Numerical Treatment of a Class of Singularly Perturbed Differential Equations with Nonlocal Boundary Conditions

Getu Mekonnen Wondimu

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Sciences

Presented in Partial Fulfillment of the Requirement for the Degree of Doctor of Philosophy in Applied Mathematics (Numerical Analysis)

Office of Graduate Studies
Adama Science and Technology University

June, 2023
Adama, Ethiopia

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DECLARATION

I hereby declare that this dissertation entitled "**Numerical Treatment of a Class of Singularly Perturbed Differential Equations with Nonlocal Boundary Conditions** " is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials used for this thesis have been duly acknowledged through appropriate citations.

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Recommendation

We, the supervisors of this dissertation, hereby certify that we have read and revised the dissertation entitled "**Numerical Treatment of a Class of Singularly Perturbed Differential Equations with Nonlocal Boundary Conditions**" prepared under our guidance by Getu Mekonnen Wondimu submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics (Numerical Analysis). Therefore, we recommend the submission of the dissertation to the department for further review and defense.

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Approval Page

We hereby certify that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the dissertation entitled "**Numerical Treatment of a Class of Singularly Perturbed Differential Equations with Nonlocal Boundary Conditions**" by Getu Mekonnen Wondimu.

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We, the undersigned, members of the Board of Examiners of the dissertation open defense by Getu Mekonnen Wondimu have read and evaluated the dissertation entitled "**Numerical Treatment of a Class of Singularly Perturbed Differential Equations with Nonlocal Boundary Conditions**" and examined the candidate during open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in Applied Mathematics (Numerical Analysis).

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Finally, approval and acceptance of the dissertation is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the candidate's Department Graduate Council (DGC) and School Graduate Committee (SGC).

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LIST OF SYMBOLS, ACRONYMS AND ABBRIVATIONS

$\bar{\Omega}$	Closed domain
$\mathbb{R}$	Set of real numbers
$\mathcal{L}$	Differential operator
$\mathcal{L}^N, \mathcal{L}^{\Delta t}$	Difference operator
$\mathcal{O}(.)$	Order symbols
Ω	Open domain
$\Omega^N, \bar{\Omega}^N$	Discrete domain
$\partial\Omega$	Boundary of the domain
ε	Singular perturbation parameter
C	Generic positive constant independent of ε, h, and Δt
$C^{(k)}(\Omega), C^{(k)}(\bar{\Omega})$	k times continuously differentiable functions in the respective domain
D^+, D^-, D^+D-	Forward, backward, and central difference operator
E_ε^N, E^N	Maximum absolute error and uniform error estimate
N, M	Number of mesh interval for space and time
R_ε^N, R^N	Rate of convergence and uniform rate of convergence
u, u_ε	Solution of continuous problem
$U, U_i, U_i^{N,M}$	Numerical solution
x, x_i, t, t_j	Independent variables
DDEs	Delay differential equations
FDM	Finite difference method
SPDDEs	Singularly perturbed delay differential equations
SPDEs	Singularly perturbed differential equations
SPODEs	Singularly perturbed ordinary differential equations
SPPPDDEs	Singularly perturbed parabolic partial delay differential equations
SPPPDEs	Singularly perturbed parabolic partial differential equations
SPPs	Singular perturbation problems

ABSTRACT

This dissertation presents a numerical treatment of a class of singularly perturbed problems with nonlocal boundary conditions. The study of these singular perturbation problems has a prominent role in many areas of science and engineering. Such problems are often described by differential equations. These differential equations become singularly perturbed when the highest order derivative is multiplied by a small perturbation parameter (ε)*. This leads to the presence of boundary layers which are essentially small regions in the neighborhood of the boundary of the domain and/or interior layer, where the solution changes rapidly. Due to the existence of the singular perturbation parameter* ε*, classical numerical methods for solving singularly perturbed problems are not well-posed and fail to give an accurate solution. To overcomes this limitations, it is essential to formulate numerical methods that are uniformly convergent independent of the values of perturbation parameter. The main aim of this dissertation is to develop, analyze and improve the* ε*-uniform numerical methods for solving a class of singularly perturbed differential equations with nonlocal boundary conditions. In this study, exponentially fitted finite difference method, fitted mesh finite difference method, nonstandard finite difference method and hybrid numerical method were formulated to solve a class of SPDEs with nonlocal boundary conditions. As a result, the developed numerical methods were shown to be stable, efficient, and uniformly convergent independent of the values of perturbation parameter. The nonlocal boundary conditions given at the right end of the spatial domain was treated using Simpson's rule of integration. The uniform stability and convergence analysis of the developed numerical methods were analyzed. In order to validate the applicabilty of the developed methods, numerical examples were considered and solved for different perturbation parameters and mesh sizes. The numerical algorithm was written using Matlab Software. The maximum pointwise error of the developed numerical methods was computed using double mesh principle. The rate of convergence of the developed numerical methods was also computed. The developed numerical methods are efficient and it achieves higher accuracy comparing with other methods. The numerical results are in agreement with the theoretical estimates. The dissertation concludes with a brief discussion of potential directions for future research.*

Keywords*: Singularly perturbed problems; Nonlocal boundary conditions; Fitted finite difference method; Shishkin Mesh; Stability analysis; Uniform convergence analysis; Error estimate; Boundary layer.*

CHAPTER ONE

INTRODUCTION

1.1 Background of the study

The study of singular perturbation problems (SPPs) is important in many fields of sciences and engineering, such as fluid mechanics, chemical engineering, and electronics. High Reynold's number flow in fluid dynamics, heat transport problems with large Péclet numbers, modeling the problems in mathematical biology, and semi-conductor devices where the edge effects are an important typical examples. These problems are often described by differential equations which are parameter-dependent problems and are known as singular perturbation problems. One of the earliest applications of singularly perturbed problems was in fluid mechanics, where the Navier-Stokes equations govern the behavior of fluids. The Naiver Stokes equation for fluid dynamics

$$\frac{\partial(u^2+p)}{\partial x}+\frac{\partial(uv)}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$

with suitable initial and boundary conditions is a good example of singular perturbation problems for large Reynold's number "Re", where u, v are velocity component along $x,\ y$ directions and p is the pressure. These equations are notoriously difficult to solve, especially in the presence of small-scale features such as boundary layers. The existence of boundary layers in fluid and gas dynamics was not known before the early years of the twentieth century when Ludwig Prandtl gave his seminar paper to the Third International Congress of Mathematicians in Heidelberg in 1904, which revolutionized the theoretical understanding of many flow phenomena (O'Malley, 2014).

Prandtl's seven page report published in the proceedings Prandtl (1904) innovated the subject called the boundary layer theory and gave an explanation that how a quantity as small as the viscosity of common fluids such as water and air could play a crucial role in determining their flow. Prandtl proved that the flow about a body can be treated by dividing it into two regions: a very small layer in proximity to the body (which he called the boundary layer) where the frictional effects are prominent, and the remaining as the outside region. The boundary layer theory became the foundation stone for modern fluid dynamics. However, the word "singular perturbation" was first used in the work of Friedrichs & Wasow (1946), a paper on nonlinear vibrations presented at New York University. Though Prandtl introduced the terminology boundary layer, it got much greater generality in the substantial work of Wasow (1942).

A differential equation becomes singularly perturbed when the magnitude of the highest derivative is dominated by the lower order terms and is often attributable to a small parameter (ε) multiplying the highest derivative (Turuna et al., 2020; Daba & Duressa, 2021). Whenever a mathematical model is related to a phenomenon, the researchers generally try and capture what's essential, retaining the important quantities and omitting the negligible ones which involve small parameters. The small parameter is called the singular perturbation parameter. So, it is necessary to constitute a question about the role of ε, where $0 < \varepsilon \ll 1$. Hence, the model that might be obtained by maintaining the small parameters is called the perturbed model, whereas the simplified model (the one that doesn't include the small parameters) is called the unperturbed (or reduced) model (Sharma et al., 2013). For example, consider the boundary value problem of the form

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \\ u(0) = u_0, u(1) = u_1 \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter. The problem (1.1) is singularly perturbed and a layer may appear depending on the given value of $a(x)$. When $a(x) > 0$, left boundary layer forms, when $a(x) < 0$, right boundary layer forms, and when $a(x) = 0$, the interior or parabolic boundary layer forms. The problem (1.1) is considered to be a reduced problem if $\varepsilon = 0$. Singular perturbation problems are manifested in the solution of differential equation as steep, small layers at the boundaries and interior of the domain across which the dependent variable undergoes very rapid changes. These SPPs commonly occur in many fields of mathematics, as boundary layers in mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics, and Stokes lines and surfaces in mathematics (Kadalbajoo & Patidar, 2003).

The solution of SPPs is well-known for having a multiscale character, as it differs rapidly within the boundary layer while acting smoothly outside of these layers. Perturbations work over a very small region in these problems, where the variable quantity undergoes very rapid change. Since the small parameter multiplies the highest derivative, these small regions sometimes adjoin the boundaries or interior stage. As a consequence, these problems have boundary and/or interior layers, which ensures that there are small areas where the solution changes rapidly. The analytical solution of the SPPs is difficult due to the presence of small boundary layer. Therefore, it is necessary to develop numerical methods, more precisely ε-uniform that solve the singularly perturbed differential equations (SPDEs) effectively. Numerical analysis and asymptotic analysis are the two main approaches to solve SPPs. The numerical analysis seeks to provide quantitative information on a particular problem, while the asymptotic analysis seeks to gain insight into the qualitative behavior of the problem domain and only semi-quantitative information about any particular domain. Numerical methods are designed to address a wide range of problems and minimize the demands of the problem solver. Asymptotic methods treat relatively restricted problem class and require the problem solver to have some understanding of the behavior of the solution (Kadalbajoo & Reddy, 1989).

Most modern numerical methods such as standard finite difference, finite element methods, and finite volume methods are not robust and layer-resolving when used to solve SPPs on a uniform mesh. Due to the presence of steep gradients (boundary layers) in the analytical solution, inaccurate numerical solutions are obtained whenever the classical numerical methods are used to solve SPPs with boundary or interior layers on uniform meshes. In this context, for careful numerical experiments, Farrell et al. (2000) and Miller et al. (1995) show that the classical methods fail to decrease the maximum point-wise error as the mesh is refined, until the mesh parameter and the perturbation parameter have the same order of magnitude. This contradicts the natural expectation that the error of a numerical method decreases while the mesh is refined. Furthermore, this assumption is unacceptable from a practical standpoint because when ε is incredibly small, an unacceptably large (ε-dependent) number of mesh points is required to attain an accurate numerical solution, leading to a large computational cost (Woldaregay et al., 2021). This limitation inspires to the construction of uniform numerical methods. Therefore, within the study of SPPs, the foremost challenging job is to develop such uniformly convergent numerical methods.

Singularly perturbed problems are mathematical problems that involve small parameters that causes the solution to change rapidly in certain regions of the domain. Fitted numerical methods are a class of numerical methods that are designed to handle such problems (Selvakumar & Lazarus, 2018). The significance of fitted numerical methods for solving singularly perturbed problems lies in their ability to accurately capture the behavior of the solution in the boundary layer regions, where the solution changes rapidly. These methods use a combination of mesh refinement and numerical approximation techniques to ensure that the solution is computed accurately in the boundary layer regions, while still maintaining efficiency and stability. Fitted numerical methods are particularly useful for solving singularly perturbed problems because traditional numerical methods, such as finite difference or finite element methods, can struggle to accurately resolve the solution in the boundary layer regions. This is because these methods typically use a uniform mesh, which may not have enough resolution in the boundary layer regions to accurately capture the behavior of the solution. In the construction of uniform numerical methods, two approaches generally have been taken into account (Bullo et al., 2021). A finite difference method has two major ingredients: the finite difference operator $\mathcal{L}^N$ that is employed to approximate the differential operator $\mathcal{L}$ and the mesh Ω^N that replaces the continual domain Ω.

The second successful approach to the development of ε-uniform numerical methods involves the utilization of ordinary finite difference operators on a special mesh condensed within the boundary or interior layers. The first of those ε-uniform methods involve replacing the classical finite difference operator with an operator that reflects the singularly perturbed nature of the differential operator. Such finite difference operators are noted as fitted operator finite difference and will be constructed by choosing their coefficient in such a way that some or all of the exponential functions are within the space of the differential operator. Such exponentially fitted operator finite difference and standard mesh (a regular mesh) are observed as fitted operator methods (Duressa & Woldaregay, 2022). In some cases, fitted operator finite difference is required only in a neighborhood of the layers, while the standard finite difference operator can be used on the mesh points in the rest part of the domain (Farrell et al., 1998).

The fitted mesh method uses special non-uniform mesh condensed around the layer region and, therefore, a recognized technique for solving SPPs that overcome the limitation of classical methods. These fitted mesh methods have an additional advantage over the fitted operator methods due to their easy implementation and extension to solve higher dimensional and nonlinear problems. Since the precision of fitted mesh methods depends only on the number of nodes and is independent of the value of the small parameter, this principle is commonly used to solve a wide range of problems.

The underlying idea is to construct a piecewise-uniform mesh, that is, a mesh that combines a finite number of uniform meshes with different mesh sizes inside and outside the layer. If the location and the width of the boundary layer are known, then a highly accurate piecewise-uniform mesh known as the Shishkin mesh can be easily constructed. These uniform fitted meshes were first introduced by Shishkin et al. (1983) and the corresponding numerical methods are further developed and shown as ε-uniform (Roos et al., 1996). The above two approaches are based on a priori knowledge of the solution before solving the original problem which is not always available. Singularly perturbed delay differential equations (SPDDEs) are a type of differential equation that involve a time or spatial delay and a small parameter that multiplies the highest derivative term (Govindarao & Mohapatra, 2019; Woldaregay & Duressa, 2020). These equations are challenging to solve because they exhibit both fast and slow dynamics, and a delay term can cause instability or oscillations in the solution. SPDDEs have many applications in a diverse area of science and engineering such as studying heat conduction in materials or mass transfer (Kaushik, 2014; Wu, 1996), biosciences (such as population dynamics) (Kuang, 1993) and fluid flow (Roos et al., 2008).

A common example from numerical control (Ansari et al., 2007) is the equation

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + v\left(g\left(u(x,t-\tau)\right)\right)\frac{\partial u}{\partial x} + c\left[f(u(x,t-\tau) - u(x,t)\right],$$

which models a furnace used to process metal sheets. Here, u is the temperature distribution in a metal sheet, moving at a velocity v and heated by a source specified by the function f; both v and f are dynamically adapted by controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length τ. SPDDEs arise when the rate of change of a time dependent process in its mathematical modeling is not only determined by its present state but also by a certain past state. If the delay parameter is smaller than the perturbation parameter, it is referred to as small delay; otherwise, it is referred to as large delay or large negative shift (Rao & Chakravarthy, 2014; Woldaregay et al., 2021; Woldaregay & Duressa, 2022; Govindarao & Mohapatra, 2019; Duressa & Woldaregay, 2022).

Boundary layers occur in the solution of SPDDEs when the singular perturbation parameter ε, tends to zero. These boundary layers are located in neighborhoods of the boundary of the domain, where the solution has a very steep gradient. Away from any corner of the domain a boundary layer of either regular or parabolic type (Miller et al., 1996; Roos et al., 2008; Govindarao et al., 2020), and/or interior layer (Devendra & Parvin, 2020; Elango et al., 2021) may occur. A boundary layer is said to be parabolic if the characteristics of the reduced problem (for $\varepsilon = 0$) are parallel to the boundary, and regular if these characteristics are not parallel to the boundary. Due to the presence of the singular perturbation parameter ε, the standard methods for solving such problems are not useful and fail to give accurate results. The necessity to resolve the boundary layers has inspired the development of uniform convergent numerical methods. Nonlocal singularly perturbed problems are mathematical models that involve a combination of nonlocal and singularly perturbed phenomena. These problems arise in many areas of science and engineering, including biology, physics, and materials science. Such types of problems are described by differential equations with or without delay involving large or small parameters. Often these mathematical problems are extremely difficult (or even impossible) to solve exactly and in these circumstances, approximate solutions are necessary. In modeling of various processes in science and engineering an integral term over the spatial domain appears interior or whole of the boundary conditions.

Over the last few years, many physical phenomena were formulated to non-local mathematical models. Problems in thermodynamics (Day, 1982), plasma physics (Bouziani, 1996), heat conduction (Cannon, 1963; Dehghan & Tatari, 2008), underground water flow and population dynamics (Kudu & Amiraliyev, 2015) can be reduced to the nonlocal problems having integral conditions. For example, consider a second order parabolic delay differential equations with nonlocal boundary condition which arise in population dynamics (Bahuguna & Dabas, 2008) with time delay of the form

$$\begin{cases} -ku_{xx}(x,t) + u_t(x,t) = -d(x,t)u(x,t) + b(x,t)u(x,t-\tau), (x,t) \in (0,\pi) \times [0,T], \\ u(x,t) = \phi(x,t),\ t \in [-\tau, 0],\ x \in (0,\pi), \\ u_x(0,t) = 0, t \in [0,T], \\ \int_0^\pi u(x,t)dx = \psi(t), t \in [0,T]. \end{cases} \tag{1.2}$$

In this problem, there is no flux condition considered in the left, and the average population size is being controlled by the function ψ(t) in the right. SPDDEs with integral condition play an important role in the mathematical modeling of various practical phenomena and also widely applied within fields like bio-sciences, control theory, economics, medicine, robotics, HIV infection models, oceanography, cellular systems, meteorology, the blood flow models, etc (Sekar & Tamilselvan, 2018).

In nonlocal SPPs, the behavior of a system at a given point depends not only on its local conditions but also on the conditions at other points in the system. This can lead to long-range interactions and complex behavior that cannot be captured by local models. The nonlocal behavior can affect the formation of the boundary layer, and the boundary layer can in turn affect the non-local behavior. SPPs with nonlocal boundary conditions contains a small parameter that causes the solution to exhibit rapid changes or transitions in a small region of the domain. This can lead to numerical difficulties and requires special techniques to accurately capture the behavior of the solution. Therefore, it is necessary to develop numerical methods that are convergent independent of perturbation parameter.

1.2 Statement of the problem

The presence of an integral term in a boundary condition can greatly complicate the application of standard numerical techniques such as finite difference method, finite element methods, spectral techniques, boundary integral methods, etc. Therefore, it is important to convert nonlocal boundary value problems to a more desirable form, to make them more applicable to problems of practical interest. In many cases this is a hard task. Because of the importance of these types of equations in science and technology, in last twenty years there has been growing interest in developing computational techniques for the numerical solution of differential equations with boundary integral conditions.

Over the past few years, various numerical techniques have been developed in order to study singularly perturbed differential equations with nonlocal boundary conditions (Kudu & Amiraliyev, 2015; Kudu et al., 2018; Raja & Tamilselvan, 2019b, 2020; Debela & Duressa, 2020c). Unfortunately, a class of singularly perturbed time-dependent differential equations with nonlocal boundary conditions has yet to be researched. The study of singularly perturbed delay differential equations with integral boundary conditions led to the development of several numerical techniques. Several researchers have explored and developed multiple numerical methods for singularly perturbed delay differential equations of convection diffusion type with integral boundary conditions (Sekar & Tamilselvan, 2018, 2019b; Debela & Duressa, 2019). However, singularly perturbed differential equations of the reaction-diffusion type with integral boundary conditions have been not investigated yet.

Elango et al. (2021) was the first, to investigate singularly perturbed partial delay differential equations of reaction-diffusion types with integral boundary condition. They developed a finite difference scheme on piecewise uniform Shishkin mesh in spatial direction and backward difference method in temporal direction. They proved that the order of convergence is almost two in space variable and one in time variable.

Several researchers (Gobena & Duressa, 2022a,b; Hailu & Duressa, 2022) have attempted to design different parameter uniform convergent numerical method for solving singularly perturbed partial delay differential equations with integral boundary conditions. Even still, the more accurate uniformly convergent layer resolving numerical methods are necessary to solve the problems. Finally, in all the cases, the well-known standard numerical methods used for solving a class of singularly perturbed differential equations are not well-posed and fail to give an analytical solution when the perturbation parameter ε is small. This is why the numerical methods for solving a class of singularly perturbed problems are of practical interest. It is necessary to develop suitable numerical methods that are uniformly convergent independent of the values of perturbation parameter.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study is to develop uniformly convergent numerical methods for a class of singularly perturbed differential equations with nonlocal boundary conditions.

1.3.2 Specific objectives

The specific objectives of the study are to:

1. formulate numerical methods that are convergent independent of the values of the perturbation parameter.
2. establish the stability of the developed numerical methods.
3. investigate the convergence of the developed numerical methods.

1.4 Significance of the study

The results of this study will contribute to researchers benefit, considering that singularly perturbed problems play an important role in sciences and engineering. The result that will be obtained in this research project can enhance the knowledge of numerical method for a class of singularly perturbed differential equation with nonlocal boundary conditions. The analysis in this research project can serve as reference for researchers. The results that will be obtained in this research project can serve as bases to extend the idea.

1.5 Delimitation of the study

Singular perturbation problems have many applications in science and engineering problems. These singularly perturbed problems are modeled by ordinary differential equations, partial differential equations or delay differential equations with an appropriate boundary conditions. Because of its wide range of applications, this study is limited to the following singularly perturbed problems.

Governing problem - I

A class of singularly perturbed delay ordinary differential equations of reaction-diffusion types with integral boundary condition is considered as

$$\begin{aligned} \mathcal{L}u(x) &= -\varepsilon\frac{d^2u(x)}{dx^2} + a(x)u(x) + b(x)u(x-1) = f(x), \quad x \in (0,2), \\ u(x) &= \phi(x), \quad x \in [-1,0], \\ \mathcal{K}u(2) &= u(2) - \varepsilon\int_0^2 g(x)u(x)dx = \phi_r, \end{aligned} \tag{1.3}$$

where ε is a small positive number $(0 < \varepsilon \ll 1)$. Assume that $a(x) \geq \alpha > 0$, $b(x) \leq \beta < 0$, $\alpha + \beta > 0$, $f(x), \phi(x), \phi_r$ are sufficiently smooth functions and $g(x)$ is monotonically non negative function and satisfy $\int_0^2 g(x)dx < 1$. Under this assumptions, equation (1.3) has a unique solution which exhibits a boundary layer at $x = 0$ and $x = 2$ and interior layer at $x = 1$.

Governing problem - II

A class of singularly perturbed parabolic partial differential equations with integral boundary condition is considered as

$$\begin{aligned}
\mathcal{L}u(x,t) &= -\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial t} + a(x,t)u(x,t) = f(x,t), \ \ (x,t) \in \Omega, \\
u(x,0) &= \phi_b(x), \ \ \forall (x,t) \in \Gamma_b = \{(x,0); 0 \le x \le 1\}, \\
u(0,t) &= \phi_l(t), \ \ \forall (x,t) \in \Gamma_l = \{(0,t); 0 \le t \le T\}, \\
\mathcal{K}u(x,t) &= u(1,t) - \varepsilon\int_0^1 g(x)u(x,t)dx = \phi_r(x,t), \ \ \forall (x,t) \in \Gamma_r = \{(1,t); 0 \le t \le T\},
\end{aligned} \tag{1.4}$$

where $(x,t) \in \Omega = (0,1) \times (0,T],\ \bar{\Omega} = [0,1] \times [0,T]$, and ε is a small positive parameter $(0 < \varepsilon \ll 1)$. Assume that $a(x,t) \ge \alpha > 0,\ f(x,t),\ \phi_l,\ \phi_r,\ \phi_b$ are sufficiently smooth and $g(x)$ is monotonically non negative function and satisfy $\int_0^1 g(x)dx < 1$. Under this assumptions, equation (1.4) has a unique solution which exhibits a parabolic boundary layer (at $x = 0$ and $x = 1$).

Governing problem - III

A class of singularly perturbed partial delay differential equations of reaction-diffusion type with integral boundary condition is considered as

$$\begin{aligned}
\mathcal{L}u(x,t) &= -\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial t} + a(x,t)u(x,t) + b(x,t)u(x-1,t) = f(x,t), \ \ (x,t) \in \Omega, \\
u(x,t) &= \phi_l(x,t),\ \forall (x,t) \in \Gamma_l = \{(x,t); -1 \le x \le 0 \text{ and } 0 \le t \le T\}, \\
\mathcal{K}u(x,t) &= u(2,t) - \varepsilon\int_0^2 g(x)u(x,t)dx = \phi_r(x,t),\ \forall (x,t) \in \Gamma_r = \{(2,t); 0 \le t \le T\}, \\
u(x,t) &= \phi_b(x,t),\ \forall (x,t) \in \Gamma_b = \{(x,0); 0 \le x \le 2\},
\end{aligned} \tag{1.5}$$

where $(x,t) \in \Omega = (0,2) \times (0,T],\ \bar{\Omega} = [0,2] \times [0,T],\ \Omega_1 = (0,1) \times [0,T],\ \Omega_2 = (1,2) \times [0,T],\ \Omega^* = \Omega_1 \cup \Omega_2$ and ε is a small positive parameter $(0 < \varepsilon \ll 1)$. Assume that $a(x,t) \ge \alpha,\ b(x,t) \le \beta < 0,\ \alpha + \beta > 0,\ f(x,t),\ \phi_l,\ \phi_r,\ \phi_b$ are sufficiently smooth and $g(x)$ is monotonically non negative function and satisfy $\int_0^2 g(x)dx < 1$. Under this assumptions, equation (1.5) has a unique solution which exhibits a boundary layer at $x = 0$ and $x = 2$ and interior layer at $x = 1$.

The numerical methods presented in this dissertation are limited to the finite difference method and the cubic spline method.

1.6 Mathematical preliminaries

This section introduces some basic definitions, notations and conventions which will be used throughout the study. The following definitions and notations are taken from the books (Farrell et al., 2000; Miller et al., 1998; Roos et al., 1996). First of all, Ω denotes a bounded open domain in $\mathbb{R}^2$ and is given by $\Omega = (0, L)\times(0, T)$. For any nonnegative integer k, $C^k(\Omega)$ denotes the space of all functions whose derivatives are continuous up to order k on Ω. In the analysis, if the standard supremum norm $||.||$ or $||.||_\Omega$ is used, which is defined by

$$||u|| = \sup_{\xi\in\Omega} |u(\xi)| \, ,$$

for a function u on some domain Ω. It is a convention that when the domain is obvious, or of no particular significance, Ω is omitted. Now, the standard finite difference operators are defined which will be used for describing the difference schemes in the subsequent studies. Consider an arbitrary nonuniform mesh $\bar{\Omega}^N = \{x\}_i^N$ given by

$$\bar{\Omega}^N = \{0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_{N-1} < x_N = L\} \, .$$

Let us denote $h_j = x_j - x_{j-1},\ j = 1, \cdots, N$. For a mesh function V_j, the forward, backward and central difference operators D^+, D^- and D^0 are respectively given as

$$D^+V_j = \frac{V_{j+1} - V_j}{h_{j+1}}, \quad D^-V_j = \frac{V_j - V_{j-1}}{h_j}, \quad D^0V_j = \frac{V_{j+1} - V_{j-1}}{h_{j+1} + h_j},$$

and the second-order finite difference operator D^+D^- is defined as

$$D^+D^-V_j = \frac{2}{h_{j+1} + h_j}\left(D^+ - D^-\right)V_j = \frac{2}{h_{j+1} + h_j}\left(\frac{V_{j+1} - V_j}{h_{j+1}} - \frac{V_j - V_{j-1}}{h_j}\right).$$

Throughout the study C denotes a generic positive constant independent of the mesh points x_j and the parameter ε and the number of mesh intervals N which can take different values at different places, even in the same argument. But a subscripted C $(i.e., C_1)$ is a fixed constant.

Definition 1. *Shishkin meshes: A Shishkin mesh is a piecewise uniform mesh. What distinguish a Shishkin mesh from any other piecewise uniform mesh is the choice of the so called transition parameter(s) which are the points at which the mesh size changes rapidly.*

Definition 2. *Boundary layer is defined as the region of the independent variables where the dependent variables change rapidly.*

Definition 3. *ε-uniform numerical method: Consider a family of mathematical problems parameterized by a parameter ε, where ε lies in the semi-open interval $0 < \varepsilon \ll 1$. Assume* *that each problem in the family has a unique solution denoted by u_ε and that of each u_ε is approximated by a sequence of numerical solution $U_\varepsilon, \bar{\Omega}^N$ where U_ε is defined on the discrete space $\bar{\Omega}^N$ and N is the discretization parameter. Now the numerical solution U_ε is said to converge ε-uniformly to the exact solution u_ε , if there exist a positive integer N_0 and positive numbers C and p, where N_0, C, p all are independent of ε such that for all $N \geq N_0$,*

$$\sup_{0<\varepsilon\ll 1} \left|\left|u_\varepsilon^N - U_\varepsilon\right|\right| \leq CN^{-p}$$

Remark 1. *Nonlocal boundary conditions are boundary conditions that depend on the behavior of the solution at points other than the boundary itself. In other words, the boundary conditions are specified in terms of integrals or derivatives of the solution over a region that includes the boundary. Nonlocal boundary conditions are often used in mathematical models of physical phenomena where the behavior of the solution at the boundary is influenced by the behavior of the solution in the interior. Examples of such phenomena include wave propagation, heat transfer, and fluid flow.*

Remark 2. *An integral boundary condition is a type of boundary condition in a differential equation that involves an integral over the boundary of the domain. It is often used to model physical systems where the behavior at the boundary is influenced by the behavior throughout the entire domain.*

1.7 Organization of the dissertation

In this dissertation, suitable numerical methods to solve a class of singularly perturbed differential equations with nonlocal boundary conditions are investigated, designed, and analyzed. Apart from this introductory chapter, this dissertation consists of nine chapters and conclusion. The first chapter is devoted to a general introduction as well as the historical background of related work in the subject of singularly perturbed problems. It provides the motivation and objectives for solving singularly perturbed problems, as well as the preliminaries to be employed in the dissertation. In the second Chapter, the literature review of the dissertation is given. In the third Chapter, a research methodology of the study is given.

In Chapter 4 and 5, numerical methods for solving a class of singularly perturbed ordinary differential equation with nonlocal boundary conditions are developed. In Chapter 6 and 7, numerical methods for solving singularly perturbed parabolic partial differential equations with nonlocal boundary conditions are presented. In Chapter 8 and 9, numerical methods for solving singularly perturbed partial delay differential equations of reaction-diffusion types with nonlocal boundary conditions are formulated.

CHAPTER TWO

LITERATURE REVIEW

This chapter describes some of the studies on numerical treatment of a class of singularly perturbed differential equations with nonlocal boundary conditions that have been documented in the literature. These class of problems play a crucial role in various problems in engineering and applied sciences. A number of recent papers have been presented to the investigation of various numerical methods to solve such types of problems.

2.1 Singularly perturbed differential equations with integral boundary conditions

Singularly perturbed problems with integral boundary conditions are a type of differential equation that involve a small parameter multiplying the highest derivative term and an integral boundary condition. These problems arise in neural network. In the literature, various methods have been proposed to solve singularly perturbed problems with integral boundary conditions. For instance, Cakir & Amiraliyev (2005) presented a finite difference method for solving singularly perturbed nonlocal boundary value problem of the form

$$\begin{aligned} Lu(x) &= \varepsilon^2 u''(x) + \varepsilon a(x)u'(x) - b(x)u(x) = f(x), 0 < x < l, \\ u(0) &= \mu_0, \\ l_1 u(l) &= u(l) - \int_{l_0}^{l_1} g(x)u(x)dx = \mu_l, 0 \le l_0 < l_1 \le l. \end{aligned} \tag{2.1}$$

Here ε is a small positive parameter, μ_0 and μ_l are given constants, $a(x) \ge \alpha$, $b(x) \ge \beta$, $g(x)$ and $f(x)$ are sufficiently smooth functions in the $[0, l]$. The solution $u(x)$ of problem (2.1) has in general a boundary layers at $x = 0$ and $x = l$ for ε near 0. They demonstrated that a presented method is ε-uniformly first order accurate in the discrete maximum norm. Amiraliyev et al. (2007) considered a uniform finite difference method on Shishkin mesh for a quasilinear first order singularly perturbed boundary value problem (BVP) with integral boundary condition of the form

$$\begin{aligned} &\varepsilon u' + f(t, u) = 0, t \in I = (0, T), T > 0, \\ &u(0) = \mu u(T) + \int_0^T b(s)u(s)ds + d, \end{aligned} \tag{2.2}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, μ and d are given constants. $b(t)$ and $f(t, u)$ are assumed to be sufficiently continuously differentiable functions in $\bar{I} = I \cup \{t = 0\}$ and

$I \times \mathbb{R}$ respectively and moreover $\frac{\partial f}{\partial u} \geq \alpha > 0$. They proved that the method is first order convergent except for a logarithmic factor, in the discrete maximum norm, independently of the perturbation parameter.

Cen & Cai (2007) developed a hybrid upwind difference scheme on a Shishkin mesh for solving singularly perturbed problem given in (2.2). Applying the discrete maximum principle and barrier function techniques, they showed that the scheme is almost second order convergent, in the discrete maximum norm, independently of singular perturbation parameter. Zhang & Xie (2009) developed the uniformly valid asymptotic solution of a singularly perturbed problem considered in (2.2), with the method of boundary layer function and the Banach fixed-point theorem. Kudu & Amiraliyev (2015) considered the singularly perturbed second order ordinary differential equation with integral boundary condition of the form

$$\begin{aligned} &Lu(x) := \varepsilon^2 u''(x) + \varepsilon a(x)u'(x) + b(x)u(x) = f(x),\ 0 < x < l, \\ &u'(0) = \frac{\mu_0}{\varepsilon}, \\ &\int_0^l d(x)u(x)dx = \mu_1. \end{aligned} \tag{2.3}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, $0 < \alpha \leq a(x)$, $f(x)$, $d(x)$ are sufficiently smooth functions in $[0, l]$, μ_i, $(i = 0, 1)$ are given constants. The function $u(x)$ has in general a boundary layer of thickness $\mathcal{O}(\varepsilon)$ near $x = 0$. The uniform convergence of the approximation solution on a uniform mesh is demonstrated. Kudu et al. (2018) constructed numerical algorithm based on upwind finite difference operator and an appropriate piecewise uniform mesh to solve problem (2.2). They established parameter-uniform error estimate for the numerical solution.

Raja & Tamilselvan (2019a) considered a class of third order singularly perturbed convection diffusion type equations with integral boundary condition of the form

$$\begin{aligned} &\varepsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x), x \in (0, 1) = \Omega, \\ &u(0) = l_1, u'(0) = l_2, u(1) = \varepsilon \int_0^l g(x)u(x)dx + l_3, \end{aligned} \tag{2.4}$$

where $0 < \varepsilon \ll 1$, $a(x) \geq \alpha > 0$, $b(x) \geq \beta \geq 0$, $\theta \leq c(x) \leq \theta_0 \leq 0$, $4\alpha + \beta + 16\theta > 0$, l_1, l_2, l_3 are real numbers, $g(x)$ is nonnegative with $\int_0^1 g(x)dx < 1$ and $a(x), b(x), c(x), f(x), g(x)$ are sufficiently smooth on $\bar{\Omega} = [0, 1]$. They presented numerical method based on a finite difference scheme on a Shishkin mesh and the method suggested is of almost first order convergent.

Raja & Tamilselvan (2019b) considered a class of system of singularly perturbed reaction diffusion equations with integral boundary condition of the form

$$\bar{L}\bar{u}(x) = \begin{cases} L_1\bar{u}(x) = -\varepsilon u_1''(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x), \\ L_2\bar{u}(x) = -\varepsilon u_2''(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x), \end{cases} \tag{2.5}$$

where $\bar{u} = (u_1(x), u_2(x))\,, x \in \Omega = (0,1)$, with the boundary conditions

$$\begin{aligned} u_1(0) = l_1, B_1u_1(1) = u_1(1) - \varepsilon \int_0^1 g(x)u_1(x)dx = l_2, \\ u_2(0) = l_3, B_2u_2(1) = u_2(1) - \varepsilon \int_0^1 g(x)u_2(x)dx = l_4, \end{aligned}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter and the functions $a_{11}(x), a_{12}(x), a_{21}(x), a_{22}(x)$, $f_1(x), f_2(x)$ are sufficiently smooth on $\Omega = [0,1]$ and g_i is nonnegative and $\int_0^1 g_i(x)dx < 1$, for $i = 1, 2, x \in \bar{\Omega}$. Here, they described a numerical method based on a finite difference scheme on a Shishkin mesh. The recommended approach is almost second order convergent.

Raja & Tamilselvan (2020) developed a finite difference scheme on uniform Shishkin mesh for solving singularly perturbed convection-diffusion problem with integral boundary condition of the form

$$\begin{aligned} Ly = -\varepsilon y''(x) + a(x)y'(x) = f(x), x \in \Omega, \\ y(0) = A, L_1y(1) = y(1) - \varepsilon \int_0^1 g(x)y(x)dx = B, \end{aligned} \tag{2.6}$$

where $0 < \varepsilon \ll 1$, A and B are constants, $a(x) \geq \alpha > 0$, $x \in \bar{\Omega} = [0,1], g(x)$ is nonnegative with $\int_0^1 g(x)dx < 1$ and $f(x), a(x)$ are continuously differentiable functions on $\bar{\Omega}$. They proved that the developed numerical method is of almost first order convergent.

Debela & Duressa (2020a) developed an accelerated numerical method for singularly perturbed problem given in (2.1). They used exponentially fitted operator method using Richardson extrapolation techniques. They treated an integral boundary condition using numerical integration techniques. They showed that the method is ε-uniformly convergent. Debela & Duressa (2020c) developed a uniformly convergent numerical method via nonstandard finite difference and numerical integration methods for a governing equation given in (2.3). They showed that the method is ε-uniformly convergent. Debela (2021) presented numerical method to solve SPDEs given in (2.6). The proposed method is a combination of the classical finite difference method for the boundary conditions and exponential fitted operator method for the differential equations at the interior points. They showed that the method is to be first-order accuracy independent of the perturbation parameter.

As a result, the problems presented above can be easily extended to time-dependent singularly perturbed parabolic partial differential equations with nonlocal boundary conditions.

2.2 Singularly perturbed delay differential equations with integral boundary conditions

Singularly perturbed delay differential equations with nonlocal boundary condition constitute a very interesting and important class of problems (Amiraliyev et al., 2007). Because of this, several scholars studied singularly perturbed delay differential equations with nonlocal boundary conditions. For example, Sekar & Tamilselvan (2018) considered a class of singularly perturbed system of delay differential equations of convection diffusion type with an integral boundary conditions of the form:

$$\begin{aligned}
&-\varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) + c_{11}(x)u_1(x-1)\\
&+ c_{12}(x)u_2(x_1) = f_1(x), \quad x \in \Omega_1 \cup \Omega_2,\\
&-\varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) + c_{21}(x)u(x-1)\\
&+ c_{22}u_2(x-1) = f_2(x), \quad x \in \Omega_1 \cup \Omega_2,\\
&u_1(x) = \phi(x), x \in [-1,0], \mathcal{K}_1 u_1(2) = u_1(2) - \varepsilon \int_0^2 g_1(x)u_1(x)dx = l_1,\\
&u_2(x) = \phi_2(x), x \in [-1,0], \;\; \mathcal{K}u_2(2) = u_2(2) - \varepsilon \int_0^2 g_2(x)u_2(x)dx = l_2,
\end{aligned} \tag{2.7}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, the functions $a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), c_{11}(x), c_{12}(x), c_{21}(x), c_{22}(x), f_1(x), f_2(x)$ are sufficiently smooth on $\Omega = [0, 2]$. They developed a finite difference scheme on an appropriate piecewise Shishkin type mesh is suggested to solve the problem. They proved that the method is almost first-order convergent. An error estimated is derived in the discrete maximum norm.

Sekar & Tamilselvan (2019a) considered a class of third order singularly perturbed delay differential equations of convection diffusion type with an integral boundary conditions:

$$\begin{aligned}
&-\varepsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) + d(x)u'(x-1) = f(x), x \in \Omega^*,\\
&u(x) = \varphi(x), x \in [-1,0], \varphi \in C_1([-1,0]),\\
&u'(2) = \varepsilon \int_0^2 g(x)u'(x)dx + l,
\end{aligned} \tag{2.8}$$

$\gamma \leq d(x) \leq \gamma_0, \alpha + \theta + \gamma > 0$, $g(x)$ is non negative and monotone with $\int_0^2 g(x)dx < 1$ and $a(x), b(x), c(x), d(x), f(x), g(x)$ are sufficiently smooth on $\Omega = [0, 2]$. They presented a numerical method based on a finite difference scheme on a Shishkin mesh. The suggested method is almost first-order convergent. An error estimate is derived in the discrete norm and they validate the theoretical estimates by presenting numerical examples.

Sekar & Tamilselvan (2019b) looked at a class of singularly perturbed delay differential equations of convection diffusion type with an integral boundary conditions of the form:

$$\begin{aligned}
\mathcal{L}u(x) &= -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-1) = f(x), x \in \Omega, \\
u(x) &= \varphi(x), \quad x \in [-1, 0], \\
\mathcal{L}u(2) &= u(2) - \varepsilon \int_0^2 g(x)u(x)dx = l,
\end{aligned} \tag{2.9}$$

where $\varphi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x), b(x)$ and $c(x)$ satisfy $a(x) > \alpha_1 > \alpha > 0, b(x) \geq \beta \geq 0, c(x) \leq \gamma \leq 0, \alpha + \beta + \gamma > 0$ and $\beta + \gamma \geq 0$. Furthermore, $g(x)$ is non negative and monotonic with $\int_0^2 g(x)dx < 1$. They developed a finite difference scheme with an appropriate piecewise Shishkin type mesh to solve the problem. They proved that the method is almost first-order convergent. An error estimated is derived in the discrete norm.

Sekar & Tamilselvan (2019c) considered a class of third order singularly perturbed delay differential equations of reaction diffusion type with an integral boundary condition of the form:

$$\begin{aligned}
&-\varepsilon u'''(x) + b(x)u'(x) + c(x)u(x) + d(x)u'(x-1) = f(x), x \in (0, 2), \\
&u(x) = \phi(x), \quad x \in [-1, 0], \quad \phi \in C^1[-1, 0], u'(2) = -\varepsilon \int_0^2 g(x)u'(x)dx + l,
\end{aligned} \tag{2.10}$$

where $0 < \varepsilon << 1, b(x) \geq \alpha \geq 0, \theta \leq c(x) \leq \theta_0, \gamma \leq d(x) \leq \gamma_0 \leq 0, \alpha + \theta + \gamma > 0, \int_0^2 g(x)dx < 1$ and $b(x), c(x), d(x), f(x), g(x)$ is non negative monotone and are sufficiently smooth on $\Omega = [0, 2]$ and l be a real number. They developed a numerical method based on a finite difference scheme on a Shishkin mesh. The method suggested is almost first-order convergent. An estimated error is derived in the discrete norm.

Debela & Duressa (2019) considered the following singularly perturbed problem

$$\begin{aligned}
Ly &= -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad x \in \Omega = (0, 2), \\
y(x) &= \phi(x), \ x \in [-1, 0], \\
Ky(2) &= y(2) - \varepsilon \int_0^2 g(x)y(x)dx = l,
\end{aligned} \tag{2.11}$$

where $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \Omega$ it is assumed that the sufficient smooth functions $a(x), b(x)$ and $c(x)$ satisfy $a(x) > \alpha_1 > \alpha > 0, b(x) \geq \beta \geq 0, c(x) \leq \gamma \leq 0$ and $\alpha + \beta + \gamma > 0$, and $g(x)$ is non-negative and monotonic with $\int_0^2 g(x)dx < 1$. They developed exponentially fitted finite difference method for solving the problem. To treat the integral boundary condition, Simpson's rule is applied. They proved that the proposed method is stable and parameter uniform convergent. To validate the applicability of the scheme, they considered two model problems for numerical experimentation and solved for different values of the perturbation parameter, ε and mesh size, h. They computed maximum absolute errors and rate of convergence and it is observed that the presented method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$.

Debela & Duressa (2020b) also developed accelerated fitted finite difference method for solving singularly perturbed delay differential equation with non-local boundary condition given in (2.11). To treat the non local boundary condition, Simpson's rule is applied and the stability and parameter uniform convergence for the proposed method was proved. To validate the applicability of the scheme, they considered two model problems for numerical experimentation and solved for different values of the perturbation parameter ε and mesh size h. They tabulated the numerical results in terms of maximum absolute errors and rate of convergence, and it is observed that the presented method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$.

Devendra & Parvin (2020) considered the following problem

$$\begin{aligned} &Lu \equiv -\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) + r(x)u(x-1) = \phi(x), x \in \Omega = (0, 2), \\ &u(x) = \phi(x), x \in [-1, 0], \\ &u(2) = k + \varepsilon \int_0^2 s(x)u(x)dx = E, \end{aligned} \tag{2.12}$$

where $\varepsilon \in (0, 1]$ and E is a constant. All the associated functions are assumed sufficiently smooth, bounded, and independent of ε. Also, $p(x), q(x)$ and $r(x)$ are such that
$p(x) \geq p^* > 0,\ q(x) \geq q^* > 0,\ r(x) \leq r^* < 0,\ q(x) + r(x) \geq \delta > 0$, where $p^* + q^* + r^* > 0$. They developed the method based on the basis of B-spline functions and efficient numerical scheme on a piecewise-uniform mesh is suggested to approximate the solution of singularly perturbed problems with an integral boundary condition and having a delay of unit magnitude. For the small diffusion parameter ε, an interior layer and a boundary layer occurred in the solution. Unlike most numerical schemes their scheme does not require the differentiation of the problem data (integral boundary condition). The parameter-uniform convergence (the second-order convergence except for a logarithmic factor) is confirmed by numerical computations of two test problems. They used double mesh principle to measure the accuracy of the method in terms of the maximum absolute error.

Based on the information provided above, it is clear that a numerical method for solving a class of singularly perturbed delay differential equations of the reaction-diffusion type has not yet been investigated.

2.3 Singularly perturbed parabolic partial delay differential equations with integral boundary condition

A numerical method for solving a class of singularly perturbed parabolic partial delay differential equations with integral boundary condition was first investigated by Elango et al. (2021). They took a below governing equations with integral boundary condition of the form

$$
\begin{aligned}
\mathcal{L}u_\varepsilon(r,t) &= \left(-\varepsilon\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial t}+a(r,t)\right)u_\varepsilon(r,t)+b(r,t)u_\varepsilon(r-1,t)=f(r,t),\ \ (r,t)\in D\\
u_\varepsilon(r,t) &= \phi_l(r,t),\ \ \forall(r,t)\in\Gamma_l=\{(r,t);-1\le r\le 0 \ \text{ and } \ 0\le t\le T\}\\
\mathcal{K}u_\varepsilon(r,t) &= u_\varepsilon(2,t)-\varepsilon\int_0^2 g(r)u_\varepsilon(r,t)dr=\phi_{\bar r}(r,t),\ \forall(r,t)\in\Gamma_{\bar r}=\{(2,t);0\le t\le T\}\,,\\
u_\varepsilon(r,t) &= \phi_b(r,t),\ \forall(r,t)\in\Gamma_b=\{(r,0)\}\,.
\end{aligned}
\tag{2.13}
$$

where $(r,t)\in D=(0,2)\in(0,T],\ \bar D=[0,2]\times[0,T], D_1=(0,1)\times[0,T],\ D_2=(1,2)\times[0,T],\ D^*=D_1\cup D_2$ and ε is a small positive parameter $(0<\varepsilon<<1)$. Assume that $a(r,t)\ge\alpha>0,\ b(r,t)\le\beta<0,\ \alpha+\beta>0,\ f(r,t),\ \phi_l,\ \phi_r,\ \phi_b$ are sufficiently smooth and $g(r)$ is monotonically non negative function and satisfy $\int_0^2 g(r)dr<1$. They proposed finite difference method on a Shishkin mesh of $N_r\times N_t$ elements, suggested to solve the given problem and proved to be parameter uniform. Also, they proved that the order of convergence is almost two in space variable and one in time variable, and the error of the proposed method is bounded by $C(N_r^{-2}\ln^2 N_r+N_t^{-1})$. Gobena & Duressa (2021) constructed a parameter-uniform numerical method via the nonstandard finite difference method for the spatial direction, and the backward Euler method for the resulting system of initial value problems in temporal direction to solve the problem considered in (2.13).

Gobena & Duressa (2022a) presented a fitted operator average finite difference scheme for singularly perturbed delay parabolic reaction diffusion problems with non-local boundary conditions given in (2.13). They proved that the developed method is shown to be second order uniformly convergent in both space and temporal directions. Gobena & Duressa (2022b) developed a numerical method for solving singularly perturbed parabolic partial differential equations with large negative shift on the spatial variable and integral boundary condition given in (2.13). They used a numerical methods combining the Crank–Nicolson difference scheme for the temporal direction on the uniform mesh and exponentially fitted operator finite difference method in spatial discretization. They proved that the developed method was shown to be second order uniformly convergent in both directions.

Hailu & Duressa (2022) developed a numerical method for solving singularly perturbed parabolic differential equations with integral boundary conditions and a large negative shift in the space variable given in (2.13). The implicit Euler method was used for the temporal direction and the cubic spline method was proposed for the spatial direction on a piecewise uniform mesh (Shishkin mesh) to formulate a parameter-uniform numerical method. The proposed scheme has been shown to be uniformly convergent with order of convergence almost two in space direction and one in time direction.

Sharma & Kaushik (2023) considered a class of singularly perturbed parabolic partial differential equations of convection diffusion types with a large delay and an integral boundary condition of the form

$$
\begin{aligned}
&-\varepsilon y_{ss}(s,t)+p(s)y_s(s,t)+q(s)y(s,t)+r(s)y(s-1,t)+y_t(s,t)=g(s,t)\in D\\
&y(s,t)=\psi_1(s,t)\; in\; \Gamma_1:=\{(s,t),s\in[-1,0],t\in[0,T]\},\\
&y(s,t)=\psi_2(s,t)\; on\; \Gamma_2:=\{(s,0),s\in[0,2]\},\\
&ky(s,t)=y(2,t)-\varepsilon\int_0^2 f(s)y(s,t)ds=\psi_3(s,t)\; on\; \Gamma_3=\{(2,t),t\in[0,T]\},
\end{aligned}
\tag{2.14}
$$

where $0<\varepsilon\ll 1$ is a small positive parameter, $g(s,t),p(s),q(s)$, and $r(s)$ are sufficiently smooth functions. Also, assume that the initial-boundary data ψ_1,ψ_2 and ψ_3 are smooth and bounded functions such that $p(s)\le p_0>p_0^*>0$, $q(s)\ge q_0>0$, $r(s)\le r_0<0$, $p_0^*+q_0+r_0>0$, $q(s)+r(s)\ge 2\eta>0$, $y(1-,t)=y(1+,t)$, $y_s(1-,t)=y_s(1+,t)$. Here, $f(s)$ is non-negative, monotonic function such that $\int_0^2 f(s)ds<1$. The problem has a weak interior layer besides a boundary layer. They employed a Crank-Nicholson difference scheme in the temporal variable, whereas an upwind difference scheme in space. The proposed method is unconditionally stable and converges uniformly independent of the perturbation parameter.

Hailu & Duressa (2023) formulated a parameter uniform numerical method to solve singularly perturbed parabolic convection–diffusion equations with integral boundary conditions and a large negative shift given in (2.14). They used the implicit Euler method for the temporal direction and the exponentially fitted finite difference scheme for the spatial direction. The Simpson's integration rule was used to handle the integral boundary condition. They showed that the method is uniformly convergent with a convergence order of two in both temporal and spatial directions after Richardson extrapolation.

As a result, a numerical methods for solving singularly perturbed parabolic delay partial differential equations with nonlocal boundary conditions has been investigated. Unfortunately, there has been minimal focus on developing higher order numerical methods based on layer adaptive meshes. In this dissertation, a more accurate numerical methods that are convergent independent of the values of perturbation parameter to solve such types of singularly perturbed problems with integral boundary conditions are formulated. The stability analysis of the formulated numerical methods are established. The convergence analysis of the developed numerical methods are investigated.

CHAPTER THREE

RESEARCH METHODOLOGY

This chapter presents the methods and materials that are used to attain the objectives of the study. For this, source of data, method of data analysis, the approach of study and mathematical procedures are briefly discussed.

3.1 Study design

Both theoretical and experimental designs were used in this investigation. The numerical experimentation was examined using the MATLAB software, and the results are given as tabulations and figures.

3.2 Source of information

In this study, the relevant sources of information from both published and unpublished documents such as books, journals, and related studies found on the internet were used to obtain the desired knowledge.

3.3 Mathematical procedures

The following mathematical procedures are used to achieve the stated objective of the study.

1. Describing the governing problem.
2. Presenting the properties of continuous solution.
3. Formulating numerical methods for the given problem.
4. Establishing the stability analysis for the developed numerical methods.
5. Investigating the convergence analysis of the developed numerical methods.
6. Writing an algorithm for the developed numerical methods.
7. Validating the theoretical results with numerical experimentation.
8. Presenting the results using tables and figures.
9. Comparing the obtained results with the findings of previous studies.

CHAPTER FOUR

EXPONENTIALLY FITTED FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITION

This chapter deals with numerical treatment of SPDDEs of reaction-diffusion types with nonlocal boundary condition. The exponential fitting factor is introduced to treat the solutions inside the boundary layer which occur due to perturbation parameter. Exponentially fitted finite difference method is proposed to solve the considered problem. The nonlocal boundary condition is treated using Composite Simpson's $\frac{1}{3}$ rule. The stability and uniform convergence analysis of the proposed method are established. The error estimation of the developed method is shown to be second-order uniform convergent.

4.1 The governing problem

Consider the following SPDDEs with nonlocal boundary condition

$$\begin{aligned}
&\mathcal{L}u(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x), \quad x \in (0,2),\\
&\quad u(x) = \phi(x), \quad x \in [-1,0],\\
&\mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 g(x)u(x)dx = \phi_r,
\end{aligned} \tag{4.1}$$

where, ε is a small positive number $(0 < \varepsilon \ll 1)$. Assume that $a(x) \geq \alpha > 0$, $b(x) \leq \beta < 0$, $\alpha + \beta > \eta > 0$, $f(x), \phi(x), \phi_r$ are sufficiently smooth functions and $g(x)$ is non negative monotone function and satisfy $\int_0^2 g(x)dx < 1$. The above assumptions ensures that $u \in X = C^0(\Omega) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$. As the perturbation parameter tends to zero, the solution manifest strong boundary layers at $x = 0$ and $x = 2$, and interior layer at $x = 1$. The problem (4.1) is equivalently written as

$$\mathcal{L}u(x) = F(x), \tag{4.2}$$

with boundary conditions

$$\begin{cases} u(x) = \phi(x), x \in (-1,0), \\ u(1^-) = u(1^+), u'(1^-) = u'(1^+), \\ \mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 g(x)u(x)dx = \phi_r. \end{cases} \tag{4.3}$$

where

$$\mathcal{L}u(x) = \begin{cases} \mathcal{L}_1 u(x) = -\varepsilon u''(x) + a(x)u(x), & x \in \Omega_1 = (0,1), \\ \mathcal{L}_2 u(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1), & x \in \Omega_2 = (1,2). \end{cases}$$

$$F(x) = \begin{cases} f(x) - b(x)\phi(x-1), & x \in \Omega_1, \\ f(x), & x \in \Omega_2. \end{cases}$$

4.2 Properties of continuous solution

The problem (4.1) fulfills the following continuous maximum principle.

Lemma 4.1. *(Continuous Maximum Principle): Assume $\theta(x)$ be any function such that $\theta(0) \geq 0$, $\mathcal{K}\theta(2) \geq 0$, $\mathcal{L}_1\theta(x) \geq 0$, $\forall x \in \Omega_1$, $\mathcal{L}_2\theta(x) \geq 0$, $\forall x \in \Omega_2$, and $[\theta'](1) \leq 0$, then $\theta(x) \geq 0$, $\forall x \in \bar{\Omega} = [0,2]$.*

Proof. We use proof by contradiction. Let us construct a test function

$$r(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & x \in [0,1], \\ \frac{3}{8} + \frac{x}{4}, & x \in [1,2]. \end{cases} \tag{4.4}$$

Note that $r(x) > 0, \forall x \in \bar{\Omega}$, $\mathcal{L}r(x) > 0$, $\forall x \in \Omega_1 \cup \Omega_2$, $r(0) > 0, \mathcal{K}r(2) > 0$ and $[r'](1) < 0$. Let $\lambda = \max\left\{\frac{-\theta(x)}{r(x)}\right\}$. Then, there exists $x_0 \in \bar{\Omega}$ such that $\theta(x_0) + \lambda r(x_0) = 0$ and $\theta(x) + \lambda r(x) \geq 0, \forall x \in \bar{\Omega}$. Therefore, the function $(\theta + \lambda r)(x)$ attains its minimum at $x = x_0$. Suppose that, the lemma doesn't hold true, then $\lambda > 0$.

Case - I : For $x_0 = 0$; $0 < (\theta + \lambda r)(0) = \theta(0) + \lambda r(0) = 0$.

Case - II : For $x_0 \in \Omega_1$,

$$0 < \mathcal{L}_1(\theta + \lambda r)(x_0) = -\varepsilon(\theta + \lambda r)''(x_0) + a(x_0)(\theta + \lambda r)(x_0) \leq 0.$$

Case - III : For $x_0 = 1$, $\quad 0 \leq [\theta + \lambda r]'(1) = [\theta'](1) + \lambda r'(1) < 0$.

Case - IV : For $x_0 \in \Omega_2$, $0 < \mathcal{L}_2(\theta + \lambda r)(x_0)$

$$= -\varepsilon(\theta + \lambda r)''(x_0) + a(x_0)(\theta + \lambda r)(x_0) + b(x_0)(\theta + \lambda r)(x_0 - 1) \leq 0.$$

Case - V : For $x_0 = 2$, $0 \leq \mathcal{K}(\theta + \lambda r)(2) = (\theta + \lambda r) - \varepsilon \int_0^2 g(x)(\theta + \lambda r)(x)dx \leq 0$.
It contradicts the assumption in all cases. Hence $\lambda > 0$ is impossible. Therefore $\theta(x) \geq 0$, $\forall x \in \bar{\Omega}$. ■

Lemma 4.2. *(Stability Result): The solution $u(x)$ for the problem (4.1) satisfies the bound*

$$\|u(x)\| \leq \max\left\{|u(0)|, |\mathcal{K}u(2)|, ||\mathcal{L}u(x)||\right\}.$$

Proof. Let us construct a barrier functions as
$\psi^{\pm}(x) = Mr(x) \pm u(x),\ x \in \bar{\Omega}$, where $r(x)$ is a test functions in (4.4) and
$M = \max\left\{|u(0)|, |\mathcal{K}u(2)|, \sup|\mathcal{L}u(x)|\right\}$.

For $x = 0$, we have, $\psi^{\pm}(0) = \|u(0)\|r(0) \pm u(0) \geq 0$.

For $0 < x \leq 1$, we have

$$\begin{aligned}
\mathcal{L}_1\psi^{\pm}(x) &= -\varepsilon\psi^{\pm}_{xx}(x) + a(x)\psi^{\pm}(x) \\
&= -\varepsilon\left(Mr(x) \pm u(x)\right)_{xx} + a(x)\left(Mr(x) \pm u(x)\right) \\
&= a(x)Mr(x) \pm \mathcal{L}_1u(x) \\
&\leq \alpha\|\mathcal{L}_1u(x)\|r(x) \pm \mathcal{L}_1u(x) \\
&\geq 0,\ \text{since } \alpha\|\mathcal{L}_1u(x)\|r(x) \geq \mathcal{L}_1u(x).
\end{aligned}$$

For $1 < x < 2$, we have

$$\begin{aligned}
\mathcal{L}_2\psi^{\pm}(x) &= -\varepsilon\psi^{\pm}_{xx}(x) + a(x)\psi^{\pm}(x) + b(x)\psi^{\pm}(x-1) \\
&= -\varepsilon\left(Mr(x) \pm u(x)\right)_{xx} + a(x)\left(Mr(x) \pm u(x)\right) + b(x)\left(Mr(x-1) \pm u(x)\right) \\
&\leq M\left(a(x) + b(x)\right)r(x) \pm \mathcal{L}_2u(x) \\
&\geq M\left(\alpha + \beta\right)r(x) \pm \mathcal{L}_2u(x) \\
&\geq M\eta r(x) \pm \mathcal{L}_2u(x),\ \ \text{since } \alpha + \beta > \eta \\
&\leq C\|\mathcal{L}_2u(x)\|r(x) \pm \mathcal{L}_2u(x) \\
&\geq 0,\ \text{since } C\|\mathcal{L}_2u(x)\|r(x) \geq \mathcal{L}_2u(x).
\end{aligned}$$

For $x = 2$, we have

$$
\begin{aligned}
\mathcal{K}\psi^{\pm}(2) &= \psi^{\pm}(2) - \varepsilon \int_0^2 g(x)\psi^{\pm}(x)dx \\
&= Mr(2) \pm u(2) - \varepsilon \int_0^2 g(x)\left(Mr(x) \pm u(x)\right)dx \\
&= Mr(2) \underbrace{\left[1 - \varepsilon \int_0^2 g(x)dx\right]}_{\geq 0} \pm \left[u(2) - \varepsilon \int_0^2 g(x)u(x)dx\right] \\
&\leq CM \pm \left[u(2) - \varepsilon \int_0^2 g(x)u(x,t)dx\right] \\
&= C\,||\mathcal{K}u|| \pm \mathcal{K}u(2) \\
&\geq 0, \text{ since } ||\mathcal{K}u|| \geq \mathcal{K}u(2,t).
\end{aligned}
$$

Hence, the problem satisfies the given bound. The proof of the lemma is completed. ■

4.3 Bounds on the solution and its derivatives

The bounds on the solution and its derivatives which are required in the convergence analysis of the numerical methods are studied in this section.

Lemma 4.3. *Let $u \in C^2(\Omega)$ be the solution of (4.1). Then, for $1 \leq k \leq 4$ the following bounds hold,*

$$
\left|\left|u^{(k)}(x)\right|\right| \leq C\left(1 + \varepsilon^{-k/2}\right). \tag{4.5}
$$

Proof. For the case $k = 0$, the result follows from Lemma 4.2. To handle for the case $k = 1$, an associated neighborhood is constructed as $N_x = (\delta, \delta + \sqrt{\varepsilon})$, such that $N_x \subset [0, 1]$, where $x \in (0, 1)$. Then, it follows from the mean value theorem, for some $c \in N_x$,

$$
|u'(c)| = \left|\frac{u(\delta + \sqrt{\varepsilon}) - u(\delta)}{\delta + \sqrt{\varepsilon} - \delta}\right| \leq 2\,||u||\,\varepsilon^{-1/2} \leq C\varepsilon^{-1/2}.
$$

Now, for any $x \in N_x$, we have

$$
\begin{aligned}
|u'(x)| = |u'(c) + u'(x) - u'(c)| &= \left|u'(c) + \varepsilon^{-1}\int_c^x u''(s)ds\right| \\
&= \left|u'(c) + \varepsilon^{-1}\int_c^x \left(f(s) - a(s)u(s) - b(s)\phi(s-1)\right)ds\right| \\
&\leq |u'(c)| + \varepsilon^{-1}\left(||f|| + ||au|| + ||b\phi||\right)\varepsilon^{-1/2} \\
&\leq C\varepsilon^{-1/2}.
\end{aligned}
$$

By similar argument we bound $u'(x)$ for $x \in (1,2)$ as $||u'(x)|| \leq C\varepsilon^{-1/2}$. For the case $k = 2$, from (4.1), we have $u''(x) = \varepsilon^{-1}\left(f(x) - a(x)u(x) - b(x)u(x-1)\right)$. By using Lemma 4.1 and 4.2, we obtained $||u''(x)|| \leq C\varepsilon^{-1}$. The bounds on the higher derivatives is obtained using the differential equations and the bounds on $u(x)$ and $u'(x)$ as

$$\left|\left|u^{(k)}\right|\right| \leq C\varepsilon^{\frac{-k}{2}}, \ \ k = 3, 4. \tag{4.6}$$

Hence, by combining the obtained results with the solution, the overall bound becomes $\left|\left|u^{(k)}\right|\right| \leq C\left(1 + \varepsilon^{1-\frac{k}{2}}\right)$, for $0 \leq k \leq 4$. Hence the proof is completed. ■

4.4 The numerical method

The problem (4.1) manifest strong boundary layers at $x = 0$ and $x = 2$ and have a weak interior layer at $x = 1$. Due to a dependence of $a(x)$ and $b(x)$ on spatial variable x, we cannot solve the problem analytically. We divide the interval $[0,2]$ into $2N$ equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \cdots, x_N = 1, x_{N+1}, \cdots, x_{2N} = 2$ be the mesh points. Then we have $x_i = ih, i = 0, 1, 2, \cdots, 2N$. If we consider the interval $x \in (0,1)$ and the coefficients of (4.1) are evaluated at the midpoint of each interval, then we obtain the differential equation

$$\begin{aligned}
&-\varepsilon u''(x) + a(x)u(x) = f(x) - b(x)\phi(x-1), \ \ x \in \Omega_1 = (0,1), \\
&u(0) = \phi(0), \\
&u(1) = \gamma,
\end{aligned} \tag{4.7}$$

where γ is any arbitrary constant. Now, an exponentially fitted operator finite difference method (FOFDM) is presented on the discretized domain $\Omega_1 = [0,1]$. From (4.7) we have

$$-\varepsilon u''(x) + a(x)u(x) = F(x), \ \ x \in \Omega_1 = (0,1), \tag{4.8}$$

where $F(x) = f(x) - b(x)\phi(x-1)$.

To find the numerical solution of (4.8), we use the theory applied in asymptotic method for solving singularly perturbed BVPs. In this scenario, the domain is separated into three subdomains, two boundary-layer subdomains near $x = 0$ and $x = 1$ and one regular subdomain, and the boundary layer problem is changed to a regular problem by proper transformations using stretching variables. The asymptotic expansion solution to the problem in (4.8) based on the theory of singular perturbations presented in (O'Malley, 1991)p.131.

$$u(x,\varepsilon) = \sum_{i=0}^{N}\left[u_i(x) + v_i(\tau) + w_i(\eta)\right]\varepsilon^i, \tag{4.9}$$

where $\tau = \frac{x}{\sqrt{\varepsilon}}$, $\eta = \frac{1-x}{\sqrt{\varepsilon}}$. Then, the zeroth order asymptotic expansion of (4.9) is given as

$$u(x) = u_0(x) + v_0(\tau) + w_0(\eta),$$

where $u_0(x) = \frac{F(x)}{a(x)}$ is a solution of a reduced problem (4.1), which does not satisfy the boundary conditions, $v_0 = Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}x}$ is the left boundary layer correction and $u_0 = Be^{-\sqrt{\frac{a(1)}{\varepsilon}}(1-x)}$ is the right boundary layer correction. Therefore, the asymptotic solution of the zeroth order of (4.7) become

$$u(x) = u_0(x) + Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}x} + Be^{-\sqrt{\frac{a(1)}{\varepsilon}}(1-x)} + \mathcal{O}(\varepsilon), \tag{4.10}$$

where A and B are determined using the given boundary conditions. Now, we separate the range $[0, 1]$ into N equal parts with uniform mesh length h. Let $0 = x_0, x_1, \cdots, x_N = 1$ be the mesh points. Then, we have $x_i = ih$; $i = 0, 1, \cdots, N$. We choose N_1 and N_2 such that $x_{N_1} = \sqrt{\varepsilon}$ and $x_{N_2} = 1 - \sqrt{\varepsilon}$. Then, in the range $[0, \sqrt{\varepsilon}\,]$ the boundary layer at $x = 0$ and in the range $[1 - \sqrt{\varepsilon}, 1]$, the boundary layer will be at $x = 1$.

At $x = x_i$, the above differential equations (4.7) can be written as

$$\begin{cases} -\varepsilon u''(x_i) + a(x_i)u(x_i) = F(x_i), \\ u(0) = \phi(0) \\ u(N) = \gamma. \end{cases} \tag{4.11}$$

For convenience, we take $a(x_i) = a_i, u(x_i) = u_i, F(x_i) = F_i$. Now, consider finite difference for $u_i'' = \dfrac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$, $i = 1, 2, 3, \cdots, N-1$ and by substituting in (4.11), we obtain

$$-\varepsilon\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}\right) + a_i u_i = F_i. \tag{4.12}$$

Case I: Left boundary layer: The problem of the form in (4.7) has left boundary layer at interval $[0, \sqrt{\varepsilon}]$. Then, the zeroth order approximation of asymptotic solution is given as

$$u(x) = u_0(x) + Ae^{\sqrt{\frac{a(0)}{\varepsilon}}x} + \mathcal{O}(\varepsilon), \tag{4.13}$$

where $u_0(x)$ is the solution of the reduced problem and we choose A as a suitable constant. Using Taylor series approximation for $u_0(i+1)h$ and $u_0(i-1)h$ up to first order, we obtain

$$u(x_{i+1}) = u_0(ih) + Ae^{-\sqrt{a(0)}(i+1)\rho}, \tag{4.14}$$
$$u(x_{i-1}) = u_0(ih) + Ae^{-\sqrt{a(0)}(i-1)\rho}, \tag{4.15}$$

where $\rho = h/\sqrt{\varepsilon}$ and $h = 1/N$. To handle the oscillation of the perturbation parameter, we multiply exponentially fitting factor σ_1 for the term with a perturbation parameter as,

$$-\varepsilon\sigma_1 u''(x) + a(x)u(x) = F(x), \tag{4.16}$$

with boundary conditions $u(0) = \phi(0)$ and $u(1) = \gamma$.

If U_i is a discrete solution for $u(x)$ at the mesh point x_i, the numerical method for (4.16) is written in operator form as

$$\mathcal{L}^h U_i = F_i, \quad i = 1, 2, 3, \cdots, N-1.$$

with boundary conditions $U(0) = \phi(0)$, $U(N) = \gamma$, where

$$\mathcal{L}^h U_i = -\varepsilon\sigma_1 \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) + a_i U_i = F_i. \tag{4.17}$$

From (4.17), we have

$$-\frac{\sigma_1}{\rho^2}\left(U_{i-1} - 2U_i + U_{i+1}\right) = F_i - a_i U_i.$$

Now, by taking the limit as $h \to 0$, it becomes

$$-\lim_{h\to} \frac{\sigma_1}{\rho^2}\left(U_{i-1} - 2U_i + U_{i+1}\right) = \lim_{h\to}\left(F_i - a_i U_i\right).$$

Using (4.13)-(4.15), we obtain

$$\sigma_1 = \frac{\rho^2 a(0)}{e^{\sqrt{a(0)}\rho} - 2 + e^{\sqrt{a(0)}\rho}} = \frac{(\frac{\rho}{2})^2 a(0)}{\left(\frac{e^{\sqrt{a(0)}\frac{\rho}{2}} - e^{\sqrt{a(0)}\frac{\rho}{2}}}{2}\right)} = \frac{(\frac{\rho}{2})^2 a(0)}{\left[\sinh\left(\sqrt{a(0)}\frac{\rho}{2}\right)\right]^2}.$$

Hence, after simplification, the exponential fitting factor is obtained as

$$\sigma_1 = \frac{\rho^2 a(0)}{4}\left(\csc h\left(\frac{\rho}{2}\sqrt{a(0)}\right)\right)^2. \tag{4.18}$$

This is a fitting factor in the left boundary layer.

Case II: Right boundary layer: In the interval $[1-\sqrt{\varepsilon}, 1]$, the right boundary layer will be on the right side near to $x = 1$. Now we introduce the exponential fitting factor as

$$-\varepsilon\sigma_2 \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) + a_i U_i = F_i, \;\; i = 1, 2, 3, \cdots, N-1. \tag{4.19}$$

with boundary condition $U(0) = \phi(0)$ and $U(N) = \gamma$.

Now, we introduce the fitting factor σ_2 for discrete problem (4.19) on the right hand side using the right boundary layer asymptotic solution with outer layer

$$u_i = u_0(x_i) + Be^{-\sqrt{\frac{a(1)}{\varepsilon}}(1-x_i)}, \tag{4.20}$$

where $u_0(x_i)$ is the solution of the reduced problem and B is arbitrary constant determined by using boundary condition. Using the same fashion as the left boundary layer case, the exponentially fitting factor is obtained as

$$\sigma_2 = \frac{\rho^2 a(0)}{4}\left(\csc h\left(\frac{\rho}{2}\sqrt{a(1)}\right)\right)^2. \tag{4.21}$$

The required discrete problem become given as

$$\mathcal{L}^h U_i = -\varepsilon\sigma_2\left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) + a_i U_i = F_i, \quad i = 1, 2, 3, \cdots, N-1$$

with boundary conditions $U(0) = \phi(0)$ and $U(N) = \gamma$.

An exponential fitting factor over $\Omega_2 = (1, 2)$ is analogously calculated as a fitting factor given in $\Omega_1 = (0, 1)$. In general, one can take an artificial viscosity (fitting factor) for the given problem on $\bar{\Omega}_i^{2N}$ as

$$\sigma(\rho) = \frac{\rho^2 a(0)}{4}\left(\csc h\left(\frac{\rho}{2}\sqrt{\alpha}\right)\right)^2. \tag{4.22}$$

Suppose that $\bar{\Omega}^{2N}$ denote a separation of $[0, 2]$ into $2N$ sub-intervals such that $0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_N = 1$, and $1 < x_{N+1} < x_{N+2} < \cdots < x_{2N} = 2$ with, $h = 2/2N = 1/N, \;\; i = 1, 2, \cdots, 2N$.

Case I: Consider equation (4.1) on the domain $\Omega_1 = (0, 1)$ which is given by

$$-\varepsilon\sigma(\rho)u''(x) + a(x)u(x) = f(x) - \phi(x-1),$$

where $\sigma(\rho)$ is a fitting factor given in (4.22).

Hence, the required difference equation becomes

$$-\frac{\varepsilon\sigma}{h^2}U_{i-1} + \left(\frac{2\varepsilon\sigma}{h^2} + a_i\right)U_i - \frac{\varepsilon\sigma}{h^2}U_{i+1} = f_i - b_i\phi(x_{i-N}), \tag{4.23}$$

for $i = 1, 2, 3, \cdots, N$. Equation (4.23) can be rewritten as

$$A_i U_{i-1} + B_i U_i + C_i U_{i+1} = H_i,$$

where $A_i = -\frac{\varepsilon\sigma}{h^2}$, $B_i = \frac{2\varepsilon\sigma}{h^2} + a_i$, $C_i = -\frac{\varepsilon\sigma}{h^2}$, $H_i = f_i - b_i\phi(x_{i-N})$.
Case II: Consider equation (4.1) on the domain Ω_2. Then, equation (4.1) also has left boundary layer near $x = 1$ and right boundary layer at $x = 2$. Then, by applying exponentially fitted finite difference scheme, we obtain

$$-\varepsilon\sigma(\rho)\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}\right) + a_i u_i + b_i u_{i-N} + \tau = f_i, \tag{4.24}$$

which is rewritten as

$$E_i U_{i-1} + F_i U_i + G_i U_{i+1} + b_i U_{i-N} = f_i,$$

where $E_i = -\frac{\varepsilon\sigma}{h^2}$, $F_i = \frac{2\varepsilon\sigma}{h^2} + a_i$, $G_i = -\frac{\varepsilon\sigma}{h^2}$.
Case III: For $i = 2N$, we approximate $\int_0^2 g(x)u(x)dx$ using the composite Simpson's $\frac{1}{3}$ rule.

$$\int_0^2 g(x)u(x)dx =$$
$$\frac{h}{3}\left(g(0)u(0) + g(2)u(2) + 2\sum_{i=1}^{2N-1} g(x_{2i})u(x_{2i}) + 4\sum_{i=1}^{2N} g(x_{2i-1})u(x_{2i-1})\right) = \phi_r. \tag{4.25}$$

Since, $u(0) = \phi(0)$, from (4.3), equation (4.25) can be rewritten as

$$-\frac{4\varepsilon h}{3}\sum_{i=1}^{2N} g(x_{2i-1})U(x_{2i-1}) - \frac{2\varepsilon h}{3}\sum_{i=1}^{2N-1} g(x_{2i})U(x_{2i}) + \left(1 - \frac{\varepsilon h g(2)}{3}\right)U(x_{2N}) = \phi_r.$$

Hence, the developed numerical method for solving (4.1) on the entire domain $\bar{\Omega} = [0, 2]$ is given by (4.23)-(4.24) and (4.25), together with the local truncation error of τ.

4.5 Uniform stability analysis

The discrete solution corresponding to equation (4.1) are given as follows:

$$\mathcal{L}_1 U_i = -\varepsilon D^+ D^- U_i + a_i U_i = f_i - b_i\phi_{i-N}, i = 1, 2, 3, \cdots N-1, \tag{4.26}$$
$$\mathcal{L}_2 U_i = -\varepsilon D^+ D^- U_i + a_i U_i + b_i U_{i-N} = f_i,\ i = N+1, N+2, N+3, \cdots 2N-1, \tag{4.27}$$

subject to the boundary conditions:

$$
\begin{aligned}
&U_i = \phi_i, \quad i = -N, -N+1, \cdots, 0 \\
&\mathcal{K}^N U_{2N} = U_{2N} - \sum_{i=1}^{2N} \frac{g_{i-1}U_{i-1} + g_i U_i + g_{i+1}U_{i+1}}{3} h_i \quad \text{and} \\
&D^- U_N = D^+ U_N,
\end{aligned}
\tag{4.28}
$$

where $h_i = x_i - x_{i-1}$.

Lemma 4.4. *(Discrete Maximum principle): Assume*

$$
\sum_{i=1}^{2N} \frac{g_{i-1} + g_i + g_{i+1}}{3} h_i = \lambda < 1
$$

and $\theta(x_i)$ be any function such that $\theta(x_0) \geq 0$, $\mathcal{K}\theta(x_{2N}) \geq 0$, $\mathcal{L}_1\theta(x_i) \geq 0$, $\forall x_i \in \Omega_1^{2N}$, $\mathcal{L}_2\theta(x_i) \geq 0$, $\forall x_i \in \Omega_2^{2N}$, and $D^+(\theta(x_N)) - D^-(\theta(x_N)) \leq 0$, then $\theta(x_i) \geq 0$, $\forall x_i \in \bar{\Omega}^{2N}$.

Proof. Define a test function

$$
r(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, & x_i \in [0,1] \cap \Omega^{2N}, \\ \frac{3}{8} + \frac{x_i}{4}, & x_i \in [1,2] \cap \Omega^{2N}, \end{cases}
\tag{4.29}
$$

Note that $r(x_i) > 0, \forall x_i \in \bar{\Omega}^{2N}$, $\mathcal{L}r(x_i) > 0$, $\forall x_i \in \Omega_1^{2N} \cup \Omega_2^{2N}$, $r(x_0) > 0$, $\mathcal{K}r(x_{2N}) > 0$ and $[r'](N) < 0$. Let $\lambda = \max\left\{\frac{-\theta(x_i)}{r(x_i)}; x_i \in \bar{\Omega}^{2N}\right\}$. Then, there exists $x_0 \in \bar{\Omega}$ such that $\theta(x_0) + \lambda r(x_0) = 0$ and $\theta(x_i) + \lambda r(x_i) \geq 0, \forall x_i \in \bar{\Omega}$. Therefore, the function $(\theta + \lambda r)(x_i)$ attains its minimum at $x = x_k$.
Suppose the lemma doesn't hold true, then $\lambda > 0$.
Case (i): If $x_k = 0$, then $0 < (\theta + \lambda r)(0) = \theta(0) + \lambda r(0) = 0$.
Case (ii): If $x_k \in \Omega_1^{2N}$, then $0 < \mathcal{L}_1(\theta+\lambda r)(x_k) = -\varepsilon(\theta+\lambda r)''(x_k) + a(x_k)(\theta+\lambda r)(x_k) \leq 0$.
Case (iii): If $x_k = x_N$, then $0 \leq D[\theta + \lambda r]'(x_N) = [\theta'](x_N) + \lambda r'(x_N) < 0$.
Case (iv): If $x_k \in \Omega_2$, then $0 < \mathcal{L}_2(\theta + \lambda r)(x_k) =$

$$
-\varepsilon(\theta + \lambda r)''(x_k) + a(x_k)(\theta + \lambda r)(x_k) + b(x_k)(\theta + \lambda r)(x_k - N) \leq 0.
$$

Case (v): If $x_k = x_{2N}$, then $0 \leq \mathcal{K}(\theta + \lambda r)(x_{2N}) = (\theta + \lambda r)x_{2N}$

$$
-\sum_{i=1}^{2N} \frac{g_{i-1}(\theta + \lambda r)x_{i-1} + g_i(\theta + \lambda r)x_i + g_{i+1}(\theta + \lambda r)x_{i+1}}{3} h_i \leq 0.
$$

In all the cases, it contradicts with assumption. Therefore $\lambda > 0$ is impossible. Hence, $\theta(x_i) \geq 0, \forall x_i \in \bar{\Omega}^{2N}$. ■

Since the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the above maximum principle, the discrete solution U_i of the (4.26)-(4.27) is unique if it exists.

Lemma 4.5. *Let $\psi(x_i)$ be any mesh function. Then for $0 \le i \le 2N$ we have the following estimate.*

$$|\psi(x_i)| \le \max\left\{|\psi(x_0)|, |\mathcal{K}\psi(x_{2N})|, \max_{i\in\Omega_1\cup\Omega_2} \left|\mathcal{L}^{2N}\psi(x_i)\right|\right\} \tag{4.30}$$

Proof. The proof is follows from Lemma 4.4 and by constructing a barrier functions

$$\theta^{\pm}(x_i) = \max\left\{|\psi(x_0)|, |\mathcal{K}\psi(x_{2N})|, \max_{i\in\Omega_1\cup\Omega_2} \left|\mathcal{L}^{2N}\psi(x_i)\right|\right\} r_i \pm \psi(x_i), \ \ \forall x_i \in \bar{\Omega}^{2N}.$$

r_i is a test function given in (4.29). ■

4.6 Uniform convergence analysis

In this section the uniform convergence analysis for the developed numerical method is investigated.

Theorem 4.1. *Let $u(x_i)$ and $U(x_i)$ be the continuous solution of (4.1) and discrete solutions of (4.23)-(4.25) respectively. Then, for sufficiently large N, the following truncation error estimate holds:*

$$\sup_{1\le i\le 2N} |U(x_i) - u(x_i)| \le CN^{-2} \tag{4.31}$$

Proof. Let us define a local truncation error as

$$\begin{aligned}
\left|\mathcal{L}^{2N}\left(U(x_i) - u(x_i)\right)\right| &= \left|\mathcal{L}^{2N}U(x_i) - \mathcal{L}^{2N}u(x_i)\right| \\
&= \left|\varepsilon\frac{d^2}{dx^2}u(x_i) - \varepsilon\sigma D^+D^-U(x_i)\right| \\
\left|\mathcal{L}^{2N}\left(U(x_i) - u(x_i)\right)\right| &\le .C\left(|\varepsilon\sigma - \varepsilon|\,|u''(\xi_i)| + \varepsilon h^2\left|u^{(4)}(\xi_i)\right|\right), \xi_i \in (0, x_i)
\end{aligned}$$

where $\sigma(\rho) = \frac{\rho^2 a(0)}{4}\csc h^2\left(\frac{\rho}{2}\sqrt{\alpha}\right)$ and $\rho = N^{-1}/\sqrt{\varepsilon}$.
According to the result given in (Doolan et al., 1980) using $0 < \sigma \le 1$, we obtain $|\varepsilon\sigma - \varepsilon| \le Ch^2$. Here, the target is to show the scheme is convergent independent of the perturbation parameter. By using the bounds for the derivatives of the solution in Lemma 4.3, we obtain

$$\begin{aligned}
\left|\mathcal{L}^{2N}\left(U(x_i) - u(x_i)\right)\right| &\le CN^{-2}\left\|\frac{d^2(u(\xi_i))}{dx^2}\right\| + CN^{-2}\varepsilon\left\|\frac{d^4(u(\xi_i))}{dx^4}\right\| \\
&\le CN^{-2}.
\end{aligned}$$

Hence by discrete maximum principle, we obtain

$$|U(x_i) - u(x_i)| \leq CN^{-2}. \tag{4.32}$$

At the point $x_i = x_{2N}$, we have

$$\begin{aligned}
\mathcal{K}^{2N}\left(U(x_i) - u(x_i)\right) &= \mathcal{K}^{2N}U(x_{2N}) - \mathcal{K}^{2N}u(x_i) = \phi_r - \mathcal{K}^{2N}U(x_{2N}) = \mathcal{K}u(x_i) - \mathcal{K}^{2N}U(x_{2N}),\\
&= u(x_{2N}) - \varepsilon\int_0^2 g(x)u(x)dx - \left(u(x_{2N}) - \varepsilon\int_{x_0}^{x_{2N}} g(x)u(x)dx\right),\\
&= \varepsilon\int_{x_0}^{x_{2N}} g(x)u(x)dx - \varepsilon\sum_{i=1}^{2N}\frac{g_{i-1}u_{i-1} + 4g_iu_i + g_{i+1}u_{i+1}}{3}h\\
&\leq C\varepsilon h^4\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2) + \cdots + u^{(4)}(\xi_{2N})\right)\\
&\leq C\varepsilon h^4\left\|\frac{d^4u(\xi_i)}{dx^4}\right\| \leq C\varepsilon h^4(1+\varepsilon^{-2}) \leq C\varepsilon h^4 + Ch^4\varepsilon^{-1} \leq Ch^2 = CN^{-2}.
\end{aligned}$$

By using the bounds for derivative of the solution in Lemma 4.3 and applying discrete maximum principle, we obtain

$$||U(x_i) - u(x_i)|| \leq CN^{-2}. \tag{4.33}$$

Thus, the results of (4.32) and (4.33) shows (4.31). Hence the proof is complete. ■

4.7 Numerical results and discussions

In this section, the suggested numerical scheme is used to test problems with constant and variable coefficients, left and right boundary layers, and interior layers in order to validate the theoretical results.

Example 4.1. *Consider the singularly perturbed delay differential equations with integral boundary condition:*

$$\begin{cases} -\varepsilon\frac{d^2u(x)}{dx^2} + 5u(x) - u(x-1) = e^{-x},\ x \in (0,2)\\ u(x) = 1,\ \ x \in [-1,0],\ \mathcal{K}u(2) = u(2) - \varepsilon\int_0^2 \frac{x}{3}u(x)dx = 0. \end{cases}$$

Example 4.2. *Consider the singularly perturbed delay differential equations with integral boundary condition:*

$$\begin{cases} -\varepsilon\frac{d^2u(x)}{dx^2} + 5u(x) - xu(x-1) = 1,\ x \in (0,2)\\ u(x) = 1,\ \ x \in [-1,0],\ \ \mathcal{K}u(2) = u(2) - \varepsilon\int_0^2 \frac{1}{6}u(x)dx = 0. \end{cases}$$

As the exact solution for Examples 4.1 and 4.2 is not available, therefore, the accuracy of its numerical solution will be computed using the double mesh principle. This principle is defined as follows: For any values of N, the maximum pointwise absolute error E_ε^N of a numerical solution $U(x_i)$ will be calculated by

$$E_\varepsilon^N = \max_i \left| U_i^N - U_i^{2N} \right|, \tag{4.34}$$

where U_i^N is the numerical solution at x_i with N number of intervals and U_i^{2N} numerical solution at x_i, on a mesh obtained by bisecting the original mesh with $2N$ number of mesh intervals. The ε-uniform error estimate is computed using the formula

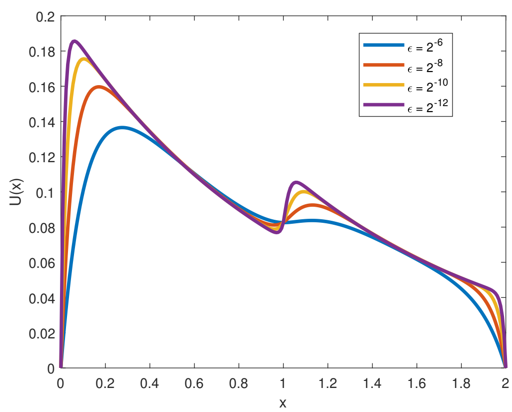
$$E^N = \max_\varepsilon \left(E^N \right). \tag{4.35}$$

The rate of convergence of the method is computed using the formula

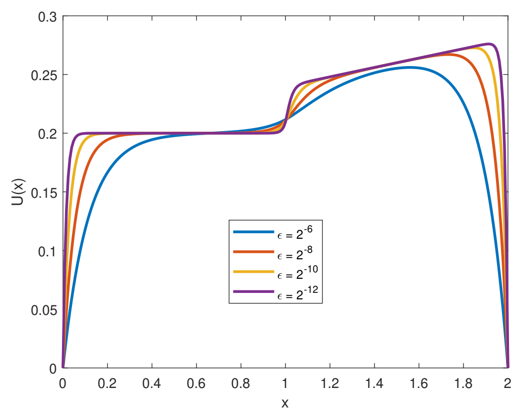
$$R_\varepsilon^N = \log_2 \left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}} \right). \tag{4.36}$$

In the same manner, the ε-uniform rate of convergence is computed using the formula

$$R^N = \log_2 \left(\frac{E^N}{E^{2N}} \right). \tag{4.37}$$



(a)



(b)

Figure 4.1: Graph of numerical solution profile: (a) for Example 4.1, for Example 4.2.

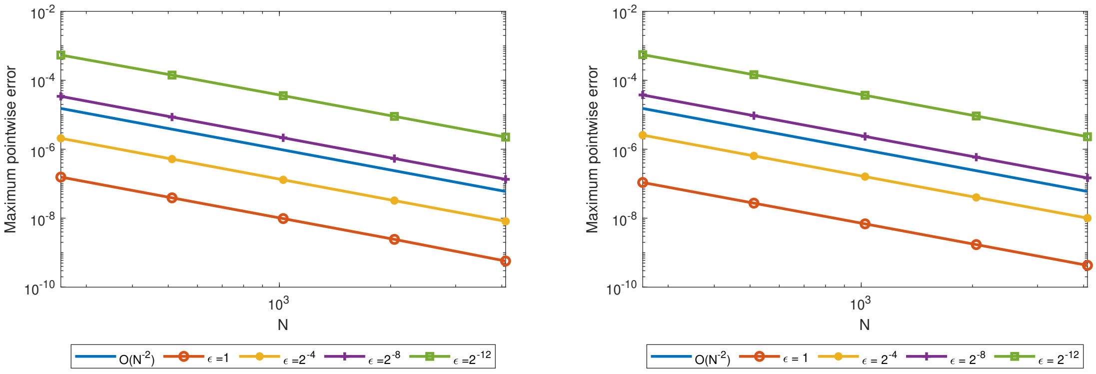
The solutions of the given examples manifest strong boundary layer of thickness $\mathcal{O}(\sqrt{\varepsilon})$ close to $x = 0$ and $x = 2$ and interior layer at $x = 1$. Tables 4.1 and 4.2 indicates the maximum absolute error and rate of convergence of the developed method for Example 4.1 and 4.2 respectively. The given tables clearly shows that the developed method is ε uniform convergence of order two. Figures 4.1(a) and 4.1(b) show a graph of a numerical solution

Table 4.1: Maximum absolute errors and rate of convergence for Example 4.1

$\varepsilon \downarrow N \mapsto$	2^8	2^9	2^{10}	2^{11}	2^{12}
2^0	1.5590e-07	3.8974e-08	9.7458e-09	2.4338e-09	5.7258e-10
	2.0000	1.9997	2.0016	2.0877	
2^{-02}	5.8936e-07	1.4735e-07	3.6837e-08	9.2184e-09	2.2931e-09
	1.9999	2.0000	1.9986	2.0072	
2^{-04}	2.0742e-06	5.1865e-07	1.2967e-07	3.2418e-08	8.1199e-09
	1.9997	1.9999	2.0000	1.9973	
2^{-06}	8.2661e-06	2.0681e-06	5.1714e-07	1.2929e-07	3.2323e-08
	1.9989	1.9997	1.9999	2.0000	
2^{-08}	3.4244e-05	8.5876e-06	2.1486e-06	5.3725e-07	1.3432e-07
	1.9955	1.9989	1.9997	1.9999	
2^{-10}	1.3923e-04	3.5237e-05	8.8363e-06	2.2108e-06	5.5280e-07
	1.9823	1.9956	1.9989	1.9997	
2^{-12}	5.3954e-04	1.4155e-04	3.5824e-05	8.9834e-06	2.2476e-06
	1.9304	1.9823	1.9956	1.9989	
2^{-14}	1.8128e-03	5.4440e-04	1.4281e-04	3.6140e-05	9.0627e-06
	1.7355	1.9306	1.9824	1.9956	
2^{-16}	3.9145e-03	1.8217e-03	5.4691e-04	1.4346e-04	3.6304e-05
	1.1035	1.7359	1.9307	1.9824	
E^N	3.9145e-03	1.8217e-03	5.4691e-04	1.4346e-04	3.6304e-05
R^N	1.1035	1.7359	1.9307	1.9824	

Table 4.2: Maximum absolute errors and rate of convergence for Example 4.2

$\varepsilon \downarrow N \mapsto$	2^8	2^9	2^{10}	2^{11}	2^{12}
2^0	1.0861e-07	2.7323e-08	6.8515e-09	1.7155e-09	4.2928e-10
	1.9910	1.9956	1.9978	1.9986	
2^{-02}	6.6233e-07	1.6559e-07	4.1397e-08	1.0372e-08	2.5643e-09
	1.9999	2.0000	1.9968	2.0161	
2^{-04}	2.5661e-06	6.4162e-07	1.6041e-07	4.0103e-08	1.0065e-08
	1.9998	2.0000	2.0000	1.9944	
2^{-06}	9.6604e-06	2.4168e-06	6.0430e-07	1.5108e-07	3.7770e-08
	1.9990	1.9998	2.0000	2.0000	
2^{-08}	3.7499e-05	9.4020e-06	2.3522e-06	5.8816e-07	1.4705e-07
	1.9958	1.9990	1.9997	1.9999	
2^{-10}	1.4619e-04	3.6986e-05	9.2740e-06	2.3202e-06	5.8017e-07
	1.9828	1.9957	1.9989	1.9997	
2^{-12}	5.5373e-04	1.4516e-04	3.6729e-05	9.2100e-06	2.3042e-06
	1.9315	1.9827	1.9956	1.9989	
2^{-14}	1.8396e-03	5.5161e-04	1.4464e-04	3.6601e-05	9.1780e-06
	1.7377	1.9312	1.9825	1.9956	
2^{-16}	3.9534e-03	1.8351e-03	5.5055e-04	1.4439e-04	3.6537e-05
	1.1072	1.7369	1.9309	1.9825	
E^N	3.9534e-03	1.8351e-03	5.5055e-04	1.4439e-04	3.6537e-05
R^N	1.1072	1.7369	1.9309	1.9825	



(a) (b)

Figure 4.2: Log-Log plot for maximum pointwise error versus N for different values of ε: (a) for Example 4.1, (b) for Example 4.2.

which displays the formation of boundary layer and interior layer as ε goes to zero for Example 4.1 and 4.2 respectively. Figures 4.2(a) and 4.2(b) indicate a Log-Log plot for maximum pointwise error versus mesh number N for different values of perturbation parameter ε for Example 4.1 and 4.2 respectively. These Log-Log plot figures show that the computed errors are decreasing approximately with the rate of $\mathcal{O}(N^{-2})$ as the number of intervals N increases.

4.8 Summary

In this chapter, an exponentially fitted numerical method for singularly perturbed delay differential equations of reaction-diffusion types with nonlocal boundary conditions is presented. Due to the presence of a perturbation parameter on the higher order derivative the solution of the problem exhibit a strong boundary layers at $x = 0$ and $x = 2$ and weak interior layer at $x = 1$. The nonlocal boundary condition is approximated using Simpson's $\frac{1}{3}$ rule. The stability and uniform convergence of the presented method are also investigated. Two test examples are considered for numerical experimentation to validate the applicability of the method. The developed numerical method is proved to be second-order uniformly convergent independent of perturbation parameter. Numerical experiments corroborate the theoretical findings. The developed numerical method is not layer resolving numerical method because of insufficient number of mesh points in the layer. To overcome this drawback, finite difference method on suitable piecewise uniform Shishkin mesh is presented in the next chapter.

CHAPTER FIVE

FITTED MESH FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITION

This chapter presents a numerical method for solving SPDDEs of reaction-diffusion type with nonlocal boundary condition. In the previous Chapter, fitted operator finite difference method is proposed to solve the considered SPPs. Here, a finite difference method on a suitable piecewise uniform Shishkin mesh is developed. The nonlocal boundary condition is approximated using Simpson's $\frac{1}{3}$ rule. Stability and uniform convergence analysis of the numerical scheme are established. The numerical method's error estimate is shown to be nearly second-order uniform convergent.

5.1 The governing problem

Consider the following SPDDEs with nonlocal boundary condition

$$\begin{aligned} \mathcal{L}u(x) &= -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x), \;\; x \in (0,2), \\ u(x) &= \phi(x), \quad x \in [-1,0], \\ \mathcal{K}u(2) &= u(2) - \varepsilon \int_0^2 g(x)u(x)dx = \phi_r. \end{aligned} \tag{5.1}$$

where, ε is a small positive number $(0 < \varepsilon \ll 1)$. Assume that $a(x) \geq \alpha > 0$, $b(x) \leq \beta < 0$, $\alpha + \beta > \eta > 0$, $f(x), \phi(x), \phi_r$ are sufficiently smooth functions and $g(x)$ is non negative monotone function and satisfy $\int_0^2 g(x)dx < 1$. The above assumptions ensures that $u \in X = C^0(\Omega) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$. As the perturbation parameter tends to zero, the solution manifest strong boundary layers at $x = 0$ and $x = 2$, and interior layer at $x = 1$.

The problem (5.1) is equivalent to

$$\mathcal{L}u(x) = F(x), \tag{5.2}$$

with boundary conditions

$$
\begin{aligned}
u(x) &= \phi(x), x \in (-1,0),\\
u(1^-) &= u(1^+), u'(1^-) = u'(1^+),\\
\mathcal{K}u(2) &= u(2) - \varepsilon \int_0^2 g(x)u(x)dx = \phi_r.
\end{aligned}
\tag{5.3}
$$

where

$$
\mathcal{L}u(x) = \begin{cases} \mathcal{L}_1 u(x) = -\varepsilon u''(x) + a(x)u(x), & x \in \Omega_1 = (0,1),\\ \mathcal{L}_2 u(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1), & x \in \Omega_2 = (1,2). \end{cases}
$$

$$
F(x) = \begin{cases} f(x) - b(x)\phi(x-1), & x \in \Omega_1,\\ f(x), & x \in \Omega_2. \end{cases}
$$

5.2 Properties of continuous solution

The problem (5.1) fulfills the following continuous maximum principle.

Lemma 5.1. *(Continuous Maximum Principle): Assume $\theta(x)$ be any function such that $\theta(0) \geq 0$, $\mathcal{K}\theta(2) \geq 0$, $\mathcal{L}_1\theta(x) \geq 0$, $\forall x \in \Omega_1$, $\mathcal{L}_2\theta(x) \geq 0$, $\forall x \in \Omega_2$, and $[\theta'](1) \leq 0$, then $\theta(x) \geq 0$, $\forall x \in \bar{\Omega} = [0,2]$.*

Proof. The proof is given in Lemma 4.1. ■

Lemma 5.2. *(Stability Result): The solution $u(x)$ for the problem in (5.1) satisfies the bound*

$$
\|u(x)\| \leq \max\{|u(0)|, |\mathcal{K}u(2)|, ||\mathcal{L}u(x)||\}.
$$

Proof. For the proof, one can refer Lemma 4.2. ■

5.3 Bounds on the solution and its derivatives

In this section, we study the bounds on the solution and its derivatives which are required in the convergence analysis of the numerical methods.

Lemma 5.3. *Let $u \in C^2(\Omega)$ be the solution of (4.1). Then, for $1 \leq k \leq 4$ the following bounds hold,*

$$
\left|\left|u^{(k)}(x)\right|\right| \leq C\left(1 + \varepsilon^{-k/2}\right). \tag{5.4}
$$

Proof. The proof is given in Lemma 4.3. ■

5.4 Decomposition of the solution

A sharpen bounds on the derivatives and its solution is used to obtain ε-uniform error estimate. The Shishkin decomposition of the form $u(x) = v(x) + w(x)$, where v and w are regular and singular components of the solution respectively, are required to get the sharpen bounds. Here, $v = v_0 + \varepsilon v_1$, where v_0 is the solution of a reduced problem, which is a trivial zero order of differential equation, $a(x)u_0(x) + b(x)u_0(x-1) = f(x),$ for all $x \in \Omega$, and v_1 satisfies

$$\begin{aligned}
\mathcal{L}v_1(x) &= v_0''(x), \ \ x \in \Omega,\\
v_1(x) &= 0, \ \ \ x \in [-1,0],\\
\mathcal{K}v_1(x) &= 0, \ \ x = 2.
\end{aligned}$$

Then, the regular component v satisfies the condition

$$\begin{aligned}
&\mathcal{L}v(x) = f(x), \ \ x \in \Omega,\\
&v(x) = v_0(x), \ \ x \in [-1,0],\\
&\mathcal{K}v(x) = \mathcal{K}v_0(x), \ \ x = 2.
\end{aligned} \tag{5.5}$$

Hence, by using the bounds on v_0'', for $0 \leq k \leq 4$, we have $\left|\left|v_1^{(k)}\right|\right| \leq C\left(1 + \varepsilon^{\frac{-k}{2}}\right)$. The singular component w is devised from

$$\begin{aligned}
&\mathcal{L}_1 w(x) = 0, \ \ x \in \Omega_1,\\
&w(x) = \phi(x) - v_0(x), \ \ x \in [-1,0],\\
&w(1^+) - w(1^-) = -[v(1^+) - v(1^-)],\\
&w(x) = 0, \ \ x = 2,
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&\mathcal{L}_2 w(x) = 0, \ \ x \in \Omega_2,\\
&w(x) = -v(x), \ \ \ x \in [-1,0],\\
&w(1^+) - w(1^-) = -[v(1^+) - v(1^-)],\\
&\mathcal{K}w(x) = \mathcal{K}w(x) - \mathcal{K}v_0(x), \ \ x = 2.
\end{aligned} \tag{5.7}$$

To bound the singular components, we decompose w as $w(x) = w_l(x) + w_r(x)$, where the boundary layer functions are defined as

$$\begin{aligned} &\mathcal{L}_1 w_l(x) = 0, \;\; x \in \Omega_1, \\ &w_l(x) = \phi(x) - v_0(x), \;\; x \in [-1, 0], \\ &w_l(x) = 0, \;\; x = 1, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\mathcal{L}_2 w_l(x) = 0, \;\; x \in \Omega_2, \\ &w_l(x) = A, \;\; x \in [-1, 0], \\ &w_l(x) = 0, \;\; x = 2, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} &\mathcal{L}_1 w_r(x) = 0, \;\; x \in \Omega_1, \\ &w_r(x) = 0, \;\; x \in [-1, 0], \\ &w_r(x) = A, \;\; x = 1, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\mathcal{L}_2 w_r(x) = 0, \;\; x \in \Omega_2, \\ &w_r(x) = 0, \;\; x \in [-1, 0], \\ &\mathcal{K} w_r(x) = \mathcal{K} u(x), \;\; x = 2. \end{aligned} \tag{5.11}$$

Here, the solution is continuous at $x = 1$ and we choose A as an arbitrary constant.

Lemma 5.4. *The regular component $v(x)$ and the singular component $w(x)$ of the solution $u(x)$ of (4.1) satisfy the following bounds*

$$\left|\left|v^{(k)}(x)\right|\right|_{\Omega} \leq C\left(1 + \varepsilon^{\frac{-k}{2}}\right), \tag{5.12}$$

$$\left|\left|w_l^{(k)}(x)\right|\right| \leq \begin{cases} C\varepsilon^{\frac{-k}{2}} e^{\frac{-x}{\sqrt{\varepsilon}}}, & x \in \Omega_1, \\ C\varepsilon^{\frac{-k}{2}} e^{\frac{-(x-1)}{\sqrt{\varepsilon}}}, & x \in \Omega_2, \end{cases} \tag{5.13}$$

$$\left|\left|w_r^{(k)}(x)\right|\right| \leq \begin{cases} C\varepsilon^{\frac{-k}{2}} e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, & x \in \Omega_1, \\ C\varepsilon^{\frac{-k}{2}} e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}, & x \in \Omega_2, \end{cases} \tag{5.14}$$

where C is a generic positive constant independent of the perturbation parameter ε, $x \in \Omega$, $0 \leq k \leq 4$.

Proof. The bounds of the derivative of regular component solution are derived as follows. First, we estimate that the solution of the reduced problem and its derivatives is bounded as

$$\left|\left|v_0^{(k)}(x)\right|\right| \leq C. \tag{5.15}$$

By using Lemma 5.2, we estimate that the derivative of the solution v_1 becomes

$$\left|\left|v_1^{(k)}(x)\right|\right| \leq C\varepsilon^{-k}, \tag{5.16}$$

since $v(x) = v_0(x) + \varepsilon v_1(x)$. Hence, by using (5.15) and (5.16), we obtained the bounds of regular component given in (5.12). To prove inequality (5.13) and (5.14), we use the procedure which is adopted in (Miller et al., 1996) and we obtained that $|w_l(x)| \leq Ce^{\frac{-x}{\sqrt{\varepsilon}}}$ on Ω_1.

To find the bound on w_l on Ω_2, from described equations of w_l, we have

$$\begin{aligned}
\mathcal{L}_2 w_l(x) &= -\varepsilon w_l''(x) + a(x)w_l(x) + b(x)w_l(x-1) = 0\\
&= -\varepsilon w_l''(x) + a(x)w_l(x) = b(x)w_l(x-1)\\
&= \mathcal{L}_1 w_l(x) = b(x)w_l(x-1)\\
|\mathcal{L}_2 w_l(x)| &\leq Ce^{\frac{-(x-1)}{\varepsilon}},
\end{aligned}$$

where C is suitable constant.

Hence, by using stability result Lemma 5.2, we have $|w_l| \leq Ce^{\frac{-(x-1)}{\varepsilon}}$ on Ω_2.

Therefore,

$$\left|\left|w_l^{(k)}(x)\right|\right| \leq \begin{cases} Ce^{\frac{-x}{\sqrt{\varepsilon}}}, & x \in \Omega_1,\\ Ce^{\frac{-(x-1)}{\sqrt{\varepsilon}}}, & x \in \Omega_2. \end{cases}$$

To prove inequality (5.14), we construct a barrier functions $\psi^{\pm}(x)$ as follows.

$$\psi^{\pm}(x) = C\begin{cases} e^{-(1-x)\sqrt{\frac{\alpha}{\varepsilon}}}, & x \in \Omega_1\\ e^{-(2-x)\sqrt{\frac{\alpha}{\varepsilon}}}, & x \in \Omega_2 \end{cases} \pm w_l(x),$$

where, $\alpha = \max\left\{0, 1 - \min_{x\in\bar{\Omega}} a(x)\right\}, \zeta = \max\left\{0, 1 - \min_{x\in\bar{\Omega}} (a(x) + b(x))\right\}.$

Note that $\psi^{\pm}(0) \geq 0$,

$$\begin{aligned}
\mathcal{L}_1\psi^{\pm}(x) &= -\varepsilon\psi^{\pm}(x) + a(x)\psi^{\pm}(x) = 0\\
&= C\left(-\alpha + a(x)\right)e^{-(1-x)\sqrt{\frac{\alpha}{\varepsilon}}}\\
&\geq 0,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_2\psi^{\pm}(x) &= -\varepsilon\psi^{\pm}(x) + a(x)\psi^{\pm}(x) + b(x)\psi^{\pm}(x-1) = 0\\
&= C\left(-\zeta + a(x) + b(x)e^{\frac{-1}{\sqrt{\varepsilon}}}\right)e^{-(2-x)\sqrt{\frac{\alpha}{\varepsilon}}}\\
&\geq \left(-\zeta + a(x) + b(x)\right)e^{-(2-x)\sqrt{\frac{\alpha}{\varepsilon}}}\\
&\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{K}\psi^{\pm}(2) &= \psi^{\pm}(2) - \varepsilon\int_0^2 g(x)\psi^{\pm}(x)dx\\
&= C\left(1-\varepsilon\int_0^2 g(x)dx\right) \pm \mathcal{K}w_r(2)\\
&= C\left(1-\varepsilon\int_0^2 g(x)dx\right) \pm \phi_r\\
&\geq 0, \qquad \text{for a choice of suitable C >0.}
\end{aligned}
$$

Using the continuous maximum principle in Lemma 5.1, we obtain the bound

$$
\left|w_r^{(k)}(x)\right| \leq \begin{cases} C\varepsilon^{\frac{-k}{2}}e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, & x\in\Omega_1,\\ C\varepsilon^{\frac{-k}{2}}e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}, & x\in\Omega_2.\end{cases}
$$

Hence, the proof is completed. ■

5.5 The numerical method

The problems in (5.1) exhibits strong boundary layers at $x = 0$ and $x = 2$ and have interior layer at $x = 1$. A finite difference method to the continuous problem (5.1) on piecewise uniform Shishkin mesh is proposed to solve the considered problem. The second order derivative is replaced by central difference $D_x^+D_x^-U$. The interval $\bar{\Omega} = [0,2]$ is divided into $\bar{\Omega}_1 = [0,1]$ and $\bar{\Omega}_2 = [1,2]$ with $\frac{N}{2}$ equal parts. The interval $\bar{\Omega}_1 = [0,1]$ is partitioned into $[0,\mu]$, $[\mu, 1-\mu]$ and $[1-\mu, 1]$. Similarly $\bar{\Omega}_2 = [1,2]$ is partitioned into $[1, 1+\mu]$, $[1+\mu, 2-\mu]$ and $[2-\mu, 2]$, where μ is a transition parameter which satisfies,

$$
\sigma = \min\left\{q, 2\sqrt{\varepsilon/\alpha}\ln(N)\right\},
$$

where N is the number of mesh points, and $q \in (0,1)$ is roughly the portion of mesh points used to resolve the layer. The discrete method corresponding to the continuous problem (5.1) is given as

$$\mathcal{L}_1^N U_i = -\varepsilon D_x^+ D_x^- U(x_i) + a(x_i)U(x_i) = f_i - b_i\phi_{i-N/2}, \text{ for } i = 1, 2, \cdots, \frac{N}{2} - 1,$$
$$\mathcal{L}_2^N U_i = -\varepsilon D_x^+ D_x^- U(x_i) + a(x_i)U(x_i) + b(x_i)U(x_i - \frac{N}{2}) = f_i, \text{ for } i = \frac{N}{2} + 1, \cdots, N-1, \tag{5.17}$$

subject to the boundary conditions

$$\begin{aligned} &U_i = \phi_i, \quad i = -\frac{N}{2}, -\frac{N}{2} + 1, -\frac{N}{2} + 2, \cdots, 0. \\ &D^- U_{\frac{N}{2}} = D^+ U_{\frac{N}{2}}, \\ &\mathcal{K}^N U_N = U_N - \varepsilon \sum_{i=1}^{N} \left(\frac{g_{i-1}U_{i-1} + 4g_iU_i + g_{i+1}U_{i+1}}{3} \right) h_i, \end{aligned} \tag{5.18}$$

For $i = N$, the integral boundary condition is solved using composite Simpson's $\frac{1}{3}$ rule, which is given as $\int_0^N g(x)u(x)dx \approx \sum_{i=1}^N \left(\frac{g_{i-1}U_{i-1} + 4g_iU_i + g_{i+1}U_{i+1}}{3} \right) h_i$.

5.6 Uniform stability analysis

This section presents the uniform stability estimate of the developed method.

Lemma 5.5. *(Discrete Maximum Principle): Assume* $\sum_{i=1}^N \left(\frac{g_{i-1} + 4g_i + g_{i+1}}{3} \right) h_i = \rho < 1$ *and* $\theta(x_i)$ *be any function such that* $\theta(x_0) \geq 0$, $\mathcal{K}\theta(x_N) \geq 0$, $\mathcal{L}_1\theta(x_i) \geq 0$, $\forall x_i \in \Omega_1^N$, $\mathcal{L}_2\theta(x_i) \geq 0$, $\forall x_i \in \Omega_2^N$, *and* $D^+(\theta(x_{\frac{N}{2}})) - D^-(\theta(x_{\frac{N}{2}})) \leq 0$, *then* $\theta(x_i) \geq 0$, $\forall x_i \in \bar{\Omega}^N$.

Proof. The proof of this lemma is similar with the proof of Lemma 4.4. ■

Lemma 5.6. *Assume* $\psi(x_i)$ *be any mesh function. Then for* $0 \leq i \leq N$ *we have the following estimate*

$$|\psi(x_i)| \leq \max\left\{ |\psi(x_0)|, |\mathcal{K}\psi(x_N)|, \left| \left| \mathcal{L}^N\psi(x_i) \right| \right| \right\}. \tag{5.19}$$

Proof. Consider a barrier function $\nu^{\pm}(x_i) = Mr_i \pm \psi(x_i), \ \forall x_i \in \bar{\Omega}^N$, where $M = \max\left\{ |\psi(x_0)|, |\mathcal{K}\psi(x_N)|, \max_{x_i \in \Omega_1 \cup \Omega_2} \left| \mathcal{L}^N\psi(x_i) \right| \right\}$ and $r(x_i)$ is a test function given in (4.29). From Lemma 5.5, we have $\nu^{\pm}(x_0) \geq 0$ and $\nu^{\pm}(x_N) \geq 0$. And then one can show

$$\begin{aligned} &\mathcal{L}_1^N \nu^{\pm}(x_i) \geq 0, \forall x_i \in \Omega_1^N, \\ &\mathcal{L}_2^N \nu^{\pm}(x_i) \geq 0, \forall x_i \in \Omega_2^N, \\ &D^+\nu^{\pm}(x_N) - D^-\nu^{\pm}(x_N) \leq 0. \end{aligned}$$

Using Lemma 5.6, $\nu^{\pm}(x_i) \geq 0$, for all $x_i \in \Omega^N$. Thus, we concluded that (5.19) holds. ■

5.7 Uniform convergence analysis

To compute the ε-uniform convergence analysis for the numerical scheme, the discrete solution $U(x_i)$ is decomposed into $V(x_i)$ and $W(x_i)$ as $U(x_i) = V(x_i) + W(x_i)$, where $V(x_i)$ satisfy the solution of

$$\begin{aligned} &\mathcal{L}^N V(x_i) = -\varepsilon\delta^2 V(x_i) + a(x_i)V(x_i) + b(x_i)V(x_i - \frac{N}{2}) = f(x_i), \quad x_i \in \Omega^N \backslash \{0, \frac{N}{2}, N\} \\ &\quad V(x_0) = v_0, [D]V(x_{\frac{N}{2}}) = [v'](1), \mathcal{K}V(x_N) = \mathcal{K}v(2) \end{aligned} \tag{5.20}$$

and $W(x_i)$ satisfy the solution of

$$\begin{aligned} &\mathcal{L}^N W(x_i) = -\varepsilon\delta^2 W(x_i) + a(x_i)W(x_i) + b(x_i)W(x_i - \frac{N}{2}) = f(x_i), \quad x_i \in \Omega^N \backslash \{0, \frac{N}{2}, N\} \\ &\quad W(x_0) = w(0), [D]W(x_{\frac{N}{2}}) = -[D]V(x_{\frac{N}{2}}), \mathcal{K}W(x_N) = \mathcal{K}w(2) \end{aligned} \tag{5.21}$$

Theorem 5.1. *Let $V(x_i)$ be an approximate solution of* (5.5) *defined by* (5.20)*. Then*

$$|(V - v)(x_i)| \leq CN^{-2} \ln N, \quad \forall x_i \in \bar{\Omega}^N.$$

Proof. The error estimates of smooth and singular components are derived separately. The error estimates for the smooth component are computed as follows
At the point $x = x_N$, we have

$$\begin{aligned} \mathcal{K}^N (V - v)(x_N) &= \mathcal{K}^N V(x_N) - \mathcal{K}^N v(x_N) \\ &= \mathcal{K}v(x_N) - \mathcal{K}^N V(x_N) \\ &= \varepsilon \int_{x_0}^{x_N} g(x)v(x)dx - \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}V_{i-1} + 4g_iV_i + g_{i+1}V_{i+1}}{3} h_i \\ \left|\mathcal{K}^N (V - v)(x_i)\right| &= \left| C_3\varepsilon \left(h_1^4 \frac{d^4 g}{dx^4}(\xi_1) \frac{d^4 v}{dx^4}(\xi_1) + \cdots h_N^4 \frac{d^4 g}{dx^4}(\xi_N) \frac{d^4 v}{dx^4}(\xi_N) \right) \right| \\ &\leq C_2\varepsilon \left(h_1^4 \frac{d^4 x}{dx^4}(\xi_1) + \cdots + h_N^4 \frac{d^4 x}{dx^4}(\xi_N) \right) \\ &\leq C_1\varepsilon^{-1} \left(e^{-\xi_1\sqrt{\frac{\alpha}{\varepsilon}}} h_1^4 + \cdots + e^{-\xi_N\sqrt{\frac{\alpha}{\varepsilon}}} h_N^4 \right) \\ &\leq CN^{-2}, \end{aligned}$$

where $x_{i-1} \leq \xi \leq x_i, 1 \leq i \leq N$, for a proper choice of C.

From the difference and differential equations, the local truncation error becomes

$$\begin{aligned}\mathcal{L}^N(V-v) &= \mathcal{L}^N V - \mathcal{L}v \\ &= \mathcal{L}v - \mathcal{L}^N v.\end{aligned}$$

By following classical argument given in (Miller et al., 2012), we have

$$|(V-v)(x_i)| \leq CN^{-2}\ln N.$$

■

Theorem 5.2. *Let $W(x_i)$ be an approximate solution of* (5.7)-(5.8) *defined by* (5.21). *Then*

$$|(W-w)(x_i)| \leq CN^{-2}\ln^2 N, \quad \forall x_i \in \Omega^N.$$

Proof. The error estimate of the singular component is equivalent to

$$\begin{aligned}(W-w)(x_i) &= (W_l - w_l)(x_i) + (W_r - w_r)(x_i) \\ \mathcal{L}^N(W-w)(x_i) &\leq -\varepsilon\left(\frac{d^2}{dx^2} - D^+D^-\right)W(x_i).\end{aligned}$$

The bound $W_l(x_i) - w_l(x_i)$ and $W_r(x_i) - w_r(x_i)$ are established separately. By using a classical estimate at each point $x_i,\ \ i = 1, 2, \cdots, N$, it follows that

$$\left|\mathcal{L}^N(W-w)(x_i)\right| \leq \begin{cases} C(N^{-1}\ln N)^2, & \text{if } x_i \in \Omega_1^N \\ C(N^{-1}\ln N)^2, & \text{if } x_i \in \Omega_2^N \end{cases}$$

The error estimate for $W_l - w_l$ is given as follows. The explanation depends on whether $\mu = \frac{1}{4}$ or $\mu = 2\sqrt{\varepsilon}\ln N$.

Case-I: $\mu = \frac{1}{4}$. In this case, the mesh is uniform and $2\sqrt{\varepsilon}\ln N \geq \frac{1}{4}$. It is known that $x_i - x_{i-1} = N^{-1}$ and $\varepsilon^{-\frac{1}{2}} \leq C\ln N$. From (Miller et al., 2012), we have

$$\left|\mathcal{K}^N(W_l - w_l)(x_i)\right| \leq C(N^{-1}\ln N)^2, \quad \forall\, x_i \in \Omega^N.$$

Further

$$
\begin{aligned}
\left|\mathcal{K}^N\left(W_l-w_l\right)\left(x_N\right)\right| &= \mathcal{K}^N W_l(x_N) - \mathcal{K}^N w_l(x_N) \\
&= \phi_r - \mathcal{K}^N w_l(x_N) \\
&= \mathcal{K} w_l(x_N) - \mathcal{K}^N w_l(x_N) \\
&\leq C\varepsilon \left(h_1^4 \frac{d^4}{dx^4} w_l(\xi_i) + \cdots + h_N^4 \frac{d^4}{dx^4} w_l(\xi_N) \right) \\
&\leq C\varepsilon^{-1} \left(h_1^4 + \cdots + h_N^4 \right) \\
&\leq CN^{-2} \ln^2 N
\end{aligned}
$$

where $x_{i-1} \leq \xi_i \leq x_i$. By applying semi-discrete uniform stability Lemma 5.5, we obtain

$$
\left|\left(W_l - w_l\right)(x_i)\right| \leq CN^{-2} \ln^2 N.
$$

Case-II: When $\mu < \frac{1}{4}$. Since the mesh is piecewise uniform in the subinterval $[\mu, 1-\mu]$, the mesh elements are $2(1-2\mu)/N$ and the rest of the intervals $[0,\mu]$ and $[1-\mu, 1]$ with $4\mu/N$ mesh elements. By Miller et al. (2012), we have

$$
\left|\mathcal{K}^N\left(W_l-w_l\right)(x_i)\right| \leq C(N^{-1}\ln N)^2, \quad \forall\, x_i \in \Omega^N
$$

Now

$$
\begin{aligned}
\left|\mathcal{K}^N\left(W_l-w_l\right)\left(x_N\right)\right| &\leq C\varepsilon \left(h_1^4 \frac{d^4}{dx^4} w_l(\xi_i) + \cdots + h_N^4 \frac{d^4}{dx^4} w_l(\xi_N) \right) \\
&\leq C\varepsilon^{-1} \left(e^{-\xi_1\sqrt{\frac{\alpha}{\varepsilon}}} h_1^4 + \cdots + e^{-\xi_N\sqrt{\frac{\alpha}{\varepsilon}}} h_N^4 \right) \\
&\leq CN^{-2} \ln^2 N
\end{aligned}
$$

where $x_{i-1} \leq \xi_i \leq x_i$. By applying the bound given in Theorem 5.4 and semi-discrete uniform stability Lemma 5.5, we obtain

$$
\left|\left(W_l - w_l\right)(x_i)\right| \leq CN^{-2} \ln^2 N. \tag{5.22}
$$

The error estimates for $(W_r - w_r)(x_i)$ is analogous to that of $(W_l - w_l)(x_i)$. This completes the proof. ■

Theorem 5.3. *Let $U(x_i)$ be an approximate solution of* (5.1) *defined by* (5.17)-(5.18)*.Then*

$$
||U - u||_{\bar{\Omega}^N} \leq CN^{-2} \ln^2 N. \tag{5.23}
$$

Proof. The proof follows by combining Theorem 5.1 and 5.2. ■

5.8 Numerical results and discussions

In this section, the presented numerical method is demonstrated by using two model examples given below.

Example 5.1. *Consider SPDDEs with integral boundary condition:*

$$
\begin{aligned}
&-\varepsilon\frac{d^2u(x)}{dx^2}+5u(x)-u(x-1)=e^{-x},\ x\in(0,2)\\
&u(x)=1,\ \ x\in[-1,0],\ \ \mathcal{K}u(2)=u(2)-\varepsilon\int_0^2\frac{x}{3}u(x)dx=0.
\end{aligned}
$$

Example 5.2. *Consider SPDDEs with integral boundary condition:*

$$
\begin{aligned}
&-\varepsilon\frac{d^2u(x)}{dx^2}+5u(x)-xu(x-1)=1,\ x\in(0,2)\\
&u(x)=1,\ \ x\in[-1,0],\ \mathcal{K}u(2)=u(2)-\varepsilon\int_0^2\frac{1}{6}u(x)dx=0.
\end{aligned}
$$

Since the exact solution of the given examples are not known, double mesh techniques is used to compute the maximum pointwise error using (4.34), uniform error estimate using (4.35), rate of convergence using (4.36), and uniform rate of convergence using the formula given in (4.37).

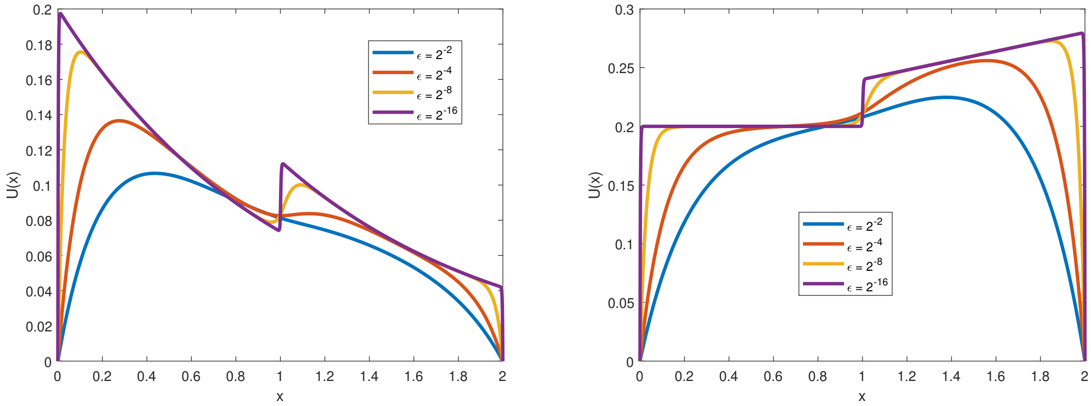
The solutions of the above two examples exhibit strong boundary layers at the left and right sides of the domain and a weak interior layer at $x=1$. The indicated boundary layers are formed because of the perturbation parameter multiplies the highest order derivative of the problems under consideration. The interior layer is also occurred because of a unit delay. Tables 5.1 and 5.2, shows the maximum pointwise error estimates of a proposed scheme for Examples 5.1 and 5.2, respectively, for different values of the perturbation parameter. In the given tables, the corresponding ε-uniform error estimate and ε-uniform rate of convergence are also computed. In these tables, in each column as the perturbation parameter goes small, the maximum absolute error becomes stable and uniform. Figures 5.1(a) and 5.1(b), indicates a graph of numerical solution that display the formation of boundary layers and interior layer for Examples 5.1 and 5.2 respectively for different values of perturbation parameter. The Log-Log scale plots of Examples 5.1 and 5.2 are given in figures 5.2(a) and 5.2(b). Figures 5.3(a) and 5.3(b) shows a plot graphs of maximum pointwise error of Examples 5.1 and 5.2 for different values of perturbation parameter. The developed numerical method is convergent independent of the perturbation parameter ε, with almost second-order of convergence.

Table 5.1: The maximum pointwise error and rate of convergence for Example 5.1.

$\varepsilon \downarrow N \longmapsto$	64	128	256	512	1024	2048
2^{-04}	3.7419e-04	9.5745e-05	2.4077e-05	6.0282e-06	1.5076e-06	3.7694e-07
	1.9665	1.9915	1.9979	1.9995	1.9998	
2^{-06}	4.4111e-04	1.2151e-04	3.1158e-05	7.8396e-06	1.9631e-06	4.9096e-07
	1.8601	1.9634	1.9908	1.9976	1.9995	
2^{-08}	6.8427e-04	1.7516e-04	4.4404e-05	1.1118e-05	2.7806e-06	6.9522e-07
	1.9659	1.9799	1.9978	1.9994	1.9999	
2^{-10}	2.2187e-03	6.9018e-04	1.7697e-04	4.4804e-05	1.1220e-05	2.8066e-06
	1.6847	1.9635	1.9818	1.9976	1.9992	
2^{-12}	2.2184e-03	7.9748e-04	2.7227e-04	8.6977e-05	2.6926e-05	8.1548e-06
	1.4760	1.5504	1.6463	1.6916	1.7233	
2^{-14}	2.2183e-03	7.9745e-04	2.7226e-04	8.6974e-05	2.6925e-05	8.1545e-06
	1.4760	1.5504	1.6463	1.6916	1.7233	
2^{-16}	2.2183e-03	7.9745e-04	2.7226e-04	8.6973e-05	2.6925e-05	8.1544e-06
	1.4760	1.5504	1.6463	1.6916	1.7233	
2^{-18}	2.2183e-03	7.9745e-04	2.7226e-04	8.6973e-05	2.6925e-05	8.1544e-06
	1.4760	1.5504	1.6463	1.6916	1.7233	
2^{-20}	2.2183e-03	7.9745e-04	2.7226e-04	8.6973e-05	2.6925e-05	8.1544e-06
	1.4760	1.5504	1.6463	1.6916	1.7233	
E^N	2.2183e-03	7.9745e-04	2.7226e-04	8.6973e-05	2.6925e-05	8.1544e-06
R^N	1.4760	1.5504	1.6463	1.6916	1.7233	

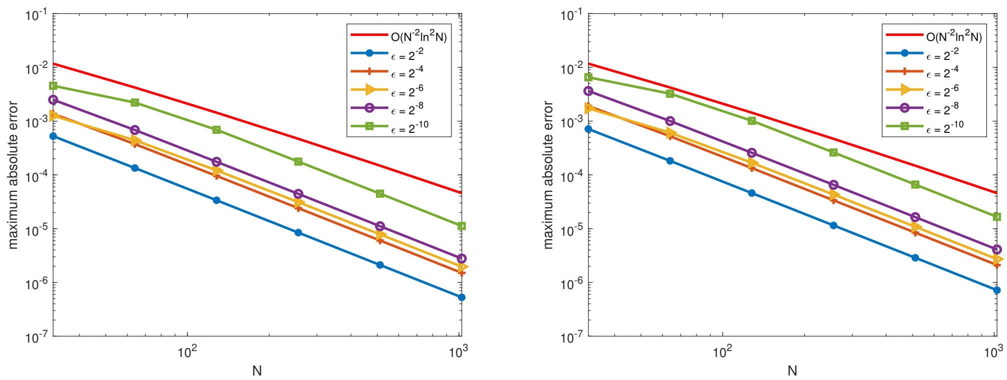
Table 5.2: The maximum pointwise error and rate of convergence for Example 5.2.

$\varepsilon \downarrow N \longmapsto$	64	128	256	512	1024	2048
2^{-04}	5.2311e-04	1.3391e-04	3.3679e-05	8.4324e-06	2.1089e-06	6.8000e-07
	1.9659	1.9913	1.9978	1.9995	1.6329	
2^{-06}	6.0930e-04	1.6818e-04	4.3147e-05	1.0858e-05	2.7189e-06	1.0246e-06
	1.8571	1.9627	1.9905	1.9977	1.4080	
2^{-08}	1.0046e-03	2.5783e-04	6.5427e-05	1.6385e-05	4.0979e-06	1.0246e-06
	1.9621	1.9785	1.9975	1.9994	1.9998	
2^{-10}	3.2426e-03	1.0158e-03	2.6117e-04	6.6227e-05	1.6588e-05	4.1489e-06
	1.6745	1.9596	1.9795	1.9973	1.9993e	
2^{-12}	3.2440e-03	1.1731e-03	4.0213e-04	1.2865e-04	3.9842e-05	1.2067e-05
	1.4674	1.5446	1.6442	1.6911	1.7232	
2^{-14}	3.2443e-03	1.1732e-03	4.0219e-04	1.2867e-04	3.9848e-05	1.2067e-05
	1.4675	1.5445	1.6442	1.6911	1.7234	
2^{-16}	3.2443e-03	1.1732e-03	4.0220e-04	1.2868e-04	3.9849e-05	1.2067e-05
	1.4675	1.5445	1.6441	1.6912	1.7235	
2^{-18}	3.2443e-03	1.1732e-03	4.0220e-04	1.2868e-04	3.9849e-05	1.2067e-05
	1.4675	1.5445	1.6441	1.6912	1.7235	
2^{-20}	3.2443e-03	1.1732e-03	4.0220e-04	1.2868e-04	3.9849e-05	1.2067e-05
	1.4675	1.5445	1.6441	1.6912	1.7235	
E^N	3.2443e-03	1.1732e-03	4.0220e-04	1.2868e-04	3.9849e-05	1.2067e-05
R^N	1.4675	1.5445	1.6441	1.6912	1.7235	



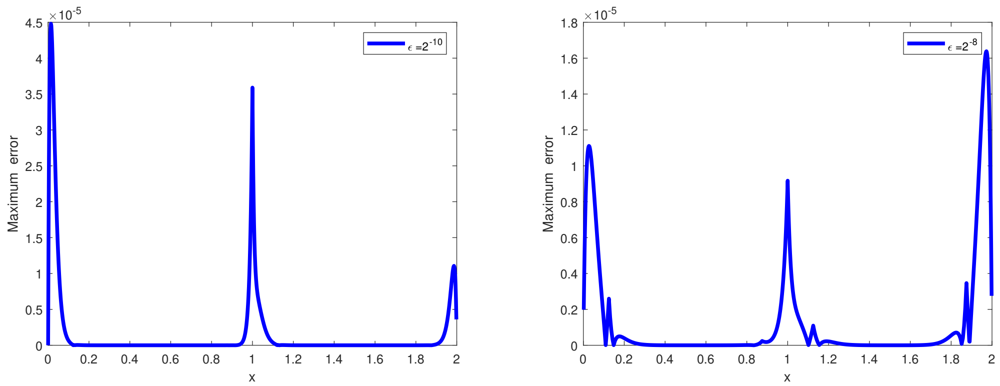
(a) (b)

Figure 5.1: Graph of numerical solution profile: (a) for Example 5.1, for Example 5.2.



(a) (b)

Figure 5.2: Log-Log plot of E^N vs N for different ε: (a) for Example 5.1, (b) for Example 5.2.



(a) (b)

Figure 5.3: Graph of pointwise absolute error: (a) for Example 5.1, (b) for Example 5.2.

Table 5.3: Comparison of uniform error estimate and uniform rate of convergence for Example 5.1.

$N \longmapsto$	256	512	1024	2048	4096
Present Method					
E^N	2.7226e-04	8.6973e-05	2.6925e-05	8.1544e-06	2.4269e-06
R^N	1.6463	1.6916	1.7233	1.7485	-
Method in Chapter 4					
E^N	3.9145e-03	1.8217e-03	5.4691e-04	1.4346e-04	3.6304e-05
R^N	1.1035	1.7359	1.9307	1.9824	-

Table 5.4: Comparison of uniform error estimate and uniform rate of convergence for Example 5.2.

$N \longmapsto$	256	512	1024	2048	4096
Present Method					
E^N	4.0220e-04	1.2868e-04	3.9849e-05	1.2067e-05	3.5916e-06
R^N	1.6441	1.6912	1.7235	1.7484	-
Method in Chapter 4					
E^N	3.9534e-03	1.8351e-03	5.5055e-04	1.4439e-04	3.6537e-05
R^N	1.1072	1.7369	1.9309	1.9825	-

Tables 5.3 and 5.4 indicates the comparison of uniform error estimate and rate of convergence of Examples 5.1 and 5.2 respectively, for exponentially fitted finite difference method and fitted mesh finite difference method. One can understand that exponentially fitted finite difference method is not a layer resolving numerical method. The fitted mesh method uses a special non-uniform mesh condensed around the layer region, and it is a recognized technique for solving SPDEs that overcomes the limitations of exponentially fitted numerical method. Fitted mesh numerical method would basically require a logarithmic factor, which affects the rate of convergence and has limitations over the rate of convergence.

5.9 Summary

A numerical solution for solving singularly perturbed delay reaction-diffusion problem with integral boundary condition is presented. The properties of the continuous solution are studied. A finite difference method on an appropriate piecewise uniform Shishkin mesh is proposed. The integral boundary condition is approximated by Simpson's $\frac{1}{3}$ rule. The uniform stability and uniform convergence analysis of the developed method are established. The developed method is uniformly convergent independent of perturbation parameter with almost second-order uniform rate of convergence. To validate the applicability of the developed numerical method, two experimental numerical examples are considered. The numerical results reflect the theoretical estimations.

CHAPTER SIX

NONSTANDARD FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITION

This chapter presents a numerical treatment of singularly perturbed parabolic partial differential equations with nonlocal boundary conditions. The problem has boundary layers at $x = 0$ and $x = 1$. The nonstandard finite difference method is developed to solve the considered problem in the spatial direction, and the implicit Euler method is proposed to solve the resulting system of IVPs in the temporal direction. The nonlocal boundary condition is approximated using Simpson's $\frac{1}{3}$ rule. The stability and uniform convergence analysis of the scheme are studied. The developed scheme is second-order uniformly convergent in the spatial direction and first-order in the temporal direction. Two test examples are carried out to validate the applicability of the developed numerical scheme. The obtained numerical results agreed with the theoretical estimate.

6.1 The governing problem

Consider a class of singularly perturbed parabolic partial differential equations of the reaction-diffusion types with nonlocal boundary condition

$$
\begin{aligned}
\mathcal{L}u(x,t) &= -\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial t} + a(x,t)u(x,t) = f(x,t), \ \ (x,t)\in\Omega,\\
u(x,0) &= \phi_b(x), \ \ \forall(x,t)\in\Gamma_b = \{(x,0); 0\le x\le 1\},\\
u(0,t) &= \phi_l(t), \ \ \forall(x,t)\in\Gamma_l = \{(0,t); 0\le t\le T\},\\
\mathcal{K}u(x,t) &= u(1,t) - \varepsilon\int_0^1 g(x)u(x,t)dx = \phi_r(x,t), \ \ \forall(x,t)\in\Gamma_r = \{(1,t); 0\le t\le T\}.
\end{aligned}
\tag{6.1}
$$

where $(x,t)\in\Omega = \Omega_x\times\Omega_t = (0,1)\times(0,T]$, $\bar{\Omega} = [0,1]\times[0,T]$, and ε is a small parameter ($0 < \varepsilon \ll 1$). Suppose that $a(x,t)\ge\alpha > 0$, $f(x,t)$, ϕ_l, ϕ_r, ϕ_b are sufficiently smooth functions and $g(x)$ is nonnegative monotone function and satisfies $\int_0^1 g(x)dx < 1$. For the problem (6.1) to have existence and uniqueness of solution, we assume that the coefficients of the given problem are Hölder continuous and also impose a proper compatibility conditions at the corners (Ladyzhenskaya et al., 1968). Note that ϕ_l and ϕ_r are only functions of t, while ϕ_b is a function of x only. The problems necessarily satisfies the following sufficient compatibility conditions $\phi_b(0,0) = \phi_l(0,0),\ \phi_b(1,0) = \phi_r(1,0)$, and

$$-\varepsilon\frac{\partial^2\phi_b(0,0)}{\partial x^2} + a(0,0)\phi_b(0,0) + \frac{\partial\phi_l(0,0)}{\partial t} = f(0,0),$$
$$-\varepsilon\frac{\partial^2\phi_b(1,0)}{\partial x^2} + a(1,0)\phi_b(1,0) + \frac{\partial\phi_r(1,0)}{\partial t} = f(1,0).$$

Note that ϕ_l, ϕ_r, and ϕ_b are assumed to be sufficiently smooth for (6.1) to make sense, namely $\phi_l, \phi_r \in C^1([0,T])$ and $\phi_b \in C^{(2,1)}(\Gamma_b)$.

6.2 Properties of continuous solution

Here, some properties of the continuous solution (6.1) which guarantee the existence and uniqueness of the analytical solution is analyzed. A replication of this belonging in semi-discrete form can be used to present the approximate solution, which we provide in the following section.

Lemma 6.1. *(Continuous Maximum Principle) Let* $\Psi(x,t) \in C^{(0,0)}(\bar{\Omega}) \cap C^{(1,0)}(\Omega) \cap C^{(2,1)}(\Omega)$ *be a sufficiently smooth function such that* $\Psi(0,t) \geq 0, \Psi(x,0) \geq 0, \mathcal{K}\Psi(1,t) \geq 0, \mathcal{L}\Psi(x,t) \geq 0, \forall(x,t) \in \Omega$. *Then* $\Psi(x,t) \geq 0,\ \forall(x,t) \in \bar{\Omega}$, *where* $\mathcal{L}\Psi(x,t) = \Psi_t(x,t) - \varepsilon\Psi_{xx}(x,t) + a\Psi(x,t)$.

Proof. Assume (x^*,t^*) be defined as $\Psi(x^*,t^*) = \min_{(x,t)\in\bar{\Omega}} \Psi(x,t)$ and suppose that $\Psi(x^*,t^*) \leq 0$. It is clear that $(x^*,t^*) \notin \partial\Omega$. Thus,
$\mathcal{L}\Psi(x^*,t^*) = \Psi_t(x^*,t^*) - \varepsilon\Psi_{xx}(x^*,t^*) + a(x,t)\Psi(x^*,t^*)$. Since $\Psi(x^*,t^*) = \min_{(x,t)\in\bar{\Omega}} \Psi(x,t)$, which indicates that $\Psi(x^*,t^*) = 0,\ \Psi_t(x^*,t^*) = 0,\ \Psi_{xx}(x^*,t^*) \geq 0$ and implies that $\mathcal{L}\Psi(x^*,t^*) < 0$, which is contradicts with the above assumption. $\mathcal{L}\Psi(x^*,t^*) > 0, \forall x \in \Omega$. So that, $\Psi(x,t) \geq 0, \forall(x,t) \in \Omega$. ■

Lemma 6.2. *(Stability Result) Assume* $u(x,t)$ *be the solution of the continuous problem in* (6.1). *Then we have the bound*

$$u(x,t) \leq \alpha^{-1}\,||f|| + \max\left\{\phi_b(x), \max\left\{\phi_l(x,t), \phi_r(x,t)\right\}\right\},$$

where $||f|| = \max\{f(x,t)\}$.

Proof. We prove by using maximum principle Lemma (6.1) and by constructing the barrier functions $\theta^{\pm}(x,t) = CM \pm u(x,t), (x,t) \in \bar{\Omega}$.
Where $M = \alpha^{-1}\,||f|| + \max\left\{\phi_b(x), \max\left\{\phi_l(x,t), \phi_r(x,t)\right\}\right\}$.

At initial, we have

$$\begin{aligned}\theta^{\pm}(x,0) &= \alpha^{-1}\,||f|| + \max\left\{\phi_b(x), \max\left\{\phi_l(x,0), \phi_r(x,0)\right\}\right\} \pm u(x,0)\\ &= \alpha^{-1}\,||f|| + \max\left\{\phi_b(x)\right\} \pm \phi_b(x) \geq 0.\end{aligned}$$

For $x = 0$, we have

$$\begin{aligned}\theta^{\pm}(0,t) &= \alpha^{-1}\,||f|| + \max\left\{\phi_b(0), \max\left\{\phi_l(0,t), \phi_r(0,t)\right\}\right\} \pm u(0,t)\\ &= \alpha^{-1}\,||f|| + \max\left\{\phi_l(t)\right\} \pm \phi_l(t) \geq 0.\end{aligned}$$

For $x = 1$, we have

$$\begin{aligned}\mathcal{K}\theta^{\pm}(1,t) &= \alpha^{-1}\,||f|| + \max\left\{\phi_b(1), \max\left\{\phi_l(1,t), \mathcal{K}\phi_r(1,t)\right\}\right\} \pm \mathcal{K}u(1,t)\\ &= \alpha^{-1}\,||f|| + \max\left\{\phi_r(1,t)\right\} \pm \phi_r(1,t) \geq 0.\end{aligned}$$

For $0 < x < 1$, we have

$$\begin{aligned}\mathcal{L}\theta^{\pm}(x,t) &= \theta^{\pm}_t(x,t) - \varepsilon\theta^{\pm}_{xx}(x,t) + a(x,t)\theta^{\pm}(x,t),\\ &= \left[\alpha^{-1}\,||f|| + \max\left\{\phi_b(x), \max\left\{\phi_l(x,t), \phi_r(x,t)\right\}\right\} \pm u(x,t)\right]_t\\ &\quad -\varepsilon\left[\alpha^{-1}\,||f|| + \max\left\{\phi_b(x), \max\left\{\phi_l(x,t), \phi_r(x,t)\right\}\right\} \pm u(x,t)\right]_{xx}\\ &\quad +a(x,t)\left(\alpha^{-1}\,||f|| + \max\left\{\phi_b(x), \max\left\{\phi_l(x,t), \phi_r(x,t)\right\}\right\} \pm u(x,t)\right)\\ &= \max\left\{\phi_{l_t}(x,t), \phi_{r_t}(x,t)\right\} \pm u_t(x,t) - \varepsilon\max\left\{\phi_{b_{xx}}(x), \phi_{l_{xx}}(x,t), \phi_{r_{xx}}(x,t)\right\}\\ &\pm -\varepsilon u_{xx}(x,t) + \alpha\alpha^{-1}\,||f|| + \alpha\max\left\{\phi_b(x), \max\left\{\phi_l(x,t), \phi_r(x,t)\right\}\right\} \pm \alpha u(x,t)\\ &\geq 0,\end{aligned}$$

where $\varepsilon > 0, a(x,t) \geq \alpha > 0$. This implies that $\mathcal{L}\theta^{\pm}(x,t) \geq 0$. Hence, by Lemma 6.1 we have, $\theta^{\pm}(x,t) \geq 0, \forall (x,t) \in \bar{\Omega}$, which indicates

$u(x,t) \leq \alpha^{-1}\,||f|| + \max\left\{\phi_b(x), \max\left\{\phi_l(x,t), \phi_r(x,t)\right\}\right\}.$ ■

6.3 Bounds on the solution and its derivatives

This section studies the bounds on the solution and its derivatives which are required in the convergence analysis of the numerical methods.

Lemma 6.3. *Suppose* $a(x,t), f(x,t) \in C^{(2,1)}(\bar{\Omega})$ *and* $\phi_l \in C^2([0,T]), \phi_b \in C^{(4,2)}(\Gamma_b), \phi_r \in C^2([0,T])$. *Then the problem (6.1) has a unique solution* u *which satisfies* $u \in C^{(4,2)}(\bar{\Omega})$. *And also the derivatives of solution* u *are bounded,* $\forall i, j \in \mathbf{Z} \geq 0$ *such that* $0 \leq i + 2j \leq 4$,

$$\left|\left|\frac{\partial^{i+j}u}{\partial x^i \partial t^j}\right|\right| \leq C\varepsilon^{\frac{-i}{2}}.$$

Proof. The boundedness of the solution and its derivative is given as follows. Under the stretched transformation $\tilde{x} = \frac{x}{\sqrt{\varepsilon}}$ problem (6.1) can be rewritten as

$$\begin{aligned}
\mathcal{L}\tilde{u}(\tilde{x},t) &= \left(-\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial}{\partial t} + \tilde{a}(\tilde{x},t)\right)\tilde{u}(\tilde{x},t) = f(\tilde{x},t), \quad (\tilde{x},t)\in\tilde{\Omega}_\varepsilon \\
\tilde{u}(\tilde{x},t) &= \phi_b(\tilde{x},t), \qquad (\tilde{x},t)\in\tilde{\Gamma}_b \\
\tilde{u}(\tilde{x},t) &= \phi_l(\tilde{x},t), \qquad (\tilde{x},t)\in\tilde{\Gamma}_l \\
\mathcal{K}\tilde{u}(\tilde{x},t) &= \tilde{u}(1,t) - \varepsilon\int_0^{1/\sqrt{\varepsilon}} g(x)\tilde{z}(\tilde{x},t)dx = \phi_r(\tilde{x},t), \qquad (\tilde{x},t)\in\tilde{\Gamma}_r
\end{aligned} \tag{6.2}$$

where $\tilde{\Omega}_\varepsilon = (0, \frac{1}{\sqrt{\varepsilon}}) \times (0,T)$, and the boundary condition $\tilde{\Gamma}$ to Γ, where equation (6.2) is independent of ε. Then, by taking the idea of estimation (10.6) on (p.352) of (Ladyzhenskaya et al., 1968), we obtained

$$\|\frac{\partial^{i+j}\tilde{u}}{\partial\tilde{x}^i\partial t^j}\|_{\tilde{N}_\delta} \le C\left(1 + \|\tilde{u}\|_{\tilde{N}_{2\delta}}\right),$$

$\forall\ \tilde{N}_\delta$ in $\tilde{\Omega}_\varepsilon$. Here $\tilde{N}_\delta$, $\delta > 0$ is a neighbourhood with diameter δ in $\tilde{\Omega}_\varepsilon$. Returning to the original variable given as

$$\|\frac{\partial^{i+j}u}{\partial\tilde{x}^i\partial t^j}\|_{\bar{\Omega}} \le C\varepsilon^{\frac{-i}{2}}\left(1 + \|u\|_{\bar{\Omega}}\right).$$

Hence, the proof is complete by using the bound on u of Lemma 6.2. ■

The bounds of the derivatives of the solution given in Lemma 6.3 were derived from classical results. They are not adequate for the proof of the ε-uniform error estimate. Stronger bounds on these derivatives are now obtained by a method originally devised in (Miller et al., 1998). The main idea is to decompose the solution u into smooth and singular components.

Lemma 6.4. *Suppose $a(x,t), f(x,t) \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{\Omega})$, and $\phi_l \in C^{(3+\alpha_1/2)}([0,T]), \phi_b \in C^{(6+\alpha_1, 3+\alpha_1/2)}(\Gamma_b)$, $\phi_r \in C^{(3+\alpha_1/2)}([0,T])$, where $\alpha_1 \in (0,1)$. Then we obtain*

$$\|\frac{\partial^{i+j}v}{\partial x^i\partial t^j}\|_{\bar{\Omega}} \le C\left(1+\varepsilon^{1-i/2}\right), \quad (x,t)\in\Omega, \tag{6.3}$$

$$|\frac{\partial^{i+j}w_l}{\partial x^i\partial t^j}| \le C\varepsilon^{\frac{-i}{2}}e^{\frac{x}{\sqrt{\varepsilon}}}, \quad (x,t)\in\Omega, \tag{6.4}$$

$$|\frac{\partial^{i+j}w_r}{\partial x^i\partial t^j}| \le C\varepsilon^{\frac{-i}{2}}e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, \quad (x,t)\in\Omega, \tag{6.5}$$

where C is a constant independent of a parameter ε, $0 \le i + 2j \le 4$.

Proof. The existence and smooth component results follows from Ladyzhenskaya et al. (1968) (Chap IV, p.320). The derivative of smooth component functions are bounded, it derived as follows. First, we estimate to reduced problem solution v_0 is bounded and its derivative are bounded

$$\left|\left|\frac{\partial^{i+j} v_0}{\partial x^i \partial t^j}\right|\right|_{\bar{\Omega}} \leq C. \tag{6.6}$$

Next, to estimate the derivative of the solution v_1 bounded form Lemma 6.2 can be applied. Then

$$\left|\left|\frac{\partial^{i+j} v_1}{\partial x^i \partial t^j}\right|\right|_{\bar{\Omega}} \leq C\varepsilon^{\frac{-i}{2}}. \tag{6.7}$$

Since

$$\frac{\partial^{i+j} v}{\partial x^i \partial t^j} = \frac{\partial^{i+j} v_0}{\partial x^i \partial t^j} + \varepsilon \frac{\partial^{i+j} v_1}{\partial x^i \partial t^j},$$

to establish a smooth component v estimates from (6.6) and (6.7). To prove the inequality (6.4) using the procedure adapted in Miller et al. (1998), it is easy to find that

$$|w_l(x,t)| \leq Ce^{\frac{-x}{\sqrt{\varepsilon}}}, \quad (x,t) \in \Omega.$$

To prove (6.5), we consider two comparison functions

$$\Theta^{\pm}(x,t) = Ce^{\alpha^* t} e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} \pm w_r(x,t),$$

where $\alpha^* = \max\left\{0, 1 - \min_{(x,t)\in\bar{\Omega}} a(x,t)\right\}$. Note that $\Theta^{\pm}(x,t) \geq 0$,

$$\begin{aligned}
\mathcal{L}\Theta^{\pm}(x,t) &= -\varepsilon\Theta^{\pm}_{xx}(x,t) + \Theta^{\pm}_t(x,t) + a(x,t)\Theta^{\pm}(x,t) \\
&= C\left(\alpha^* - 1 + a(x,t)\right) e^{\alpha^* t} e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} \\
&\geq 0. \\
\mathcal{K}\Theta^{\pm}(1,t) &= \Theta^{\pm}(1,t) - \varepsilon \int_0^1 \Theta^{\pm}(x,t)dx \\
&= C - C\varepsilon \int_0^1 g(x) e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} dx \\
&\geq C(1 - \varepsilon \int_0^1 g(x)dx) \\
&\geq 0.
\end{aligned}$$

choice suitable $C > 0$. Using maximum principle Lemma 6.1, we have

$$\left|\frac{\partial^{i+j} w_r}{\partial x^i \partial t^j}\right| \leq Ce^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, \quad (x,t) \in \Omega.$$

Let $\tilde{x} = \frac{x}{\sqrt{\varepsilon}}$, Clearly, under the transformation, stretched variable $\tilde{x}$ domain is $(0, \frac{1}{\sqrt{\varepsilon}})$, note that $(x,t) \Rightarrow (\tilde{x},t)$ and the parameter ε is independent of problem (6.2), then a suitable estimates in Ladyzhenskaya et al. (1968) (Section 4.10) solution of $\tilde{w}_r$. The position of $\tilde{x}$ argument partition into two cases. First case, neighborhood of $\tilde{N}_\delta$ in $(0, \frac{1}{\sqrt{\varepsilon}}) \times (0, T]$, by applying an estimate in Ladyzhenskaya et al. (1968) (Section 4.10), we have

$$\left|\left|\frac{\partial^{i+j}\tilde{w}_r}{\partial\tilde{x}^i\partial t^j}\right|\right|_{\tilde{N}_\delta} \le C\left(||\tilde{u}||_{\tilde{N}_\delta}\right),$$

and the required bound follows by transforming back to the variable x and using the bound just obtained on w_r. Likewise, for each neighborhood $\tilde{N}_\delta$ in $(0,1] \times (0,T]$, from (Ladyzhenskaya et al., 1968) (Section 4.10) we have

$$\left|\left|\frac{\partial^{i+j}\tilde{w}_r}{\partial\tilde{x}^i\partial t^j}\right|\right|_{\tilde{N}_\delta} \le C\left(1 + ||\tilde{w}_r||_{\tilde{N}_\delta}\right),$$

and the required bound follows by transforming back to the variable x, using the bound on w_l and noting that $e^{\frac{-x}{\varepsilon}} \ge e^{-1} = C$ for $\tilde{x} \le 1$. Hence the proof is complete. ■

6.4 The numerical method

This section presents a numerical method using a method of lines. Nonstandard finite difference method is used to discretize the spatial direction and implicit Euler method is used for temporal direction.

6.4.1 Spatial semi-discretization

The fundamental idea of non-standard discrete modeling techniques is the development of the exact finite difference technique. Micken presented methods and rules for developing nonstandard FDMs for various types of problems (Mickens, 2005). To develop a discrete scheme in keeping with Mickens' guidelines, the denominator characteristic for the discrete derivatives should be described in terms of more complicated functions with larger step sizes than those used in classical methods. These complicated functions are a general property of the method that may be useful when constructing a dependable methods for such problems.

To construct a genuine finite difference scheme for the problem of the form in (6.1), we use the procedures described in (Woldaregay & Duressa, 2022). The constant coefficient given in equation (6.8) without the time variable is considered as follows.

$$-\varepsilon\frac{d^2u(x)}{dx^2} + au(x) = 0. \tag{6.8}$$

By solving (6.8), we obtain two independent solutions $e^{\mu_1 x}$ and $e^{\mu_2 x}$, where

$$\mu_{1,2} = \pm\sqrt{a/\varepsilon}.$$

The spatial domain $[0,1]$ is discretized on uniform mesh length $\Delta x = h$ as follows. $\Omega^N = \{x_i = x_0 + ih, \;\; i = 1(1)N, x_0 = 0, x_N = 1, h = 1/N\}$, N is taken as number of mesh points in the spatial discretization. The approximate solution of $u(x_i)$ will be denoted by U_i. Here, the main aim is to compute difference equations which have similar results with the problem (6.1) at the mesh point x_i which is given by $U_i = B_1 e^{\mu_1 x_i} + B_2 e^{\mu_2 x_i}$. Applying the procedures given in (Mickens, 2005), we obtain

$$\det \begin{bmatrix} U_{i-1} & \exp(\mu_1 x_{i-1}) & \exp(\mu_2 x_{i-1}) \\ U_i & \exp(\mu_1 x_i) & \exp(\mu_2 x_i) \\ U_{i+1} & \exp(\mu_1 x_{i+1}) & \exp(\mu_2 x_{i+1}) \end{bmatrix} = 0. \tag{6.9}$$

After simplification, equation (6.9) becomes

$$U_{i-1} - 2\cosh\left(\sqrt{\frac{\alpha}{\varepsilon}}h\right) U_i + U_{i+1} = 0, \tag{6.10}$$

which is an exact difference scheme for (6.8). By performing some arithmetic manipulation and making rearrangement on (6.10) for the variable coefficient problem, we obtain

$$-\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\lambda_i^2} + a_i U_i = 0. \tag{6.11}$$

The denominator function λ_i^2 becomes

$$\lambda_i^2 = \frac{4}{\beta_i^2}\sinh^2\left(\frac{\beta_i}{2}h\right), \tag{6.12}$$

where λ^2 is a function of ε, β_i, h and $\beta_i = \sqrt{\frac{a_i}{\varepsilon}}$.

By using (6.12), and applying the nonstandard finite difference method on semi-discrete problem, we have

$$\frac{dU_i(t)}{dt} - \varepsilon \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\lambda_i^2(\varepsilon, h, t)} + a_i U_i(t) = f(x_i, t). \tag{6.13}$$

with boundary conditions

$$\begin{cases} U_i = \phi_{li}(t), \;\; i = 0, \\ U_i = \phi_b, \qquad\qquad i = 1(1)N-1, \\ \mathcal{K}^N U_N = U_N - \varepsilon \sum_{i=1}^N \dfrac{g_{i-1}U_{i-1}^{j+1} + 4g_i U_i^{j+1} + g_{i+1}U_{i+1}^{j+1}}{3} h = \phi_{rN}, \;\; i = N. \end{cases} \tag{6.14}$$

Here, for $i = N$, the integral boundary condition $\int_0^1 g(x)u(x,t)dx$ approximated by composite Simpson's integration rule.

$$\begin{aligned}\int_0^1 g(x)u(x,t)dx &= \frac{h}{3}\left(g(0)u(0)+g(2)u(2)+2\sum_{i=1}^{N-1}g(x_{2i})u(x_{2i})+4\sum_{i=1}^{N}g(x_{2i-1})u(x_{2i-1})\right)\\ &= \phi_r.\end{aligned} \tag{6.15}$$

Substituting (6.15) in to (6.14), we obtain

$$u(N,t)-\frac{\varepsilon h}{3}\left(g(0)u(0)+g(N)u(N)+2\sum_{i=1}^{N-1}g(x_{2i})u(x_{2i})+4\sum_{i=1}^{N}g(x_{2i-1})u(x_{2i-1})\right)=\phi_r. \tag{6.16}$$

Equation (6.16) can be rewritten as

$$-\frac{4\varepsilon h}{3}\sum_{i=1}^{N}g(x_{2i-1})u(x_{2i-1})-\frac{2\varepsilon h}{3}\sum_{i=1}^{N-1}g(x_{2i})u(x_{2i})+\left(1-\frac{\varepsilon h}{3}g(N)\right)u(N)=\phi_r+\frac{\varepsilon h}{3}g(0)u(0).$$

Assume that the approximation of $u(x_i,t)$ be denoted as $U_i(t)$, by using the non-standard finite difference approximation. At this level, equation (6.1) is reduced in the form of semi-discrete as follows.

$$\begin{cases}\mathcal{L}^h U_i(t)=\frac{dU_i(t)}{dt}-\varepsilon\dfrac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{\lambda_i^2(\varepsilon,h,t)}+a_iU_i(t)=f(x_i,t),\\ U_i(0)=\phi_b(x_i),\\ U_0(t)=\phi_l(0,t),\\ \mathcal{K}U_N(t)=\phi_r(N,t).\end{cases} \tag{6.17}$$

Equation (6.17) is the system of IVPs and its compact form is written as

$$\frac{dU_i(t)}{dt}+BU_i(t)=F_i(t), \tag{6.18}$$

where B is $(N-1)\times(N-1)$ tridiagonal matrix, $U_i(t)$ and $F_i(t)$ are $(N-1)$ entries of column vector. The entries of B and F respectively given as

$$\begin{cases}b_{i,i}=\frac{2\varepsilon}{\lambda_i^2(\varepsilon,h,t)}+a(x_i), & i=1(1)N-1\\ b_{i,i-1}=-\frac{2\varepsilon}{\lambda_i^2(\varepsilon,h,t)}, & i=2(1)N-1\\ b_{i,i+1}=-\frac{2\varepsilon}{\lambda_i^2(\varepsilon,h,t)}, & i=1(1)N-1,\end{cases}$$

and

$$\begin{cases} F_1(t) = f_1(t) - \left(a(x_1) + \frac{2\varepsilon}{\lambda_1^2(\varepsilon,h,t)}\right)\phi_l(0,t), \\ F_i(t) = f_i(t), \quad i = 2(1)N-1 \\ F_{N-1}(t) = f_{N-1}(t) - \frac{2\varepsilon}{\lambda_{N-1}^2(\varepsilon,h,t)}\phi_{rN}(t) \end{cases}$$

6.4.2 Uniform stability analysis

Here, the uniform stability estimate of the semi-discrete operator $\mathcal{L}^h$ is presented.

Lemma 6.5. *(Semi-discrete Maximum Principle): Assume that* $U_0(t) \geq 0$, $\mathcal{K}U_N(t) \geq 0$. *Then* $\mathcal{L}^h U_i(t) \geq 0, \forall\, i = 1(1)N-1$, *implies that* $U_i(t) \geq 0, \forall\, i = 0(1)N$.

Proof. Assume there exists $q \in \{0, \cdots, N\}$ such that $U_q(t) = \min_{0\leq i\leq N} U_i(t)$. Suppose $U_q(t) \leq 0$, which implies $q \neq 0, N$. Also, we have $U_{q+1} - U_q > 0$ and $U_q - U_{q-1} < 0$. Here, we have

$$\mathcal{L}^h U_q(t) = \frac{dU_q(t)}{dt} - \varepsilon\frac{U_{q+1}(t) - 2U_q(t) + U_{q-1}(t)}{\lambda_q^2} + a_q U_q(t).$$

By using the above assumption, we get that $\mathcal{L}^h U_i(t) < 0$, for $i = 1(1)N-1$. Thus the assumption $U_i(t) < 0$, $i = 0(1)N$ is not correct. Thus, $U_i(t) \geq 0\;\forall\, i = 0(1)N$. ■

This Lemma 6.5 is used to obtain the bounds of the discrete solution given in Lemma 6.6. In general, the discrete maximum principle is widely used to show the boundedness and positivity of a discrete solution.

Lemma 6.6. *The solution* $U_i(t)$ *of the semidiscrete problem in* (6.17) *or* (6.18) *satisfy the following bound.*

$$|U_i(t)| = \frac{1}{\alpha}\max_i \left|\mathcal{L}^h U_i(t)\right| + \max_i \left\{\phi_b(x_i), \max_i \{\phi_l(x_i,t), \phi_r(x_i,t)\}\right\}.$$

Proof. Suppose $q = \frac{1}{\alpha}\max_i \left|\mathcal{L}^h U_i(t)\right| + \max_i \{\phi_b(x_i), \max_i \{\phi_l(x_i,t), \phi_r(x_i,t)\}\}$ and define a comparison function $\theta_i^{\pm}(t)$ as

$$\theta_i^{\pm}(t) = q \pm Z_i(t).$$

For the points on the boundary, we have

$$\theta_0^{\pm}(t) = q \pm U_0(t) = q \pm \phi_l(0,t) \geq 0,$$
$$\mathcal{K}^N\theta_N^{\pm}(t) = q \pm \mathcal{K}^N U_N(t) = q \pm \phi_r(1,t) \geq 0.$$

For $1 \leq i \leq N-1$, we have

$$\begin{aligned}
\mathcal{L}^h\theta_i^{\pm}(t) &= \frac{d(q \pm U_i(t))}{dt} - \varepsilon\frac{(q \pm U_{i-1}(t) - 2(q \pm U_i(t)) + q \pm U_{i+1}(t))}{\lambda^2} + a_i(q \pm U_i(t)) \\
&= a_i q \pm \mathcal{L}^h U_i(t) \\
&= a_i\left(\alpha^{-1}\max_i\left|\mathcal{L}^h U_i(t)\right| + \max_i\left\{\phi_b(x_i), \max_i\left\{\phi_l(x_i,t), \phi_r(x_i,t)\right\}\right\}\right) \pm f_{i,j} \\
&\geq 0, \quad \text{since } a_i \geq \alpha.
\end{aligned}$$

From Lemma 6.5, we get, $\theta_i^{\pm}(t) \geq 0, \forall\, (x_i,t) \in \bar{\Omega}_x^N \times (0,T)$. ■

6.4.3 Uniform convergence analysis

Here, we present the convergence analysis of the spatial discretization. Here, $U_i(t)$ denoted as numerical solution for the spatial semidiscretization to the exact solution $u(x,t)$ at $x = x_i$, $i = 0(1)N$. Let us define the backward and forward finite differences in space as:

$$D^-u(x_i,t) = \frac{u(x_i,t) - u(x_{i-1},t)}{h}, \quad D^+u(x_i,t) = \frac{u(x_{i+1},t) - u(x_i,t)}{h},$$

respectively, and the second order central finite difference operator as

$$\delta^2 u(x_i,t) = D^+D^-u(x_i,t) = \frac{D^+u(x_i,t) - D^-u(x_i,t)}{h}.$$

Lemma 6.7. *Let N be a fixed mesh. Then, for $\varepsilon \to 0$, we have*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\max_{1\leq i\leq N-1}\frac{\exp(-px_i/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = 0 \quad \textit{and} \\
&\lim_{\varepsilon\to 0}\max_{1\leq i\leq N-1}\frac{\exp(-p(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = 0.
\end{aligned}$$

where $m = 1,2,3,\cdots$.

Proof. Let us consider the partition of the interval $[0,1]$, with points $x_i = ih$, $h = 1/N$ for $i = 1,2,3,\cdots,N$. By using $x_1 = h, 1 - x_{N-1} = h$, we have

$$\begin{aligned}
&\max_{1\leq i\leq N-1}\frac{\exp(-px_i/\sqrt{\varepsilon})}{\varepsilon^{m/2}} \leq \frac{\exp(-px_1/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = \frac{\exp(-ph/\sqrt{\varepsilon})}{\varepsilon^{m/2}} \quad \text{and} \\
&\max_{1\leq i\leq N-1}\frac{\exp(-p(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{m/2}} \leq \frac{\exp(-p(1-x_{N-1})/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = \frac{\exp(-ph/\sqrt{\varepsilon})}{\varepsilon^{m/2}}
\end{aligned}$$

Then, by applying L'Hopital's rule repeatedly, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\exp(-ph/\sqrt{\varepsilon})}{\varepsilon^{m/2}} &= \lim_{r=1/\sqrt{\varepsilon}\to\infty} r^m\exp(-phr)\\ &= \lim_{r=1/\sqrt{\varepsilon}\to\infty}\frac{r^m}{\exp(phr)}\\ &= \lim_{r\to\infty}\frac{m!}{(ph)^m\exp(phr)}=0.\end{aligned}$$

■

Theorem 6.1. *Let the coefficient function $a(x)$ and the function $f(x,t)$ in (6.17) be sufficiently smooth and $u(x,t)\in C^4\left(\bar{\Omega}\right)$. Then the semidiscrete solution $U_i(t)$ of (6.17) satisfies*

$$\left|\mathcal{L}^h\left(u(x_i,t)-U_i(t)\right)\right|\leq Ch^2.$$

Proof. The truncation error in spatial direction is considered as

$$\begin{aligned}\mathcal{L}^h\left(u(x_i,t)-U_i(t)\right) &= \mathcal{L}^h u(x_i,t)-\mathcal{L}^h U_i(t)\\ &= -\varepsilon\frac{d^2}{dx^2}u(x_i,t)+\varepsilon\frac{D_x^+D_x^+h^2}{\lambda^2}u(x_i,t)\\ &= -\varepsilon\frac{d^2}{dx^2}u(x_i,t)+\frac{\varepsilon}{\lambda^2}\left(h^2\frac{d^2}{dx^2}u(x_i,t)+\frac{h^4}{12}\frac{d^4}{dx^4}u(x_i,t)\right).\end{aligned}\tag{6.19}$$

Note that we have used a Taylor expansions of $u_{i-1}(t)$ and $u_{i+1}(t)$. A truncated Taylor expansion of $\frac{1}{\lambda^2}$ of order five becomes

$$\frac{1}{\lambda^2}=\frac{\beta^2}{4}\left(\frac{4}{\beta^2h^2}-\frac{1}{3}+\frac{\beta^2h^2}{60}\right).\tag{6.20}$$

Using (6.20) in (6.19), we obtain

$$\begin{aligned}\mathcal{L}^h\left(u(x_i,t)-U_i(t)\right) &= \frac{\varepsilon}{12}\left(\frac{d^4}{dx^4}u(x_i,t)-\beta^2\frac{d^2}{dx^2}u(x_i,t)\right)h^2\\ &+\ \varepsilon\beta^2h^4\left(\frac{\beta^2}{240}\frac{d^2u(x_i,t)}{dx^2}-\frac{1}{144}\frac{d^4u(x_i,t)}{dx^4}\right)+h^6\frac{\varepsilon\beta^4}{2880}\frac{d^4u(x_i,t)}{dx^4}.\end{aligned}\tag{6.21}$$

Using Lemma 6.7 and Theorem 6.3, we obtain

$$\left|\mathcal{L}^h\left(u(x_i,t)-U_i(t)\right)\right|\leq CN^{-2}.$$

The truncation error at $x = x_N$, become

$$
\begin{aligned}
\mathcal{K}^N\left(U(x_i)-u(x_i)\right) &= \mathcal{K}^N U(x_N) - \mathcal{K}^N u(x_i),\\
&= \phi_r - \mathcal{K}^N U(x_N),\\
&= \mathcal{K}u(x_i) - \mathcal{K}^N U(x_N),\\
&= u(x_N) - \varepsilon \int_0^1 g(x)u(x)dx - \left(U(x_N) - \varepsilon \int_{x_0}^{x_N} g(x)u(x)dx\right),\\
&= \varepsilon \int_{x_0}^{x_N} g(x)u(x)dx - \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}u_{i-1} + 4g_i u_i + g_{i+1}u_{i+1}}{3}h,\\
&= \varepsilon \left[\int_{x_0}^{x_1} g(x)u(x)dx + \int_{x_1}^{x_2} g(x)u(x)dx + \cdots + \int_{x_N}^{x_{N+1}} g(x)u(x)dx\right]\\
&- \varepsilon \left[\frac{g_0u_0 + 4g_1u_1 + g_2u_2}{3}h + \cdots + \frac{g_{N-1}u_{N-1} + 4g_Nu_N + g_{N+1}u_{N+1}}{3}h\right],\\
\left|\mathcal{K}^N\left(U(x_i)-u(x_i)\right)\right| &= \left|C\varepsilon\left(h^4u^{(4)}(\xi_1) + h^4u^{(4)}(\xi_2) + \cdots + h^4u^{(4)}(\xi_N)\right)\right|,\\
\left|\mathcal{K}^N\left(U(x_i)-u(x_i)\right)\right| &\leq C\varepsilon h^4\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2) + \cdots + u^{(4)}(\xi_N)\right),\\
&\leq C\varepsilon h^4 \left|\left|\frac{d^4u(\xi_i)}{dx^4}\right|\right| \leq Ch^2 = CN^{-2}.
\end{aligned}
$$

■

Theorem 6.2. *The semidiscrete solutions satisfies the uniform error bound*

$$\sup_{0<\varepsilon\leq 1} ||u(x_i,t) - U_i(t)||_{\bar{\Omega}} \leq CN^{-2}. \tag{6.22}$$

Proof. The proof is follows from Lemma 6.3 and Lemma 6.7 under the properties of boundedness of semi-discrete solution and the required bound is satisfied. ■

6.4.4 Temporal semi-discretization

A mesh with length $\Delta t = t_{j+1} - t_j$, $j = 0(1)M-1$ is constructed on the time domain $\Omega_t = [0,T]$, where M is a positive integer. The IVPs (6.18) is discretized using the implicit Euler method on a uniform mesh. By denoting the approximation of $u_i(t_j)$ as U_i^j, we construct the time discretization as follows.

$$\frac{U_i^j - U_i^{j-1}}{\Delta t} = BU_i^j + F_i^j \tag{6.23}$$

with the initial condition $U_0(t) = \phi_l(t_j)$, and by rearranging equation (6.23), we obtain

$$U_i^j = [I + \Delta t B]^{-1}\left[\Delta t F_i^j + U_i^{j-1}\right]. \tag{6.24}$$

Lemma 6.8. *Suppose* $\left|\frac{\partial^i u(x,t_j)}{\partial t^i}\right| \leq C,\ \forall (x,t) \in \bar{\Omega},\ i = 0,1,2.$ *Then the local truncation* *error associated with the time direction satisfies* $|e_j| \leq C(\Delta t)^2$.

Proof. The local truncation error is defined as

$$\begin{aligned} e_j &= u(t_j) - U_i^j \\ &= u(t_j) - [I + \Delta t B]^{-1} \left[\Delta t F_i^j + U_i^{j-1}\right]. \end{aligned}$$

Using Taylor expansion, we obtain $u(t_{j-1})$ as

$$u(t_{j-1}) = u(t_j) - \Delta t u_t(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \frac{(\Delta t)^3}{3!} u_{ttt}(t_j) + \mathcal{O}((\Delta t)^4).$$

However, $u_t(t_j) = F(t_j) - B(t_j)u(t_j)$. Thus,

$$u(t_{j-1}) = u(t_j) - \Delta t[F(t_j) - B(t_j)u(t_j)] + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \frac{(\Delta t)^3}{3!} u_{ttt}(t_j) + \mathcal{O}((\Delta t)^4).$$

Now, the local truncation error e_j becomes

$$\begin{aligned} e_j &= u(t_j) - [I + \Delta t B]^{-1} \left[\Delta t F_i^j + U_i^{j-1}\right] \\ &= u(t_j) - [I + \Delta t B]^{-1} \left[[I + \Delta t B(t_j)]u(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \cdots\right] \\ &= [I + \Delta t B]^{-1} \left[\frac{(\Delta t)^2}{2} u_{tt}(t_j) - \frac{(\Delta t)^3}{3!} u_{ttt}(t_j) + \mathcal{O}((\Delta t)^4)\right]. \end{aligned}$$

Since the matrix B is invertible, using the relation $(\Delta t)^2 > (\Delta t)^3$ for small Δt and $u(t_j) \leq C$, we obtain

$$\begin{aligned} ||e_j|| &\leq \left|\left|[I + \Delta t B]^{-1}\right|\right| \left|\left|\frac{(\Delta t)^2}{2} u_{tt}(t_j) - \frac{(\Delta t)^3}{3!} u_{ttt}(t_j) + \mathcal{O}((\Delta t)^4)\right|\right| \\ &\leq \left|\left|[I + \Delta t B]^{-1}\right|\right| (\Delta t)^2 \leq C(\Delta t)^2. \end{aligned}$$

■

Lemma 6.9. *The global error estimate in time direction is given by* $||E_{j+1}|| \leq C\Delta t, \quad \forall j \leq T/\Delta t$, *where* $E_{j+1} = \max_i |U_i(t_{j+1}) - U_{i,j+1}|_\Omega$.

Proof. The global error estimate at $(j+1)^{th}$ time step is obtained by using the local error estimate up to j^{th} time step as follows.

$$\begin{aligned}
||E_{j+1}|| &= \left|\left|\sum_{i=1} e_j\right|\right|, \qquad j \leq T/\Delta t \\
&\leq ||e_1|| + ||e_2|| + ||e_3|| + ||e_4|| + \cdots + ||e_j|| \\
&\leq C_1(j\Delta t)\Delta t \\
&\leq C_1 T\Delta t, \qquad \text{since } \ j\Delta t \leq T \\
&\leq C\Delta t.
\end{aligned}$$

Hence,

$$\left|\left|E^{j+1}\right|\right| = \max_i \left|U_i(t_{j+1}) - u_i^{j+1}\right|_\Omega \leq C\Delta t. \tag{6.25}$$

where C is a positive constant independent of ε and Δt. By taking the supremum $\forall\, \varepsilon \in (0,1]$, we obtained

$$\sup_{0<\varepsilon\leq 1} \left|\left|U_i(t_{j+1}) - U_i^{j+1}\right|\right|_\Omega \leq C\Delta t. \tag{6.26}$$

■

We summarizes the results of this work by considering the error estimate obtained in (6.22) and (6.26) and we conclude by the following theorem.

Theorem 6.3. *The error estimate for the solution of continuous and fully discrete problem is given by*

$$\sup_{0<\varepsilon\leq 1} \left|\left|u(x,t) - U_i^{j+1}\right|\right| \leq C\left(N^{-2} + \Delta t\right),$$

where $u(x,t)$ and U_i^{j+1} are the solutions of the problem (6.1) and (6.17) respectively.

Proof. The error estimation of the fully discrete scheme is given as follows.

$$\begin{aligned}
\sup_{0<\varepsilon\leq 1} \left|\left|u(x_i,t_j) - U_i^j\right|\right| &= \sup_\varepsilon \max_{i,j} \left|u(x_i,t_j) - U_i(t_j) + U_i(t_j) - U_i^j\right| \\
&\leq \sup_\varepsilon \max_{i,j} \left|u(x_i,t_j) - U_i(t_j)\right| + \sup_\varepsilon \max_{i,j} \left|U_i(t_j) - U_i^j\right|. \\
&\leq C\left(N^{-2} + \Delta t\right).
\end{aligned}$$

Then, by combining the bound given in Theorem 6.1 and Lemma 6.9, the theorem gets proved. ■

6.5 Numerical results and discussions

In this section, the developed numerical method is demonstrated by using two model examples given below. Since the exact solution of the given examples are not known, double mesh techniques is used to obtain the maximum pointwise error of the developed numerical method. Now, let $U^{N,\Delta t}$ be a computed solution of a problem using mesh points N and time step size Δt. Again, $U_{i,j}^{2N,\Delta t/2}$ be a computed solution on double mesh points of $2N$ and half of time step size $\Delta t/2$. The maximum absolute error for the developed numerical method is calculated as

$$E_{\varepsilon}^{N,\Delta t} = \max_{i,j} \left| U_{i,j}^{N,\Delta t} - U_{i,j}^{2N,\Delta t/2} \right|. \tag{6.27}$$

The parameter uniform error estimate is computed as

$$E^{N,\Delta t} = \max_{\varepsilon} \left(E_{\varepsilon}^{N,\Delta t} \right). \tag{6.28}$$

The rate of convergence of the developed method is calculated using

$$R_{\varepsilon}^{N,\Delta t} = \log_2(E_{\varepsilon}^{N,\Delta t}) - \log_2(E_{\varepsilon}^{2N,\Delta t/2}). \tag{6.29}$$

The parameter uniform rate of convergence is also calculated as

$$R^{N,\Delta t} = \log_2(E^{N,\Delta t}) - \log_2(E^{2N,\Delta t/2}). \tag{6.30}$$

Example 6.1. *Consider SPPPDEs with nonlocal boundary conditions of the form:*

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1+x^2}{2} u(x,t) = e^{-t} - 1 + \sin(\pi\, x), \quad (x,t) \in (0,1) \times (0,1] \\
&u(x,0) = 0, \quad x \in (0,1), u(0,t) = 0, \quad t \in (0,1], \\
&\mathcal{K}u(1,t) = u(1,t) - \varepsilon \int_0^1 \frac{x}{6} u(x,t) dx = 0, \quad t \in (0,1].
\end{aligned}$$

Example 6.2. *Consider SPPPDEs with nonlocal boundary conditions of the form:*

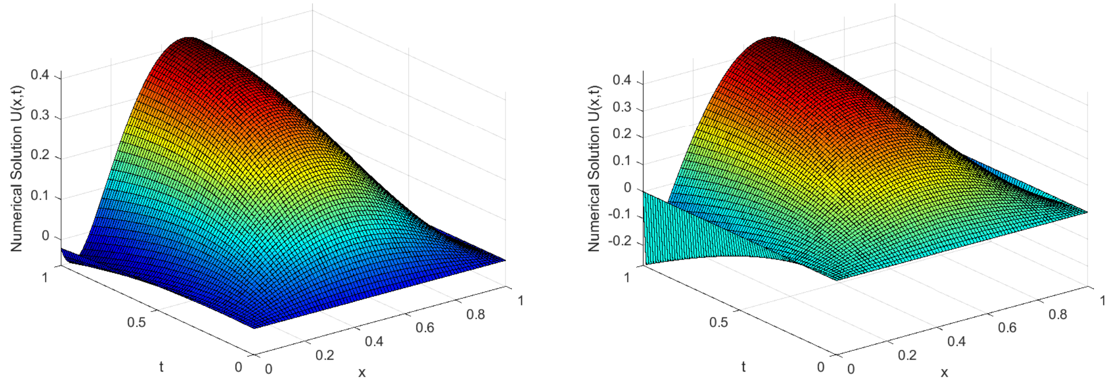
$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1+x^2}{2} u(x,t) = t^3, \qquad (x,t) \in (0,1) \times (0,1] \\
&u(x,0) = 0, \qquad x \in (0,1),\ u(0,t) = 0, \qquad t \in (0,1], \\
&\mathcal{K}u(1,t) = u(1,t) - \varepsilon \int_0^1 \cos(x) u(x,t) dx = 0, \quad t \in (0,1].
\end{aligned}$$

Table 6.1: Maximum absolute error and the corresponding rate of convergence for Example 6.1

ε $\downarrow$	$N=32$ $\Delta t=0.1$	$N=64$ $\Delta t=0.1/4$	$N=128$ $\Delta t=0.1/4^2$	$N=256$ $\Delta t=0.1/4^3$	$N=512$ $\Delta t=0.1/4^4$
10^{-6}	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
	1.8951	1.9645	1.9859	1.9938	-
10^{-8}	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
	1.8951	1.9645	1.9859	1.9938	-
10^{-10}	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
	1.8951	1.9645	1.9859	1.9938	-
10^{-12}	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
	1.8951	1.9645	1.9859	1.9938	-
10^{-14}	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
	1.8951	1.9645	1.9859	1.9938	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
	1.8951	1.9645	1.9859	1.9938	-
$E^{N,\Delta t}$	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
$R^{N,\Delta t}$	1.8951	1.9645	1.9859	1.9938	-

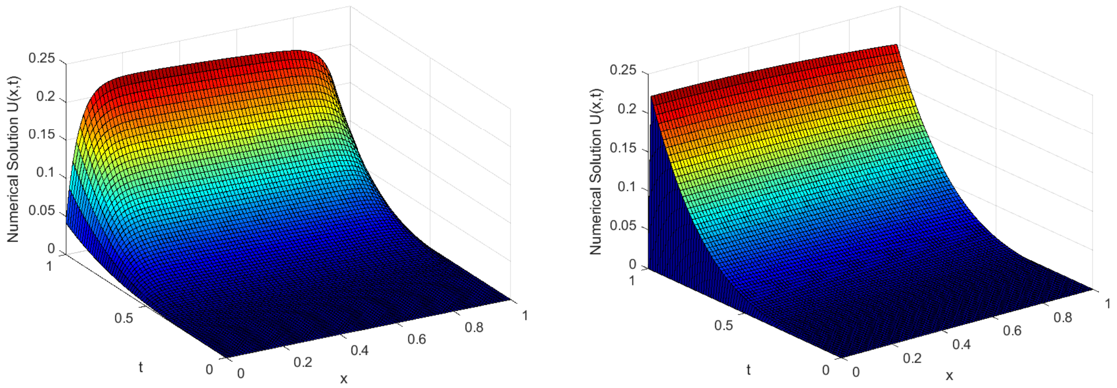
Table 6.2: Maximum absolute error and the corresponding rate of convergence for Example 6.2

ε $\downarrow$	$N=32$ $\Delta t=0.1$	$N=64$ $\Delta t=0.1/4$	$N=128$ $\Delta t=0.1/4^2$	$N=256$ $\Delta t=0.1/4^3$	$N=512$ $\Delta t=0.1/4^4$
10^{-6}	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
	1.5354	1.8919	1.9736	1.9935	-
10^{-8}	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
	1.5354	1.8919	1.9736	1.9935	-
10^{-10}	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
	1.5354	1.8919	1.9736	1.9935	-
10^{-12}	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
	1.5354	1.8919	1.9736	1.9935	-
10^{-14}	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
	1.5354	1.8919	1.9736	1.9935	-
$E^{N,\Delta t}$	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
$R^{N,\Delta t}$	1.5354	1.8919	1.9736	1.9935	-



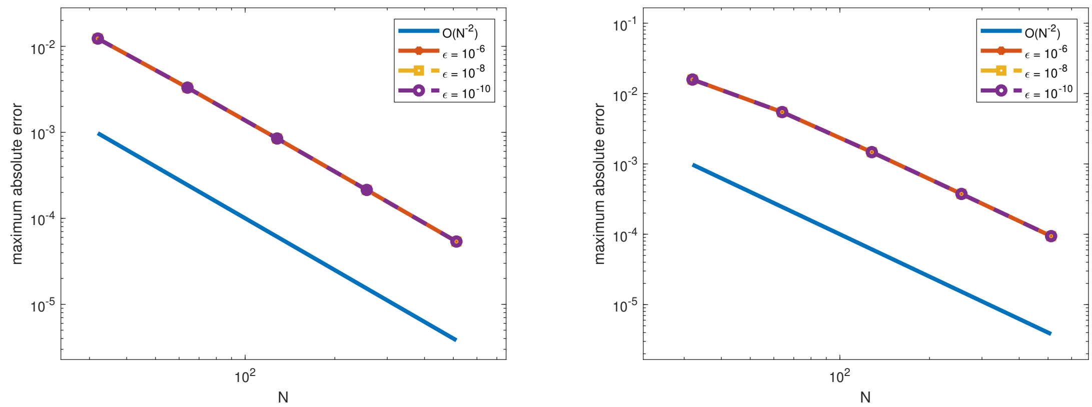
(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-12}$

Figure 6.1: Graph of numerical solution which displays the existing layer for different values of perturbation parameter for Example 6.1: (a) for $\varepsilon = 10^{-2}$, (b) for $\varepsilon = 10^{-12}$



(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-12}$

Figure 6.2: Graph of numerical solution which displays the existing layer for different values of perturbation parameter for Example 6.2: (a) for $\varepsilon = 10^{-2}$, (b) for $\varepsilon = 10^{-12}$



(a) (b)

Figure 6.3: Log-Log plot of maximum absolute error versus mesh numbers for different values of ε (a) for Example 6.1, (b) for Example 6.2.

The solutions of the above two examples exhibit a boundary layers near $x = 0$ and $x = 1$. Figures 6.1(a) and 6.1(b) indicates graph of numerical solutions of Examples 6.1 which displays the presence of boundary layers on the left and right side of spatial domain for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-12}$ respectively. Figures 6.2(a) and 6.2(b) indicates graph of numerical solutions of Examples 6.2 which shows the existence of boundary layers on the left and right side of spatial domain for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-12}$ respectively. The maximum pointwise error and rate of convergence of the proposed method for Examples 6.1 and 6.2 are given in Tables 6.1 and 6.2 respectively for different values of perturbation parameter ε, mesh number N and time step size Δt. From these tables, one can observe that the developed method is parameter uniform convergent, which supports the theoretical results. Figures 6.3(a) and 6.3(b) indicates the Log-Log plots for the maximum absolute error versus mesh number N for the singular perturbation parameter ε for the given examples. One can observe that as ε goes very small, the developed method converges uniformly independent of the perturbation parameter ε.

6.6 Summary

A numerical treatment of singularly perturbed parabolic partial differential equations of the reaction-diffusion type with nonlocal boundary conditions is investigated. A nonstandard finite difference scheme is used to semi-discretize the spatial direction, and the implicit Euler method is used to discretize the results of initial value problems. Simpson's $\frac{1}{3}$ rule is applied to treat integral boundary condition. The stability and convergence analysis of the proposed scheme are established. The formulated method proved to be second-order uniformly convergent in the spatial direction and first-order in the temporal direction. Two numerical experiments are employed to validate the applicability of the developed numerical method. The numerical results justify the theoretical estimations. Further, nonstandard finite difference method is not layer resolving numerical method because of insufficient of mesh number in the layer region. To overcomes this limitations, a standard finite difference method on a suitable piecewise uniform Shishkin mesh is suggested to solve the problem in the next Chapter.

CHAPTER SEVEN

A FITTED MESH FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED PARTIAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITION

This chapter presents a numerical computation of SPPPDEs with nonlocal boundary condition. The solution of the considered problem exhibits a boundary layers on the left and right sides of the spatial domain. A finite difference method on a piecewise uniform Shishkin mesh is proposed to discretize the problem in the spatial direction, and the implicit Euler method is used in the temporal direction. The integral boundary condition is approximated using Simpson's $\frac{1}{3}$ rule. The proposed method gives almost second-order uniform convergence in the space derivative discretization and first-order in the time derivative discretization. To validate the applicability of the method, two numerical examples are considered. The computational result accurately reflects the theoretical estimation.

7.1 The governing problem

Consider a class of SPPPDEs of the reaction-diffusion type with IBC

$$\begin{aligned}\mathcal{L}u(x,t) &= \frac{\partial u(x,t)}{\partial t} - \varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)u(x,t) = f(x,t) \ \ (x,t) \in \Omega,\\ u(x,0) &= \phi_b(x), \ \ \forall (x,t) \in \Gamma_b = \{(x,0); 0 \le x \le 1\},\\ u(0,t) &= \phi_l(t), \ \ \forall (x,t) \in \Gamma_l = \{(0,t); 0 \le t \le T\},\\ \mathcal{K}u(x,t) &= u(1,t) - \varepsilon\int_0^1 g(x)u(x,t)dx = \phi_r(x,t), \ \ \forall (x,t) \in \Gamma_r = \{(1,t); 0 \le t \le T\}.\end{aligned} \tag{7.1}$$

Where $(x,t) \in \Omega = \Omega_x \times \Omega_t = (0,1) \times (0,T]$, $\bar{\Omega} = [0,1] \times [0,T]$, and ε is a small parameter $(0 < \varepsilon \ll 1)$. Suppose that $a(x,t) \ge \alpha > 0$, $f(x,t)$, ϕ_l, ϕ_r, ϕ_b are sufficiently smooth functions and $g(x)$ is nonnegative monotone function and satisfies $\int_0^1 g(x)dx < 1$. The existence and uniqueness of the problem (7.1) can be established under the assumption that the data are Hölder continuous and imposing proper compatibility conditions at the corners (Ladyzhenskaya et al., 1968). Note that ϕ_l and ϕ_r are only functions of t, while ϕ_b is a function of x only. The problem necessarily satisfies the following sufficient compatibility conditions $\phi_b(0,0) = \phi_l(0,0),\ \phi_b(1,0) = \phi_r(1,0)$, and

$$-\varepsilon\frac{\partial^2\phi_b(0,0)}{\partial x^2} + a(0,0)\phi_b(0,0) + \frac{\partial\phi_l(0,0)}{\partial t} = f(0,0),$$
$$-\varepsilon\frac{\partial^2\phi_b(1,0)}{\partial x^2} + a(1,0)\phi_b(1,0) + \frac{\partial\phi_r(1,0)}{\partial t} = f(1,0).$$

Note that ϕ_l, ϕ_r, and ϕ_b are assumed to be sufficiently smooth for (7.1) to make sense, namely $\phi_l, \phi_r \in C^1([0,T])$ and $\phi_b \in C^{(2,1)}(\Gamma_b)$.

7.2 Properties of continuous solution

In this section, the properties of the continuous solution (6.1) which guarantee the existence and uniqueness of the analytical solution is analyzed.

Lemma 7.1. *(Maximum Principle.) Let $\psi(x,t) \in C^{(0,0)}(\bar{\Omega}) \cap C^{(1,0)}(\Omega) \cap C^{(2,1)}(\Omega)$ be a sufficiently smooth function such that $\psi(0,t) \geq 0, \psi(x,0) \geq 0, \mathcal{K}\psi(1,t) \geq 0, \mathcal{L}\psi(x,t) \geq 0, \forall(x,t) \in \Omega$. Then $\psi(x,t) \geq 0$, $\forall(x,t) \in \bar{\Omega}$, where $\mathcal{L}\psi(x,t) = \psi_t(x,t) - \varepsilon\psi_{xx}(x,t) + a\psi(x,t)$.*

Proof. Define $v_1(x,t) = 1 + x$. Note that $v_1(x,t) > 0$, $x \in \Omega$, $\mathcal{L}v_1(x,t) > 0$, $v_1(0,t) > 0$, and $\mathcal{K}v_1(2,t) > 0$. Further we define

$$\lambda_1 = \max\left\{\frac{-\psi(x,t)}{v_1(x,t)} : \quad (x,t) \in \bar{\Omega}\right\} \tag{7.2}$$

Then there exists $(x_0,t_0) \in \bar{\Omega}$ such that $\psi(x_0,t_0)+\lambda_1 v_1(x_0,t_0) = 0$ and $\psi(x,t)+\lambda_1 v_1(x,t) \geq 0, \forall(x,t) \in \bar{\Omega}$. Then the functions $(\psi + \lambda_1 v_1)$ has a minimum value at $(x,t) = (x_0,t_0)$.

Assume $\lambda_1 > 0$, we prove by contradiction.

Case 1: When $(x_0,t_0) = (0,t_0)$,

$$0 \leq (\psi + \lambda_1 v_1)(0,t_0) = \psi(0,t_0) + \lambda_1 v_1(0,t_0) < 0.$$

Case 2: When $(x_0,t_0) \in \Omega$,

$$\begin{aligned} 0 < \mathcal{L}(\psi + \lambda_1 v_1) &= -\varepsilon(\psi + \lambda_1 v_1)_{xx}(x_0,t_0) + (\psi + \lambda_1 v_1)_t(x_0,t_0) \\ &\quad + a(x_0,t_0)(\psi + \lambda_1 v_1)(x_0,t_0) \leq 0 \end{aligned}$$

Case 3: When $(x_0,t_0), = (1,t_0)$

$$0 < \mathcal{K}(\psi + \lambda_1 v_1)(1,t_0) = (\psi + \lambda_1 v_1)(1,t_0) - \varepsilon\int_0^1 g(x)(\psi + \lambda_1 v_1)(x,t)dx \leq 0.$$

In all the cases, the proof is contradictory. Hence $\lambda_1 > 0$ is impossible. Therefore $\psi(x,t) \geq 0$. ■

Lemma 7.2. *(Stability Result) Suppose $u(x,t)$ be the solution of (7.1). Then it satisfies the bound*

$$u(x,t) \leq \alpha^{-1}\|f\| + \max\left\{\phi_b(x), \phi_l(x,t), \phi_r(x,t)\right\},$$

where $\|f\| = \max_{(x,t)\in\Omega}|f(x,t)|$.

Proof. The required bound is obtained by constructing a comparison functions

$$\theta^{\pm}(x,t) = \alpha^{-1}\|f\| + \max\left\{\phi_b(x), \phi_l(x,t), \phi_r(x,t)\right\} \pm u(x,t), (x,t) \in \bar{\Omega}$$

and using the maximum principle in Lemma 7.1. ■

7.3 Bounds on the solution and its derivatives

In this section, we study the bounds on the solution and its derivatives which are required in the convergence analysis of the numerical methods.

Lemma 7.3. *Suppose $f(x,t), a(x,t) \in C^{(2+\alpha_1,1+\alpha_1/2)}(\bar{\Omega})$, and $\phi_l \in C^{2+\alpha_1/2}([0,T]), \phi_b \in C^{(4+\alpha_1,2+\alpha_1/2)}(\Gamma_b), \phi_r \in C^{2+\alpha_1/2}([0,T])$, where $\alpha_1 \in (0,1)$. Then the problem in (7.1) has a unique solution u which satisfies $u \in C^{(4+\alpha_1,2+\alpha_1/2)}(\bar{\Omega})$. And the derivatives of u are bounded as*

$$\left\|\frac{\partial^{i+j} z}{\partial x^i \partial t^j}\right\| \leq C\varepsilon^{\frac{-i}{2}}, \quad \textit{for } 0 \leq i + 2j \leq 4.$$

Proof. The proof of the lemma is given in Lemma 6.3. ■

7.4 Decomposition of the solution

The bounds on the derivatives of the solution given in Lemma 7.3 were derived from classical results. They are not adequate for the proof of the ε -uniform error estimate. Stronger bounds on these derivatives are now obtained by a method originally given in (Shishkin, 1989). The key step is to decompose the solution u into smooth and singular components.

The solution u of (7.1) was decomposed into v-smooth and w- singular components. Further, v is separated into

$$v = v_0 + \varepsilon v_1,$$

where v_0 and v_1 are solution of the following differential equations

$$\begin{aligned} &\frac{\partial v_0}{\partial t}(x,t) + a(x,t)v_0(x,t) = f(x,t), && (x,t) \in \Omega, \\ &v_0(x,t) = 0, && (x,t) \in \Gamma_b, \end{aligned} \tag{7.3}$$

and

$$
\begin{aligned}
&\mathcal{L}v_1(x,t) = \frac{\partial^2 v_0}{\partial x^2}(x,t), && (x,t) \in \Omega \\
&v_1(x,t) = 0, && (x,t) \in \Gamma_l, \\
&\mathcal{K}v_1(x,t) = 0, && (x,t) \in \Gamma_r, \\
&v_1(x,t) = 0, && (x,t) \in \Gamma_b
\end{aligned} \tag{7.4}
$$

Then v satisfy

$$
\begin{aligned}
&\mathcal{L}v(x,t) = f(x,t), && (x,t) \in \Omega \\
&v(x,t) = v_0(x,t), && (x,t) \in \Gamma_l, \\
&\mathcal{K}v(x,t) = \mathcal{K}v_0(1,t), && (x,t) \in \Gamma_r, \\
&v(x,t) = \phi_b(x,t), && (x,t) \in \Gamma_b
\end{aligned} \tag{7.5}
$$

The singular component w is determined from

$$
\begin{aligned}
&\mathcal{L}w(x,t) = 0, && (x,t) \in \Omega \\
&w(x,t) = \phi_l(x,t) - v_0(x,t), && (x,t) \in \Gamma_l, \\
&w(x,t) = 0, && (x,t) \in \Gamma_b, \\
&\mathcal{K}w(x,t) = \mathcal{K}u(x,t) - \mathcal{K}v_0(x,t), && (x,t) \in \Gamma_r
\end{aligned} \tag{7.6}
$$

The reaction diffusion problem occurs boundary layers on both boundary points Γ_l and Γ_r , then we separate w as $w = w_l + w_r$, where w_l and w_r are satisfy the following problems

$$
\begin{aligned}
&\mathcal{L}w_l(x,t) = 0, && (x,t) \in \Omega \\
&w_l(x,t) = \phi_l(x,t) - v_0(x,t), && (x,t) \in \Gamma_l, \\
&w_l(x,t) = 0, && (x,t) \in \Gamma_r, \\
&w_l(x,t) = 0, && (x,t) \in \Gamma_b
\end{aligned} \tag{7.7}
$$

and

$$
\begin{aligned}
&\mathcal{L}w_r(x,t) = 0, && (x,t) \in \Omega \\
&w_r(x,t) = 0, && (x,t) \in \Gamma_l, \\
&w_r(x,t) = 0, && (x,t) \in \Gamma_b, \\
&\mathcal{K}w_r(x,t) = \mathcal{K}w(x,t), && (x,t) \in \Gamma_r.
\end{aligned} \tag{7.8}
$$

Lemma 7.4. *Suppose* $a(x,t), f(x,t) \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{\Omega})$*, and* $\phi_l \in C^{(3+\alpha_1/2)}([0,T]), \phi_b \in$

$C^{(6+\alpha_1,3+\alpha_1/2)}(\Gamma_b)$, $\phi_r \in C^{(3+\alpha_1/2)}([0,T])$, *where* $\alpha_1 \in (0,1)$. *Then we obtain*

$$\|\frac{\partial^{i+j}v}{\partial x^i \partial t^j}\|_{\bar{\Omega}} \leq C\left(1+\varepsilon^{1-i/2}\right), \quad (x,t)\in\Omega \tag{7.9}$$

$$|\frac{\partial^{i+j}w_l}{\partial x^i \partial t^j}| \leq C\varepsilon^{\frac{-i}{2}} e^{\frac{x}{\sqrt{\varepsilon}}}, \qquad (x,t)\in\Omega \tag{7.10}$$

$$|\frac{\partial^{i+j}w_r}{\partial x^i \partial t^j}| \leq C\varepsilon^{\frac{-i}{2}} e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, \qquad (x,t)\in\Omega \tag{7.11}$$

where C *is a constant independent of* ε, $0 \leq i+2j \leq 4$.

Proof. The proof of the lemma is given in Lemma 6.4. ■

7.5 The numerical method

In this section, the time derivative is first discretized using the implicit Euler method, and then a layer-adapted mesh of the Shishkin type is introduced to discretize the space derivative using the standard finite difference method.

7.5.1 Time semi-discretization

Let $\{t_j\}_0^M$ be a uniform mesh which is used in the time semi-discretization and defined as $\{t_j\}_0^M = \{t_j = j\Delta t, j = 0,1,\cdots,M, \ \ \Delta t = T/M\}$, where M is a non-negative integer. Using implicit Euler method the problem in (7.1) is semi-discretized as

$$\mathcal{L}^{\Delta t}U^{j+1}(x) = g(x,t_{j+1}), \ \ j = 0,1,\cdots,M-1, \tag{7.12}$$

with discretized boundary condition

$$\begin{aligned} U^{j+1}(x) &= \phi_l^{j+1}(x), \quad \phi_l^{j+1}(x)\in\Gamma_l, \\ \mathcal{K}U^{j+1}(x) &= U^{j+1}(1) - \varepsilon\int_0^1 g(x)u(x)dx = \phi_r^{j+1}(x), \quad \phi_r^{j+1}(x)\in\Gamma_r, \\ U^{j+1}(x) &= \phi_b^{j+1}(x), \qquad \phi_b^{j+1}(x)\in\Gamma_b, \end{aligned} \tag{7.13}$$

where $\mathcal{L}^{\Delta t}U^{j+1}(x) = -\varepsilon U_{xx}^{j+1} + r(x)U^{j+1}(x), g(x,t_{j+1}) = \frac{U^j(x)}{\Delta t} + f(x,t_{j+1})$, and $r(x) = \frac{1}{\Delta t} + a(x)$. Here, $U^{j+1}(x)$ is denoted for the approximation of $u(x,t_{j+1})$ at the $(j+1)^{th}$ time level.

The semi-discrete operator $\mathcal{L}^{\Delta t}$ satisfies the following Lemma.

Lemma 7.5. *For* $j = 0,1,2,\cdots,M-1$, *suppose that* $\Psi^{j+1}(x)$ *be a sufficiently smooth function in* Ω *such that* $\Psi^{j+1}(0) \geq 0$ *and* $\mathcal{K}\Psi^{j+1}(1) \geq 0$, *then* $\mathcal{L}\Psi^{j+1}(x) \geq 0, \forall x \in (0,1)$, *implies* $\Psi^{j+1}(x) \geq 0$.

Proof. Assume $x^* \in [0,1]$ be such that $\Psi^{j+1}(x^*) = \min_{x\in\Omega}\Psi^{j+1}(x) < 0$, from the above assumption, it is known that $x^* \notin \{0,1\}$ implies that $x^* \in (0,1)$. By using properties from calculus, we have $\frac{d^2}{dx^2}\Psi^{j+1}(x) \geq 0$. Which implies that $\mathcal{L}^{\Delta t}\Psi^{j+1}(x) < 0$, which is contradict with $\mathcal{L}^{\Delta t}\Psi^{j+1}(x) \geq 0$, $\forall\, x \in (0,1)$. Therefore, we conclude that $\Psi^{j+1}(x) \geq 0$, $\forall$ $x \in [0,1]$. ∎

The local truncation error estimation for time semi-discretization which is denoted as $e^j := u(x,t_{j+1}) - U^{j+1}(x)$, for $j = 0,1,2,\cdots,M$ is analyzed as follows.

Lemma 7.6. *The local truncation error associated with the implicit Euler method satisfies the bound*

$$|e^j| \leq C(\Delta t)^2.$$

Proof. Using Taylor expansion to $u(x,t_{j+1})$, we get

$$u(x,t_{j+1}) = u(x,t_j) + \Delta t\frac{\partial u(x,t_j)}{\partial t} + \mathcal{O}((\Delta t)^2). \tag{7.14}$$

Substituting (7.14) into (7.1), we obtain

$$\begin{aligned}\frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} &= \frac{\partial u(x,t)}{\partial t} + \mathcal{O}((\Delta t)^2)\\ &= -\left(-\varepsilon\frac{\partial^2}{\partial x^2}u(x,t) - f(x,t_j)\right) + \mathcal{O}((\Delta t)^2).\end{aligned}$$

Since the error satisfies the given problem, then the local truncation error satisfy $\mathcal{L}^{\Delta t}e^{j+1}(x) = \mathcal{O}((\Delta t)^2)$. Using the discrete maximum principle gives

$$\|e^{j+1}\| \leq C(\Delta t)^2. \tag{7.15}$$

∎

The bound for the global truncation error of the semi-discrete scheme are given as follows.

Lemma 7.7. *The global error in the temporal direction satisfies the estimate*

$$\|TE^{j+1}\| \leq C(\Delta t), \qquad \forall j \leq T/\Delta t. \tag{7.16}$$

Proof. The global error estimate at $(j+1)^{th}$ time step is obtained using the local error estimate up to j^{th} time step as follows.

$$\begin{aligned}
\|E_{j+1}\| &= \|\sum_{i=1} e_j\|, \qquad j \leq T/\Delta t \\
&\leq \|e_1\| + \|e_2\| + \|e_3\| + \|e_4\| + \cdots + \|e_j\| \\
&\leq C_1(j\Delta t)\Delta t \\
&\leq C_1 T\Delta t, \qquad \text{since } \; j\Delta t \leq T \\
&\leq C\Delta t.
\end{aligned}$$

Hence, $\|E^{j+1}\| = \max_i|U(x,t_{j+1}) - U^{j+1}(x)|_\Omega \leq C\Delta t$, where C is a positive constant independent of ε and Δt. ■

The solution $U^{j+1}(x)$ of the problem (7.12) can be decomposed as

$$U^{j+1}(x) = V^{j+1}(x) + W^{j+1}(x), \tag{7.17}$$

where $V^{j+1}(x)$ and $W^{j+1}(x)$ are the regular and singular component respectively, and $V^{j+1}(x)$ is the solution of

$$\begin{cases}
-\varepsilon\frac{d^2V^{j+1}(x)}{dx^2} + r(x)V^{j+1}(x) = g_1^{j+1}(x), \quad x \in (0,1), \\
V^{j+1}(0) = V_0^{j+1}(0), \\
V^{j+1}(1) = r(x)^{-1}(1)\left(g^{j+1}(1)\right), \\
\mathcal{K}V^{j+1}(1) = \mathcal{K}V_0^{j+1}(1).
\end{cases}$$

and $W^{j+1}(x)$ is the solution of

$$\begin{cases}
-\varepsilon\frac{d^2W^{j+1}(x)}{dx^2} + r(x)W^{j+1}(x) = 0 \qquad x \in (0,1), \\
W^{j+1}(x) = U^{j+1}(x) - V_0^{j+1}(x), \\
\mathcal{K}W^{j+1}(1) = \mathcal{K}U^{j+1}(1) - \mathcal{K}V_0^{j+1}(1),
\end{cases}$$

where $V_0^{j+1}(x)$ is the solution of the reduced problem.

Lemma 7.8. *The derivatives of $V^{j+1}(x)$ and $W^{j+1}(x)$ satisfy the estimates*

$$\begin{aligned}
|\frac{d^kV^{j+1}(x)}{dx^k}| &\leq C(1 + \varepsilon^{-\frac{(k-2)}{2}}d_1(x,\alpha)), \;\; x \in (0,1), \\
|\frac{d^kW^{j+1}(x)}{dx^k}| &\leq \varepsilon^{-\frac{k}{2}}d_1(x,\alpha), \;\; x \in (0,1),
\end{aligned}$$

for $k = 0,1,2,3$, where $d_1(x,\alpha) = e^{-x\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}} + e^{-(1-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}$.

Proof. The proof of the lemma follows using the estimates of Lemma 7.4 and the decomposition in (7.17).

■

7.5.2 Spatial discretization

The piecewise-uniform mesh having $N \geq 4$ mesh elements on $[0,1]$ is generated by dividing into three subintervals as $\Omega_1 = \Omega_l \cup \Omega_c \cup \Omega_r$, where $\Omega_1 = [0,1]$, $\Omega_l = [0,\mu]$, $\Omega_c = (\mu, 1-\mu]$, and $\Omega_r = (1-\mu, 1]$. To obtain a piecewise-uniform mesh, we take $\frac{N}{2}$ mesh elements in Ω_c and $\frac{N}{4}$ mesh elements in each of the subintervals Ω_l and Ω_r. Hence, the piecewise-uniform mesh is given by $x_i = \begin{cases} 0, & i=0 \\ x_{i-1}+h_i, & i=1,2,3,\cdots,N, \end{cases}$ where h_i's are given by $h_i = \begin{cases} \frac{4\mu}{N}, & i=1,2,3,\cdots,\frac{N}{4}, \\ \frac{2(1-2\mu)}{N}, & i=\frac{N}{4}+1,\cdots,\frac{3N}{4}, \\ \frac{4\mu}{N}, & i=\frac{3N}{4}+1,\cdots,N, \end{cases}$ where the transition parameter μ separates the non-uniform mesh into uniform meshes and is given by $\mu = \min\left\{\frac{1}{4},\ 2\sqrt{\varepsilon}\ln N\right\}$, where N denotes the number of mesh elements in the spatial direction. Let U_i^{j+1} be denoted for the numerical approximation for the exact solution $u(x,t)$ at (x_i, t_{j+1}).

Here, a standard finite difference method to the spatial semi-discrete is applied by replacing the second-order derivative with a central difference scheme. From (7.12), the discrete operator $\mathcal{L}^{\Delta t}$ is defined as

$$\mathcal{L}^{\Delta t}U^{j+1}(x_i) = -\varepsilon D_x^+ D_x^- U_i^{j+1} + r_i U_i^{j+1}, \qquad i=1,2,\cdots,N-1, \tag{7.18}$$

with initial and boundary conditions

$$\begin{cases} U_0 = \phi_l(0), \\ U_i^0 = \phi_b, & i=1,2,3,\cdots,N-1, \\ \mathcal{K}^N U_N^{j+1} = U_N^{j+1} - \varepsilon\sum_{i=1}^{N} \dfrac{g_{i-1}U_{i-1}^{j+1} + 4g_iU_i^{j+1} + g_{i+1}U_{i+1}^{j+1}}{3}h_i = \phi_r, \end{cases} \tag{7.19}$$

where $U_i = U(x_i), r_i = r(x_i), g_i = g(x_i)$.
Here, for $(i = N)$, the integral boundary condition $\int_0^1 g(x)u(x,t)dx$ is approximated using composite Simpson's $\frac{1}{3}$ integration rule which is given as

$$\int_{i=1}^{N} g(x)u(x,t)dx \approx \sum_{i=1}^{N} \frac{g_{i-1}U_{i-1}^{j+1} + 4g_iU_i^{j+1} + g_{i+1}U_{i+1}^{j+1}}{3}h_i.$$

$$= \frac{g_0U_0^{j+1} + 4g_1U_1^{j+1} + g_2U_2^{j+1}}{3}h_1 + \cdots + \frac{g_{N-2}U_{N-2}^{j+1} + 4g_{N-1}U_{N-1}^{j+1} + g_NU_N^{j+1}}{3}h_N. \tag{7.20}$$

Substituting (7.20) into (7.19), we obtain

$$U_N^{j+1} - \varepsilon\left[\frac{g_0U_0^{j+1}+4g_1U_1^{j+1}+g_2U_2^{j+1}}{3}h_1+\cdots+\frac{g_{N-2}U_{N-2}^{j+1}+4g_{N-1}U_{N-1}^{j+1}+g_NU_N^{j+1}}{3}h_N\right]=\phi_r$$

which implies that

$$U_N^{j+1}\left(1-\varepsilon\frac{g_N}{3}h_N\right) - \varepsilon\left[\frac{g_0U_0^{j+1}+4g_1U_1^{j+1}+g_2U_2^{j+1}}{3}h_1+\cdots+\frac{g_{N-2}U_{N-2}^{j+1}+4g_{N-1}U_{N-1}^{j+1}}{3}h_N\right]=\phi_r.$$

Thus, the integral boundary condition at the right end spatial domain is given as

$$U_N^{j+1} = \frac{\varepsilon}{1-\frac{\varepsilon g_N}{3}h_N}\left[\frac{g_0U_0^{j+1}+4g_1U_1^{j+1}+g_2U_2^{j+1}}{3}h_1+\cdots+\frac{g_{N-2}U_{N-2}^{j+1}+4g_{N-1}U_{N-1}^{j+1}}{3}h_N+\phi_r\right]. \tag{7.21}$$

The proposed method satisfies the following semi-discrete maximum principle.

Lemma 7.9. *(Semi-discrete Maximum Principle): Assume that* $\sum_{i=1}^{N}\dfrac{g_{i-1}+4g_i+g_{i+1}}{3}h_i = \rho < 1$ *and* ψ^{j+1} *is any mesh function satisfying* $\psi_0^{j+1} \geq 0$, $\psi_i^{j+1} \geq 0$, $\mathcal{K}^N\psi_N^{j+1} \geq 0$, $\mathcal{L}\Psi_i^{j+1} \geq 0, \forall i \in \Omega^N$*, then* $\psi_i^{j+1} \geq 0$*, for all* $i = 0, 1, 2, \cdots, N$.

Proof. Define a test function $v_1^{j+1}(x_i) = 1 + x_i$. Note that $v_1^{j+1}(x_i) \geq 0 \ \ \forall x_i \in \bar{\Omega}^N$, $\mathcal{L}^N v_1^{j+1}(x_i) > 0, \forall x_i \in \Omega^N$, $v_1^{j+1}(x_0) > 0$, and $\mathcal{K}^N v_1^{j+1}(x_N) > 0$. Let $\lambda_1 = \max\left\{\frac{-\psi^{j+1}(x_i)}{v_1^{j+1}(x_i)}\right\}, i = 1, 2, 3, \cdots, N$. Then there exist x_i^* such that $\psi^{j+1}(x_i^*)+\lambda_1 v_1^{j+1}(x_i^*) = 0$ and $\psi^{j+1}(x_i) + \lambda_1 v_1^{j+1}(x_i) \geq 0$, for all $i = 1, 2, 3, \cdots, N, j = 1, 2, 3, \cdots, M-1$. Hence the function $\psi^{j+1}(x_i) + \lambda_1 v_1^{j+1}(x_i)$ attains a minimum value at x_i^*. Assume that the lemma does not hold true, then $\lambda_1 > 0$.

Case 1- $x_i^* = x_0, \quad 0 < (\psi^{j+1} + \lambda_1 v_1^{j+1})(x_0) = 0.$

Case 2- $x_i^* = x_i, \quad i = 1, 2, 3, \cdots, N,$

$$0 < \mathcal{L}(\psi^{j+1} + \lambda_1 v_1^{j+1})(x_i^*) = \left(-\frac{\varepsilon}{2}D_x^+D_x^- + \frac{r_i}{2}\right)\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(x_i) \leq 0.$$

Case 3- $x_i^* = x_N,$

$$\begin{aligned}
0 < \mathcal{K}\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(x_N) &= \left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(x_N)\\
&- \frac{\varepsilon}{3}\sum_{i=1}^{N}\left[g_{i-1}\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(x_{i-1}) + 4g_i\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(x_i)\right]h_i +\\
&- \frac{\varepsilon}{3}\sum_{i=1}^{N} g_{i+1}\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(x_{i+1})h_i \leq 0.
\end{aligned}$$

Which is contradicted to our assumption. Therefore $\lambda_1 > 0$ is not possible. Hence $\psi_i^{j+1} \geq 0$, $\forall x_i \in \bar{\Omega}^N$. ■

An immediate results of the above semi-discrete maximum principle, given in Lemma 7.9 is the following discrete stability result.

Lemma 7.10. *Let ψ_i^{j+1} be any mesh function for $i = 0, 1, \cdots, N$. Then*

$$\|\psi_i^{j+1}\| \leq \max\left\{\|\psi_0^{j+1}\|, \|\mathcal{K}\psi_N^{j+1}\|, \|\mathcal{L}\psi_i^{j+1}\|\right\}.$$

Proof. One can prove this Lemma 7.10, by constructing a barrier function $\left(\Theta_i^{j+1}\right)^{\pm} = \max\left\{|\psi_0^{j+1}|, |\mathcal{K}\psi_N^{j+1}|, |\mathcal{L}\psi_i^{j+1}|\right\} \pm \psi_i^{j+1}$. Then, the results followed by applying semi-discrete maximum principle. ■

7.6 Uniform convergence analysis

The ε-uniform error estimated by decomposing the solution U_i^{j+1} into smooth and singular components as $U_i^{j+1} = V_i^{j+1} + W_i^{j+1}$, where V_i^{j+1} is the solution of

$$\begin{cases} \mathcal{L}V_i^{j+1} = g_i^{j+1}, \quad i = 1, 2, \cdots, N \\ V_0^{j+1} = {\phi_l}_0^{j+1}, \\ V_i^{j+1} = {\phi_b}_i^{j+1}, \\ \mathcal{K}^N V_N^{j+1} = \mathcal{K}V_0^{j+1}(x_N), \end{cases} \tag{7.22}$$

and W_i^{j+1} must satisfy

$$\begin{cases} \mathcal{L}^N W_i^{j+1} = 0, \quad i = 1, 2, \cdots, N \\ W_i^{j+1} = U_i^{j+1} - v_i^{j+1}, \\ \mathcal{K}^N W_N^{j+1} = \mathcal{K}^N U_N^{j+1} - \mathcal{K}^N V_N^{j+1}. \end{cases} \tag{7.23}$$

Theorem 7.1. *Suppose $U^{j+1}(x)$ and U_i^{j+1} are the solutions of the problem (7.12)-(7.13) and (7.18)-(7.19) respectively, and assume that the coefficients $r(x), a(x), g(x) \in C^{4+\alpha_1}(0,1)$, and the boundary conditions satisfy $\phi_l^{j+1} \in C^{3+\alpha_1/2}(0,T)$, $\phi_b^{j+1} \in C^{6+\alpha_1}(\Gamma_b)$, $\phi_r^{j+1} \in (0,T)$, where $\alpha_1 \in (0,1)$, then we have*

$$\sup_{0<\varepsilon\ll 1} \|U^{j+1}(x) - u_i^{j+1}\| \leq CN^{-2}\ln^2 N.$$

The error can be written in the form of

$$U_i^{j+1} - U^{j+1}(x_i) = \left(V_i^{j+1} - V^{j+1}(x_i)\right) + \left(W_i^{j+1} - W^{j+1}(x_i)\right)$$

Proof. We prove the error estimates for smooth and singular components separately. The following are the error estimates for the smooth component.

At the point $x_i = x_N$,

$$\begin{aligned}
\mathcal{K}^N\left(V_i^{j+1} - V^{j+1}(x_i)\right) &= \mathcal{K}^N V_i^{j+1} - \mathcal{K}^N V^{j+1}(x_i) = \phi_r - \mathcal{K}^N V_i^{j+1} \\
&= \mathcal{K} V^{j+1}(x_i) - \mathcal{K}^N V^{j+1}(x_i) \\
&= V^{j+1}(x_i) - \varepsilon \int_{x_0}^{x_N} g(x) V^{j+1}(x) dx - V^{j+1}(x_i) \\
&+ \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}V_{i-1}^{j+1} + 4g_i V_i^{j+1} + g_{i+1}V_{i+1}^{j+1}}{3} h_i \\
&= \varepsilon \frac{g_0 V_0^{j+1} + 4g_1 V_1^{j+1} + g_2 V_2^{j+1}}{3} h_1 + \cdots + \\
&\varepsilon \frac{g_{N-1}V_{N-1}^{j+1} + 4g_N V_N^{j+1} + g_{N+1}V_{N+1}^{j+1}}{3} h_N \\
&- \varepsilon \int_{x_0}^{x_1} g(x) V^{j+1}(x) dx - \cdots - \varepsilon \int_{x_N}^{x_{N+1}} g(x) V^{j+1}(x) dx \\
&= -\varepsilon \frac{h_1^4}{90} g^{iv}(\xi_1) \frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_1) - \cdots - \frac{h_1^4}{90} g^{iv}(\xi_N) \frac{\partial^4 V^{j+1}(\xi_N)}{\partial x^4} \\
\left|\mathcal{K}^N\left(V_i^{j+1} - V^{j+1}(x_i)\right)\right| &= \left|C\varepsilon\left(h_1^4 \frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_1) + \cdots + h_N^4 \frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_N)\right)\right| \\
&\le C\varepsilon\left(h_1^4 \frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_1) + \cdots + h_N^4 \frac{\partial^4 V^{j+1}}{\partial x^4}(\xi_N)\right) \\
&\le C N^{-2}
\end{aligned}$$

where $x_{i-1} \le \xi_i \le x_i, \quad 1 \le i \le N$, and C is chosen to be an arbitrary positive constant. From the differential and difference equations, we have

$$\mathcal{L}\left(V_i^{j+1} - V^{j+1}(x_i)\right) \;=\; g^{j+1} - \mathcal{L}V^{j+1}(x_i) = \left(\mathcal{L} - \mathcal{L}^N\right) V^{j+1}(x_i).$$

Then, it implies that

$$\mathcal{L}^N\left(V_i^{j+1} - V^{j+1}(x_i)\right) = -\varepsilon\left(\frac{\partial^2}{\partial x^2} - D_x^+ D_x^-\right) V^{j+1}(x_i).$$

By using the result obtained in (Miller et al., 2012), we have the following classical estimates at $x_i \in \Omega^N$.

$$\mathcal{L}^N\left(V_i^{j+1} - V^{j+1}(x_i)\right) \leq \begin{cases} \frac{\varepsilon}{3}\left(x_{i+1} - x_{i-1}\right)\|\frac{\partial^3 V^{j+1}(x)}{\partial x^3}\|, \text{if } x_i = \mu \text{ or } x_i = 1-\mu \\ \frac{\varepsilon}{12}\left(x_i - x_{i-1}\right)^2\|\frac{\partial^4 V^{j+1}(x)}{\partial x^4}\|, \quad \text{otherwise} \end{cases}$$
$$\leq C\begin{cases} \sqrt{\varepsilon}\, N^{-1}, & \text{if } \ x_i = \mu \ \text{ or } \ x_i = 1-\mu \\ N^{-2}, & \text{otherwise.} \end{cases}$$

Now, let us define a barrier function as

$$\Phi^{j+1}(x_i) = C\frac{\mu}{\sqrt{\varepsilon}}\vartheta(x_i)N^{-2} + N^{-2}$$

where ϑ is a piecewise linear polynomial

$$\vartheta(x) = \begin{cases} \frac{x}{\mu} & \text{for } \ 0 \leq x \leq \mu, \\ 1 & \text{for } \ \mu \leq x \leq 1-\mu, \\ \frac{1-x}{\mu} & \text{for } \ 1-\mu \leq x \leq 1. \end{cases}$$

Then, $\forall\, x_i, i = 1, 2, 3, \cdots, N$,

$$0 \leq \Phi^{j+1}(x_i) \leq CN^{-2}\ln N.$$

And

$$\mathcal{L}^N\Phi^{j+1}(x_i) \geq \begin{cases} C\sqrt{\varepsilon}N^{-1} + N^{-2} & \text{if } \ x_i = \mu \ \text{ or } \ x_i = 1-\mu, \\ CN^{-2} & \text{otherwise,} \end{cases}$$

where the observations that $\frac{\mu}{\sqrt{\varepsilon}} \leq 2\ln N$ and

$$\mathcal{L}^N\vartheta(x_i) = \begin{cases} \frac{\varepsilon N}{\mu} + r(x_i), & \text{if } \ x_i = \mu \ \text{ or } \ x_i = 1-\mu, \\ r(x_i)\vartheta(x_i), & \text{otherwise.} \end{cases}$$

And for all $x_i \in \Gamma^N$, we obtain $\phi_r^{j+1}(x_i) \geq 0$. Which implies that $\mathcal{K}^N\vartheta^{j+1}(x_i) \geq 0$.

Define a comparison function

$$\pi^{\pm}(x_i) = \Phi(x_i) \pm \left(V_i^{j+1} - V^{j+1}(x_i)\right).$$

For each point $x_i, i = 1, 2, \cdots, N$, we have $\mathcal{L}^N\pi^{\pm}(x_i) \geq 0$. Then, from the semi-discrete maximum principle, $\pi^{\pm}(x_i) \geq 0, \forall i = 1, 2, \cdots, N$.

Then, we have $V_i^{j+1} - V^{j+1}(x_i) \leq \Phi(x_i) \leq CN^{-2}\ln N$. Hence,

$$|V_i^{j+1} - V^{j+1}(x_i)| \leq CN^{-2}\ln N. \tag{7.24}$$

To estimate the singular component error, we decompose $W^{j+1}(x)$ into $W_l^{j+1}(x)$ and $W_r^{j+1}(x)$ as follows.

$$\begin{cases} \mathcal{L}^N W_l^{j+1}(x_i) = 0, \quad i = 1, 2, \cdots, N, \\ W_l^{j+1}(x_i) = \phi_l^{j+1}(x_i) - v_0^{j+1}(x_i), & x_i \in \Gamma_l^N, \\ W_l^{j+1}(x_i) = 0, & x_i \in \Gamma_r^N, \\ W_l^{j+1}(x_i) = 0, & x_i \in \Gamma_b^N, \end{cases}$$

and

$$\begin{cases} \mathcal{L}^N W_r^{j+1}(x_i) = 0, \quad i = 1, 2, \cdots, N, \\ W_r^{j+1}(x_i) = 0, & x_i \in \Gamma_l^N, \\ \mathcal{K}^N W_r^{j+1}(x_i) = \mathcal{K}^N W^{j+1}(x_i), & x_i \in \Gamma_r^N, \\ W_r^{j+1}(x_i) = 0, & x_i \in \Gamma_b^N. \end{cases}$$

The error of singular component is equivalent to

$$W_i^{j+1} - W^{j+1}(x_i) = \left({W_l}_i^{j+1} - {W_l}^{j+1}(x_i)\right) + \left({W_r}_i^{j+1} - {W_r}^{j+1}(x_i)\right).$$

$$\mathcal{L}^N\left(W_i^{j+1} - W^{j+1}(x_i)\right) \leq -\varepsilon\left(\frac{\partial^2}{\partial x^2} - D^+D^-\right)W^{j+1}(x_i).$$

By using a classical estimate given in (Miller et al., 2012), at each point $x_i, \ i = 1, 2, \cdots, N$, it follows that

$$\mathcal{L}^N\left(W_i^{j+1} - W^{j+1}(x_i)\right) \leq C(N^{-1}\ln N)^2, \qquad \text{for } i = 1, 2, \cdots, N.$$

The error estimate for ${W_r}_i^{j+1} - {W_r}^{j+1}(x_i)$ is given as follows. The explanation depends on whether $\mu = \frac{1}{4}$ or $\mu = 2\sqrt{\varepsilon}\ln N$.

Case-I. $\mu = \frac{1}{4}$. The mesh in this case is uniform and $\mu = 2\sqrt{\varepsilon}\ln N \geq \frac{1}{4}$. It is clear that

$x_i - x_{i-1} = N^{-1}$ and $\varepsilon^{-\frac{1}{2}} \leq C \ln N$. From (Miller et al., 2012) we have

$$\begin{aligned}
\mathcal{K}^N \left(W_{ri}^{j+1} - W_r{}^{j+1}(x_i) \right) &= \mathcal{K}^N W_{ri}^{j+1} - \mathcal{K}^N W_r{}^{j+1}(x_i) \\
&= \phi_r - \mathcal{K}^N W_r{}^{j+1}(x_i) \\
&= \mathcal{K} W_r{}^{j+1}(x_i) - \mathcal{K}^N W_r{}^{j+1}(x_i) \\
\left| \mathcal{K}^N \left(W_{ri}^{j+1} - W_r{}^{j+1}(x_i) \right) \right| &\leq C\varepsilon \left(h_1^4 \frac{\partial^4}{\partial x^4} W_r{}^{j+1}(\xi_i) + \cdots + h_N^4 \frac{\partial^4}{\partial x^4} W_r{}^{j+1}(\xi_N) \right) \\
&\leq C\varepsilon^{-1} \left(h_1^4 + \cdots + h_N^4 \right) \\
&\leq C N^{-2} \ln^2 N
\end{aligned}$$

where $x_{i-1} \leq \xi_i \leq x_i$. By applying semi-discrete uniform stability estimate of Lemma 7.10, we obtain

$$|W_{ri}^{j+1} - W_r{}^{j+1}(x_i)| \leq C N^{-2} \ln^2 N.$$

Case-II. When $\mu < \frac{1}{4}$. Because the mesh is piecewise uniform in the subinterval $[\mu, 1-\mu]$, the mesh elements are $2(1-2\mu)/N$, while the rest of the intervals $[0, \mu]$ and $[1-\mu, 1]$ have mesh elements of $4\mu/N$. By (Miller et al., 2012), we have

$$\mathcal{K}^N \left(W_{ri}^{j+1} - W_r{}^{j+1}(x_i) \right) \leq C N^{-2} \ln^2 N$$

and

$$\begin{aligned}
\left| \mathcal{K}^N \left(W_{ri}^{j+1} - W_r{}^{j+1}(x_i) \right) \right| &\leq C\varepsilon \left(h_1^4 \frac{\partial^4}{\partial x^4} W_r{}^{j+1}(\xi_i) + \cdots + h_N^4 \frac{\partial^4}{\partial x^4} W_r{}^{j+1}(\xi_N) \right) \\
&\leq C\varepsilon^{-1} \left(h_1^4 + \cdots + h_N^4 \right) \\
&\leq C N^{-2} \ln^2 N
\end{aligned}$$

where $x_{i-1} \leq \xi_i \leq x_i$. By applying Lemma 7.8 and semi-discrete uniform stability given in Lemma 7.10, we obtain

$$|W_{ri}^{j+1} - W_r{}^{j+1}(x_i)| \leq C N^{-2} \ln^2 N. \tag{7.25}$$

Similar arguments are used to compute W_l's error estimate. By combining equations (7.24) and (7.25), we have

$$\begin{aligned}
|U_i^{j+1} - U^{j+1}(x_i)| &\leq C \left(N^{-2} \ln N + N^{-2} \ln^2 N \right) \\
&\leq C N^{-2} \ln^2 N.
\end{aligned} \tag{7.26}$$

■

The semidiscrete error estimate produced in (7.16) and (7.26) is used to summarize the results of this work, which is concluded by the following theorem.

Theorem 7.2. *The error estimate for the solution of the continuous and fully discrete problems is provided by*

$$\sup_{0<\varepsilon\leq 1}\left|\left|u(x,t)-U_i^{j+1}\right|\right|\leq C\left(N^{-2}\ln^2 N+\Delta t\right),$$

where $u(x,t)$ and U_i^{j+1} are the solutions of problem (7.1) and (7.18)-(7.19) respectively.

Proof. The proof follows from Lemma 7.7 and Theorem 7.1. ■

7.7 Numerical results and discussions

In this section, the proposed numerical method is demonstrated by using two model examples given below. Since the exact solution for the presented problems is unknown, the double mesh technique is employed to determine the maximum pointwise error of the proposed method using (6.27), uniform error estimate using (6.28), rate of convergence of the developed numerical scheme using (6.29), and uniform rate of convergence by using (6.30).

Example 7.1. *Consider SPPPDEs with integral boundary conditions of the form*

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t}-\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{1+x^2}{2}u(x,t)=e^{-t}-1+\sin(\pi\, x), && (x,t)\in(0,1)\times(0,1]\\
&u(x,0)=0, && x\in(0,1),\\
&u(0,t)=0, && t\in(0,1],\\
&\mathcal{K}u(1,t)=u(1,t)-\varepsilon\int_0^1\frac{x}{6}u(x,t)dx=0, \quad t\in(0,1].
\end{aligned}$$

Example 7.2. *Consider SPPPDEs with integral boundary conditions of the form*

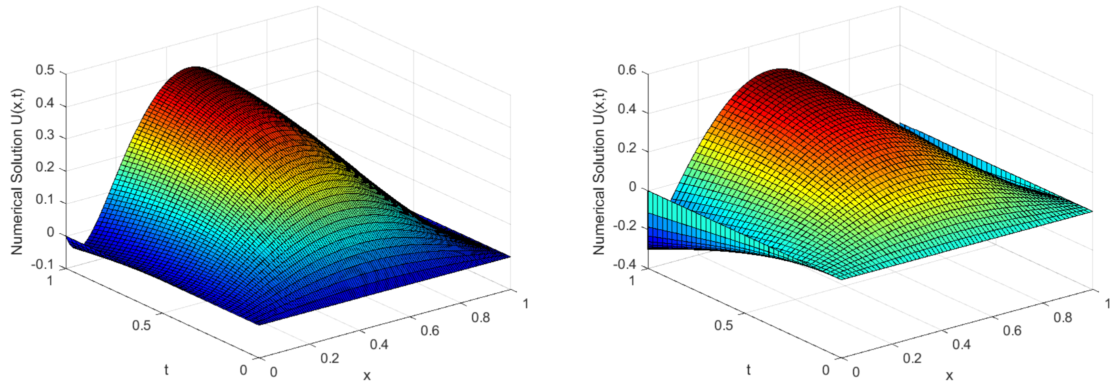
$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t}-\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{1+x^2}{2}u(x,t)=t^3, && (x,t)\in(0,1)\times(0,1]\\
&u(x,0)=0, && x\in(0,1),\\
&u(0,t)=0, && t\in(0,1],\\
&\mathcal{K}u(1,t)=u(1,t)-\varepsilon\int_0^1\cos(x)u(x,t)dx=0, \quad t\in(0,1].
\end{aligned}$$

Table 7.1: Maximum absolute error and uniform rate of convergence for Example 7.1

ε $\downarrow$	$N=32$ $\Delta t=0.1$	$N=64$ $\Delta t=0.1/4$	$N=128$ $\Delta t=0.1/4^2$	$N=256$ $\Delta t=0.1/4^3$	$N=512$ $\Delta t=0.1/4^4$
10^{-3}	1.5452e-02	4.7360e-03	1.2406e-03	3.1235e-04	7.8226e-05
10^{-4}	1.6278e-02	5.0844e-03	1.4961e-03	4.3023e-04	1.2322e-04
10^{-5}	1.6535e-02	5.1259e-03	1.5084e-03	4.3288e-04	1.2391e-04
10^{-6}	1.6616e-02	5.1390e-03	1.5123e-03	4.3372e-04	1.2412e-04
	1.6930	1.7647	1.8019	1.8050	
10^{-7}	1.6642e-02	5.1431e-03	1.5135e-03	4.3398e-04	1.2419e-04
	1.6941	1.7647	1.8022	1.8051	
10^{-8}	1.6650e-02	5.1444e-03	1.5139e-03	4.3406e-04	1.2421e-04
	1.6944	1.7647	1.8023	1.8051	
10^{-9}	1.6653e-02	5.1448e-03	1.5140e-03	4.3409e-04	1.2422e-04
	1.6946	1.7647	1.8023	1.8051	
10^{-10}	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
	1.6946	1.7647	1.8024	1.8051	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
	1.6946	1.7647	1.8024	1.8051	-
$E^{N,\Delta t}$	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
$R^{N,\Delta t}$	1.6946	1.7647	1.8024	1.8051	-

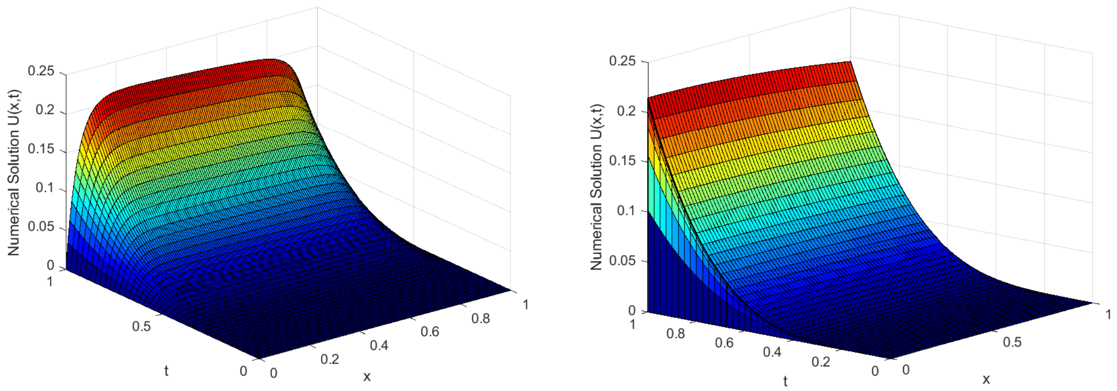
Table 7.2: Maximum absolute error and uniform rate of convergence for Example 7.2

ε $\downarrow$	$N=32$ $\Delta t=0.1$	$N=64$ $\Delta t=0.1/4$	$N=128$ $\Delta t=0.1/4^2$	$N=256$ $\Delta t=0.1/4^3$	$N=512$ $\Delta t=0.1/4^4$
10^{-3}	1.8099e-02	7.1575e-03	2.0069e-03	5.1704e-04	1.3036e-04
10^{-4}	1.8099e-02	7.2697e-03	2.2771e-03	6.7485e-04	1.9289e-04
10^{-5}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
10^{-6}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
	1.3159	1.6747	1.7545	1.8068	
10^{-7}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
	1.3159	1.6747	1.7545	1.8068	
10^{-8}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
	1.3159	1.6747	1.7545	1.8068	
10^{-8}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
	1.3159	1.6747	1.7545	1.8068	
10^{-9}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
	1.3159	1.6747	1.7545	1.8068	
10^{-10}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
	1.3159	1.6747	1.7545	1.8068	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
	1.3159	1.6747	1.7545	1.8068	
$E^{N.\Delta t}$	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
$R^{N,\Delta t}$	1.3159	1.6747	1.7545	1.8068	-



(a) (b)

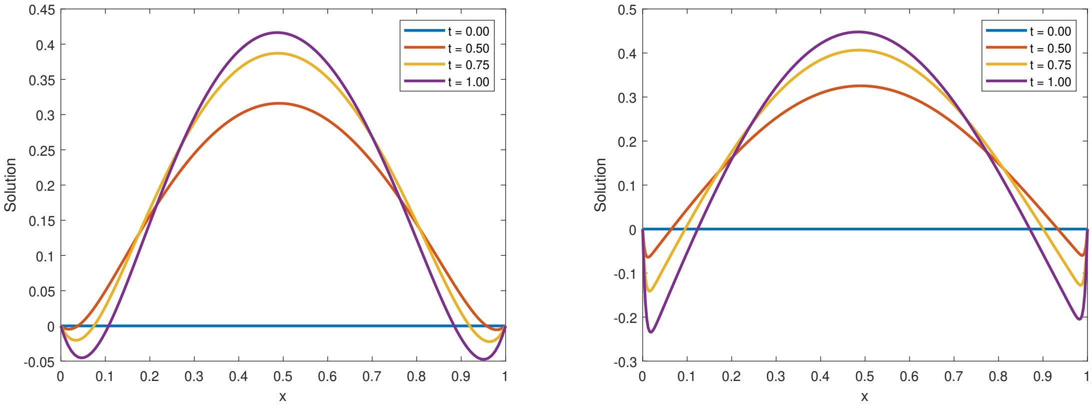
Figure 7.1: Graph of numerical solution profiles for Example 7.1: (a) for $\varepsilon = 10^{-2}$, for $\varepsilon = 10^{-12}$.



(a) (b)

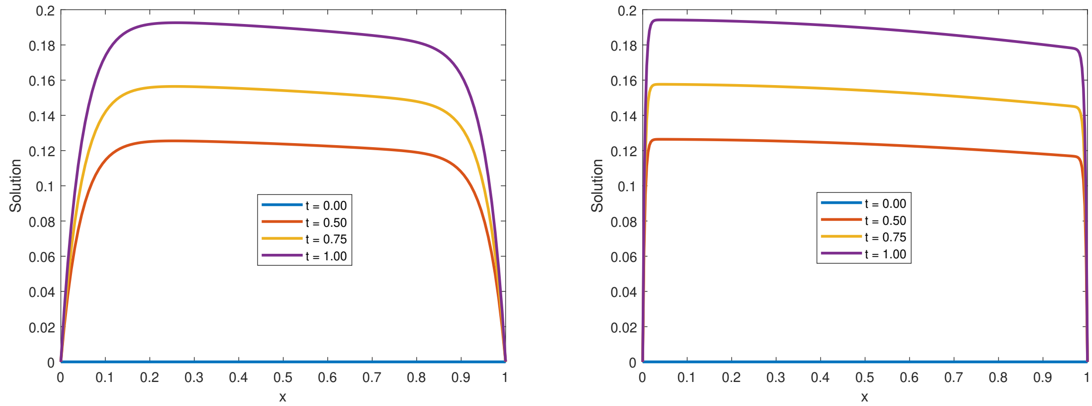
Figure 7.2: Graph of numerical solution profiles for Example 7.1: (a) for $\varepsilon = 10^{-2}$, for $\varepsilon = 10^{-12}$.

The solutions of the above two examples exhibit strong boundary layers near $x = 0$ and $x = 1$. Because the small perturbation parameter ε operates on the problem's highest derivative. Figures 7.1(a) and 7.1(b) depicts the 3-D graphs for the numerical solution of Examples 7.1 which demonstrating the presence of boundary layer formation on the left and right sides of the spatial domain for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-12}$ respectively. Figures 7.2(a) and 7.2(b) depicts the 3-D graphs for the numerical solution of Examples 7.2 which demonstrating the presence of boundary layer formation on the left and right sides of the spatial domain for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-12}$ respectively. Figures 7.3 and 7.4 shows the numerical solution profiles for different time level respectively for Examples 7.1 and 7.2. The maximum pointwise error and the parameter uniform rate of convergence for the proposed method for Examples 7.1 and 7.2 are given in Tables 7.1 and 7.2, respectively, for different values of the perturbation parameter ε, the mesh number N, and the time step size Δt. From these tables, one can



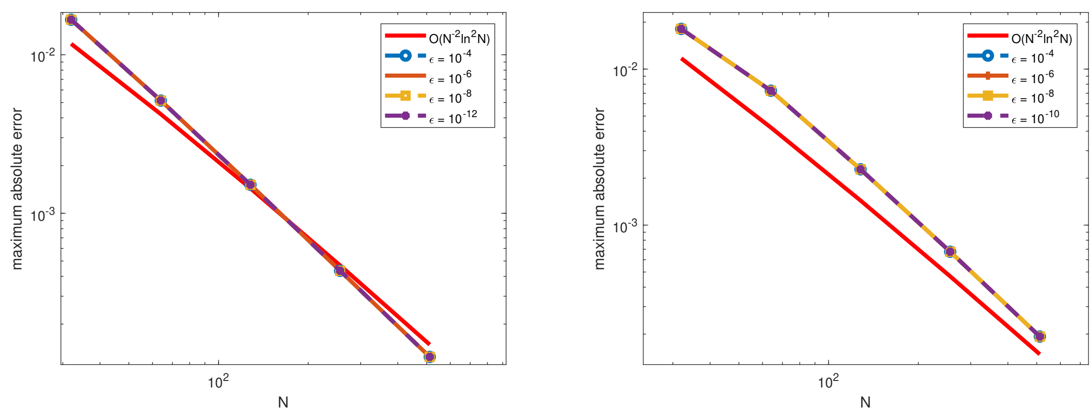
(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-4}$

Figure 7.3: Graph of numerical solution profiles for Example 7.1 for different time level



(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-4}$

Figure 7.4: Graph of numerical solution profiles for Example 7.2 for different time level



(a) (b)

Figure 7.5: Log-Log plot of maximum absolute error versus mesh numbers for different values of ε (a) for Example 7.1, (b) for Example 7.2.

Table 7.3: Comparison of uniform error estimate for nonstandard FDM and standard FDM on Shishkin mesh for Example 7.1

	$N = 32$ $\Delta t = 0.1$	$N = 64$ $\Delta t = 0.1/4$	$N = 128$ $\Delta t = 0.1/4^2$	$N = 256$ $\Delta t = 0.1/4^3$	$N = 512$ $\Delta t = 0.1/4^4$
Present Method					
$E^{N,\Delta t}$	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
$R^{N,\Delta t}$	1.6946	1.7647	1.8024	1.8051	-
Method in Chapter 6					
$E^{N,\Delta t}$	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
$R^{N,\Delta t}$	1.8951	1.9645	1.9859	1.9938	-

Table 7.4: Comparison of uniform error estimate for nonstandard FDM and standard FDM on Shishkin mesh for Example 7.2

	$N = 32$ $\Delta t = 0.1$	$N = 64$ $\Delta t = 0.1/4$	$N = 128$ $\Delta t = 0.1/4^2$	$N = 256$ $\Delta t = 0.1/4^3$	$N = 512$ $\Delta t = 0.1/4^4$
Present Method					
$E^{N.\Delta t}$	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
$R^{N,\Delta t}$	1.3159	1.6747	1.7545	1.8068	-
Method in Chapter 6					
$E^{N,\Delta t}$	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
$R^{N,\Delta t}$	1.5354	1.8919	1.9736	1.9935	-

understand that the proposed method is a parameter uniform convergent independent of the perturbation parameter ε, with almost second-order uniform convergence in the spatial direction. Tables 7.3 and 7.4 are indicates a comparison of uniform error estimate and uniform rate of convergence for a nonstandard FDM on uniform mesh versus a standard FDM on piecewise uniform Shishkin mesh. A standard FDM on Shishkin mesh has a limitations over order of convergence due to a logarithmic factor. Further, the log-log plot for the maximum absolute error for the two examples are plotted in figures 7.5. From these figures, one can observe that the error decreases monotonically as N increases.

7.8 Summary

In this chapter, a uniformly convergent numerical method for singularly perturbed partial differential equation with integral boundary condition is developed. Finite difference method on a rectangular piecewise uniform Shishkin mesh is proposed for the spatial direction and the implicit Euler method is used for the temporal direction. The composite Simpson's $\frac{1}{3}$ rule is used to approximate the integral boundary condition. The uniform stability and uniform convergence of the method are investigated. The developed numerical method is shown to be almost second-order uniformly convergent in spatial direction and first-order in temporal direction. Two test examples are carried out to validate the applicability of the proposed method.

CHAPTER EIGHT

A FITTED MESH FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED PARABOLIC PARTIAL DELAY DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITION

This chapter presents a numerical treatment of singularly perturbed parabolic partial delay differential equations (SPPPDDEs) of reaction diffusion type with nonlocal boundary condition. In the considered problem, a small parameter ε multiplies the higher order derivative term. The presence of this parameter causes the existence of boundary layers in the solution. The solution also exhibits an interior layer because of the large spatial delay. A standard finite difference scheme on piecewise uniform Shishkin mesh is proposed to discretize the problem in the spatial direction, and the Crank-Nicolson method is used in the temporal direction. Simpson's $\frac{1}{3}$ rule is applied to approximate the integral boundary condition given on the right end plane. The developed numerical scheme is parameter uniform convergent, with almost second order of convergence in space and second-order of convergence in time. Two numerical examples are considered to validate the applicabilty of the developed numerical method.

8.1 The governing problem

Consider the SPPPDDE of reaction diffusion type with integral boundary condition

$$\begin{aligned}
&\mathcal{L}u(x,t) = -\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial t} + a(x,t)u(x,t) + b(x,t)u(x-1,t) = f(x,t), \ \ (x,t) \in \Omega, \\
&\quad u(x,t) = \phi_l(x,t), \ \ \forall (x,t) \in \Gamma_l = \{(x,t); -1 \le x \le 0 \ \text{ and } \ 0 \le t \le T\}, \\
&\mathcal{K}u(x,t) = u(2,t) - \varepsilon\int_0^2 g(x)u(x,t)dx = \phi_r(x,t), \ \forall (x,t) \in \Gamma_r = \{(2,t); 0 \le t \le T\}, \\
&\quad u(x,t) = \phi_b(x,t), \ \forall (x,t) \in \Gamma_b = \{(x,0); 0 \le x \le 2\},
\end{aligned} \tag{8.1}$$

where $(x,t) \in \Omega = (0,2) \times (0,T]$, $\bar{\Omega} = [0,2] \times [0,T]$, $\Omega_1 = (0,1) \times [0,T]$, $\Omega_2 = (1,2) \times [0,T]$, $\Omega^* = \Omega_1 \cup \Omega_2$ and ε is a small positive parameter ($0 < \varepsilon \ll 1$). Assume that $a(x,t) \ge \alpha > 0$, $b(x,t) \le \beta < 0$, $\alpha + \beta \ge 2\gamma > 0$, $f(x,t)$, ϕ_l, ϕ_r, ϕ_b are sufficiently smooth and $g(x)$ is monotonically non negative function and satisfy $\int_0^2 g(x)dx < 1$. The above problem (8.1) is equivalent to

$$\mathcal{L}u(x,t) = F(x,t)$$

where

$$\mathcal{L}u(x,t) = \begin{cases} \mathcal{L}_1 u(x,t) = \left(-\varepsilon \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + a(x,t)\right) u(x,t), & (x,t) \in \Omega_1, \\ \mathcal{L}_2 u(x,t) = \left(-\varepsilon \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} + a(x,t)\right) u(x,t) + b(x,t)u(x-1,t), & (x,t) \in \Omega_2, \end{cases} \tag{8.2}$$

$$F(x,t) = \begin{cases} f(x,t) - b(x,t)\phi_l(x-1,t), & (x,t) \in \Omega_1, \\ f(x,t), & (x,t) \in \Omega_2, \end{cases} \tag{8.3}$$

with boundary conditions

$$\begin{aligned} & u(x,t) = \phi_l(x,t), \ \ \phi_l(x,t) \in \Gamma_l = \{(x,t); -1 \le x \le 0 \ \text{ and } \ 0 \le t \le T\}, \\ & u(1^-,t) = u(1^+,t), \ \ \frac{\partial u}{\partial x}(1^-,t) = \frac{\partial u}{\partial x}(1^+,t), \\ & \mathcal{K}u(x,t) = u(2,t) - \varepsilon \int_0^2 g(x)u(x,t)dx = \phi_r(x,t), \ \phi_r(x,t) \in \Gamma_r = \{(2,t); 0 \le t \le T\}, \\ & u(x,t) = \phi_b(x,t), \ \phi_b(x,t) \in \Gamma_b = \{(x,0)\}. \end{aligned} \tag{8.4}$$

For the problem (8.1) - (8.4) to have a unique solution, we assume that the coefficients of the given problem are Hölder continuous and also impose a proper compatibility conditions at the corners $(0,0), (2,0), (-1,0)$ and $(1,0)$ (Ladyzhenskaya et al., 1968). The problem necessarily satisfy the following conditions:

$$\phi_b(0,0) = \phi_l(0,0), \phi_b(2,0) = \phi_r(2,0)$$

and

$$\begin{aligned} -\varepsilon \frac{\partial^2 \phi_b(0,0)}{\partial x^2} + a(0,0)\phi_b(0,0) + \frac{\partial \phi_l(0,0)}{\partial t} + b(0,0)\phi_l(-1,0) = f(0,0) \\ -\varepsilon \frac{\partial^2 \phi_b(2,0)}{\partial x^2} + a(2,0)\phi_b(2,0) + \frac{\partial \phi_l(2,0)}{\partial t} + b(2,0)\phi_l(1,0) = f(2,0) \end{aligned}$$

8.2 Properties of continuous solution

Some properties of the continuous solution (8.1) which guarantee the existence and uniqueness of the exact solution is presented here. A replication of this property in the semi-discrete form will be used to analyze the numerical method which presented in the next section.

Lemma 8.1. *(Continuous Maximum Principle) If* $\psi(x,t) \in C^{(0,0)}(\bar{\Omega}) \cap C^{(1,0)}(\Omega) \cap C^{(2,1)}(\Omega_1 \cup \Omega_2)$ *such that* $\psi(0,t) \geq 0, \psi(x,0) \geq 0, \mathcal{K}\psi(2,t) \geq 0, \mathcal{L}_1\psi(x,t) \geq 0, \forall (x,t) \in \Omega_1, \mathcal{L}_2\psi(x,t) \geq 0, \forall (x,t) \in \Omega_2$ *and* $[\psi_x](1,t) = \psi_x(1^+,t) - \psi_x(1^-,t) \leq 0$*, then* $\psi(x,t) \geq 0$*, for all* $(x,t) \in \bar{\Omega}$.

Proof. Define a test function as

$$s(x,t) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & (x,t) \in \Omega_1, \\ \frac{3}{8} + \frac{x}{4}, & (x,t) \in \Omega_2. \end{cases} \tag{8.5}$$

Since $s(x,t) \geq 0, \;\; \forall (x,t) \in \bar{\Omega}$, $s(0,t) \geq 0$, we have

$$\begin{aligned}
\mathcal{L}_1 s(x,t) &= \frac{\partial}{\partial t}s(x,t) - \varepsilon \frac{\partial^2}{\partial x^2}s(x,t) + a(x,t)s(x,t) \geq 0, \;\; \forall (x,t) \in \Omega_1 \\
\mathcal{L}_2 s(x,t) &= \frac{\partial}{\partial t}s(x,t) - \varepsilon \frac{\partial^2}{\partial x^2}s(x,t) + a(x,t)s(x,t) + b(x,t)s(x-1,t) \geq 0, \;\; \forall (x,t) \in \Omega \\
\mathcal{K}s(x,t) &= s(2,t) - \varepsilon \int_0^2 g(x)s(x,t)dx \geq 0. \\
[s_x](1,t) &= s_x(1^+,t) - s_x(1^-,t) \leq 0.
\end{aligned}$$

Now, let

$$\lambda_1 = \max\left\{ \frac{-\psi(x,t)}{s(x,t)} : \;\; (x,t) \in \bar{\Omega} \right\}. \tag{8.6}$$

Then, there exists $(x_0,t_0) \in \bar{\Omega}$ such that $\psi(x_0,t_0) + \lambda_1 s(x_0,t_0) = 0$ and $\psi(x,t) + \lambda_1 s(x,t) \geq 0, \forall (x,t) \in \bar{\Omega}$. Then the functions $(\psi + \lambda_1 s)$ has a minimum value at $(x,t) = (x_0,t_0)$. Assume $\lambda_1 > 0$, we prove by contradiction.

Case-I: When $(x_0,t_0) = (0,t_0)$,

$$0 \leq (\psi + \lambda_1 s)(0,t_0) = \psi(0,t_0) + \lambda_1 s(0,t_0) < 0.$$

Case-II: When $(x_0,t_0) \in \Omega_1$,

$$0 < \mathcal{L}_1(\psi+\lambda_1 s) = -\varepsilon(\psi+\lambda_1 s)_{xx}(x_0,t_0) + (\psi+\lambda_1 s)_t(x_0,t_0) + a(x_0,t_0)(\psi+\lambda_1 s)(x_0,t_0) \leq 0.$$

Case-III: When $(x_0,t_0) = (1,t_0)$,

$$0 \leq [\psi + \lambda_1 s]_x (1,t) = (\psi + \lambda_1 s)(1^+,t) - (\psi + \lambda_1 s)(1^-,t) \leq 0$$

Case-IV: When $(x_0, t_0) \in \Omega_2$,

$$0 < \mathcal{L}_2(\psi + \lambda_1 s) = -\varepsilon(\psi + \lambda_1 s)_{xx}(x_0, t_0) + (\psi + \lambda_1 s)_t(x_0, t_0)$$
$$+a(x_0, t_0)(\psi + \lambda_1 s)(x_0, t_0) + b(x_0, t_0)(\psi + \lambda_1 s)(x_0 - 1, t_0) \leq 0.$$

Case-V: When $(x_0, t_0) = (2, t_0)$,

$$0 < \mathcal{K}(\psi + \lambda_1 s)(2, t_0) = (\psi + \lambda_1 s)(2, t_0) - \varepsilon \int_0^2 g(x)(\psi + \lambda_1 s)(x, t)dx \leq 0.$$

In all the cases, we arrived at contradiction. Hence $\lambda_1 > 0$ is impossible. Therefore $\psi(x) \geq 0, \forall x \in \bar{\Omega}$. Hence the proof of the lemma is completed. ■

Lemma 8.2. *(Stability Result) The solution $u(x,t)$ for the problems (8.1)–(8.3) satisfies the bound*

$$||u||_{\bar{\Omega}} = C \max \left\{ ||u||_{\Gamma_l}, ||u||_{\Gamma_b}, ||\mathcal{K}u||_{\Gamma_r}, ||\mathcal{L}u||_{\Omega^*} \right\}.$$

Proof. The lemma is proved using maximum principle given in Lemma 4.1 and by constructing the barrier functions $\Theta^{\pm}(x,t) = CMs(x,t) \pm u(x,t), (x,t) \in \bar{\Omega}$, where $M = \max \left\{ ||u||_{\Gamma_l}, ||u||_{\Gamma_b}, ||u||_{\Gamma_r}, ||\mathcal{L}u||_{\Omega^*} \right\}$ and $s(x,t)$ is the test function in (8.5). For $x = 0$, we have

$$\begin{aligned}
\Theta^{\pm}(x,t) &= CMs(0,t) \pm u(0,t) \\
&= C\, ||u||_{\Gamma_l} s(0,t) \pm u(0,t) \\
&\geq 0 \;\; \text{Since} \quad ||u||_{\Gamma_l} \geq u(0,t).
\end{aligned}$$

For $0 < x \leq 1$, we have

$$\begin{aligned}
\mathcal{L}_1 \Theta^{\pm}(x,t) &= -\varepsilon \Theta^{\pm}_{xx}(x,t) + \Theta^{\pm}_t(x,t) + a(x,t)\Theta^{\pm}(x,t) \\
&= -\varepsilon \left(CMs(x,t) \pm u(x,t) \right)_{xx} + \left(CMs(x,t) \pm u(x,t) \right)_t \\
&+ a(x,t) \left(CMs(x,t) \pm u(x,t) \right) \\
&= a(x,t)CMs(x,t) \pm \mathcal{L}_1 u(x,t) \\
&\leq \alpha C\, ||\mathcal{L}_1 u|| \pm \mathcal{L}_1 u(x,t) \\
&\geq 0. \quad \text{Since} \quad \alpha C\, ||\mathcal{L}_1 u|| \geq \mathcal{L}_1 u(x,t)
\end{aligned}$$

For $1 \leq x < 2$, we have

$$\begin{aligned}
\mathcal{L}_2\Theta^{\pm}(x,t) &= -\varepsilon\Theta^{\pm}_{xx}(x,t) + \Theta^{\pm}_t(x,t) + a(x,t)\Theta^{\pm}(x,t) + b(x,t)\Theta^{\pm}(x-1,t) \\
&= -\varepsilon\left(CMs(x,t) \pm u(x,t)\right)_{xx} + \left(CMs(x,t) \pm u(x,t)\right)_t \\
&+ a(x,t)\left(CMs(x,t) \pm u(x,t)\right) + b(x,t)\left(CMs(x-1) \pm u(x,t)\right) \\
&= CM\left(a(x,t)s(x,t) + b(x,t)s(x-1)\right) \pm \mathcal{L}_2u(x,t) \\
&\leq CM\left(a(x,t) + b(x,t)\right)s(x,t) \pm \mathcal{L}_2u(x,t) \\
&\leq CMs(x,t) \pm \mathcal{L}_2u(x,t) \\
&\leq C\,||\mathcal{L}_2u|| \pm \mathcal{L}_2u(x,t) \\
&\geq 0. \quad \text{Since} \quad C\,||\mathcal{L}_2u|| \geq \mathcal{L}_2u(x,t)
\end{aligned}$$

For $x = 2$, we have

$$\begin{aligned}
\mathcal{K}\Theta^{\pm}(2,t) &= \Theta^{\pm}(2,t) - \varepsilon\int_0^2 g(x)\Theta^{\pm}(x,t)dx \\
&= CMs(2,t) \pm u(2,t) - \varepsilon\int_0^2 g(x)\left(CMs(x,t) \pm u(x,t)\right)dx \\
&= CM\left[s(x,t) - \varepsilon\int_0^2 g(x)s(x,t)dx\right] \pm \left[u(2,t) - \varepsilon\int_0^2 g(x)u(x,t)dx\right] \\
&\geq CM \pm \left[u(2,t) - \varepsilon\int_0^2 g(x)u(x,t)dx\right] \\
&= C\,||\mathcal{K}u||_{\Gamma_r} \pm \mathcal{K}u(2,t) \\
&\geq 0, \qquad \text{since} \qquad ||\mathcal{K}u||_{\Gamma_r} \geq \mathcal{K}u(2,t).
\end{aligned}$$

Hence, the proof is completed. ■

8.3 Bounds on the solution and its derivatives

To estimate an error for the fitted mesh finite difference method under the assumption that the solution of (8.1) is more regular than the one guaranteed by the result in Lemma 8.3. To obtain this greater regularity, stronger compatibility conditions are imposed at the corners as given above.

Lemma 8.3. *Suppose* $a(x,t), b(x,t), f(x,t) \in C^{(2+\alpha_1,1+\alpha_1/2)}(\bar{\Omega})$*, and* $\phi_l \in C^{2+\alpha_1/2}([0,T])$, $\phi_b \in C^{(4+\alpha_1,2+\alpha_1/2)}(\Gamma_b)$, $\phi_r \in C^{2+\alpha_1/2}([0,T])$*, where* $\alpha_1 \in (0,1)$*. Then, the problem (8.1) - (8.3) has a unique solution* u *which satisfy* $u \in C^{(4+\alpha_1,2+\alpha_1/2)}(\bar{\Omega})$*. And also the derivatives of solution* u *are bounded,* $\forall i,j \in \mathbb{Z} \geq 0$ *such that* $0 \leq i+2j \leq 4$,

$$\left|\left|\frac{\partial^{i+j}u}{\partial x^i\partial t^j}\right|\right| \leq C\varepsilon^{\frac{-i}{2}}.$$

Proof. The solution and its derivative are bounded as follows. Under the stretched transformation $\tilde{x} = \frac{x}{\sqrt{\varepsilon}}$ problem (8.1) can be rewritten as

$$\begin{aligned}
&\mathcal{L}\tilde{u}(\tilde{x},t) = \left(-\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial}{\partial t} + \tilde{a}(\tilde{x},t)\right)\tilde{u}(\tilde{x},t) + \tilde{b}(\tilde{x},t)\tilde{u}(\tilde{x}-1,t) = f(\tilde{x},t), && (\tilde{x},t) \in \tilde{\Omega}\\
&\tilde{u}(\tilde{x},t) = \phi_l(\tilde{x},t), && (\tilde{x},t) \in \tilde{\Gamma}_l\\
&\mathcal{K}\tilde{u}(\tilde{x},t) = \tilde{u}(2,t) - \varepsilon\int_0^{\frac{2}{\sqrt{\varepsilon}}} g(x)\tilde{u}(\tilde{x},t)dx = \phi_r(\tilde{x},t), && (\tilde{x},t) \in \tilde{\Gamma}_r\\
&\tilde{u}(\tilde{x},t) = \phi_b(\tilde{x},t), && (\tilde{x},t) \in \tilde{\Gamma}_b
\end{aligned} \tag{8.7}$$

The problem is equivalently written as

$$\begin{aligned}
&\mathcal{L}_1\tilde{u}(\tilde{x},t) = \left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial}{\partial t} + \tilde{a}(\tilde{x},t)\right)\tilde{u}(\tilde{x},t) = f(\tilde{x},t) - \tilde{b}(\tilde{x},t)\tilde{\phi}_l(\tilde{x}-1,t) \;\; (\tilde{x},t) \in \tilde{\Omega}_1\varepsilon\\
&\mathcal{L}_2\tilde{u}(\tilde{x},t) = \left(-\varepsilon\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial}{\partial t} + \tilde{a}(\tilde{x},t)\right)\tilde{u}(\tilde{x},t) + \tilde{b}(\tilde{x},t)\tilde{u}(\tilde{x}-1,t) = f(\tilde{x},t) \;\; (\tilde{x},t) \in \tilde{\Omega}_{2,\varepsilon}
\end{aligned} \tag{8.8}$$

$$\begin{aligned}
&\tilde{u}(\tilde{x},t) = \phi_l(\tilde{x},t), && (\tilde{x},t) \in \tilde{\Gamma}_l\\
&\tilde{u}(1^-,t) = \tilde{u}(1^+,t), \quad \frac{\partial \tilde{u}}{\partial \tilde{x}}(1^-,t) = \frac{\partial \tilde{u}}{\partial \tilde{x}}(1^+,t), &&\\
&\mathcal{K}\tilde{u}(\tilde{x},t) = \tilde{u}(2,t) - \varepsilon\int_0^{\frac{2}{\sqrt{\varepsilon}}} g(\tilde{x})\tilde{u}(\tilde{x},t)d\tilde{x} = \phi_r(\tilde{x},t), && (\tilde{x},t) \in \tilde{\Gamma}_r\\
&\tilde{u}(\tilde{x},t) = \phi_b(\tilde{x},t), && (\tilde{x},t) \in \tilde{\Gamma}_b
\end{aligned} \tag{8.9}$$

where $\tilde{\Omega}_\varepsilon = (0, \frac{2}{\sqrt{\varepsilon}}) \times (0,T)$ and the boundary conditions $\tilde{\Gamma}$ to Γ.
Note that (8.7) is independent of ε. Then, by using the idea of estimation (10.6) of (Ladyzhenskaya et al., 1968) (p.352), we obtain

$$\left|\left|\frac{\partial^{i+j}\tilde{u}}{\partial \tilde{x}^i \partial t^j}\right|\right|_{\tilde{N}_\delta} \leq C\left(1 + ||\tilde{u}||_{\tilde{N}_{2\delta}}\right),$$

for all $\tilde{N}_\delta$ in $\tilde{\Omega}_\varepsilon$. Here $\tilde{N}_\delta$, $\delta > 0$ is a neighborhood with diameter δ in $\tilde{\Omega}_\varepsilon$. Returning to the original variable

$$\left|\left|\frac{\partial^{i+j}u}{\partial \tilde{x}^i \partial t^j}\right|\right|_{\bar{\Omega}} \leq C\varepsilon^{\frac{-i}{2}}\left(1 + ||u||_{\bar{\Omega}}\right)$$

Hence, the proof is complete by using the bound on u of Lemma 8.2. ■

8.4 Decomposition of the solution

The bounds on the derivatives of the solution given in Lemma 8.3 were derived from classical results. They are not adequate for the proof of the ε -uniform error estimate. Stronger bounds on these derivatives are now obtained by a method originally given in (Shishkin, 1989). The key step is to decompose the solution u into smooth and singular components.

The solution u of (8.1)-(8.4) is decomposed into v-smooth and w- singular components. Further, v is separated into

$$v = v_0 + \varepsilon v_1,$$

where v_0 and v_1 are solution of the following differential equations,

$$\begin{aligned} &\frac{\partial v_0}{\partial x}(x,t) + a(x,t)v_0(x,t) + b(x-1,t) = f(x,t), \ (x,t) \in \Omega, \\ &v_0(x,t) = 0, \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Gamma_b, \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} &\mathcal{L}v_1(x,t) = \frac{\partial^2 v_0}{\partial x^2}(x,t), && (x,t) \in \Omega \\ &v_1(x,t) = 0, && (x,t) \in \Gamma_l, \\ &\mathcal{K}v_1(x,t) = 0, && (x,t) \in \Gamma_r, \\ &v_1(x,t) = 0, && (x,t) \in \Gamma_b \end{aligned} \tag{8.11}$$

Then v satisfy

$$\begin{aligned} &\mathcal{L}v(x,t) = f(x,t), && (x,t) \in \Omega \\ &v(x,t) = v_0(x,t), && (x,t) \in \Gamma_l, \\ &\mathcal{K}v(x,t) = \mathcal{K}v_0(2,t), && (x,t) \in \Gamma_r, \\ &v(x,t) = \phi_b(x,t), && (x,t) \in \Gamma_b \end{aligned} \tag{8.12}$$

The singular component w is determined from

$$\begin{aligned} &\mathcal{L}_1 w(x,t) = 0, && (x,t) \in \Omega_1 \\ &w(x,t) = \phi_l(x,t) - v_0(x,t), && (x,t) \in \Gamma_l, \\ &[w]_x(1,t) = -[v]_x(1,t), && (x,t) \in \Gamma_r, \\ &w(x,t) = 0, && (x,t) \in \Gamma_b \end{aligned} \tag{8.13}$$

and

$$
\begin{aligned}
&\mathcal{L}_2 w(x,t) = 0, && (x,t) \in \Omega_2\\
&w(x,t) = \phi_l(x,t) - v_0(x,t), && (x,t) \in \Gamma_l,\\
&[w]_x(1,t) = -[v]_x(1,t), && (x,t) \in \Gamma_r,\\
&\mathcal{K}w(x,t) = \mathcal{K}u(x,t) - \mathcal{K}v_0(x,t), && (x,t) \in \Gamma_r\\
&w(x,t) = 0, && (x,t) \in \Gamma_b
\end{aligned}
\tag{8.14}
$$

The reaction diffusion problem occurs boundary layers on both boundary points Γ_l and Γ_r, then we separate w as $w = w_l + w_r$, where w_l and w_r are satisfy the following problems

$$
\begin{aligned}
&\mathcal{L}_1 w_l(x,t) = 0, && (x,t) \in \Omega_1\\
&w_l(x,t) = \phi_l(x,t) - v_0(x,t), && (x,t) \in \Gamma_l,\\
&w_l(x,t) = 0, && (x,t) \in \Gamma_r,\\
&w_l(x,t) = 0, && (x,t) \in \Gamma_b
\end{aligned}
\tag{8.15}
$$

$$
\begin{aligned}
&\mathcal{L}_2 w_l(x,t) = 0, && (x,t) \in \Omega_2\\
&w_l(x,t) = A, && (x,t) \in \Gamma_l,\\
&w_l(x,t) = 0, && (x,t) \in \Gamma_r,\\
&w_l(x,t) = 0, && (x,t) \in \Gamma_b
\end{aligned}
\tag{8.16}
$$

and

$$
\begin{aligned}
&\mathcal{L}_1 w_r(x,t) = 0, && (x,t) \in \Omega_1\\
&w_r(x,t) = 0, && (x,t) \in \Gamma_l,\\
&w_r(x,t) = A, && (x,t) \in \Gamma_r,\\
&w_r(x,t) = 0, && (x,t) \in \Gamma_b
\end{aligned}
\tag{8.17}
$$

$$
\begin{aligned}
&\mathcal{L}_2 w_r(x,t) = 0, && (x,t) \in \Omega_2\\
&w_r(x,t) = 0, && (x,t) \in \Gamma_l,\\
&\mathcal{K}w_r(x,t) = \mathcal{K}w(x,t), && (x,t) \in \Gamma_r,\\
&w_r(x,t) = 0, && (x,t) \in \Gamma_b
\end{aligned}
\tag{8.18}
$$

Here, A is a constant to be chosen in such a way that the solution is continuous at $x = 1$ satisfied.

Lemma 8.4. *Suppose* $a(x,t), b(x,t), f(x,t) \in C^{(4+\alpha_1,2+\alpha_1/2)}(\bar{\Omega})$, *and* $\phi_l \in C^{(3+\alpha_1/2)}([0,T]), \phi_b \in C^{(6+\alpha_1,3+\alpha_1/2)}(\Gamma_b), \phi_r \in C^{(3+\alpha_1/2)}([0,T])$, *where* $\alpha_1 \in (0,1)$. *Then, we have*

$$\left\|\frac{\partial^{i+j} v}{\partial x^i \partial t^j}\right\|_{\bar{\Omega}} \leq C\left(1+\varepsilon^{1-i/2}\right), \tag{8.19}$$

$$\left|\frac{\partial^{i+j} w_l}{\partial x^i \partial t^j}\right| \leq \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{x}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(x-1)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_2, \end{cases} \tag{8.20}$$

$$\left|\frac{\partial^{i+j} w_r}{\partial x^i \partial t^j}\right| \leq \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_2, \end{cases} \tag{8.21}$$

where C *is constant independent of perturbation parameter* ε, $(x,t) \in \bar{\Omega}$ $i,j \geq 0, 0 \leq i+2j \leq 4$.

Proof. The existence and smooth component results follows from (Ladyzhenskaya et al., 1968) (Chap IV, p.320). The derivative of smooth component functions are bounded, its derived as follows. First, we estimate that the reduced problem solution v_0 is bounded and its derivative is bounded

$$\left\|\frac{\partial^{i+j} v_0}{\partial x^i \partial t^j}\right\|_{\bar{\Omega}} \leq C. \tag{8.22}$$

Next, to estimate the derivative of the solution v_1 bounded from Lemma 4.2 can be applied. Then

$$\left\|\frac{\partial^{i+j} v_1}{\partial x^i \partial t^j}\right\|_{\bar{\Omega}} \leq C\varepsilon^{\frac{-i}{2}}, \tag{8.23}$$

since

$$\frac{\partial^{i+j} v}{\partial x^i \partial t^j} = \frac{\partial^{i+j} v_0}{\partial x^i \partial t^j} + \varepsilon \frac{\partial^{i+j} v_1}{\partial x^i \partial t^j}$$

to establish smooth component v estimates from (8.22) and (8.23). To prove the inequality (8.20) following the procedure adapted in (Miller et al., 1998), it is easy to find that

$$|w_l(x,t)| \leq Ce^{\frac{-x}{\sqrt{\varepsilon}}}, \quad (x,t) \in \Omega_1.$$

Now, we derive the bound on w_l on Ω_2 . From the defining equations for w_l, we have

$$\begin{aligned}
\mathcal{L}_2 w_l(x,t) &= -\varepsilon w_{lxx}(x,t) + w_{lt}(x,t) + a(x,t)w_l(x,t) + b(x,t)w_l(x-1,t) = 0 \quad \text{or} \\
\mathcal{L}_2 w_l(x,t) &= -\varepsilon w_{lxx}(x,t) + w_{lt}(x,t) + a(x,t)w_l(x,t) + b(x,t)w_l(x-1,t) \\
\mathcal{L}_2 w_l(x,t) &= \mathcal{L}_1 w_l(x,t) = -b(x,t)w_l(x-1,t) \\
|\mathcal{L}_2 w_l(x,t)| &\le Ce^{\frac{-(x-1)}{\sqrt{\varepsilon}}}
\end{aligned}$$

choice suitable $C > 0$. Using stability result Lemma 8.2, we have $|w_l(x,t)| \le Ce^{\frac{-(x-1)}{\sqrt{\varepsilon}}}$

$$\text{Hence,} \qquad |w_l(x,t)| \le \begin{cases} Ce^{\frac{-(x)}{\sqrt{\varepsilon}}}, \\ Ce^{\frac{-(x-1)}{\sqrt{\varepsilon}}}. \end{cases}$$

Since $\tilde{x} = \frac{x}{\sqrt{\varepsilon}}$, clearly, under the transformation, stretched variable $\tilde{x}$ domain is $(0, \frac{2}{\sqrt{\varepsilon}})$, note that $(x,t) \Rightarrow (\tilde{x},t)$ and the parameter ε is independent of problem (8.15) and (8.16), then suitable estimates in (Ladyzhenskaya et al., 1968) (Section 4.10) solution of $\tilde{w}_l$ is obtained. The position of $\tilde{x}$ argument partition into two cases. First case, neighborhood of $\tilde{N}_\delta$ in $(0, \frac{2}{\sqrt{\varepsilon}}) \times (0,T]$, by applying an estimate in (Ladyzhenskaya et al., 1968) (Section 4.10), we have

$$\left|\left| \frac{\partial^{i+j}\tilde{w}_l}{\partial\tilde{x}^i\partial t^j} \right|\right|_{\tilde{N}_\delta} \le C\left(||\tilde{w}_l||_{\tilde{N}_{2\delta}}\right),$$

and the required bound follows by transforming back to the variable x and using the bound just obtained on w_l. Likewise, for each neighborhood $\tilde{N}_\delta$ in $(0,2] \times (0,T]$, from (Ladyzhenskaya et al., 1968) (Section 4.10), we have

$$\left|\left| \frac{\partial^{i+j}\tilde{w}_l}{\partial\tilde{x}^i\partial t^j} \right|\right|_{\tilde{N}_\delta} \le C\left(1 + ||\tilde{w}_l||_{\tilde{N}_{2\delta}}\right),$$

and the required bound follows by again transforming back to the variable x, using the bound on w_l and noting that $e^{\frac{-x}{\varepsilon}} \ge e^{-2} = C$ for $\tilde{x} \le 2$. In the similar manner, to prove (8.21), consider two comparison functions

$$\Theta^{\pm}(x,t) = C\begin{cases} e^{\alpha^* t}e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_1 \\ e^{\beta^* t}e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_2 \end{cases} \pm w_r(x,t)$$

where

$$\alpha^* = \max\left\{0, 1 - \min_{(x,t)\in\bar{\Omega}} a(x,t)\right\} \text{ and } \beta^* = \max\left\{0, 1 - \min_{(x,t)\in\bar{\Omega}} (a(x,t) + b(x,t))\right\}$$

Note that $\Theta^{\pm}(x,t) \geq 0$,

$$\begin{aligned}
\mathcal{L}_1\Theta^{\pm}(x,t) &= -\varepsilon\Theta^{\pm}_{xx}(x,t) + \Theta^{\pm}_{t}(x,t) + a(x,t)\Theta^{\pm}(x,t) \\
&= C\left(\alpha^* - 1 + a(x,t)\right)e^{\alpha^* t}e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} \\
&\geq 0.
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_2\Theta^{\pm}(x,t) &= -\varepsilon\Theta^{\pm}_{xx}(x,t) + \Theta^{\pm}_{t}(x,t) + a(x,t)\Theta^{\pm}(x,t) + b(x,t)\Theta^{\pm}(x-1,t) \\
&= C\left(\beta^* - 1 + a(x,t) + b(x,t)e^{\frac{-1}{\varepsilon}}\right)e^{\beta^* t}e^{\frac{-(2-x)}{\sqrt{\varepsilon}}} \\
&\geq C\left(\beta^* - 1 + a(x,t) + b(x,t)\right)e^{\beta^* t}e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}\Theta^{\pm}(2,t) &= \Theta^{\pm}(2,t) - \varepsilon\int_0^2 \Theta^{\pm}(x,t)dx \\
&= C - C\varepsilon\int_0^1 \Theta^{\pm}(x,t)dx - C\varepsilon\int_1^2 \Theta^{\pm}(x,t)dx \\
&= C - C\varepsilon\int_0^1 g(x)e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}dx - C\varepsilon\int_1^2 g(x)e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}dx \\
&\geq C(1-\varepsilon\int_0^2 g(x)dx) \\
&\geq 0.
\end{aligned} \tag{8.24}$$

choice suitable $C > 0$. Using maximum principle Lemma 8.1, we have

$$\left|\frac{\partial^{i+j}w_r}{\partial x^i\partial t^j}\right| \leq \begin{cases} Ce^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, & (x,t)\in\Omega_1 \\ Ce^{\frac{-(2-x)}{\sqrt{\varepsilon}}}, & (x,t)\in\Omega_2 \end{cases}$$

Let $\tilde{x} = \frac{x}{\sqrt{\varepsilon}}$. Clearly, under the transformation, stretched variable $\tilde{x}$ domain is $(0, \frac{2}{\sqrt{\varepsilon}})$, note that $(x,t) \Rightarrow (\tilde{x},t)$ and the parameter ε is independent of problem (8.17, 8.18) then suitable estimates in (Ladyzhenskaya et al., 1968) (Section 4.10) solution of $\tilde{w}_r$. The position of $\tilde{x}$ argument partition into two cases. First case, neighborhood of $\tilde{N}_\delta$ in $(0, \frac{2}{\sqrt{\varepsilon}}) \times (0,T]$, by applying an estimate in (Ladyzhenskaya et al., 1968) (Section 4.10), we have

$$\left|\left|\frac{\partial^{i+j}\tilde{w}_r}{\partial\tilde{x}^i\partial t^j}\right|\right|_{\tilde{N}_\delta} \leq C\left(||\tilde{u}||_{\tilde{N}_{2\delta}}\right),$$

and the required bound follows by transforming back to the variable x and using the bound just obtained on w_r . Likewise, for each neighborhood $\tilde{N}_\delta$ in $(0,2]\times(0,T]$, from (Ladyzhenskaya et al., 1968) (Section 4.10) we have

$$\left|\left|\frac{\partial^{i+j}\tilde{w}_r}{\partial\tilde{x}^i\partial t^j}\right|\right|_{\tilde{N}_\delta} \leq C\left(1+||\tilde{w}_r||_{\tilde{N}_{2\delta}}\right),$$

and the required bound follows by again transforming back to the variable x, using the bound on w_l and noting that $e^{\frac{-x}{\varepsilon}} \geq e^{-2} = C$ for $\tilde{x} \leq 2$. Hence the proof is complete. ■

8.5 The numerical method

In this section, Crank-Nicholson method is proposed for time semi-discretization and a finite difference method on a rectangular piecewise uniform Shishkin mesh is proposed for spatial semi-discretization to solve the problem under consideration.

8.5.1 The time semi-discretization

Let M be a positive integer. The uniform mesh Ω^M which is used in the time semi-discretization is defined as

$$\Omega^M = \{t_j = j\Delta t, j = 0, 1, \cdots, M, \ \ \Delta t = T/M\}. \tag{8.25}$$

Using the Crank-Nicolson scheme on Ω^M the discretized problem in the temporal direction associated with the continuous problem (8.1) is given by

$$\begin{aligned}&\frac{U^{j+1}(x)-U^j(x)}{\Delta t} - \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}U^{j+1}(x) + \frac{a(x,t_{j+1})}{2}U^{j+1}(x) + \frac{b(x,t_{j+1})}{2}U^{j+1}(x-1)\\ &-\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}U^j(x) + \frac{a(x,t_j)}{2}U^j(x) + \frac{b(x,t_j)}{2}U^j(x-1) = \frac{1}{2}\left(f(x)^{j+1}+f(x)^j\right), \ \ (x,t)\in\Omega,\end{aligned}$$

which implies that

$$\mathcal{L}U^{j+1}(x) = g(x, t_{j+1}), \tag{8.26}$$

where the operator $\mathcal{L}$ and function $g(x, t_{j+1})$ is defined as

$$\begin{aligned}\mathcal{L}U^{j+1}(x) &= -\frac{\varepsilon}{2}U^{j+1}_{xx} + \frac{p(x)}{2}U^{j+1}(x) + \frac{b(x)}{2}U^{j+1}(x-1),\\ g(x,t_{j+1}) &= \frac{\varepsilon}{2}U^j_{xx} + \frac{q(x)}{2}U^j(x) - \frac{b(x)}{2}U^j(x-1) + \frac{1}{2}\left[f(x)^{j+1}+f(x)^j\right],\\ \text{and} \quad p(x) &= \frac{2}{\Delta t} + a(x), \qquad q(x) = \frac{2}{\Delta t} - a(x).\end{aligned}$$

Hence, the problem (8.1)-(8.4) can be written as the following.

$$\mathcal{L}_k U^{j+1}(x) = g_k(x,t_{j+1}), \quad k=1,2, \quad x\in\Omega, \quad j=0,1,\cdots,M-1, \tag{8.27}$$
$$U^{j+1}(x) = \phi_l^{j+1}(x), \quad \phi_l^{j+1}(x)\in\Gamma_l, \tag{8.28}$$
$$\mathcal{K}U^{j+1}(x) = U^{j+1}(2) - \varepsilon\int_0^2 g(x)U^{j+1}(x)dx = \phi_r^{j+1}(x),\ \phi_r^{j+1}(x)\in\Gamma_r, \tag{8.29}$$
$$U^{j+1}(x) = \phi_b^{j+1}(x), \qquad \phi_b^{j+1}(x)\in\Gamma_b, \tag{8.30}$$

where

$$\begin{aligned}
\mathcal{L}_1 U^{j+1}(x) &= -\frac{\varepsilon}{2}U_{xx}^{j+1} + \frac{p(x)}{2}U^{j+1}(x), \quad x\in\Omega_1,\\
\mathcal{L}_2 U^{j+1}(x) &= -\frac{\varepsilon}{2}U_{xx}^{j+1} + \frac{p(x)}{2}U^{j+1}(x) + \frac{b(x)}{2}U^{j+1}(x-1),\ x\in\Omega_2,\\
g_1(x,t_{j+1}) &= \frac{\varepsilon}{2}U_{xx}^{j} + \frac{q(x)}{2}U^{j}(x) - \frac{b(x)}{2}U^{j}(x-1) + \frac{1}{2}\left[f(x)^{j+1}+f(x)^{j}\right] - \frac{b(x)}{2}\phi_l^{j+1}(x-1),\\
g_2(x,t_{j+1}) &= \frac{\varepsilon}{2}U_{xx}^{j} + \frac{q(x)}{2}U^{j}(x) - \frac{b(x)}{2}U^{j}(x-1) + \frac{1}{2}\left[f(x)^{j+1}+f(x)^{j}\right].
\end{aligned}$$

The discrete operator $\mathcal{L}$ satisfies the following lemma which can be proved by dividing the domain Ω into two parts Ω_1 and Ω_2.

Lemma 8.5. *For $j=0,1,2,\cdots M-1$, assume that $\psi^{j+1}(x)$ be any function in Ω such that $\psi^{j+1}(0)\ge 0$ and $\mathcal{K}\psi^{j+1}(2)\ge 0$, then $\mathcal{L}\psi^{j+1}\ge 0, \forall x\in\Omega$, implies $\psi^{j+1}(x)\ge 0$.*

Proof. Assume that the function $[\psi^{j+1}+\lambda_1 s^{j+1}](x)$ attains a minimum value at $x=x_0$. Where s and λ_1 defined in Eqs.(8.5), and (8.6) respectively. Without loss of generality we have $[\psi^{j+1}+\lambda_1 s^{j+1}](x_0)=0$ and $[\psi^{j+1}+\lambda_1 s^{j+1}](x)\ge 0$. Suppose the lemma does not hold true, then $\lambda_1>0$.

Case-I: For $x_0\in\Omega_1$, $\mathcal{L}_1[\psi^{j+1}+\lambda_1 s^{j+1}](x_0) = -\frac{\varepsilon}{2}[\psi^{j+1}+\lambda_1 s^{j+1}]_{xx}(x_0) + \frac{r(x)}{2}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0)\le 0$.

Case-II: For $x_0\in\Omega_2$,

$$\begin{aligned}
&\mathcal{L}_2[\psi^{j+1}+\lambda_1 s^{j+1}](x_0) =\\
&= -\frac{\varepsilon}{2}[\psi^{j+1}+\lambda_1 s^{j+1}]_{xx}(x_0) + \frac{p(x)}{2}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0) + \frac{b(x)}{2}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0-1),\\
&\le -\frac{\varepsilon}{2}[\psi^{j+1}+\lambda_1 s^{j+1}]_{xx}(x_0) + \frac{p(x)}{2}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0) + \frac{b(x)}{2}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0),\\
&= -\frac{\varepsilon}{2}[\psi^{j+1}+\lambda_1 s^{j+1}]_{xx}(x_0) + \left(\frac{a(x)}{2}+\frac{1}{\Delta t}\right)[\psi^{j+1}+\lambda_1 s^{j+1}](x_0) + \frac{b(x)}{2}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0),
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\varepsilon}{2}[\psi^{j+1}+\lambda_1 s^{j+1}]_{xx}(x_0)+\frac{a(x)+b(x)}{2}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0)+\frac{1}{\Delta t}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0),\\
&\le -\frac{\varepsilon}{2}[\psi^{j+1}+\lambda_1 s^{j+1}]_{xx}(x_0)+\gamma\,[\psi^{j+1}+\lambda_1 s^{j+1}](x_0)+\frac{1}{\Delta t}[\psi^{j+1}+\lambda_1 s^{j+1}](x_0),\\
&\le 0.
\end{aligned}
$$

Case-III: For $x_0 = 2$, $\mathcal{K}[\psi^{j+1}+\lambda_1 s^{j+1}](2) = [\psi^{j+1}+\lambda_1 s^{j+1}](2) - \varepsilon\int_0^2 g(x)[\psi^{j+1}+\lambda_1 s^{j+1}](x)dx \le 0$. Observed that in all the cases we arrived at a contradiction. Therefore $\lambda_1 > 0$ is not possible. This implies that $\psi^{j+1}(x) \ge 0$. ■

An application of the above lemma is the following uniform stability estimate.

Lemma 8.6. *The solution $U^{j+1}(x)$ of (8.1) satisfies the bound*

$$
\left|\left|U^{j+1}(x)\right|\right|_{\bar{\Omega}} \le C\max\left\{\left|\left|U^{j+1}(0)\right|\right|_{\Gamma_l}, \left|\left|\mathcal{K}U^{j+1}(2)\right|\right|_{\Gamma_r}, \frac{\Delta t}{\gamma\Delta t+1}\,||g||_{\bar{\Omega}}\right\}.
$$

Proof. The proof can be done by considering Ω_1 and Ω_2 separately. Consider two comparison functions as follows

$$
\Pi^{\pm}(x,t_{j+1}) = CMs(x,t_{j+1}) \pm U^{j+1}(x),
$$

where $M = \sup\left\{||U^{j+1}(0)||_{\Gamma_l}, ||\mathcal{K}U^{j+1}(2)||_{\Gamma_r}, \frac{\Delta t}{\gamma\Delta+1}\,||g||_{\bar{\Omega}_1}\right\}$ and $s(x,t_{j+1})$ is a test function given in (8.5).

For $x = 0$, $\Pi^{\pm}(0,t_{j+1}) = CMs(0,t_{j+1}) \pm U^{j+1}(0) \ge 0$.

For $x \in \Omega_1$,

$$
\begin{aligned}
\mathcal{L}_1\Pi^{\pm}(x,t_{j+1}) &= -\frac{\varepsilon}{2}\Pi^{\pm}_{xx}(x,t_{j+1}) + \frac{p(x)}{2}\Pi^{\pm}(x,t_{j+1}) \pm U^{j+1}(x)\\
&= \frac{p(x)}{2}CMs(x,t_{j+1}) \pm \mathcal{L}_1U^{j+1}(x)\\
&= \frac{p(x)}{2}C\frac{\Delta t}{\gamma\Delta t+1}\,||g_1||_{\Omega_1}\,s(x,t_{j+1}) \pm \mathcal{L}_1U^{j+1}(x)\\
&\ge \frac{p(x)}{2}C\frac{\Delta t}{\gamma\Delta t+1}\,||g_1||_{\Omega_1}\,s(x,t_{j+1}) \pm \mathcal{L}_1U^{j+1}(x)\\
&= \frac{a(x)\Delta t+2}{2\Delta t}C\frac{\Delta t}{\gamma\Delta t+1}\,||g_1||_{\Omega_1}\,s(x,t_{j+1}) \pm g_1(x,t_{j+1})\\
&\ge \frac{(a(x)+b(x))\Delta t+2}{2\Delta t}C\frac{\Delta t}{\gamma\Delta t+1}\,||g_1||_{\Omega_1}\,s(x,t_{j+1}) \pm g_1(x,t_{j+1})\\
&= \frac{2\gamma\Delta t+2}{\Delta t}C\frac{\Delta t}{2\gamma\Delta t+2}\,||g_1||_{\Omega_1}\,s(x,t_{j+1}) \pm g_1(x,t_{j+1})\\
&= C\,||g_1||_{\Omega_1}\,s(x,t_{j+1}) \pm g_1(x,t_{j+1})\\
&\ge ||g_1||_{\Omega_1} \pm g_1(x,t_{j+1})\\
&\ge 0.
\end{aligned}
$$

For $x \in \Omega_2$,

$$\begin{aligned}
\mathcal{L}_2\Pi^{\pm}(x,t_{j+1}) &= -\frac{\varepsilon}{2}\Pi^{\pm}_{xx}(x,t_{j+1}) + \frac{p(x)}{2}\Pi^{\pm}(x,t_{j+1}) + \frac{b(x)}{2}\Pi^{\pm}(x-1,t_{j+1}) \pm u^{j+1}(x) \\
&= \frac{p(x)}{2}CMs(x,t_{j+1}) + \frac{b(x)}{2}CMs(x-1,t_{j+1}) \pm \mathcal{L}_2u^{j+1}(x) \\
&\leq \left(\frac{p(x)}{2} + \frac{b(x)}{2}\right)C\frac{\Delta t}{\gamma\Delta t+1}\,||g_2||_{\bar{\Omega}}\,s(x,t_{j+1}) \pm \mathcal{L}_2u^{j+1}(x) \\
&= \frac{p(x)+b(x)}{2}C\frac{\Delta t}{\gamma\Delta t+1}\,||g_2||_{\bar{\Omega}}\,s(x,t_{j+1}) \pm g_2(x,t_{j+1}) \\
&= \frac{a(x)\Delta t+2+b(x)\Delta t}{2\Delta t}C\frac{\Delta t}{\gamma\Delta t+1}\,||g_2||_{\Omega_2}\,s(x,t_{j+1}) \pm g_2(x,t_{j+1}) \\
&\geq \frac{2\gamma\Delta t+2}{\Delta t}C\frac{\Delta t}{2\gamma\Delta t+2}\,||g_2||_{\Omega_2}\,s(x,t_{j+1}) \pm g_2(x,t_{j+1}) \\
&= C\,||g_2||_{\Omega_2}\,s(x,t_{j+1}) \pm g_2(x,t_{j+1}) \\
&\geq ||g_2||_{\Omega_2} \pm g_2(x,t_{j+1}) \\
&\geq 0.
\end{aligned} \tag{8.31}$$

For $x = 2$, we have

$$\begin{aligned}
\mathcal{K}\Pi^{\pm}(2,t_{j+1}) &= \Pi^{\pm}(2,t_{j+1}) - \varepsilon\int_0^2 g(x)\Pi^{\pm}(x,t_{j+1})dx \\
&= CMs(2,t_{j+1}) \pm U^{j+1}(2) - \varepsilon\int_0^2 g(x)[CMs(2,t_{j+1}) \pm U^{j+1}(x)]dx \\
&= CM[S(2,t_{j+1}) - \varepsilon\int_0^2 g(x)S(2,t_{j+1})dx] \pm [U^{j+1}(2) - \varepsilon\int_0^2 g(x)u^{j+1}(x)] \\
&= CM\mathcal{K}s(2,t_{j+1}) \pm \mathcal{K}U^{j+1}(x) \\
&\geq \mathcal{K}\,||U||_{\Gamma_r} \pm \mathcal{K}U^{j+1}(2) \\
&\geq 0.
\end{aligned}$$

Hence, the result follows by the maximum principle in Lemma 8.5. ■

Lemma 8.7. *Let* $\left|\frac{\partial^j u(x,t)}{\partial t^j}\right| \leq C,\ j = 0,1,2.$ *Then the local truncation error associated with temporal direction satisfies*

$$\parallel e^{j+1} \parallel \ \leq \ C(\Delta t)^3$$

where C *is a constant independent of* ε *and* j

The local error is defined as $e^{j+1} = u(x,t_{j+1}) - U^{j+1}(x)$, where $u(x,t_{j+1})$ is the exact solution of (8.1)-(8.3)) and U^{j+1} is the numerical solution.

A truncated Taylor series expansions of $u(x, t_j)$ centered at $(j + 1/2)$ takes the form

$$u(x,t_{j+1}) = u(x,t_{j+1/2}) + \frac{\Delta t}{2}u_t(x,t_{j+1/2}) + \frac{(\Delta t)^2}{8}u_{tt}(x,t_{j+1/2}) + \frac{(\Delta t)^3}{48}u_{ttt}(x,t_{j+1/2}) + \mathcal{O}((\Delta t)^4). \tag{8.32}$$

$$u(t_j) = u(x,t_{j+1/2}) - \frac{(\Delta t)}{2}u_t(x,t_{j+1/2}) + \frac{(\Delta t)^2}{8}u_{tt}(x,t_{j+1/2}) - \frac{(\Delta t)^3}{48}u_{ttt}(x,t_{j+1/2}) + \mathcal{O}((\Delta t)^4). \tag{8.33}$$

From (8.32) and (8.33), we have

$$\frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} = u_t(x,t_{j+1/2}) + \mathcal{O}((\Delta t)^2). \tag{8.34}$$

Now an expansion of $u_t(x, t_{j+1})$ also yields

$$u_t(x,t_{j+1}) = u_t(x,t_{j+1/2}) + \frac{\Delta t}{2}u_{tt}(x,t_{j+1/2}) + \frac{(\Delta t)^2}{8}u_{ttt}(x,t_{j+1/2}) + \mathcal{O}((\Delta t)^3). \tag{8.35}$$

$$u_t(x,t_j) = u_t(x,t_{j+1/2}) - \frac{\Delta t}{2}u_{tt}(x,t_{j+1/2}) + \frac{(\Delta t)^2}{8}u_{ttt}(x,t_{j+1/2}) + \mathcal{O}((\Delta t)^3). \tag{8.36}$$

which reduces to

$$\frac{u_t(x,t_{j+1}) + u_t(x,t_j)}{2} = u_t(x,t_{j+1/2}) + \mathcal{O}((\Delta t)^2). \tag{8.37}$$

Equation (8.37) can be written as

$$u_t(t_{j+1/2}) = \frac{1}{2}[u(t_{j+1}) + u(t_j)] + \mathcal{O}((\Delta t)^2). \tag{8.38}$$

substituting (8.38) into (8.26) and further simplifying yields

$$\mathcal{L}U^{j+1}(x) = g^{j+1}(x) + \mathcal{O}((\Delta t)^3)$$

The local error $e^{j+1} = u(x, t_{j+1}) - U^{j+1}(x)$ is satisfies the semi-discrete differential operator

$$[\mathcal{L}]e^{j+1}(x) = \mathcal{O}((\Delta t)^3) \tag{8.39}$$

Since the matrix $[\mathcal{L}]$ is invertible, we obtain

$$\left|\left|e^{j+1}\right|\right| = \left|\left|\mathcal{L}\right|\right|^{-1}(\Delta t)^3$$

Now by using stability estimate, we obtain

$$||e^{j+1}|| \leq C(\Delta t)^3$$

which completes the proof.

Theorem 8.1. *The global error estimate $E^j = u(x,t_j) - U(x,t_{j+1})$ in the temporal direction satisfies*

$$||E^j|| \leq C(\Delta t)^2 = CM^{-2}, \qquad \forall j \leq T/\Delta t \tag{8.40}$$

Proof. Using the local error up to the $(j+1)^{th}$ time step given in the above lemma, we obtain the following global error at the $(j+1)^{th}$ time step as

$$\begin{aligned}
||E^j|| &\leq \left|\left|\sum_{i=1}^{j+1} E^j\right|\right| = ||E_1|| + ||E_2|| + ||E_3|| + ||E_4|| + \cdots + ||E_{j+1}|| \\
&\leq c((j+1)\Delta t)(\Delta t)^2, \qquad \text{since} \quad (j+1)\Delta t = T \\
&= cT(\Delta t)^2 \\
&\leq C(\Delta t)^2 \\
&= CM^{-2},
\end{aligned}$$

where C is a constant independent of ε and Δt. ■

The solution $U^{j+1}(x)$ of the problem (8.26) can be written in its decomposition form as

$$U^{j+1}(x) = V^{j+1}(x) + W^{j+1}(x), \tag{8.41}$$

where $V^{j+1}(x)$ and $W^{j+1}(x)$ are regular and singular component respectively, and $V^{j+1}(x)$ is the solution of the differential equation in $(0,1)$

$$\begin{aligned}
&-\frac{\varepsilon}{2}\frac{d^2V^{j+1}(x)}{dx^2} + p(x)V^{j+1}(x) = g_1^{j+1}(x) - \frac{b(x)}{2}\phi_l^{j+1}(x), \quad x \in (0,1), \\
&V^{j+1}(0) = V_0^{j+1}(0), \\
&V^{j+1}(1) = 2p(x)^{-1}(1)\left(g_1^{j+1}(1) - \frac{b(1)}{2}\phi_l^{j+1}(0)\right),
\end{aligned}$$

and also it is a solution of differential equation in $(1,2)$,

$$
\begin{aligned}
&-\frac{\varepsilon}{2}\frac{d^2V^{j+1}(x)}{dx^2}+p(x)V^{j+1}(x)+\frac{b(x)}{2}V^{j+1}(x-1)=g_2^{j+1}(x),\\
&V^{j+1}(1)=2p(x)^{-1}(1)\left(g_2^{j+1}(1)-\frac{b(1)}{2}V_0^{j+1}(0)\right),\\
&\mathcal{K}V^{j+1}(2)=\mathcal{K}V_0^{j+1}(2),
\end{aligned}
$$

where $V_0^{j+1}(x)$ is the solution of the reduced problem. In the other case $W^{j+1}(x)$ is the solution of

$$
\begin{aligned}
&-\frac{\varepsilon}{2}\frac{d^2W^{j+1}(x)}{dx^2}+p(x)W^{j+1}(x)=0, \qquad x\in(0,1),\\
&-\frac{\varepsilon}{2}\frac{d^2W^{j+1}(x)}{dx^2}+p(x)W^{j+1}(x)+\frac{b(x)}{2}W^{j+1}(x-1),\ \ x\in(1,2),\\
&W^{j+1}=U^{j+1}(x)-V_0^{j+1}(x),\\
&\mathcal{K}W^{j+1}(2)=\mathcal{K}U^{j+1}(2)-\mathcal{K}V_0^{j+1}(2).
\end{aligned}
$$

Lemma 8.8. *The derivatives of $V^{j+1}(x)$ and $W^{j+1}(x)$ satisfy the following estimates for $0\le k\le 4$.*

$$
\left|\frac{d^kV^{j+1}(x)}{dx^k}\right|\le C\begin{cases}1+\varepsilon^{-\frac{(k-2)}{2}}d_1(x,\alpha), & x\in(0,1),\\ 1+\varepsilon^{-\frac{(k-2)}{2}}d_2(x,\alpha), & x\in(1,2),\end{cases}
$$

$$
\left|\frac{d^kW^{j+1}(x)}{dx^k}\right|\le C\begin{cases}\varepsilon^{-\frac{k}{2}}d_1(x,\alpha), & x\in(0,1),\\ \varepsilon^{-\frac{k}{2}}d_2(x,\alpha), & x\in(1,2),\end{cases}
$$

where $d_1(x,\alpha)=e^{-x\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}+e^{-(1-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}$ and $d_2(x,\alpha)=e^{-(x-1)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}+e^{-(2-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}$

Proof.

The proof of the lemma follows using the estimates of Lemma 8.4 and the decomposition in (8.41). ■

8.5.2 The spatial discretization

The piecewise-uniform mesh having $N(\ge 8)$ mesh elements on $[0,2]$ is generated by dividing the first half interval $[0,1]$ into three subintervals as $\Omega_1=\Omega_l\cup\Omega_c\cup\Omega_r$, where $\Omega_1=[0,1]$, $\Omega_l=[0,\mu]$, $\Omega_c=(\mu,1-\mu]$, and $\Omega_r=(1-\mu,1]$. To obtain a piecewise-uniform mesh, we place $\frac{N}{4}$ mesh elements in Ω_c and $\frac{N}{8}$ mesh elements in each of the subintervals Ω_l and Ω_r. Hence, the piecewise-uniform mesh is given by

$$x_i = \begin{cases} 0, & i = 0 \\ x_{i-1} + h_i, & i = 1, 2, 3, \cdots, N/2, \end{cases}$$

where h_i's are given by

$$h_i = \begin{cases} \frac{8\mu}{N}, & i = 1, 2, 3, \cdots, \frac{N}{8}, \\ \frac{4(1-2\mu)}{N}, & i = \frac{N}{8} + 1, \cdots, \frac{3N}{8}, \\ \frac{8\mu}{N}, & i = \frac{3N}{8} + 1, \cdots, \frac{N}{2} \end{cases}$$

Similarly $\Omega_2 = [1, 2]$ divided into three subintervals, $\Omega_2 = \Omega_l \cup \Omega_c \cup \Omega_r$, where $\Omega_l = (1, 1 + \mu]$, $\Omega_c = (1 + \mu, 2 - \mu]$, and $\Omega_r = (2 - \mu, 2]$. The nodal points are then given by

$$x_i = x_{i-1} + h_i, \;\; i = \frac{N}{2}, \cdots, N$$

where h_i's are given by

$$h_i = \begin{cases} \frac{8\mu}{N}, & i = \frac{N}{8}, \cdots, \frac{5N}{8}, \\ \frac{4(1-2\mu)}{N}, & i = \frac{5N}{8} + 1, \cdots, \frac{7N}{8}, \\ \frac{8\mu}{N}, & i = \frac{7N}{8} + 1, \cdots, N, \end{cases}$$

where the transition parameter μ separates the non-uniform mesh into uniform meshes and is given by

$$\mu = \min\left\{\frac{1}{4}, \; 2\sqrt{\varepsilon}\ln N\right\},$$

where N denotes the number of mesh elements in the space-direction.
Now, a standard finite difference method is applied for semi-discrete problem (8.26) and replacing the second order derivative by central difference scheme, we obtain

$$\mathcal{L}U^{j+1}(x_i) = G^{j+1}(x_i),$$

where

$$\begin{aligned} \mathcal{L}U^{j+1}(x_i) &= \begin{cases} -\frac{\varepsilon}{2}U_{xx}^{j+1}(x_i) + \frac{p(x_i)}{2}U^{j+1}(x_i), & x_i \in \Omega_1^N \\ -\frac{\varepsilon}{2}U_{xx}^{j+1}(x_i) + \frac{p(x_i)}{2}U^{j+1}(x_i) + \frac{b(x_i)}{2}U^{j+1}(x_i - \frac{N}{2}), & x_i \in \Omega_2^N \end{cases} \\ G^{j+1}(x_i) &= \begin{cases} g^{j+1}(x_i) - \frac{b(x_i)}{2}\phi_l^{j+1}(x_i - \frac{N}{2}), & x_i \in \Omega_1^N \\ g^{j+1}(x_i), & x_i \in \Omega_2^N, \end{cases} \end{aligned}$$

and the discrete operator $\mathcal{L}^N$ is defined as

$$\mathcal{L}U(x_i) = \begin{cases} -\frac{\varepsilon}{2}D_x^+D_x^-U_i + \frac{p_i}{2}U_i, & i = 0,1,2,\cdots,\frac{N}{2} \\ -\frac{\varepsilon}{2}D_x^+D_x^-U_i + \frac{p_i}{2}U_i + \frac{b_i}{2}U_{i-\frac{N}{2}}, & i = \frac{N}{2}+1, \frac{N}{2}+2,\cdots,N \end{cases} \tag{8.42}$$

$$G^{j+1}(x_i) = \begin{cases} g_i - \frac{b_i}{2}\phi_{l\,i-\frac{N}{2}}, & i = 0,1,2,\cdots,\frac{N}{2} \\ g_i, & i = \frac{N}{2}+1, \frac{N}{2}+2,\cdots,N \end{cases} \tag{8.43}$$

with boundary conditions

$$\begin{cases} U_i = \phi_{l_i}, & i = -\frac{N}{2}, -\frac{N}{2}, \cdots 0. \\ D_x^- U_{\frac{N}{2}} = D_x^+ U_{\frac{N}{2}}, \\ \mathcal{K}^N U_N = U_N - \varepsilon \sum_{i=1}^{N} \dfrac{g_{i-1}U_{i-1}^{j+1} + 4g_iU_i^{j+1} + g_{i+1}U_{i+1}^{j+1}}{3} h_i = \phi_{rN} \\ U_i = \phi_b, & i = 0,1,2,3,\cdots,N. \end{cases} \tag{8.44}$$

where $U_i = U(x_i), r_i = r(x_i), g_i = g(x_i), b_i = b(x_i)$.
Here, for $i = N$, Simpson's $\frac{1}{3}$ rule given in (7.21) is used to approximate the integral $\int_0^2 g(x)u(x,t)dx$.

8.6 Uniform stability estimate

The uniform stability estimate for the developed numerical method is discussed as follows.

Lemma 8.9. *(Semi-discrete Maximum Principle): Assume that*

$$\sum_{i=1}^{N} \frac{g_{i-1} + 4g_i + g_{i+1}}{3} h = \rho < 1,$$

and Z^{j+1} be any mesh function satisfying $Z_0^{j+1} \geq 0$, $Z_i^{j+1} \geq 0$, $\mathcal{K}^N Z_N^{j+1} \geq 0$, $\mathcal{L}_1 Z_i^{j+1} \geq 0, \forall i \in \Omega_1^N$, $\mathcal{L}_2 Z_i^{j+1} \geq 0, \forall i \in \Omega_2^N$, and $[D_x]\, Z_{\frac{N}{2}}^{j+1} = D_x^+ Z_{\frac{N}{2}}^{j+1} - D_x^- Z_{\frac{N}{2}}^{j+1} \leq 0$, then $Z_i^{j+1} \geq 0$, for all $i = 0,1,2,\cdots,N$.

Proof. Define a test function

$$s^{j+1}(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2} \\ \frac{3}{8} + \frac{x_i}{4} \end{cases}$$

Note that $s^{j+1}(x_i) \geq 0 \;\; \forall x_i \in \bar{\Omega}^N$, $\mathcal{L}^N s^{j+1}(x_i) > 0, \forall x_i \in \Omega_1^N \cup \Omega_2^N$, $s^{j+1}(x_0) > 0, \mathcal{K}^N s^{j+1}(x_N) > 0$ and $[D_x]\, s^{j+1}(x_{\frac{N}{2}}) < 0$. Let

$$\lambda = \max\left\{ \frac{-Z^{j+1}(x_i)}{s^{j+1}(x_i)};\;\; i = 1,2,3,\cdots,N \right\}.$$

Then there exist x_i^* such that $Z^{j+1}(x_i^*) + \lambda s^{j+1}(x_i^*) = 0$ and $Z^{j+1}(x_i) + \lambda s^{j+1}(x_i) \geq 0$, for all $i = 1, 2, 3, \cdots, N, j = 1, 2, 3, \cdots, M-1$. Hence the function $Z^{j+1}(x_i) + \lambda s^{j+1}(x_i)$ attains a minimum value at x_i^*. Assume that the lemma does not hold true, then $\lambda > 0$.

Case-I: When $x_i^* = x_0, \quad 0 < (Z^{j+1} + \lambda s^{j+1})(x_0) = 0.$

Case-II: When $x_i^* = x_i, \quad i = 1, 2, 3, \cdots . \frac{N}{2},$

$$0 < \mathcal{L}_1(Z^{j+1} + \lambda s^{j+1})(x_i^*) = \left(-\frac{\varepsilon}{2} D_x^+ D_x^- + \frac{p_i}{2}\right)\left(Z^{j+1} + \lambda s^{j+1}\right)(x_i) \leq 0.$$

Case-III: When $x_i^* = x_{\frac{N}{2}}, \;\; 0 \leq [D_x]\,(Z^{j+1} + \lambda s^{j+1})(x_{\frac{N}{2}}) < 0.$

Case-IV: When $x_i^* = x_i, \quad i = \frac{N}{2}+1, \frac{N}{2}+2, \cdots, N,$

$$\begin{aligned} 0 < \mathcal{L}_2(Z^{j+1} + \lambda s^{j+1})(x_i^*) = \left(-\frac{\varepsilon}{2} D_x^+ D_x^- + \frac{p_i}{2}\right)\left(Z^{j+1} + \lambda s^{j+1}\right)(x_i) \\ + b\left(Z^{j+1} + \lambda s^{j+1}\right)(x_i - \frac{N}{2}) \leq 0. \end{aligned}$$

Case-V: When $x_i^* = x_N$, $0 < \mathcal{K}\left(Z^{j+1} + \lambda s^{j+1}\right)(x_N) = \left(Z^{j+1} + \lambda s^{j+1}\right)(x_N)$

$$-\varepsilon \sum_{i=1}^{N} \frac{g_{i-1}\left(Z^{j+1} + \lambda s^{j+1}\right)(x_{i-1}) + 4g_i\left(Z^{j+1} + \lambda s^{j+1}\right)(x_i) + g_{i+1}\left(Z^{j+1} + \lambda s^{j+1}\right)(x_{i+1})}{3} h_i \leq 0.$$

In all the cases we arrived at a contradiction. Therefore $\lambda > 0$ is not possible. This implies that $Z^{j+1}(x_i) \geq 0$. ■

An application of the above lemma is the following uniform stability estimate.

Lemma 8.10. *Let $U^{j+1}(x_i),\ i = 1, 2,$ be any mesh functions. Then,*

$$\left|\left|U_i^{j+1}\right|\right| \leq C \max\left\{\left|\left|U_0^{j+1}\right|\right|, \;\; \left|\left|\mathcal{K} U_N^{j+1}\right|\right|, \;\; \max_{1 \leq i \leq N} \left|\left|\mathcal{L}^N U_i^{j+1}\right|\right|\right\}.$$

Proof. Consider two barrier functions $\Psi_i^{\pm} = CMs_i^{j+1} \pm U_i^{j+1}$. We consider four different cases.

Case-I: For $i = 0, \quad \Psi_0^{\pm} = CMs_0^{j+1} \pm U_0^{j+1} = C\left|\left|U_0^{j+1}\right|\right| s_0^{j+1} \pm U_0^{j+1} \geq 0.$

Case-II: For $i = 1, 2, \cdots, \frac{N}{2},$

$$\begin{aligned} \mathcal{L}_1 \Psi_i^{\pm} &= -\frac{\varepsilon}{2}(\Psi_i^{\pm})_{xx} + \frac{p_i}{2}\Psi_i^{\pm} \\ &= \frac{p_i}{2} CMs_i^{j+1} \pm \mathcal{L}_1^N U_i^{j+1} \\ &= \frac{p_i}{2} C\left|\left|\mathcal{L}_1^N U_i^{j+1}\right|\right| s_i^{j+1} \pm \mathcal{L}_1^N U_i^{j+1} \\ &\geq 0. \end{aligned}$$

Case-III: For $\quad i = \frac{N}{2}+1, \cdots, N-1,$

$$
\begin{aligned}
\mathcal{L}_2\Psi_i^{\pm} &= -\frac{\varepsilon}{2}(\Psi_i^{\pm})_{xx} + \frac{p_i}{2}\Psi_i^{\pm} + \frac{b_i}{2}\Psi_{i-\frac{N}{2}}^{\pm} \\
&\geq \frac{p_i+b_i}{2}CMs_i^{j+1} \pm \mathcal{L}_2^N U_i^{j+1} \\
&\leq \gamma\, C\left|\left|\mathcal{L}_2^N U_i^{j+1}\right|\right| s_i^{j+1} \pm \mathcal{L}_2^N U_i^{j+1} \\
&\geq 0.
\end{aligned}
$$

Case IV: For $\quad i = N$

$$
\begin{aligned}
\mathcal{K}\Psi_i^{\pm} &= \Psi_N^{\pm} - \varepsilon\sum_{i=1}^{N}\frac{g_{i-1}\Psi_{i-1}^{\pm} + 4g_i\Psi_i^{\pm} + g_{i+1}\Psi_{i+1}^{\pm}}{2}h_i \\
&= C\left|\left|\mathcal{K}U_i^{j+1}\right|\right| s_i^{j+1} \pm \mathcal{K}U_i^{j+1} \\
&\geq 0.
\end{aligned} \tag{8.45}
$$

■

Hence, the result is obtained by applying the discrete maximum principle.

8.7 Uniform convergence analysis

The uniform convergence analysis is presented by decomposing the solution U_i^{j+1} into the smooth and singular components as

$$U_i^{j+1} = V_i^{j+1} + W_i^{j+1},$$

where V_i^{j+1} is the solution of

$$
\begin{aligned}
&\mathcal{L}V_i^{j+1} = g_i^{j+1}, \quad i = 1, 2, \cdots, N, j = 0, 1, \cdots, M-1. \\
&V_0^{j+1} = \phi_{l0}^{j+1}, \\
&\mathcal{K}^N V_N^{j+1} = \mathcal{K}V_0^{j+1}(x_N), \\
&V_i^{j+1} = \phi_{bi}^{j+1},
\end{aligned} \tag{8.46}
$$

and therefore W_i^{j+1} must satisfy

$$
\begin{aligned}
&\mathcal{L}^N W_i^{j+1} = 0, \quad i = 1, 2, \cdots, N, j = 0, 1, \cdots, M-1, \\
&W_i^{j+1} = U_i^{j+1} - v_i^{j+1}, \\
&\mathcal{K}^N W_N^{j+1} = \mathcal{K}^N U_N^{j+1} - \mathcal{K}^N V_N^{j+1}.
\end{aligned} \tag{8.47}
$$

Theorem 8.2. *Let $U^{j+1}(x)$ and U_i^{j+1} are the solutions of the problem* (8.27)-(8.30) *and*

(8.42)-(8.43) *respectively, and assume that the coefficients* $p(x), b(x), g(x) \in C^{4+\alpha_1}(\bar{\Omega})$, *and the boundary conditions satisfy* $\phi_l^{j+1} \in C^{3+\alpha_1/2}(0,T), \phi_b^{j+1} \in C^{6+\alpha_1}(\Gamma_b), \phi_r^{j+1}(0,T)$, *where* $\alpha_1 \in (0,1)$, *then we have*

$$\sup_{0<\varepsilon\ll 1} \left|\left|U^{j+1}(x) - U_i^{j+1}\right|\right| \le CN^{-2}\ln^2 N.$$

The error can be written in the form

$$U_i^{j+1} - U^{j+1}(x_i) = \left(V_i^{j+1} - V^{j+1}(x_i)\right) + \left(W_i^{j+1} - W^{j+1}(x_i)\right).$$

Proof. We prove the error estimates of the smooth and singular components separately. First, we derive the error estimate for the smooth component using the following classical argument.

At the point $x_i = x_N$,

$$\begin{aligned}
\mathcal{K}^N\left(V_i^{j+1} - V^{j+1}(x_i)\right) &= \mathcal{K}^N V_i^{j+1} - \mathcal{K}^N V^{j+1}(x_i) \\
&= \phi_r - \mathcal{K}^N V_i^{j+1} \\
&= \mathcal{K}V^{j+1}(x_i) - \mathcal{K}^N V^{j+1}(x_i) \\
&\le CN^{-2}.
\end{aligned}$$

The detailed proof is given in Theorem 7.1.
From the difference and discrete equations, we have

$$\mathcal{L}_1\left(V_i^{j+1} - V^{j+1}(x_i)\right) = \qquad g_1^{j+1} - \mathcal{L}_1 V^{j+1}(x_i) = \left(\mathcal{L}_1 - \mathcal{L}_1^N\right) V^{j+1}(x_i).$$

Then, it implies that

$$\mathcal{L}_1^N\left(V_i^{j+1} - V^{j+1}(x_i)\right) = -\frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial x^2} - D_x^+ D_x^-\right) V^{j+1}(x_i).$$

It follows from classical estimates in (Miller et al., 2012), at each point $x_i \in \Omega_1^N$,

$$\begin{aligned}
\mathcal{L}_1^N\left(V_i^{j+1} - V^{j+1}(x_i)\right) &\le \begin{cases} \frac{\varepsilon}{6}\left(x_{i+1} - x_{i-1}\right)\left|\left|\frac{\partial^3 V^{j+1}(x)}{\partial x^3}\right|\right|, & \text{if } \ x_i = \mu \ \text{ or } \ x_i = 1-\mu, \\ \frac{\varepsilon}{12}\left(x_i - x_{i-1}\right)^2\left|\left|\frac{\partial^4 V^{j+1}(x)}{\partial x^4}\right|\right|, & \text{otherwise}, \end{cases} \\
&\le \qquad C\begin{cases} \sqrt{\varepsilon}\, N^{-1}, & \text{if } \ x_i = \mu \ \text{ or } \ x_i = 1-\mu, \\ N^{-2}, & \text{otherwise}. \end{cases}
\end{aligned}$$

Similarly,

$$\mathcal{L}_2^N\left(V_i^{j+1}-V^{j+1}(x_i)\right) \leq \begin{cases} \frac{\varepsilon}{6}\left(x_{i+1}-x_{i-1}\right)\left|\left|\frac{\partial^3 V^{j+1}(x)}{\partial x^3}\right|\right|, & \text{if } \ x_i = 1+\mu \ \text{ or } \ x_i = 2-\mu, \\ \frac{\varepsilon}{12}\left(x_i-x_{i-1}\right)^2\left|\left|\frac{\partial^4 V^{j+1}(x)}{\partial x^4}\right|\right|, & \text{otherwise.} \end{cases}$$

$$\leq C\begin{cases} \sqrt{\varepsilon}\, N^{-1}, & \text{if } \ x_i = 1+\mu \ \text{ or } \ x_i = 2-\mu, \\ N^{-2}, & \text{otherwise.} \end{cases}$$

Define

$$\Phi^{j+1}(x_i) = C\begin{cases} \frac{\mu}{\varepsilon}\theta_1(x_i)N^{-2}+N^{-2}, \\ \frac{(1+\mu)}{\varepsilon}\theta_2(x_i)+N^{-2}, \end{cases}$$

where θ_1 and θ_2 are the piecewise linear polynomial

$$\theta_1 = \begin{cases} \frac{x}{\mu} & \text{for } \ 0 \leq x \leq \mu, \\ 1 & \text{for } \ \mu \leq x \leq 1-\mu, \\ \frac{1-x}{\mu} & \text{for } \ 1-\mu \leq x \leq 1. \end{cases}$$

and

$$\theta_2 = \begin{cases} \frac{x}{1+\mu} & \text{for } \ 1 \leq x \leq 1+\mu, \\ 1 & \text{for } \ 1+\mu \leq x \leq 2-\mu, \\ \frac{2-x}{1+\mu} & \text{for } \ 2-\mu \leq x \leq 2. \end{cases}$$

Then, for all $x_i, i = 1, 2, 3, \cdots, N$,

$$0 \leq \Phi^{j+1}(x_i) \leq CN^{-2}\ln N,$$

and also

$$\mathcal{L}_1^N\Phi^{j+1}(x_i) = \begin{cases} C\sqrt{\varepsilon}N^{-1}+N^{-2} & \text{if } \ x_i = \mu \ \text{ or } \ x_i = 1-\mu, \\ CN^{-2} & \text{otherwise.} \end{cases}$$

Observe that $\frac{\mu}{\sqrt{\varepsilon}} \leq 2\ln N$ and

$$\mathcal{L}_1^N\theta(x_i) = \begin{cases} \frac{\varepsilon N}{\mu}+p(x_i), & \text{if } \ x_i = \mu \ \text{ or } \ x_i = 1-\mu, \\ r(x_i)\theta(x_i), & \text{otherwise.} \end{cases}$$

Similarly,

$$\mathcal{L}_2^N \Phi^{j+1}(x_i) = \begin{cases} C\sqrt{\varepsilon}N^{-1} + N^{-2} & \text{if } \; x_i = 1+\mu \; \text{ or } \; x_i = 2-\mu, \\ CN^{-2} & \text{otherwise}. \end{cases}$$

Note that $\frac{1+\mu}{\sqrt{\varepsilon}} \leq 2\ln N$ and

$$\mathcal{L}_2^N \theta(x_i) = \begin{cases} \frac{\varepsilon N}{1+\mu} + p(x_i), & \text{if } \; x_i = 1+\mu \; \text{ or } \; x_i = 2-\mu. \\ p(x_i)\theta(x_i) + b(x_i)\theta(x_i - \frac{N}{2}), & \text{otherwise}, \end{cases}$$

and for all $x_i \in \Gamma^N$, then $\Phi^{j+1}(x_i) \geq 0$. Observe that $\mathcal{K}^N \theta^{j+1}(x_i) \geq 0$.

Define a barrier functions

$$\Xi^{\pm}(x_i) = \Phi(x_i) \pm \left(V_i^{j+1} - V^{j+1}(x_i)\right),$$

it follows that $\forall x_i, i = 1, 2, \cdots, N$,

$$\mathcal{L}_1^N \Xi^{\pm}(x_i) \geq 0 \quad \text{and} \quad \mathcal{L}_2^N \Xi^{\pm}(x_i) \geq 0.$$

Then, from the semi-discrete maximum principle, $\Xi^{\pm}(x_i) \geq 0, \forall i = 1, 2, \cdots, N$.
Then, we have

$$\begin{aligned} V_i^{j+1} - V^{j+1}(x_i) &\leq \Phi(x_i) \leq CN^{-2}\ln N \\ \left|V_i^{j+1} - V^{j+1}(x_i)\right| &\leq CN^{-2}\ln N. \end{aligned} \tag{8.48}$$

To estimate the singular component of the error, we decompose $W^{j+1}(x)$ into $W_l^{j+1}(x)$ and $W_r^{j+1}(x)$.

$$\begin{aligned} &\mathcal{L}^N W_l^{j+1}(x_i) = 0, \quad i = 1, 2, \cdots, N, j = 0, 1, \cdots, M-1, \\ &W_l^{j+1}(x_i) = \phi_l^{j+1}(x_i) - v_0^{j+1}(x_i), \qquad x_i \in \Gamma_l^N, \\ &W_l^{j+1}(x_i) = 0, \qquad x_i \in \Gamma_r^N, \\ &W_l^{j+1}(x_i) = 0, \qquad x_i \in \Gamma_b^N, \end{aligned} \tag{8.49}$$

$$\begin{aligned}
&\mathcal{L}^N W_l^{j+1}(x_i) = 0, && i = \frac{N}{2}+1, \cdots, N.\\
&W_l^{j+1}(x_i) = A, && x_i \in \Gamma_l^N\\
&\mathcal{K}^N W_l^{j+1}(x_i) = 0, && x_i \in \Gamma_r^N\\
&W_l^{j+1}(x_i) = 0,, && x_i \in \Gamma_b^N
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}^N W_r^{j+1}(x_i) = 0, && i = 1, \cdots, N.\\
&W_r^{j+1}(x_i) = 0, && x_i \in \Gamma_l^N\\
&\mathcal{K}^N W_r^{j+1}(x_i) = A, && x_i \in \Gamma_r^N\\
&W_r^{j+1}(x_i) = 0,, && x_i \in \Gamma_b^N
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}^N W_r^{j+1}(x_i) = 0, && i = \frac{N}{2}+1, \cdots, N.\\
&W_r^{j+1}(x_i) = 0, && x_i \in \Gamma_l^N\\
&\mathcal{K}^N W_r^{j+1}(x_i) = \mathcal{K}^N W^{j+1}(x_i), && x_i \in \Gamma_r^N\\
&W_r^{j+1}(x_i) = 0, && x_i \in \Gamma_b^N
\end{aligned}$$

The singular component error is equivalent to

$$W_i^{j+1} - W^{j+1}(x_i) = \left(W_{l_i}^{j+1} - W_l^{j+1}(x_i)\right) + \left(W_{r_i}^{j+1} - W_r^{j+1}(x_i)\right).$$

$$\mathcal{L}^N\left(W_i^{j+1} - W^{j+1}(x_i)\right) \le -\frac{\varepsilon}{2}\left(\frac{\partial^2}{\partial x^2} - D^+D^-\right)W^{j+1}(x_i).$$

It follows from classical estimates given in (Miller et al., 2012), at each point x_i, $i = 1, 2, \cdots, N$,

$$\mathcal{L}^N\left(W_i^{j+1} - W^{j+1}(x_i)\right) \le C\begin{cases}(N^{-1}\ln N)^2, & \text{if } i = 1, 2, \cdots, \frac{N}{2},\\ (N^{-1}\ln N)^2, & \text{if } i = \frac{N}{2}+1, \cdots, N.\end{cases}$$

First, the estimate for $W_{l_i}^{j+1} - W_l^{j+1}(x_i)$ is given. The argument depends on whether $\mu = \frac{1}{4}$ or $\mu = 2\sqrt{\varepsilon}\ln N$.

Case-I: $\mu = \frac{1}{4}$

In this case the mesh is uniform and $\mu = 2\sqrt{\varepsilon}\ln N \ge \frac{1}{4}$. It is clear that $x_i - x_{i-1} = N^{-1}$

and $\varepsilon^{-\frac{1}{2}} \leq C \ln N$. By (Miller et al., 2012) we have

$$\begin{aligned}
\mathcal{K}^N \left(W_{l_i}^{j+1} - W_l^{j+1}(x_i)\right) &= \mathcal{K}^N W_{l_i}^{j+1} - \mathcal{K}^N W_l^{j+1}(x_i) \\
&= \phi_r - \mathcal{K}^N W_l^{j+1}(x_i) \\
&= \mathcal{K} W_l^{j+1}(x_i) - \mathcal{K}^N W_l^{j+1}(x_i) \\
\left|\mathcal{K}^N \left(W_{l_i}^{j+1} - W_l^{j+1}(x_i)\right)\right| &\leq C\varepsilon \left(h_1^4 \frac{\partial^4}{\partial x^4} W_l^{j+1}(\xi_i) + \cdots + h_N^4 \frac{\partial^4}{\partial x^4} W_l^{j+1}(\xi_N)\right) \\
&\leq C\varepsilon^{-1} \left(h_1^4 + \cdots + h_N^4\right) \\
&\leq CN^{-2} \ln^2 N,
\end{aligned}$$

where $x_{i-1} \leq \xi_i \leq x_i$. By applying semi-discrete uniform stability, Lemma 8.10, we obtain

$$\left|W_{l_i}^{j+1} - W_l^{j+1}(x_i)\right| \leq C(N^{-2} \ln^2 N).$$

Case-II: $\mu < \frac{1}{4}$
Since the mesh is piecewise uniform with the subinterval $[\mu, 1-\mu]$, mesh elements are $4(1-2\mu)/N$ and the rest of the intervals $[0, \mu]$ and $[1-\mu, 1]$ with $8\mu/N$ mesh elements. By (Miller et al., 2012), we have

$$\mathcal{K}^N \left(W_{l_i}^{j+1} - W_l^{j+1}(x_i)\right) \leq CN^{-2} \ln^2 N,$$

and

$$\begin{aligned}
\left|\mathcal{K}^N \left(W_{l_i}^{j+1} - W_l^{j+1}(x_i)\right)\right| &\leq C\varepsilon \left(h_1^4 \frac{\partial^4}{\partial x^4} W_l^{j+1}(\xi_i) + \cdots + h_N^4 \frac{\partial^4}{\partial x^4} W_l^{j+1}(\xi_N)\right) \\
&\leq C\varepsilon^{-1} \left(h_1^4 + \cdots + h_N^4\right) \\
&\leq CN^{-2} \ln^2 N,
\end{aligned}$$

where $x_{i-1} \leq \xi_i \leq x_i$. By applying semi-discrete uniform stability, Lemma (8.10), we obtain

$$\left|W_{l_i}^{j+1} - W_l^{j+1}(x_i)\right| \leq CN^{-2} \ln^2 N, \tag{8.50}$$

by combining equation (8.48) and (8.50), we have

$$\left|U_i^{j+1} - U^{j+1}(x_i)\right| \leq CN^{-2} \ln^2 N. \tag{8.51}$$

■

The results of this work is summarized by considering the semidiscrete error estimate obtained in (8.40) and (8.50) and we conclude by the following theorem.

Theorem 8.3. *The error estimate for the solution of the continuous and fully discrete problem is given by*

$$\sup_{0<\varepsilon\leq 1} \left|\left|u(x,t) - U_i^{j+1}\right|\right| \leq C\left(N^{-2}\ln^2 N + (\Delta t)^2\right),$$

where $u(x,t)$ and U_i^{j+1} are the solutions of the problem (8.1)-(8.4) and (8.27)-(8.30) respectively.

Proof. The proof follows from Theorems (8.1) and (8.2). ■

8.8 Numerical results and discussions

Two model examples are presented to validate the results of this study. Since the exact solution of the problem is not known, double mesh principle is used to estimate maximum point wise error using (6.27), uniform error estimate using (6.28), rate of convergence using (6.29), and uniform rate of convergence using (6.30). The obtained results are displayed in tables for different values of $N, \Delta t$ and ε.

Example 8.1. *Consider a class of SPPDDEs with integral boundary condition of the form Elango et al. (2021)*

$$\begin{cases} -\varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + 5u(x,t) - u(x-1,t) = e^{-x}, \quad (x,t) \in (0,2)\times(0,2] \\ u(x,t) = 0, \quad \forall (x,t) \in \Gamma_L \\ \mathcal{K}u(2,t) = u(2,t) - \varepsilon\int_0^2 \frac{x}{3}u(x,t)dx = 0, \quad \forall\ (x,t) \in \Gamma_r \\ u(x,t) = 0, \quad \forall\ \ (x,t) \in \Gamma_b. \end{cases} \tag{8.52}$$

Example 8.2. *Consider a class of SPPDDEs with integral boundary condition of the form Elango et al. (2021)*

$$\begin{cases} -\varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + 5u(x,t) - xu(x-1,t) = 1, \quad (x,t) \in (0,2)\times(0,2] \\ u(x,t) = 0, \quad \forall (x,t) \in \Gamma_L \\ \mathcal{K}u(2,t) = u(2,t) - \varepsilon\int_0^2 \frac{1}{6}u(x,t)dx = 0, \quad \forall\ (x,t) \in \Gamma_r \\ u(x,t) = \sin(\pi x), \quad \forall\ \ (x,t) \in \Gamma_b. \end{cases} \tag{8.53}$$

Figures 8.1 and 8.2 indicate the numerical solution profiles of Examples 8.1 and 8.2 respectively. One can observe that for a small values of ε, the solution of the problem exhibit a

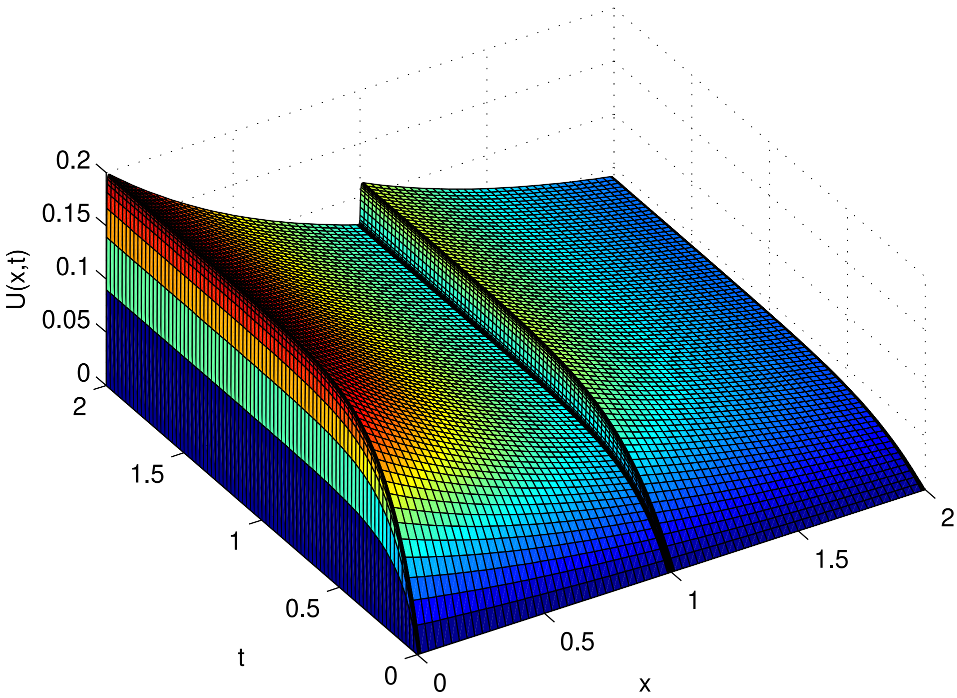


Figure 8.1: Graph of numerical solution for Example 8.1

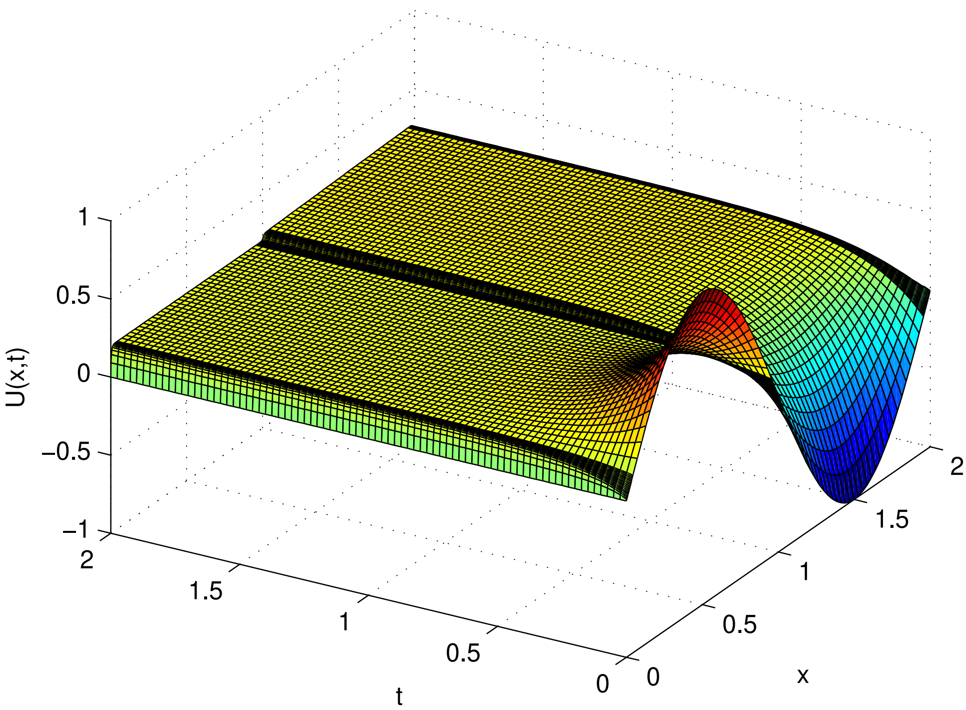
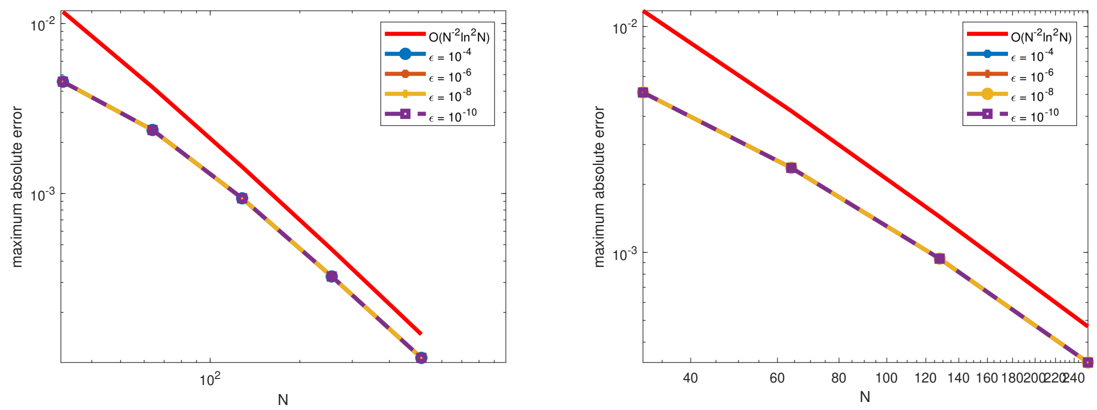


Figure 8.2: Graph of numerical solution for Example 8.2

Table 8.1: Maximum pointwise error and rate of convergence for Example 8.1.

ε	N=32	64	128	256	512
$\downarrow$	Δt =0.1/4	0.1/8	0.1/16	0.1/32	0.1/64
10^{-3}	4.5524e-03	2.3661e-03	7.8475e-04	2.1043e-04	5.3542e-05
	0.9441	1.5922	1.8989	1.9746	
10^{-4}	4.5511e-03	2.3657e-03	9.3776e-04	3.2531e-04	1.0796e-04
	0.9439	1.3350	1.5274	1.5913	
10^{-5}	4.5508e-03	2.3657e-03	9.3775e-04	3.2531e-04	1.0796e-04
	0.9439	1.3350	1.5274	1.5913	
10^{-6}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
	0.9439	1.3349	1.5274	1.5913	
10^{-7}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
	0.9439	1.3349	1.5274	1.5913	
10^{-8}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
	0.9439	1.3349	1.5274	1.5913	
10^{-9}	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
	0.9439	1.3349	1.5274	1.5913	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	4.5507e-03	2.3656e-03	9.3775e-04	3.2531e-04	1.0796e-04
	0.9439	1.3349	1.5274	1.5913	
$E^{N,\Delta t}$	4.5524e-03	2.3661e-03	9.3776e-04	3.2531e-04	1.0796e-04
$R^{N,\Delta t}$	0.9441	1.3334	1.5274	1.5913	-



(a) (b)

Figure 8.3: Log-Log plot of maximum absolute error versus mesh numbers for different values of ε (a) for Example 8.1, (b) for Example 8.2.

boundary layer at $x = 0$ and $x = 2$ and interior layer occurred at $x = 1$, because of the delay term in spatial direction. One can also observe that the width of the layers decreases as the parameter ε decreases. Tables 8.1 and 8.2 indicates ε-uniform maximum pointwise error $E^{N,\Delta t}$ and the rate of convergence $R^{N,\Delta t}$ for both Examples 8.1 and 8.2. Figures 8.3 show the Log-Log plot of the maximum absolute error verses N for singular perturbation parameter ranging from $\varepsilon = 10^{-5}$ to 10^{-8} for Examples 8.1 and 8.2 respectively. In these figures the

Table 8.2: Maximum pointwise errors and rate of convergence for Example 8.2.

ε	$N = 32$	64	128	256	512
$\downarrow$	$\Delta t = 0.1/4$	0.1/8	0.1/16	0.1/32	0.1/64
10^{-3}	5.0995e-03	2.3698e-03	7.8500e-04	4.4835e-04	3.1910e-04
	1.1056	1.5940	0.8081	0.4906	
10^{-4}	5.1011e-03	2.3669e-03	9.3786e-04	3.2534e-04	2.6596e-04
	1.1078	1.3356	1.5274	0.2907	
10^{-5}	5.1010e-03	2.3660e-03	9.3778e-04	3.2531e-04	2.6602e-04
	1.1083	1.3351	1.5274	0.2903	
10^{-6}	5.1009e-03	2.3658e-03	9.3776e-04	3.2531e-04	2.6604e-04
	1.1084	1.3350	1.5274	0.2902	
10^{-7}	5.1009e-03	2.3657e-03	9.3775e-04	3.2531e-04	2.6604e-04
	1.1085	1.3350	1.5274	0.2902	
10^{-8}	5.1009e-03	2.3657e-03	9.3775e-04	3.2531e-04	2.6604e-04
	1.1085	1.3350	1.5274	0.2902	
10^{-9}	5.1009e-03	2.3657e-03	9.3775e-04	3.2531e-04	2.6604e-04
	1.1085	1.3350	1.5274	0.2902	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	5.1009e-03	2.3657e-03	9.3775e-04	3.2531e-04	2.6604e-04
	1.1085	1.3350	1.5274	0.2902	
$E^{N,\Delta t}$	5.2949e-03	2.8424e-03	1.4907e-03	7.6522e-04	2.6604e-04
$R^{N,\Delta t}$	0.8975	0.9311	0.9620	1.5242	-

Table 8.3: Comparison of uniform error estimate and rate of convergence for Example 8.1 at N and Δt.

	$N = 32$	64	128	256	512
	$\Delta t = 0.1/4$	0.1/8	0.1/16	0.1/32	0.1/64
Present Method					
$E^{N,\Delta t}$	4.5524e-03	2.3661e-03	9.3776e-04	3.2531e-04	1.0796e-04
$R^{N,\Delta t}$	0.9441	1.3334	1.5274	1.5913	-
Hailu & Duressa (2022)					
$E^{N,\Delta t}$	1.2974e-02	8.7628e-03	4.6313e-03	2.0155e-03	7.6160e-04
$R^{N,\Delta t}$	0.56616e	0.91997	1.2003	1.4040	-
Gobena & Duressa (2021)					
$E^{N,\Delta t}$	2.0026e-03	1.0723e-03	5.5505e-04	2.8241e-04	1.4245e-04
$R^{N,\Delta t}$	0.90117	0.95002	0.97483	0.98734	-
Elango et al. (2021)					
$E^{N,\Delta t}$	1.2534e-02	6.9738e-03	3.6873e-03	1.8972e-03	9.6241e-04
$R^{N,\Delta t}$	0.84584	0.91937	0.95873	0.97912	

graphs are parallel and overlapped as ε goes small, this indicate that the proposed scheme converges independent of the values of perturbation parameter. Tables 8.3 and 8.4, shows the comparison of maximum absolute error and rate of convergence for the proposed scheme with the methods existed in the literature. From those tables, one can see that the developed numerical method is a more accurate parameter uniform convergent than the methods

Table 8.4: Comparison of uniform error estimate and rate of convergence for Example 8.2 at N and Δt.

	$N = 32$ $\Delta t = 0.1/4$	64 0.1/8	128 0.1/16	256 0.1/32	512 0.1/64
Present Method					
$E^{N,\Delta t}$	5.2949e-03	2.8424e-03	1.4907e-03	7.6522e-04	2.6604e-04
$R^{N,\Delta t}$	0.8975	0.9311	0.9620	1.5242	-
Gobena & Duressa (2021)					
$E^{N,\Delta t}$	1.5980e-02	8.3421e-03	4.2681e-03	2.1581e-03	1.0853e-03
$R^{N,\Delta t}$	0.93778	0.96682	0.98383	0.99180	
Hailu & Duressa (2022)					
$E^{N,\Delta t}$	3.1547e-02	1.7445e-02	9.2639e-03	4.7759e-03	2.4253e-03
$R^{N,\Delta t}$	0.85469	0.91312	0.95585	0.97761	-
Elango et al. (2021)					
	1.4776e-01	9.7571e-02	5.7092e-02	3.1057e-02	1.6222e-02
$R^{N,\Delta t}$	0.59873	0.77316	0.87837	0.93697	

presented in (Elango et al., 2021), (Gobena & Duressa, 2021), and (Hailu & Duressa, 2022).

8.9 Summary

A class of singularly perturbed reaction diffusion problem of partial delay differential equations with nonlocal boundary conditions was considered. We developed a finite difference scheme on a piecewise uniform Shishkin mesh for spatial discretization and Crank-Nicolson method to discretize the time derivative. The properties of the continuous solution were studied and satisfies the continuous stability estimate. To show the boundedness of the solution and its derivatives, the solution decomposed into smooth and regular components. Simpsons 1/3 rule is used to treat the integral boundary conditions given at the right end plane of the domain. The stability estimate and parameter uniform convergence analysis for numerical method are also studied. The developed numerical method is ε-uniform convergent with almost second-order in spatial direction and second-order in temporal direction. The numerical results agree with the theoretical estimates.

CHAPTER NINE

HYBRID NUMERICAL METHOD FOR SINGULARLY PERTURBED PARTIAL DELAY DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITION

This chapter deals with a numerical method for SPPDDEs with nonlocal boundary conditions. Since a small perturbation parameter ε multiplies the highest derivatives, solution of the problem exhibit a boundary layers at left and right end plane of the domain and interior layer which forms because of the presence of large delay on the space variable. A hybrid numerical method is proposed in the spatial direction, and an implicit Euler method is used in temporal direction. The proposed hybrid method constitutes cubic spline method in the boundary layer region and a classical finite difference method in the outer layer region. The nonlocal boundary condition is treated using Simpson's $\frac{1}{3}$ rule. The uniform stability and convergence analysis for the proposed scheme are studied. The developed method is uniformly convergent with second-order in space and first-order in time. Two numerical model examples are considered to validate the applicabilty of the proposed method. The numerical results agreed with the theoretical estimations.

9.1 The governing problem

Consider SPPPDDEs of reaction-diffusion types with an integral boundary condition:

$$\begin{aligned}
&\mathcal{L}u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)u(x,t) + b(x,t)u(x-1,t) = f(x,t), (x,t) \in \Omega,\\
&\quad u(x,t) = \phi_l(x,t),\ \ \phi_l(x,t) \in \Gamma_l = \{(x,t); -1 \le x \le 0 \ \text{ and } \ 0 \le t \le T\},\\
&\mathcal{K}u(x,t) = u(2,t) - \varepsilon\int_0^2 g(x)u(x,t)dx = \phi_r(x,t),\ \phi_r(x,t) \in \Gamma_r = \{(2,t); 0 \le t \le T\},\\
&\quad u(x,t) = \phi_b(x,t),\ \phi_b(x,t) \in \Gamma_b = \{(x,0)\},
\end{aligned} \tag{9.1}$$

where $(x,t) \in \Omega = (0,2) \times (0,T],\ \bar{\Omega} = [0,2] \times [0,T],\ \Omega_1 = (0,1) \times [0,T],\ \Omega_2 = (1,2) \times [0,T],\ \Omega^* = \Omega_1 \cup \Omega_2$ and ε is a small positive parameter $(0 < \varepsilon \ll 1)$. Assume that $a(x,t) \ge \alpha > 0,\ b(x,t) \le \beta < 0,\ \alpha + \beta > 0,\ f(x,t),\ \phi_l,\ \phi_r,\ \phi_b$ are sufficiently smooth and $g(x)$ is monotonically non negative function and satisfy $\int_0^2 g(x)dx < 1$. These assumptions assure that the problem (9.1) has a unique solution (Bahuguna & Dabas, 2008).

The aforementioned problem emerges in the research of population density with time lag. In the left, no flux condition is considered, and the average population size is regulated by the function $\phi_r(x,t)$ in the right. For a very small value of ε, the solution $u(x,t)$ has a boundary and interior layers. Because of the presence of these layers, classical numerical approaches are ineffective for solving such types singularly perturbed problems. Special techniques, in which the error is independent of the perturbation parameter are required to obtain a good numerical approximations to such problems.
The problem (9.1) is equivalently written as

$$\mathcal{L}u(x,t) = F(x,t) \tag{9.2}$$

where

$$\mathcal{L}u(x,t) = \begin{cases} \mathcal{L}_1 u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)u(x,t), \ (x,t) \in \Omega_1, \\ \mathcal{L}_2 u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)u(x,t) + b(x,t)u(x-1,t), (x,t) \in \Omega_2, \end{cases}$$

$$F(x,t) = \begin{cases} f(x,t) - b(x,t)\phi_l(x-1,t), & (x,t) \in \Omega_1, \\ f(x,t), & (x,t) \in \Omega_2. \end{cases}$$

with boundary conditions

$$\begin{aligned} & u(x,t) = \phi_l(x,t), \ \ \phi_l(x,t) \in \Gamma_l = \{(x,t); -1 \le x \le 0 \ \text{ and } \ 0 \le t \le T\}, \\ & u(1^-,t) = u(1^+,t), \quad \frac{\partial u}{\partial x}(1^-,t) = \frac{\partial u}{\partial x}(1^+,t), \\ & \mathcal{K}u(x,t) = u(2,t) - \varepsilon \int_0^2 g(x)u(x,t)dx = \phi_r(x,t), \ \phi_r(x,t) \in \Gamma_r = \{(2,t); 0 \le t \le T\}, \\ & u(x,t) = \phi_b(x,t), \ \phi_b(x,t) \in \Gamma_b = \{(x,0)\}. \end{aligned} \tag{9.3}$$

For the problems (9.2) - (9.3) to have existence and uniqueness of a solution, we assume that the coefficients of the problem are Hölder continuous and imposing correct compatibility conditions at the corners $(0,0), (2,0), (-1,0)$ and $(1,0)$ (Ladyzhenskaya et al., 1968). The following conditions must be satisfied.

$$\phi_b(0,0) = \phi_l(0,0), \ \ \phi_b(2,0) = \phi_r(2,0)$$

and

$$
\begin{aligned}
&-\varepsilon\frac{\partial^2\phi_b(0,0)}{\partial x^2}+a(0,0)\phi_b(0,0)+\frac{\partial\phi_l(0,0)}{\partial t}+b(0,0)\phi_l(-1,0)=f(0,0),\\
&-\varepsilon\frac{\partial^2\phi_b(2,0)}{\partial x^2}+a(2,0)\phi_b(2,0)+\frac{\partial\phi_r(2,0)}{\partial t}+b(2,0)\phi_l(1,0)=f(2,0).
\end{aligned}
$$

9.2 Properties of the continuous solution

Properties of the governing problem (9.1) which guarantee the uniqueness and existence of the solution are presented here.

Lemma 9.1. *(Continuous Maximum Principle) If* $\psi(x,t)\in C^{(0,0)}(\bar{\Omega})\cap C^{(1,0)}(\Omega)\cap C^{(2,1)}(\Omega_1\cup\Omega_2)$ *such that* $\psi(0,t)\geq 0,\psi(x,0)\geq 0,\mathcal{K}\psi(2,t)\geq 0,\mathcal{L}_1\psi(x,t)\geq 0,\forall(x,t)\in\Omega_1,\mathcal{L}_2\psi(x,t)\geq 0,\forall(x,t)\in\Omega_2$ *and* $[\psi_x](1,t)=\psi_x(1^+,t)-\psi_x(1^-,t)\leq 0$, *then* $\psi(x,t)\geq 0$, *for all* $(x,t)\in\bar{\Omega}$.

Proof. The proof is given in Lemma 8.1. ■

Lemma 9.2. *(Stability Result) The solution* $u(x,t)$ *of the problem* (1.3) *satisfies the bound*

$$
\|u\|_{\bar{\Omega}}=C\max\left\{\|u\|_{\Gamma_l},\|u\|_{\Gamma_b},\|\mathcal{K}\|_{\Gamma_r},\|\mathcal{L}u\|_{\Omega^*}\right\}.
$$

Proof. We prove by using the maximum principle in Lemma 9.1 and by constructing the barrier functions $\Theta^{\pm}(x,t)=\max\left\{\|u\|_{\Gamma_l},\|u\|_{\Gamma_b},\|u\|_{\Gamma_r},\|\mathcal{L}u\|_{\Omega^*}\right\}s(x,t)\pm u(x,t),(x,t)\in\bar{\Omega}$, where $s(x,t)$ is the test function in (8.5). ■

Lemma 9.3. *Suppose* $a(x,t),b(x,t),f(x,t)\in C^{(2+\alpha_1,1+\alpha_1/2)}(\bar{\Omega})$, *and* $\phi_l\in C^{2+\alpha_1/2}([0,T]),\phi_b\in C^{(4+\alpha_1,2+\alpha_1/2)}(\Gamma_b),\phi_r\in C^{2+\alpha_1/2}([0,T])$, *where* $\alpha_1\in(0,1)$. *Then, the problem* (9.1) *has a unique solution* u *which satisfy* $u\in C^{(4+\alpha_1,2+\alpha_1/2)}(\bar{\Omega})$. *And also the derivatives of a solution* u *are bounded.* $\forall i,j\in\mathbf{Z}\geq 0$ *such that* $0\leq i+2j\leq 4$,

$$
\left\|\frac{\partial^{i+j}z}{\partial x^i\partial t^j}\right\|\leq C\varepsilon^{\frac{-i}{2}}. \tag{9.4}
$$

.

Proof. The proof is given in Lemma 8.3. ■

9.3 Decomposition of the solution

The bounds on the derivatives of the solution given in Lemma 9.3 were derived from classical results. They are not adequate for the proof of the ε -uniform error estimate. Stronger bounds on these derivatives are now obtained by a method originally given in (Shishkin, 1989). The key step is to decompose the solution u into smooth and singular components. The solution

u of (9.1) is decomposed as $u = v + w$, where v is smooth and w is singular components. The smooth and singular component satisfy the following bound.

Lemma 9.4. *Suppose* $a(x,t), b(x,t), f(x,t) \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{\Omega})$*, and* $\phi_l \in C^{(3+\alpha_1/2)}([0,T])$*,* $\phi_b \in C^{(6+\alpha_1, 3+\alpha_1/2)}(\Gamma_b)$*,* $\phi_r \in C^{(3+\alpha_1/2)}([0,T])$*, where* $\alpha_1 \in (0,1)$*. Then, we have*

$$\left\|\frac{\partial^{i+j} v}{\partial x^i \partial t^j}\right\|_{\bar{\Omega}} \leq C\left(1 + \varepsilon^{1-i/2}\right), \tag{9.5}$$

$$\left|\frac{\partial^{i+j} w_l}{\partial x^i \partial t^j}\right| \leq \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{x}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(x-1)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_2, \end{cases} \tag{9.6}$$

$$\left|\frac{\partial^{i+j} w_r}{\partial x^i \partial t^j}\right| \leq \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(2-x)}{\sqrt{\varepsilon}}}, & (x,t) \in \Omega_2, \end{cases} \tag{9.7}$$

where C *is constant independent of parameter* ε*,* $(x,t) \in \bar{\Omega}$ $i, j \geq 0, 0 \leq i + 2j \leq 4$.

Proof. The proof the lemma is given in Lemma 8.4. ■

9.4 The numerical method

In this section, the time derivative is discretized first using the implicit Euler method and then a layer-adapted mesh of Shishkin type is introduced to discretize the space domain using the hybrid numerical scheme comprising cubic spline scheme in the boundary layer regions and classical finite difference method in the regular layer region.

9.4.1 The time semi-discretization

Let Ω^M be a uniform mesh which is used in the time semi-discretization and defined as $\Omega^M = \{t_j = j\Delta t, j = 0, 1, \cdots, M, \ \ \Delta t = T/M\}$, where M is a non-negative integer. Using implicit Euler method, the problem (9.1) is semi-discretized as

$$\mathcal{L}^{\Delta t} U^{j+1}(x) = g(x, t_{j+1}), \ \ j = 0, 1, \cdots, M-1, \tag{9.8}$$

with discretized the boundary conditions as

$$\begin{aligned} &U^{j+1}(x) = \phi_l^{j+1}(x), \quad \phi_l^{j+1}(x) \in \Gamma_l, \\ &\mathcal{K}U^{j+1}(x) = U^{j+1}(2) - \varepsilon \int_0^2 g(x)u(x,t)dx = \phi_r^{j+1}(x), \quad \phi_r^{j+1}(x) \in \Gamma_r, \\ &U^{j+1}(x) = \phi_b^{j+1}(x), \qquad \phi_b^{j+1}(x) \in \Gamma_b, \end{aligned} \tag{9.9}$$

where $\mathcal{L}^{\Delta t} U^{j+1}(x) = -\varepsilon U_{xx}^{j+1} + r(x)U^{j+1}(x) + b(x)U(x-1)^{j+1}, \ \ g(x, t_{j+1}) = \frac{U^j(x)}{\Delta t} + f(x, t_{j+1})$, and $r(x) = \frac{1}{\Delta t} + a(x)$.

The local truncation error estimation for time semi-discretization which is denoted as $e^j := u(x, t_{j+1}) - U^{j+1}(x)$, for $j = 0, 1, 2, \cdots, M$ is analyzed as follows.

Lemma 9.5. *The local truncation error associated with the implicit Euler method satisfies the bound*

$$|e^j| \leq C(\Delta t)^2.$$

Proof. Using Taylor series expansion to $u(x, t_{j+1})$, we get

$$u(x, t_{j+1}) = u(x, t_j) + \Delta t \frac{\partial u(x, t_j)}{\partial t} + \mathcal{O}((\Delta t)^2). \tag{9.10}$$

Substituting (9.8) into (9.1), we obtain

$$\begin{aligned} \frac{u(x, t_{j+1}) - u(x, t_j)}{\Delta t} &= \frac{\partial u(x, t)}{\partial t} + \mathcal{O}((\Delta t)^2) \\ &= -\left(-\varepsilon \frac{\partial^2}{\partial x^2} u(x, t) - f(x, t_j)\right) + \mathcal{O}((\Delta t)^2). \end{aligned}$$

Since the error satisfies the given problem, then the local truncation error satisfy $\mathcal{L}^{\Delta t} e^{j+1}(x) = \mathcal{O}((\Delta t)^2)$. Using the discrete maximum principle gives

$$\|e^{j+1}\| \leq C(\Delta t)^2. \tag{9.11}$$

■

The bound for the global truncation error of the time semi-discrete scheme are satisfies the estimate

$$\|TE^{j+1}\| \leq C(\Delta t), \qquad \forall j \leq T/\Delta t. \tag{9.12}$$

where C is a positive constant independent of ε and Δt.

9.4.2 The spatial discretization

The piecewise-uniform mesh having $(N \geq 8)$ mesh elements on $[0, 2]$ is generated by dividing the first half interval $[0, 1]$ into three sub-intervals as $[0, \mu]$, $(\mu, 1 - \mu]$, $(1 - \mu, 1]$ and the second half interval $[1, 2]$ into three sub-intervals as $(1, 1 + \mu]$, $(1 + \mu, 2 - \mu]$, and $(2 - \mu, 2]$. To obtain a piecewise-uniform mesh, $\frac{N}{4}$ mesh elements placed in regular region and $\frac{N}{8}$ mesh elements placed in each of the boundary layer regions. Hence, the piecewise-uniform mesh is given by $x_i = \begin{cases} 0, & i = 0, \\ x_{i-1} + h_i, & i = 1, 2, 3, \cdots, N, \end{cases}$ where h_i's are given by

$$h_i = \begin{cases} \frac{8\mu}{N}, & 1 \le i \le \frac{N}{8}, \frac{N}{2} \le i \le \frac{5N}{8} \\ \frac{4(1-2\mu)}{N}, & \frac{N}{8}+1 \le i \le \frac{3N}{8}, \frac{5N}{8}+1 \le i \le \frac{7N}{8} \\ \frac{8\mu}{N}, & i = \frac{3N}{8}+1 \le i \le \frac{N}{2}, \frac{7N}{8} \le i \le N, \end{cases}$$

where μ is a transition parameter which satisfies the following condition:

$$\mu = \min\left\{\frac{1}{4}, 2\sqrt{\varepsilon}\ln N\right\},$$

where N denotes the number of mesh elements in the spatial-direction. Further, we denote the mesh size in the boundary layer regions $h = \frac{8\mu}{N}$ and in the regular region $H = \frac{4(1-2\mu)}{N}$. Here, a classical finite difference method is applied for the spatial semi-discrete on outer layer region $(\mu, 1-\mu]$ and $(1+\mu, 2-\mu]$, by replacing the second-order derivative with a central difference scheme. From (9.6), the discrete operator $\mathcal{L}^{\Delta t}$ is defined as

$$\mathcal{L}^{\Delta t} U^{j+1}(x_i) = \begin{cases} -\varepsilon D_x^+ D_x^- U_i^{j+1} + r_i U_i^{j+1} & i = 1, 2, \cdots, \frac{N}{2}, \\ -\varepsilon D_x^+ D_x^- U_i^{j+1} + r_i U_i^{j+1} + b_i U_{i-\frac{N}{2}}^{j+1}, & i = \frac{N}{2}+1, \cdots, N-1, \end{cases} \tag{9.13}$$

with initial and boundary conditions

$$\begin{cases} U_0 = \phi_l(0), \\ U_i^0 = \phi_b, & i = 1, 2, 3, \cdots, N-1, \\ \mathcal{K}^N U_N^{j+1} = U_N^{j+1} - \varepsilon \sum_{i=1}^{N} \dfrac{g_{i-1} U_{i-1}^{j+1} + 4 g_i U_i^{j+1} + g_{i+1} U_{i+1}^{j+1}}{3} h_i = \phi_{rN}, \end{cases} \tag{9.14}$$

where $U_i^{j+1} = U(x_i, t_{j+1}), r_i = r(x_i), g_i = g(x_i)$.
Here, for $(i = N)$, the integral boundary condition $\int_0^2 g(x)u(x,t)dx$ is approximated using composite Simpson's $\frac{1}{3}$ integration rule given in (7.21).
By rearranging (9.13), the following system of equations is obtained in the outer region.

$$\mathcal{L}^{\Delta t} U^{j+1}(x_i) = \begin{cases} R_i^- U_{i-1}^{j+1} + R_i^0 U_i^{j+1} + R_i^+ U_{i-1}^{j+1} = G_i, & \text{for } i = 1, \cdots, \frac{N}{2}, \\ R_i^- U_{i-1}^{j+1} + R_i^0 U_i^{j+1} + U_i^+ U_{i-1}^{j+1} + B_i U_{i-N}^{j+1} = G_i, \\ \quad \text{for } i = \frac{N}{2}+1, \cdots, N-1, \end{cases} \tag{9.15}$$

where the coefficients are given as

$$R_i^- = \frac{-2\varepsilon}{h_{i-1}(h_i+h_{i-1})}, R_i^0 = \frac{2\varepsilon}{h_i h_{i-1}} + r_i, \quad R_i^+ = \frac{-2\varepsilon}{h_i(h_i+h_{i-1})}, B_i = q_i,$$

$$G_i = \begin{cases} g_i^{j+1} - q_i\eta_{i-N}, & \text{for } i = \frac{N}{8}+1,\cdots,\frac{3N}{8}, \\ g_i^{j+1}, & \text{for } i = \frac{5N}{8}+1,\cdots,\frac{7N}{8}, \end{cases} \qquad g_i^{j+1} = \frac{U_i^j}{\Delta t} + f_i^{j+1}.$$

Again, the problem (9.8) is discretized in the inner regions, $[0,\mu]$, $(1-\mu,1]$, $(1,1+\mu)$, and $(2-\mu,2]$, using cubic spline method in Shishkin mesh as follows. From (9.13), we have

$$\mathcal{L}^{\Delta t}U^{j+1}(x_i) = -\varepsilon M_k^{j+1}U_k^{j+1} + r_k U_k^{j+1} + b_k U_{k-\frac{N}{2}}^{j+1} = g_k^{j+1}, \quad k = i, i\pm 1. \tag{9.16}$$

Where $M_k^{j+1} = U_{xx}^{j+1}$ for $k = i, i\pm 1$. From cubic spline method for variable mesh, we have

$$\frac{h_{i-1}}{6}M_{i-1}^{j+1} + \frac{(h_i+h_{i-1})}{3}M_i^{j+1} + \frac{h_i}{6}M_{i+1}^{j+1} = \left(\frac{U_{i+1}^{j+1}-U_i^{j+1}}{h_i}\right) - \left(\frac{U_i^{j+1}-U_{i-1}^{j+1}}{h_{i-1}}\right),$$
$$i = 1,\cdots,N-1. \tag{9.17}$$

Substituting (9.16) into (9.17) and after arranging some terms, the following difference scheme is obtained in the boundary layer region.

$$\mathcal{L}^{\Delta t}U^{j+1}(x_i) = \begin{cases} R_i^- U_{i-1}^{j+1} + R_i^0 U_i^{j+1} + R_i^+ U_{i-1}^{j+1} = G_i, & \text{for } i = 1,\cdots,\frac{N}{2}, \\ R_i^- U_{i-1}^{j+1} + R_i^0 U_i^{j+1} + R_i^+ U_{i-1}^{j+1} + B_i U_{i-N}^{j+1} = G_i, \\ \text{for } i = \frac{N}{2}+1,\cdots,N-1, \end{cases} \tag{9.18}$$

where the coefficients are given by

$$R_i^- = \frac{-3\varepsilon}{h_{i-1}(h_i+h_{i-1})} + \frac{h_{i-1}}{2(h_i+h_{i-1})}r_{i-1}, \qquad R_i^0 = \frac{3\varepsilon}{h_i h_{i-1}} + r_i,$$
$$R_i^+ = \frac{-3\varepsilon}{h_i(h_i+h_{i-1})} + \frac{h_i}{2(h_i+h_{i-1})}r_{i+1},$$
$$G_i = \begin{cases} l_i^- F_{i-1} + l_i^0 F_i + l_i^+ F_{i+1}, & i = 1,\cdots,N, \\ l_i^- g_{i-1} + l_i^0 g_i + l_i^+ g_{i+1}, & i = N+1,\cdots,N-1, \end{cases}$$
$$B_i = l_i^- q_{i-1} + l_i^0 q_i + l_i^{i+1} q_{i+1}, F_i = g_i^{j+1} - q_i\eta_{i-1}, \; g_i^{j+1} = \frac{Z_i^j}{\Delta t} + f_i^{j+1}, \quad l_i^- = \frac{h_{i-1}}{2(h_i+h_{i-1})},$$
$$l_i^0 = 1, l_i^+ = \frac{h_i}{2(h_i+h_{i-1})}.$$

Now by combining the difference scheme given in (9.15) and (9.18), we obtain the following hybrid difference scheme.

$$\mathcal{L}^{\Delta t}U^{j+1}(x_i)=\begin{cases}R_i^-U_{i-1}^{j+1}+R_i^0U_i^{j+1}+R_i^+U_{i-1}^{j+1}=G_i, & \text{for } i=1,\cdots,\frac{N}{2},\\ R_i^-U_{i-1}^{j+1}+R_i^0U_i^{j+1}+R_i^+U_{i-1}^{j+1}+B_iU_{i-N}^{j+1}=G_i, & \text{for } i=\frac{N}{2}+1,\cdots,N-1,\end{cases} \tag{9.19}$$

for $i=1,2,\cdots,N-1,\quad j=1,2,\cdots,M-1$ with the following interval and integral boundary conditions

$$\begin{cases}U_0=\phi_l(0),\\ U_i^0=\phi_b, \qquad\qquad i=1,2,3,\cdots,N-1,\\ \mathcal{K}^NU_N^{j+1}=U_N^{j+1}-\varepsilon\sum_{i=1}^{N}\dfrac{g_{i-1}U_{i-1}^{j+1}+4g_iU_i^{j+1}+g_{i+1}U_{i+1}^{j+1}}{3}h_i=\phi_{rN},\end{cases} \tag{9.20}$$

where the coefficients for $i=1,2,\cdots,\frac{N}{8}$, $i=\frac{3N}{8}+1,\cdots\frac{N}{2}$, $i=\frac{N}{2},\cdots,\frac{5N}{8}$, and for $i=\frac{7N}{8}+1,\cdots,N-1$ are given by

$$\begin{aligned}
R_i^-&=\frac{-3\varepsilon}{h_{i-1}(h_i+h_{i-1})}+\frac{h_{i-1}}{2(h_i+h_{i-1})}r_{i-1},\\
R_i^0&=\frac{3\varepsilon}{h_ih_{i-1}}+r_i,\\
R_i^+&=\frac{-3\varepsilon}{h_i(h_i+h_{i-1})}+\frac{h_i}{2(h_i+h_{i-1})}r_{i+1},\\
G_i&=\begin{cases}l_i^-F_{i-1}+l_i^0F_i+l_i^+F_{i+1}, & \text{for } i=1,\cdots,\frac{N}{8}, i=\frac{3N}{8}+1,\cdots,\frac{N}{2},\\ l_i^-g_{i-1}+l_i^0g_i+l_i^+g_{i+1}, & \text{for } i=\frac{N}{2}+1,\cdots,\frac{5N}{8},\ \text{for } i=\frac{7N}{8}+1,\cdots,N-1,\end{cases}\\
B_i&=l_i^-q_{i-1}+l_i^0q_i+l_i^{i+1}q_{i+1},\\
F_i&=g_i^{j+1}-b_i\phi_{i-1},\ g_i^{j+1}=\frac{Z_i^j}{\Delta t}+f_i^{j+1},\\
l_i^-&=\frac{h_{i-1}}{2(h_i+h_{i-1})},l_i^0=1,l_i^+=\frac{h_i}{2(h_i+h_{i-1})}.
\end{aligned}$$

Again, for $i=\frac{N}{8}+1,\cdots,\frac{3N}{8}$ and for $i=\frac{5N}{8}+1,\cdots,\frac{7N}{8}$, the coefficients for the hybrid scheme are given by

$$R_i^-=\frac{-2\varepsilon}{h_{i-1}(h_i+h_{i-1})},R_i^0=\frac{2\varepsilon}{h_ih_{i-1}}+r_i,\ \ R_i^+=\frac{-2\varepsilon}{h_i(h_i+h_{i-1})},B_i=q_i,$$

$$G_i=\begin{cases}g_i^{j+1}-b_i\phi_{i-N}, & \text{for } i=\frac{N}{8}+1,\cdots,\frac{3N}{8},\\ g_i^{j+1}, & \text{for } i=\frac{5N}{8}+1,\cdots,\frac{7N}{8},\end{cases}\qquad g_i^{j+1}=\frac{U_i^j}{\Delta t}+f_i^{j+1}$$

9.5 Uniform stability analysis

Lemma 9.6. *(Discrete comparison principle) Let there exist a comparison function* V_i^{j+1} *such that if* $U_0^{j+1} \leq V_0^{j+1}$ *and* $\mathcal{K}U_i^{j+1} \leq \mathcal{K}V_i^{j+1}$*, then* $\mathcal{L}^{\Delta t}U^{j+1}(x_i) \leq \mathcal{L}^{\Delta t}V^{j+1}(x_i)$*, for* $i = 1, 2, \cdots, N-1$ *implies that* $U_i^{j+1} \leq V_i^{j+1}$ *for* $i = 1, 2, \cdots, N$.

Proof. It is easy to verify that the associated matrix $\mathcal{L}^{\Delta t}U^{j+1}(x_i)$ is a size of $(N+1) \times (N+1)$ with its interies for $i = 1, \cdots, N-1$ which are $R_i^- < 0$, $R_i^0 > 0$, and $R_i^0 < 0$. As a result a coefficient matrix satisfy the M-matrix property. Therefore, there is a non-singular matrix which is non-negative. This prove shows that the discrete solution is exists and unique. ■

Lemma 9.7. *(Uniform stability estimate): The solution* U_i^{j+1} *of the hybrid scheme in* (9.19) *satisfies the bounds*

$$|U_i^{j+1}| \leq \frac{\|\mathcal{L}^{\Delta t}U_i^{j+1}\|}{\kappa} + \max\left\{|U_0^{j+1}|, |\mathcal{K}U_{2N}^{j+1}|\right\}, \textit{ for } i = 1, \cdots 2N, j = 0, 1, \cdots, M,$$

where $r(x_i) \geq \kappa$.

Proof. Consider the comparison functions

$$\left(\Upsilon_i^{j+1}\right)^{\pm} = \frac{\|\mathcal{L}^{\Delta t}U_i^{j+1}\|}{\kappa} + \max\left\{|U_0^{j+1}|, |\mathcal{K}U_{2N}^{j+1}|\right\} \pm U_i^{j+1}. \tag{9.21}$$

Applying discrete comparison principle given on Lemma 9.6, we obtain the required estimate. ■

9.6 Uniform convergence analysis

In this section, the truncation error at inner and outer region is derived. Now, the truncation error for the hybrid numerical scheme at the boundary layer regions on the subintervals $0 < i < \frac{N}{8}$ and $\frac{3N}{8} < i < \frac{N}{2}$ is

$$\tau_{i,U} = \left(R_i^- U_{i-1}^{j+1} + U_i^0 R_i^{j+1} + R_i^+ U_{i-1}^{j+1}\right) - \left(l_i^- F_{i-1} + l_i^0 F_i + l_i^+ F_{i+1}\right), \tag{9.22}$$

and for the subintervals is $\frac{N}{2} < i < \frac{5N}{2}$, and $\frac{7N}{8} < i < N-1$

$$\tau_{i,U} = \left(R_i^- U_{i-1}^{j+1} + U_i^0 R_i^{j+1} + R_i^+ U_{i-1}^{j+1} + B_i U_{i-\frac{N}{2}}^{j+1}\right) - \left(l_i^- g_{i-1} + l_i^0 g_i + l_i^+ g_{i+1}\right). \tag{9.23}$$

The truncation error for the hybrid scheme for the regular region on the interval $\frac{N}{8} < i < \frac{3N}{8}$ become

$$\tau_{i,U} = \left(R_i^- U_{i-1}^{j+1} + U_i^0 R_i^{j+1} + R_i^+ U_{i-1}^{j+1}\right) - \left(g_i^{j+1} - b_i \phi_{i-N}\right). \tag{9.24}$$

and on the interval $\frac{5N}{8} < i < \frac{7N}{8}$ is given as

$$\tau_{i,U} = \left(R_i^- U_{i-1}^{j+1} + U_i^0 R_i^{j+1} + R_i^+ U_{i-1}^{j+1} + B_i U_{i-\frac{N}{2}}^{j+1}\right) - \left(g_i^{j+1}\right). \tag{9.25}$$

Now using Taylor expansion, we have

$$U^{j+1}(x_{i-1}) = U^{j+1}(x_i) - h_{i-1}\frac{dU^{j+1}}{dx}(x_i) + \frac{h_{i-1}^2}{2!}\frac{d^2U^{j+1}}{dx^2}(x_i) - \frac{h_{i-1}^3}{3!}\frac{d^3U^{j+1}}{dx^3}(x_i) + \cdots.$$
$$U^{j+1}(x_{i+1}) = U^{j+1}(x_i) + h_{i-1}\frac{dU^{j+1}}{dx}(x_i) + \frac{h_{i-1}^2}{2!}\frac{d^2U^{j+1}}{dx^2}(x_i) + \frac{h_{i-1}^3}{3!}\frac{d^3U^{j+1}}{dx^3}(x_i) + \cdots.$$

Using the values of $U^{j+1}(x_{i-1})$ and $U^{j+1}(x_{i+1})$ into (9.22)-(9.25), we obtain the following truncation error of an hybrid numerical scheme for $i = 1, \cdots, \frac{N}{8}, \frac{3N}{8}, \cdots, \frac{N}{2}, \frac{N}{2}, \cdots, \frac{5N}{8}$ and $\frac{7N}{8}, \cdots, N-1$.

$$\tau_{i,U} = T_{0,i}U(x_i) + T_{1,i}\frac{dU^{j+1}}{dx}(x_i) + T_{2,i}\frac{d^2U^{j+1}}{dx^2}(x_i) + T_{3,i}\frac{d^3U^{j+1}}{dx^3}(x_i) + T_{4,i}\frac{d^4U^{j+1}}{dx^4}(x_i) + \cdots.$$

where

$$\begin{aligned}
T_{0,i} &= R_i^- + R_i^0 + R_i^+ - \left(l_i^- r_{i-1} + l_i^0 r_i + l_i^+ r_{i+1}\right), \\
T_{1,i} &= R_i^+ h_i - R_i^- h_{i-1} + \left(l_i^+ r_{i+1} h_i - l_i^- r_{i-1} h_{i-1}\right), \\
T_{2,i} &= \frac{1}{2}\left(R_i^+ h_i^2 - R_i^- h_{i-1}^2\right) + \varepsilon\left(l_i^- + l_i^0 + l_i^+\right) - \left(\frac{h_{i-1}^2}{2!} l_i^- r_{i-1} + \frac{h_i^2}{2!} l_i^+ r_{i+1}\right), \\
T_{3,i} &= \frac{h_i^3}{3!} R_i^+ - \frac{h_{i-1}^3}{3!} R_i^- + \varepsilon\left(l_i^+ h_i - l_i^- h_{i-1}\right) - \left(\frac{h_i^3}{3!} l_i^+ a_{i+1} - \frac{h_{i-1}^3}{3!} l_{i-1}^+ r_{i-1}\right), \\
T_{4,i} &= \frac{h_{i-1}^4}{4!} R_i^- + \frac{h_i^4}{4!} R_i^+ + \varepsilon\left(l_i^- \frac{h_{i-1}^2}{2!} + l_i^+ \frac{h_i^2}{2!}\right) - \left(\frac{h_{i-1}^4}{4!} l_i^- r_{i-1} + \frac{h_i^4}{4!} l_i^+ r_{i+1}\right),
\end{aligned}$$

After performing some mathematical calculations, we obtain

$$T_{0,i} = T_{1,i} = T_{2,i} = T_{3,i} = 0, T_{4,i} = -3\varepsilon\left(\frac{h_{i-1}^3 + h_i^3}{h_i + h_{i-1}}\right)\left(\frac{1}{4!} - \frac{1}{2!6}\right).$$

As a result, the truncation error of hybrid scheme in a boundary layer region is given as

$$\tau_{i,U} = -3\varepsilon\left(\frac{h_{i-1}^3 + h_i^3}{h_i + h_{i-1}}\right)\left(\frac{1}{4!} - \frac{1}{2!6}\right) + \mathcal{O}(N^{-3}). \tag{9.26}$$

Similarly, the truncation error of a hybrid scheme in a regular region that is for $i = \frac{N}{8} + 1, \cdots, \frac{3N}{8}$ and for $i = \frac{5N}{8} + 1, \cdots, \frac{7N}{8}$ is given as

$$\tau_{i,U} = -\varepsilon\left(\frac{h_i - h_{i-1}}{3}\right) U^{(3)}(x_i) + \frac{2\varepsilon}{4!}\left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right) U^{(4)}(x_i) + \mathcal{O}\left(N^{-3}\right). \tag{9.27}$$

Theorem 9.1. *Let $U^{j+1}(x)$ and U_i^{j+1} be respectively the solutions of (9.8)-(9.9) and (9.19)-(9.20). Then, the local truncation error satisfies the following bounds:*

$$|\tau_{i,U}| \leq \begin{cases} CN^{-2}\ln^2 N, \textit{ for } 1 \leq i \leq \frac{N}{8}, \frac{3N}{8}+1 \leq \frac{N}{2}, \\ C\left(N^{-2}\varepsilon + N^{-2}\right), \textit{for } \frac{N}{8}+1 \leq i \leq \frac{3N}{8}, \\ CN^{-2}\ln^2 N, \textit{ for } \frac{N}{2} \leq i \leq \frac{5N}{8}, \frac{7N}{8}+1 \leq N-1, \\ C\left(N^{-2}\varepsilon + N^{-2}\right), \textit{ for } \frac{5N}{8}+1 \leq i \leq \frac{7N}{8}, \\ CN^{-2}, \textit{for } i = N. \end{cases} \tag{9.28}$$

Proof. Several cases are distinguished depending on the location of the mesh points.
Case-I: When $x_i \in (0,\mu) \cup (1-\mu, 1)$ and $x_i \in (1, 1+\mu) \cup (2-\mu, 2)$. Let us state first the bounds for the derivatives of the continuous solution given in (9.1), which is the solution $u(x,t)$ satisfies the bound $|\frac{d^k U^{j+1}(x)}{dx^k}| \leq C\left(1+\varepsilon^{-\frac{k}{2}} d(x,x,\alpha,\varepsilon)\right)$ where, $d(x,x,\alpha,\varepsilon) = \exp\left(-x\sqrt{\alpha/\varepsilon}\right) + \exp\left(-(1-x)\sqrt{\alpha/\varepsilon}\right)$, for $x \in \Omega_1$ and $d(x,x,\alpha,\varepsilon) = \exp\left(-(x-1)\sqrt{\alpha/\varepsilon}\right) + \exp\left(-(2-x)\sqrt{\alpha/\varepsilon}\right)$, for $x \in \Omega_2$. Now, from (9.26), we obtain a truncation error as

$$|\tau_{i,U}| \leq C\left[h^2\varepsilon + h^2\varepsilon^{-1}\left[d(x_i, x_{i+1}, \alpha, \varepsilon) + d(x_{i-1}, x_i, \alpha, \varepsilon)\right]\right].$$

Now, using $h = 4N^{-1}\sqrt{\varepsilon}\ln N$, and bounding the exponential functions by constants, we obtain that $|\tau_{i,U}| \leq CN^{-2}\ln^2 N$, for $1 \leq i \leq \frac{N}{8}$, $\frac{3N}{8}+1 \leq \frac{N}{2}$, $\frac{N}{2}+1 \leq i \leq \frac{5N}{8}$, and $\frac{7N}{8}+1 \leq i \leq N-1$.
Case-II: When $x_i \in (\mu, 1-\mu)$ and $(1+\mu, 2-\mu)$, we must distinguish two cases. First, $H^2 \leq \varepsilon$, we obtain

$$|\tau_{i,U}| \leq C\left(H^2\varepsilon + \frac{H}{\varepsilon}\left(d(x_i, x_{i+1}, \alpha, \varepsilon) + d(x_{i-1}, x_i, \alpha, \varepsilon)\right)\right) \leq C\left(N^{-2}\varepsilon + N^{-2}\right)$$

On the other hand, if $H^2 \geq \varepsilon$, it can been shown that

$$\begin{aligned} |\tau_{i,U}| &\leq C\left(H^2\varepsilon + \int_{x_i}^{x_{i+1}} (x_{i+1}-\xi)\, d(\xi,\xi,\alpha,\varepsilon)\varepsilon^{-1} d\xi\right) + C\int_{x_{i-1}}^{x_i} (\xi - x_{i-1})\,\varepsilon^{-1} d(\xi,\xi,\alpha,\varepsilon) d\xi \\ &\leq C\left(H^2\varepsilon + H\varepsilon^{-1/2} d(x_i, x_i, \alpha, \varepsilon) + \int_{x_i}^{x_{i+1}} \varepsilon^{-1/2} d(x_i, x_i, \alpha, \varepsilon) dx\right) \\ &+ C\left(H^2\varepsilon + H\varepsilon^{-1/2} d(x_{i-1}, x_{i-1}, \alpha, \varepsilon) + \int_{x_{i-1}}^{x_i} \varepsilon^{-1} d(\xi,\xi,\alpha,\varepsilon) d\xi\right) \\ &\leq C\left(H^2\varepsilon\right) + Cd(x_i, x_i, \alpha, \varepsilon) + Cd(x_i, x_{i+1}, \alpha, \varepsilon) \\ &+ Cd(x_{i-1}, x_{i-1}, \alpha, \varepsilon) + Cd(x_{i-1}, x_i, \alpha, \varepsilon) \\ &\leq CH^2\varepsilon + CN^{-2} + CN^{-2}. \end{aligned}$$

In general, using the fact that $H < 2N^{-1}$, we obtain, the truncation error $\tau_{i,U}$ as

$$|\tau_{i,U}| \leq C\left(N^2\varepsilon + N^{-2}\right). \tag{9.29}$$

The ε-uniform error estimates for the integral boundary conditions which approximated using Simpson's $\frac{1}{3}$ rule at $i = N$, becomes

$$\begin{aligned}
\|\mathcal{K}^N\left(U^{j+1}(x_i) - U_i^{j+1}\right)\| &= |\varepsilon \int_{x_0}^{x_N} g(x)zu(x,t)dx - \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}U_{i-1}^{j+1} + 4g_iU_i^{j+1} + g_{i+1}U_{i+1}^{j+1}}{3} h_i| \\
&\leq C_3\varepsilon\left(h_1^4|\frac{d^4g}{dx^4}(\xi_1)\frac{d^4u}{dx^4}(\xi_1)| + \cdots h_N^4|\frac{d^4g}{dx^4}(\xi_N)\frac{d^4u}{dx^4}(\xi_N)|\right) \\
&\leq C_2\varepsilon\left(h_1^4\frac{d^4u}{dx^4}(\xi_1) + \cdots + h_N^4\frac{d^4u}{dx^4}(\xi_N)\right) \\
&\leq C_1\varepsilon^{-1}\left(e^{-\xi_1\sqrt{\frac{\alpha}{\varepsilon}}}h_1^4 + \cdots + e^{-\xi_N\sqrt{\frac{\alpha}{\varepsilon}}}h_N^4\right) \\
&\leq CN^{-2},
\end{aligned}$$

where $x_{i-1} \leq \xi \leq x_i, 1 \leq i \leq N$, for a proper choice of C. Combining all the previous results, the required error estimates is obtained. ■

Theorem 9.2. *Let $U^{j+1}(x_i)$ be the solution of (9.8)-(9.9) and $U_i^{j+1}, i = 0, 1, \cdots, N$ be the numerical solution of the hybrid numerical scheme in (9.19)-(9.20) at the mesh points of the Shishkin mesh. Then, the error estimate satisfies*

$$|U^{j+1}(x_i) - U_i^{j+1}| \leq C\left(N^{-2}\ln^2 N + N^{-\sqrt{\alpha}\mu_0}\right),\ i = 0, 1, \cdots, N-1, \tag{9.30}$$

where the constant μ_0 satisfies $\mu_0 \geq 2\sqrt{\alpha}$, and hence the hybrid numerical scheme given in (9.19)-(9.20) is uniformly convergent of almost order two.

Proof. Defining the discrete barrier function

$$\Psi_i = C\begin{cases} N^{-2}\ln^2 N + N^{-2}\varepsilon + N^{-2} + \frac{\sigma}{\sqrt{\varepsilon}}N^{-2}\psi_1(x_i) \\ N^{-2}\ln^2 N + N^{-2}\varepsilon + N^{-2} + \frac{1+\sigma}{\sqrt{\varepsilon}}N^{-2}\psi_2(x_i) \end{cases}, \tag{9.31}$$

where ψ_1 and ψ_2 are piecewise linear polynomials defined as

$$\psi_1(t) = \begin{cases} \frac{x}{\mu},\ \text{for } 0 \leq x \leq \mu \\ 1,\ \text{for } \mu \leq x \leq 1-\mu \\ \frac{1-x}{\mu},\ \text{for } 1-\mu \leq x \leq 1 \end{cases} \quad \text{and} \quad \psi_2(t) = \begin{cases} \frac{x}{1+\mu},\ \text{for } 1 \leq x \leq 1+\mu \\ 1,\ \text{for } 1+\mu \leq x \leq 1-\mu \\ \frac{2-x}{1+\mu},\ \text{for } 2-\mu \leq x \leq 2 \end{cases}. \tag{9.32}$$

Then for all $i = 1, 2, \cdots, N$, we have

$$0 \leq \Psi(x_i) \leq CN^{-2} \ln^2 N,$$

and by choosing a sufficiently large C, using Theorem 9.1 and the discrete maximum principle, it is simple to see that

$$\mathcal{L}^N \left(\Psi_i \pm \left(U^{j+1}(x_i) - u_i^{j+1}\right)\right) \geq 0.$$

Equivalently,

$$\mathcal{L}^N \Psi_i \geq |\tau_{i,U}|.$$

Therefore it follows that

$$|U^{j+1}(x_i) - U_i^{j+1}| \leq \Psi_i \leq CN^{-2} \ln^2 N, \quad \forall x_i \in \bar{\Omega}^N. \tag{9.33}$$

■

Thus, a required ε-uniform error bound is obtained. The results of this work is summarized by considering the semidiscrete error estimate obtained in (9.12) and (9.33) and concluded by the following theorem.

Theorem 9.3. *The ε-uniform error estimate for the solution of continuous and fully discrete problem is given by*

$$\sup_{0<\varepsilon\leq 1} \left|\left|u(x,t) - U_i^{j+1}\right|\right| \leq C\left(N^{-2} \ln^2 N + \Delta t\right), \tag{9.34}$$

where $u(x,t)$ and U_i^{j+1} are the solutions of the problem (9.2)-(9.3) and (9.19)-(9.20) respectively.

Proof. The proof follows from Lemma 9.5 and Theorem 9.2. Therefore, the theorem concluded that the order of convergence is almost two in the spatial direction and one in the temporal direction. ■

9.7 Numerical results and discussions

Two model examples are presented to validate the results of this study. Since the exact solution of the problem is not known, double mesh principle is used to estimate maximum point wise error using (6.27), uniform error estimate using (6.28), rate of convergence using (6.29), and uniform rate of convergence using (6.30). The obtained results are displayed in tables for different values of $N, \Delta t$ and ε.

Example 9.1. *Consider a class of SPPDDEs with integral boundary condition of the form*

Elango et al. (2021)

$$\begin{cases} -\varepsilon\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t}+5u(x,t)-u(x-1,t)=e^{-x}, \quad (x,t)\in(0,2)\times(0,2] \\ u(x,t)=0, \quad \forall (x,t)\in\Gamma_L \\ \mathcal{K}u(2,t)=u(2,t)-\varepsilon\int_0^2 \frac{x}{3}u(x,t)dx=0, \quad \forall\,(x,t)\in\Gamma_r \\ u(x,t)=0, \quad \forall\;(x,t)\in\Gamma_b. \end{cases} \tag{9.35}$$

Example 9.2. *Consider a class of SPPDDEs with integral boundary condition of the form Elango et al. (2021)*

$$\begin{cases} -\varepsilon\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t}+5u(x,t)-xu(x-1,t)=1, \quad (x,t)\in(0,2)\times(0,2] \\ u(x,t)=0, \quad \forall (x,t)\in\Gamma_L \\ \mathcal{K}u(2,t)=u(2,t)-\varepsilon\int_0^2 \frac{1}{6}u(x,t)dx=0, \quad \forall\,(x,t)\in\Gamma_r \\ u(x,t)=\sin(\pi x), \quad \forall\;(x,t)\in\Gamma_b. \end{cases} \tag{9.36}$$

The maximum pointwise error for different values of perturbation parameter ε and uniform rate of convergence for Example 9.1 is given in Table 9.1. Again, Table 9.2 shows a maximum pointwise error for various values of ε and uniform rate of convergence for Example 9.2. A comparison of uniform error estimate and uniform rate of convergence of the proposed method with a method existed in the literature is given in tables 9.3 and 9.4 for Examples 9.1 and 9.2 respectively. The presence of both boundary layer and interior is apparent from the surface plots of numerical solution for Examples 9.1 and 9.2 are given in Figures. 9.1 and 4.2 respectively. The Log-Log plot for maximum pointwise error for Example 9.1 and 9.2 are given in Figure. 9.3. We proved that a proposed numerical scheme is of order $O\left(N^{-2}\ln^2 N+\Delta t\right)$. From Tables 9.3 and 9.4, one can observed that the developed numerical scheme is a more accurate uniform convergent than the methods presented in (Hailu & Duressa, 2022), Gobena & Duressa (2021), and Elango et al. (2021).

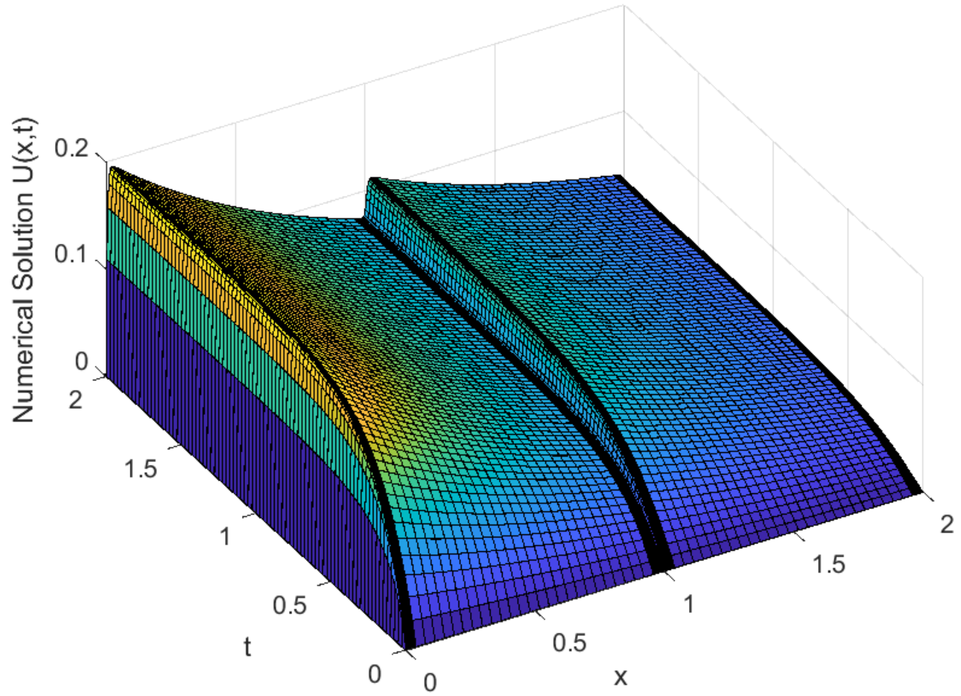


Figure 9.1: Graph of numerical solution of Example 9.1 for $\varepsilon = 10^{-4}$.

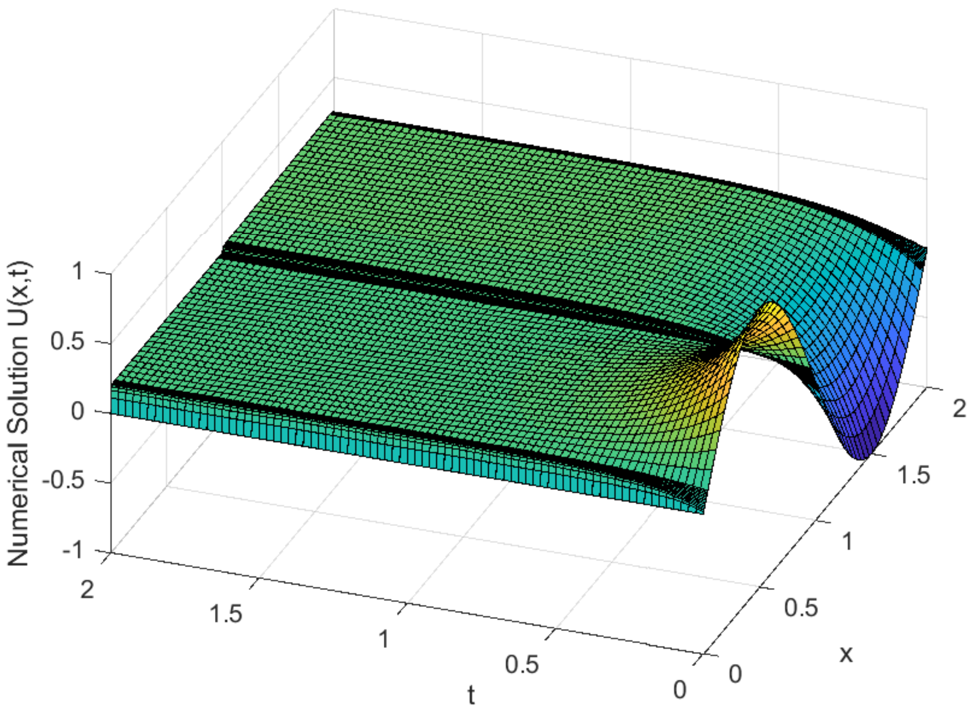
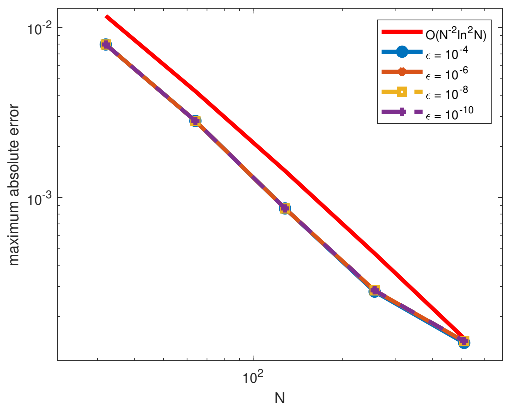


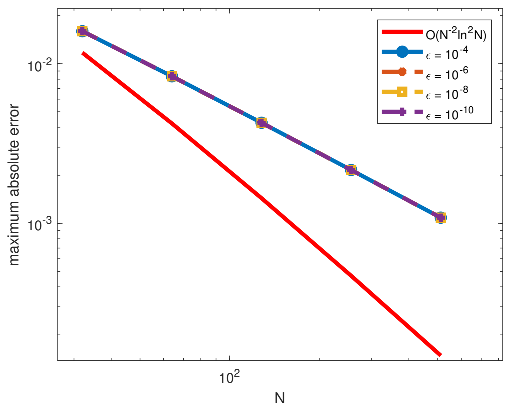
Figure 9.2: Graph of numerical solution of Example 9.2 for $\varepsilon = 10^{-4}$.

Table 9.1: Values of $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$, $R_\varepsilon^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 9.1

$\varepsilon \downarrow N \rightarrow$	$N \longmapsto 16$	32	64	128	256	512
	$\Delta t \rightarrow$0.1/2	0.1/4	0.1/8	0.1/16	0.1/32	0.1/64
10^{-02}	4.5138e-03	1.8872e-03	9.7931e-04	5.5490e-04	4.1685e-04	1.2638e-04
	1.2581	0.9464	0.8195	0.4127	0.7012	
10^{-03}	1.7283e-02	7.9562e-03	2.8246e-03	7.2260e-04	2.6566e-04	1.3570e-04
	1.1192	1.4940	1.9668	1.4436	0.9692	
10^{-04}	1.7276e-02	7.9546e-03	2.8240e-03	8.6335e-04	2.7970e-04	1.3963e-04
	1.1189	1.4940	1.7097	1.6261	1.0023	
10^{-05}	1.7275e-02	7.9544e-03	2.8240e-03	8.6334e-04	2.8216e-04	1.4184e-04
	1.1189	1.4940	1.7097	1.6134	0.9922	
10^{-06}	1.7275e-02	7.9544e-03	2.8240e-03	8.6333e-04	2.8380e-04	1.4261e-04
	1.1189	1.4940	1.7098	1.6050	0.9928	
10^{-07}	1.7275e-02	7.9544e-03	2.8240e-03	8.6333e-04	2.8435e-04	1.4287e-04
	1.1189	1.4940	1.7098	1.6022	0.9930	
10^{-08}	1.7275e-02	7.9544e-03	2.8240e-03	8.6333e-04	2.8453e-04	1.4296e-04
	1.1189	1.4940	1.7098	1.6013	0.9930	
10^{-09}	1.7275e-02	7.9544e-03	2.8240e-03	8.6333e-04	2.8459e-04	1.4299e-04
	1.1189	1.4940	1.7098	1.6010	0.9927	
10^{-10}	1.7275e-02	7.9544e-03	2.8240e-03	8.6333e-04	2.8462e-04	1.4300e-04
	1.1189	1.4940	1.7098	1.6009	0.9930	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	1.7275e-02	7.9544e-03	2.8240e-03	8.6333e-04	2.8462e-04	1.4300e-04
	1.1189	1.4940	1.7098	1.6009	0.9930	
$E^{N,\Delta t}$	1.7283e-02	7.9562e-03	2.8246e-03	8.6335e-04	4.1685e-04	1.4300e-04
$R^{N,\Delta t}$	1.1192	1.4940	1.7100	1.0504	1.5435	-



(a)



(b)

Figure 9.3: Log-Log plot of maximum absolute error versus mesh numbers for different values of ε (a) for Example 9.1, (b) for Example 9.2.

Table 9.2: Values of $E_{\varepsilon}^{N,\Delta t}$, $E^{N,\Delta t}$, $R_{\varepsilon}^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 9.2

$\varepsilon \downarrow N \rightarrow$	$N \longmapsto 16$	32	64	128	256	512
	$\Delta t \rightarrow$0.1/2	0.1/4	0.1/8	0.1/16	0.1/32	0.1/64
10^{-02}	2.9468e-02	1.5966e-02	8.3385e-03	4.2672e-03	2.1578e-03	1.0851e-03
	0.8841	0.9371	0.9665	0.9837	0.9917	
10^{-03}	2.9506e-02	1.5977e-02	8.3395e-03	4.2672e-03	2.1578e-03	1.0852e-03
	0.8850	0.9380	0.9667	0.9837	0.9916	
10^{-04}	2.9519e-02	1.5980e-02	8.3402e-03	4.2654e-03	2.1580e-03	1.0851e-03
	0.8854	0.9381	0.9674	0.9830	0.9919	
10^{-05}	2.9520e-02	1.5980e-02	8.3403e-03	4.2646e-03	2.1581e-03	1.0852e-03
	0.8854	0.9381	0.9677	0.9826	0.9918	
10^{-06}	2.9520e-02	1.5980e-02	8.3403e-03	4.2643e-03	2.1581e-03	1.0852e-03
	0.8854	0.9381	0.9678	0.9825	0.9918	
10^{-07}	2.9520e-02	1.5980e-02	8.3403e-03	4.2642e-03	2.1581e-03	1.0852e-03
	0.8854	0.9381	0.9678	0.9825	0.9918	
10^{-08}	2.9520e-02	1.5980e-02	8.3403e-03	4.2641e-03	2.1581e-03	1.0852e-03
	0.8854	0.9381	0.9678	0.9825	0.9918	
10^{-09}	2.9520e-02	1.5980e-02	8.3403e-03	4.2641e-03	2.1581e-03	1.0852e-03
	0.8854	0.9381	0.9678	0.9825	0.9918	
10^{-10}	2.9520e-02	1.5980e-02	8.3403e-03	4.2641e-03	2.1581e-03	1.0852e-03
	0.8854	0.9381	0.9678	0.9825	0.9918	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	2.9520e-02	1.5980e-02	8.3403e-03	4.2641e-03	2.1581e-03	1.0852e-03
	0.8854	0.9381	0.9678	0.9825	0.9918	
$E^{N,\Delta t}$	2.9520e-02	1.5980e-02	8.3403e-03	4.2672e-03	2.1581e-03	1.0852e-03
$R^{N,\Delta t}$	0.8854	0.9381	0.9668	0.9835	0.9918	-

Table 9.3: Comparison of $E^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 9.1 at mesh number N and Δt.

$\varepsilon \downarrow$	$N \longmapsto 32$	64	128	256	512
	$\Delta t \rightarrow$0 0.1/4	0.1/8	0.1/16	0.1/32	0.1/64
Present Method					
$E^{N,\Delta t}$	7.9562e-03	2.8246e-03	8.6335e-04	4.1685e-04	1.4300e-04
$R^{N,\Delta t}$	1.4940	1.7100	1.0504	1.5435	-
Hailu & Duressa (2022)					
$E^{N,\Delta t}$	1.2974e-02	8.7628e-03	4.6313e-03	2.0155e-03	7.6160e-04
$R^{N,\Delta t}$	0.56616e	0.91997	1.2003	1.4040	-
Gobena & Duressa (2021)					
$E^{N,\Delta t}$	2.0026e-03	1.0723e-03	5.5505e-04	2.8241e-04	
$R^{N,\Delta t}$	0.90117	0.95002	0.97483	0.98734	-
Elango et al. (2021)					
$E^{N,\Delta t}$	1.2534e-02	6.9738e-03	3.6873e-03	1.8972e-03	9.6241e-04
$R^{N,\Delta t}$	0.84584	0.91937	0.95873	0.97912	

Table 9.4: Comparison of $E^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 9.2 at mesh number N and Δt.

$\varepsilon \downarrow N \longmapsto$ $\Delta t \rightarrow$	32 0.1/4	64 0.1/8	128 0.1/16	256 0.1/32	512 0.1/64
Present Method					
$E^{N,\Delta t}$	1.5980e-02	8.3403e-03	4.2672e-03	2.1581e-03	1.0852e-03
$R^{N,\Delta t}$	0.9381	0.9668	0.9835	0.9918	-
Hailu & Duressa (2022)					
$E^{N,\Delta t}$	3.1547e-02	1.7445e-02	9.2639e-03	4.7759e-03	2.4253e-03
$R^{N,\Delta t}$	0.85469	0.91312	0.95585	0.97761	-
Gobena & Duressa (2021)					
$E^{N,\Delta t}$	1.5980e-02	8.3421e-03	4.2681e-03	2.1581e-03	1.0853e-03
$R^{N,\Delta t}$	0.93778	0.96682	0.98383	0.99180	
Elango et al. (2021)					
$E^{N,\Delta t}$	1.4776e-01	9.7571e-02	5.7092e-02	3.1057e-02	
$R^{N,\Delta t}$	0.59873	0.77316	0.87837	0.93697	

9.8 Summary

A class of singularly perturbed partial differential equations of reaction-diffusion type with a large spatial delay and integral boundary condition is solved numerically. The presented method consists of a hybrid numerical scheme on Shishkin mesh in space and implicit Euler method on a uniform mesh in the time variable. The hybrid numerical scheme consists of a cubic spline method used in the boundary layer region, and classical finite difference method used in outer layer region. The integral boundary condition is treated using Simpson's $\frac{1}{3}$ rule. The uniform stability and convergence analysis of the proposed method are analyzed. An error estimate of the proposed scheme shows a parameter uniform convergence of almost second-order in space and first-order in time variable. Two numerical test examples are considered to validate applicability of a proposed scheme. Numerical results agree with the theoretical estimation.

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

This dissertation contributes to the investigation of numerical treatment of a class of singularly perturbed differential equations with nonlocal boundary conditions. Uniformly convergent numerical methods in which the error is independent of perturbation parameter were developed. The primary findings of this dissertation, along with some relevant observations, are summarized here.

Exponentially fitted finite difference method is proposed and analyzed for SPDODEs with integral boundary condition. In this method, an exponential fitting factor is introduced to treat the solutions inside the boundary layer which occur due to perturbation parameter. The nonlocal boundary condition is treated using Simpson's rule of integration. The stability and uniform convergence analysis of the proposed approach are established. The error estimation of the developed method is shown to be second-order uniform convergent. The obtained numerical results reflects the theoretical estimations. A fitted mesh numerical scheme on a suitable piecewise uniform Shishkin mesh is proposed and analyzed for SPDODEs with integral boundary condition. The uniform stability and uniform convergence analysis of the method are studied. The developed scheme is uniformly convergent with almost second-order accurate in the discrete maximum norm. The numerical results are in agreement with theoretical justifications.

A nonstandard FDM is formulated and analyzed for SPPPDEs with nonlocal boundary conditions. The basic idea for nonstandard method is the construction of the exact finite difference technique, which the denominator characteristic for the discrete derivatives should be described in terms of more complicated functions with larger step sizes than those used in classical methods. The method of lines is employed. The stability analysis and convergence analysis for the proposed method are demonstrated. The developed numerical method is uniformly convergent independent of perturbation parameter with second-order in spatial direction and first-order in time direction. The numerical results validated with theoretical justifications. A uniformly convergent numerical method is formulated and analyzed for SPPPDEs with integral boundary conditions. The method involves a FDM on a suitable piecewise uniform Shishkin mesh for the spatial direction and implicit Euler method for the temporal direction. The stability and error estimates for the evolved method are analyzed. The developed method is uniformly convergent with almost second-order in space and first-order in time. The numerical results corresponds to theoretical estimations.

Fitted mesh numerical method is developed and analyzed for SPPDDEs with integral boundary conditions. The developed method used FDM on a suitable piecewise uniform

Shishkin mesh for the spatial direction and Crank-Nicholson method is used for the temporal direction. The uniform stability analysis and error estimation for the proposed method are analyzed. The developed method is more accurate uniformly convergent independent of perturbation parameter with almost second-order in spatial direction and second-order in temporal direction. The numerical results reflects the theoretical estimations. A hybrid numerical method is also proposed for SPPDDEs with integral boundary condition. The proposed hybrid scheme constitute of cubic spline method in the boundary layer region and a classical finite difference method in the outer layer region. This hybrid numerical method is used in spatial direction and implicit Euler method is used in temporal direction. The integral boundary condition is treated using Simpson's $\frac{1}{3}$ rule. The uniform stability analysis and uniform convergence analysis for the proposed method is established. The proposed method is more accurate uniformly convergent with almost second-order in spatial direction and first-order in time direction. The numerical results are in agreement with theoretical estimations.

Recommendations

The findings in this dissertation have numerous applications in mathematical physics and engineering, particularly in fluid dynamics and population dynamics. The numerical methods developed in this dissertation can be utilized to solve a class of singularly perturbed problems in fluid dynamics and other related topics. A uniformly convergent numerical methods for a class of SPPs with nonlocal boundary conditions are discussed. The following highlights are the research perspectives resulting from this dissertation.

- To produce more accurate uniformly convergent numerical methods, layer adaptive mesh numerical methods based on Backhavalov's mesh can be employed to address the problems considered in this dissertation.

- The numerical methods proposed in this dissertation can be employed to a class of SPPDEs of convection-diffusion type with integral boundary conditions.

- The numerical methods considered in this dissertation can be extended to higher dimensional problems with nonlocal boundary conditions.

- The numerical methods considered in this dissertation can be extended to a coupled system of a class of singularly perturbed problems with nonlocal boundary conditions.

- Numerical solution for a class of singularly perturbed differential equations with two nonlocal boundary conditions can be considered.

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APPENDIX A

LIST OF PUBLISHED AND SUBMITTED PAPERS

1. Wondimu, G. M., Woldaregay, M. M., Dinka, T. G., & Duressa, G. F. (2022). Numerical treatment of singularly perturbed parabolic partial differential equations with nonlocal boundary condition. Frontiers in Applied Mathematics and Statistics, 8, https://doi.org/10.3389/fams.2022.1005330.

2. Wondimu, G. M., Dinka, T. G., Woldaregay, M., & Duressa, G. F. (2023). Fitted mesh numerical scheme for singularly perturbed delay reaction diffusion problem with integral boundary condition. Computational Methods for Differential Equations, https://dx.doi.org/10.22034/cmde.2023.49239.2054.

3. Wondimu, G. M., Woldaregay, M. M., Duressa, G. F., & Dinka, T. G. (2023). Exponentially fitted numerical method for solving singularly perturbed delay reaction-diffusion problem with nonlocal boundary condition, BMC Research Notes, Springer, https://doi.org/10.1186/s13104-023-06347-6.

4. A uniformly convergent numerical scheme for solving singularly perturbed parabolic partial differential equations with integral boundary condition: **Under Review**.

5. Fitted numerical scheme for solving singularly perturbed delay reaction-diffusion problem with integral boundary condition: **Under Review**.

6. Hybrid numerical scheme for solving singularly perturbed partial delay reaction-diffusion problem with integral boundary condition: **Under Review**.

APPENDIX B

MATLAB CODES

Everyone can use the following URLs to get the Matlab Algorithm that was written in the dissertation.

Matlab code for the proposed method in Chapter 4

https://drive.google.com/file/d/11r0WB1d2D6alyoWtrUSLmSdCQvlKxZIm/view?usp=sharing

Matlab code for the proposed method in Chapter 5

https://drive.google.com/file/d/1WbR3EnUbCo_o7DhEiTn_1EFFciw10w76/view?usp=sharing

Matlab code for the proposed method in Chapter 6

https://drive.google.com/file/d/1wxMYS2U8W_EEGKaRja_vvz3WKvbm-vYW/view?usp=sharing

Matlab code for the proposed method in Chapter 7

https://drive.google.com/file/d/1LtzJ5niTtwkS1X1UonvKYPxvOi50hLTw/view?usp=sharing

Matlab code for the proposed method in Chapter 8

https://drive.google.com/file/d/10Wrryrv9cIP3zuDJNSGLXqonEDEVqmS2/view?usp=sharing

Matlab code for the proposed method in Chapter 9

https://drive.google.com/file/d/15F9dp-PSzoJUpLl0MvZqjNKVwNynQsQ9/view?usp=sharing