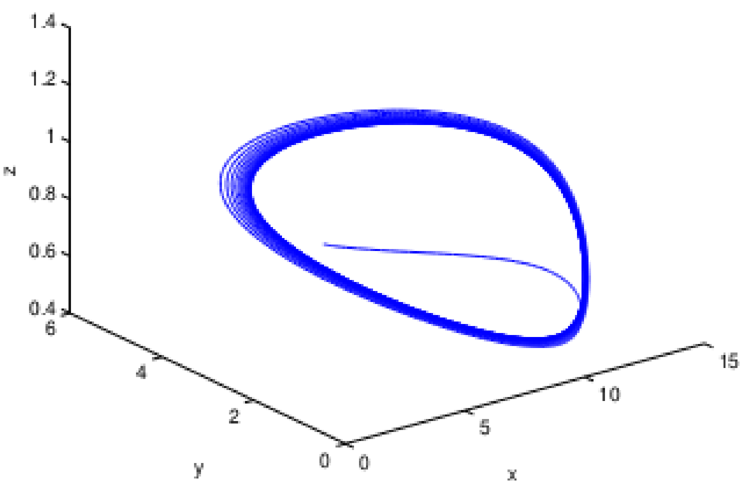

Figure 3.5: Phase graph for the Model B, around the equilibrium point $\bar{E_{13}}$, showing the bifurcation behavior.

Chapter 4

Conclusion

In this thesis we have proposed and analyzed a nonlinear mathematical model to study the effect of toxicant on a top predator and the effects on prey and intermediate populations with food-chain system. The local stability analysis of all the equilibrium points of the Model B has been carried out. The global stability analysis of only the non-trivial positive equilibrium points of the Model B has been conducted. The equilibrium level of prey population reduces due to the presence of intermediate predators,From the stability of $\tilde{E_{11}}$ of Model B, it is concluded that only the prey population will survive and both the predator populations would tend to extinction.(see Fig.3.1). From the stability of $\hat{E_{12}}$ of Model B, it is concluded that only prey and intermediate predator populations would survive and the top predator population may die out. However, in this case also the equilibrium level of prey and predator populations decrease due to the limited food(see Fig. 3.2).

The interior equilibrium points of the Model is locally stable showing the same dynamical behavior and co-existence of all the three populations of prey and predator species. However, from the equilibrium values it is seen that the equilibrium density of top predator reduces due to the presence of toxicant in the environment but intermediate predator is increasing(see fig.3.3). It may be also noted from the equilibrium of the intermediate predator population that the level of intermediate predator population may increase due to the presence of toxicant in the top predator.

The interior equilibrium points of the Model is globally asymptotically stable in the region Ω. Looking at Ω, it may be concluded that the region of global stability shrinks when the toxicant is introduced in the underlying system of top predator populations. It is noted from the stability conditions of the equilibrium of the Model that the system with toxicant seems to be more stable than that of the system with no toxicant effects. It is further concluded that the system with toxicant moves faster towards equilibrium after given perturbation than that of the system with out toxicant for same parametric values. Finally, we have demonstrated the dynamical behavior of a three species food chain system with toxicant with the help of numerical simulation to support analytical results.

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