HIGHER ORDER COMPACT FINITE DIFFERENCE METHOD AND FINITE ELEMENT METHOD FOR SOLVING 2D POISSON'S EQUATIONS

By

Debisa Muleta

A Thesis Submitted to Department of Applied Mathematics

School of Applied Natural Science

Office of Graduate Studies

Adama Science and Technology University

August, 2020

Adama, Ethiopia

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Approval Page

This is to certify that the thesis prepared by **Debisa Muleta** entitle " **Higher Order Compact Finite Difference Method and Finite Element Method For Solving 2D Elliptic Boundary Value Problems** " submitted in fulfillment of the requirement for the degree of master of science in Numerical with the regulation of the University and meets the accepted standards with respect to originality.

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Abstract

The main purpose of this thesis is to solve the 2D Elliptic Boundary Value Problems using higher-order CFDM and FEM. We derived eight-order compact finite difference schemes to solve the 2D-Poisson equation subject to Dirichlet boundary condition on a bounded two dimensional region by using Taylor series expansion. In eight order compact finite difference for solving 2D-Poisson equations, the suggested scheme has the stencil of twenty five points. By using steps of finite element method in rectangular elements we obtained 9-point finite difference method in assembling elements. The numerical solutions were obtained by these two methods were compared with each other graphically in two dimension. Accuracy of the finite difference technique can be improved by representing the partial differential equation by a higher order approximation developed to reduce the Taylor series truncation error. Therefore, eight-order compact finite difference schemes have higher accuracy than six-order compact finite difference schemes and finite element method for solving 2D Poisson equations.

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Notations and Abbreviations

FDM	Finite difference method.
FEM	Finite element method.
CFDM	Compact finite difference method.
HOC	Higher order compact
$O(h) = O(\triangle x)$	Errors of order h.
$O(h^2) = O(\triangle x)^2$	Errors of order h^2 etc.
$\triangle x = h$	The length of mesh points.
U(x;y)	Is the analytical solution of the problems.
U(i;j)	Is the initial mesh point of finite difference equation.
PDE	Partial differential equation.

Chapter 1

Introduction

1.1 Background of study

Differential equations appear in many areas of applications such as in sciences and engineering. Using partial differential equations most of the problems in the scientific and engineering areas are modeled. Partial differential equations occur when a dependent variable depends on two or more independent variables. Usually analytic solutions for these partial differential equations are difficult to compute and hence numerical methods are used [1].

One of the PDE is an elliptic partial differential equation that frequently emerges when modeling electromagnetic systems. However, like many other partial differential equations, exact solutions are difficult to obtain for complex geometries. This motivates the use of numerical methods in order to provide accurate results for real-world systems.

The general linear elliptic equation is given as follows:

$$\bigtriangledown (- \bigtriangledown U(x)) + p \bigtriangledown U(x) + qU = f(x) \quad x \in \ \Omega \in R^n \tag{1.1}$$

where $\Omega = [0, L_1] \times [0, L_2] \times [0, L_3] \times ... \times [0, L_n]$ is a domain with suitable boundary conditions defined on $\partial\Omega$ and p,q are constant.

Where $\bigtriangledown$ is defined as :

$$\bigtriangledown^2 U(x) = \sum_{i,j=1}^{n} U_{x_i x_j}$$

From (1.1), when $p = q = 0$ and $x \in \Omega = R^2$,then particularly two-dimensional Poisson's equation with Dirichlet boundary condition can be written in the form of [2]

$$\bigtriangledown^2 U = -f \qquad or \qquad -(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}) = f(x, y) \quad (x, y) \in \Omega \tag{1.2}$$

where $\Omega = [0, L_x] \times [0, L_y]$ is a rectangular region domain with suitable boundary conditions defined on $\partial\Omega$. The solution $U(x, y)$ and the forcing function f(x,y) are assumed

to be sufficiently smooth and to have the required continuous partial derivatives. The spatial domain Ω is discretized with uniform grid sizes $h = x_{i+1} - x_i$ and $k = y_{j+1} - y_j$ in the x-and y-coordinates, respectively. The grid points are (x_i, y_j) in the x- and y-coordinate directions with the number of grid points M×N, $0 \leq i \leq M, 0 \leq j \leq N$. The quantity $U(x_i, y_j)$ represents the exact solution at (x_i, y_j), while $U_{i,j}$ represents the numerical solution at (x_i, y_j).
The development and application of the implicit finite difference formula to solve initial value problem and boundary value Problem modeled by partial differential equations is of more recent appearance. They are called Hermitian finite difference schemes [4]. This name is due to the analogy with Hermite's interpolation formula which, in addition to the values of the function, also uses the value of the derivatives at several points. In view of their smaller stencils, the Hermitian finite difference equations are mainly known as **compact finite difference schemes**. The compact finite difference schemes are one of numerical methods that used to approximate partial differential equation. Higher order compact finite difference method was first introduced by Kreiss and Oliger [33] and implemented by R.S Hirsh [14], then popularized by Sanjiva K. Lele [34]. Compact finite differencing expressions have been known for very recent years. Some particular formulas were reported by, [4]; however, their implementation as difference schemes approximating partial differential equations began in the early years for some fluid mechanics problems . Since that time, several distinct classes of compact schemes have been developed. In recent years, due to the appearance of faster and more powerful computing possibilities as well as the development of algorithms for fast matrix inversion, compact schemes are proving more advantageous.

High-order finite difference schemes can be classified into two main categories [5]:
i. **Explicit schemes** and
ii. **Pade-type or compact schemes**.
Explicit schemes compute the numerical derivatives directly at each grid by using large stencils, while **compact schemes** obtain all the numerical derivatives along a grid line using smaller stencils and solving a linear system of equations. Experience has

shown that compact schemes are much more accurate than the corresponding explicit scheme of the same order. High-order compact finite difference schemes give higher order accuracy and better resolution characteristics as compared to finite difference schemes for the same number of grid points[6].

The finite element method is another numerical technique frequently used to obtain approximate solutions of problems governed by differential equations. The finite element method (FEM), sometimes referred to as finite element analysis (FEA), is a computational technique used to obtain approximate solutions of boundary value problems in engineering. Simply stated, a boundary value problem is a mathematical problem in which one or more dependent variables must satisfy a differential equation everywhere within a known domain of independent variables and satisfy specific conditions on the boundary of the domain[23]. When the mathematical study of the finite element method started in the mid 60's, it soon became clear that the method is a general technique for the numerical solution of partial differential equations with roots in the century. The FEM dates back to 1909 when Ritz developed an effective method for the approximate solution of problems in the mechanics of deformable solids. It includes an approximation of energy functional by the known functions with unknown coefficients. Minimization of the functional in relation to each unknown leads to a system of equations from which the unknown coefficients may be determined. One of the main restrictions in the Ritz method is that functions used should satisfy the boundary conditions of the problem[24]. Finite element methods are essentially methods for finding approximate solutions of partial differential equations(2D-Poisson equation)defined in a finite region or domain[2] .

In this thesis we apply eight order CFDM for solving 2D elliptic boundary value problem and compare the result with FEM.

1.2 Statement of Problem

In this thesis ,we derive eight order CFDM scheme and compare with finite element method for solving the two dimensional Poisson's equations. The FDM with the second order accuracy is an explicit method and it is established on a stencil of 5-points for 2D space. However, the significant shortcoming of the second order FDM is the solution of Poisson's equation has lower accuracy and higher order accuracy method requires a larger stencil. A large stencil requires some modifications near the boundaries where the points needed in the stencil are not available, but this problems could be eliminated if a compact method is used. Finite element method (FEM) is the most known numerical method used for approximating the solution of partial differential equations on domain with irregular shapes. The power of the Finite Element method becomes more evident, because the Finite Difference method will have much more difficulty in solving problems in a domain with complex geometries. FEM is more efficient method than FDM for approximating the solution of Poisson equation on irregular domain. However, until to day the researcher did not compare accuracy of eight order CFDM with FEM for solving 2D-Poisson equations.
So, this study is intended to consider the following:

- ✓ Deriving eighth order compact finite difference method
- ✓ Comparing eight order compact finite difference method solution with finite element method solution for solving 2D Poisson's equations
- ✓ Formulation of FEM for solving 2D Poisson's equations

1.3 Objective of the study

1.3.1 General Objective

The main objective of this study is using eight Order Compact Finite Difference Method and Finite Element Method for solving Boundary-value problems of 2D elliptic equations.

1.3.2 specific objectives

The specific objectives of this study are to:

- ✓ compare higher order CFDM with FEM for solving Poisson's equation.
- ✓ develop Finite Element method for solving Poisson's equation.
- ✓ derive the eight order compact finite method.
- ✓ develop matlab code for each of the Methods .
- ✓ Solve some illustrative examples.

1.4 Significance of the study

The main importance of the study are:

- ✓ To develop skills in solving 2D Poisson equation using higher order compact finite difference method and finite element method.
- ✓ To identify the method which is considered as one of the best method to solve Poisson equation.
- ✓ Moreover it provides guides to natural science students, teachers and anyone who works on this area.

Chapter 2

Review of Related Literature

Poisson's equation is often encountered in mechanical engineering applications, theoretical physics and other fields as described in [7]. Finite-difference methods (FDM) are efficient tools for solving the partial differential equation, which work by replacing the continuous derivative operators with approximate finite differences directly according to [7]. The FDM with the second order accuracy is an explicit method and it is established on a stencil of 5-points for 2D space and 7-points for 3D space. The significant shortcoming of the second order FDM is the solution of Poisson's equation has lower accuracy and higher order accuracy method requires a larger stencil as in [8]. A large stencil requires some modifications near the boundaries where the points needed in the stencil are not available, but this problems could be eliminated if a compact method is used [9].

The compact finite difference method (CFDM) is one of the methods used to increase the accuracy of the numerical solutions of the partial differential equation. Therefore, we can use many methods to derive the compact finite difference scheme for any problem. According to Spotz.W.F [10] was used the Taylor series to drive a fourth and sixth compact finite difference for 2D and 3D Poisson equation. By Zhang.J [11] was derive a fourth order compact finite difference for 2D Poisson equation by using compact operators method. Recently , as Shuvam Sen. [13] discussed a new family of compact finite difference schemes applicable for unsteady Navier-Stokes equations.

Compact schemes can provide numerical solutions with spectral-like resolution and very low numerical dissipation. Compared with explicit finite difference methods, this method is implicit and obtains the evaluation of derivatives with higher accuracy for the same number of grid points in [12]. Many studies on compact finite difference schemes have been conducted for solving Poisson's equation [15]. Shiferaw and Mittal [16] have obtained the numerical solutions of three dimensional Poisson's equation with the Dirichlet's boundary conditions. The Poisson's equation was approximated by a 19-points fourth order compact finite difference approximation schemes. Kyei and Tang [18] were derived a family of sixth-order compact finite difference schemes for the three-dimensional Poisson's equation, they considered the discretization of source function on a compact finite difference stencil. By Richardson and Li [20] were studied the multigrid method combined with a fourth order compact scheme for the 2D Poisson's equation. The results showed that the new method was of higher accuracy and less computational time. According to [22] the accuracy of the sixth order schemes is very high compare with the fourth order schemes; also the accuracy of the fourth order schemes is very high compare with the central finite difference scheme.

Finite Element Method (FEM) is the most known numerical method used for approximating the solution of partial differential equations on domain with irregular shapes in [27]. The power of the Finite Element method becomes more evident, because the Finite Difference method will have much more difficulty in solving problems in a domain with complex geometries. The FDM is very simple and efficient method for approximating the solution of the BVP when the domain has regular shape. On the other hand FEM is more efficient method than FDM for approximating the solution of Poisson equation on irregular domain. Numerical result show that finite difference method more efficient than the finite element method for regular domains, whereas finite element method is more accurate for complex and irregular domains in [27]. Ezeh and Enem [31], were used triangular and rectangular finite element models in a comparative study to determine the nodal displacements in the numerical analysis of a deep beam. The results of the analysis show that the use of rectangular elements yielded results that are closer to the exact solutions than the triangular element model.

In this thesis, we compare eight order compact finite difference with finite element method for solving Poisson's equation, which can improve the flexibility and increase the accuracy of numerical solution.

Chapter 3

Research Methodology

This section consists of methods and materials used to undertake the study. These are study design, source of information procedure of the study and instrumentation.

3.1 Study Design

The design of this study is by discretized Domain of partial differential equation using CFDM and FEM , and solving system of linear equation. We also use graphical representation and mat-lab as a tool.

3.2 Source of Information

The study depends on various sources of information such as:
- Journals
- Reference books
- Other materials from the internet and research papers

3.3 Procedures of the study

To attain the objective of the study, the following procedures were undertaken:

- Defining the problem/formulating the method.
- Discretizing the domain/interval for CFDM and FEM.
- Replacing the Poisson's equation by the CFDM .

- Variational and weak formulation for FEM.
- Choice of trial and base function.
- solving system of equations
- Validating the methods using numerical examples.
- Writing a code for the problem by using matlab language.
- Compare the method within the result.
- Discussing the methods from the graphs.

3.4 Instrumentation of analysis

Solving the discrete system and approximate solution using mat-lab codes and shows the result by tables and graphs.

Chapter 4

Higher Order CFDM and FEM For Solving 2D Poisson's Equations

In this chapter we discuss finite element method and higher order compact finite difference method for solving 2D Poisson equation, and derive the eight order compact finite difference method.

4.1 The Finite Element Method For 2D Poisson Equations

The finite element method (FEM) discretizes the global solution domain Ω (x,y) into a number of sub-domains $\Omega_i(x, y)$ (i = 1, 2 ... N), called elements, and applies either the Rayleigh-Ritz method or the Galerkin weighted residual method to the discretized global solution domain. In this section the finite element method is developed to solve the following 2D Poisson equation with appropriate boundary conditions:

$$U_{xx} + U_{yy} = f(x, y) \tag{4.1}$$

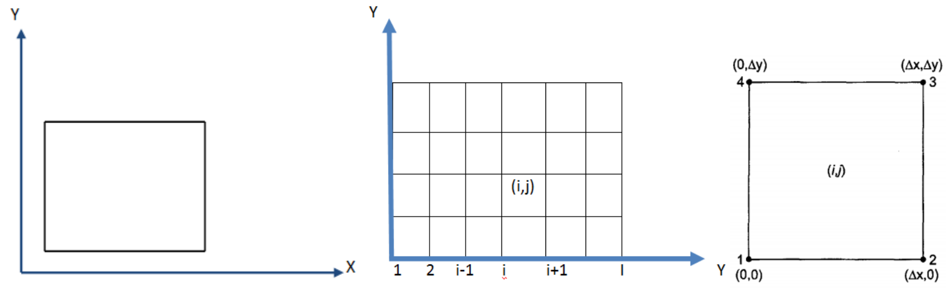
The steps in the finite element approach are as follows[29]:

1. Formulate the problem. Using Galerkin weighted residual approach ,determining the differential equation to be solved.

2. Discretize the global solution domain Ω(x,y) into sub-domains (i.e., elements) $\Omega_i(x, y)$ (i = 1, 2 ... N). Specify the type of element to be used (i.e., linear, quadratic,etc.).

3. Assume the functional form of the approximate solution $U^{*(i,j)}(x, y)$ within each element, and choose the interpolating functions for the elements.

4. For the Galerkin weighed residual approach, substitute the approximate solution



(a) (b) (c)

Figure 4.1: Rectangular solution domain Ω (x,y) and Discretized global solution domain Ω (x,y)

$U^*(x,y)$ into the differential equation to determine the residual R(x,y), weighted residual with the weighting functions $W_k(x,y)$,and form the weighted residual integral .

5. Determine the element equations. Using the Galerkin weighted residual approach ,evaluate the partial derivatives of the weighted residual integral $I(C_{i,j})$ with respect to the nodal values $C_{i,j}$, and equate them to zero.

6. Assemble the element equations to determine the system equation.

7. Adjust the system equation to account for the boundary conditions.

8. Solve the adjusted system equation for the nodal values $C_{i,j}$.

4.1.1 Domain Discretization and the Interpolating Polynomials

In this section, we'll discretize the solution domain Ω (x, y) illustrated in Figure 4.1(a) into rectangular elements, as illustrated in Figure 4.1(b),4.1(c)[29].

The global solution domain Ω(x, y) is covered by a two-dimensional grid of lines. There are I lines perpendicular to the x axis, which are denoted by the subscript i. There are J lines perpendicular to the y axis, which are denoted by the subscript j. There are $(I-1)\times(J-1)$ elements, which are denoted by the superscript (i,j). Element(i,j) starts at node i,j and ends at node i+1, j+1.

Let the global exact solution U(x,y) be approximated by the global approximate solution $U^*(x, y)$, which is the sum of a series of local interpolating polynomials $U^{*(i,j)}(x, y)$ (i =1,2 ...,I-1, j =1,2...,J-1) that are valid within each element. Thus,

$$U^*(x, y) = \sum_{i=1}^{I-1}\sum_{j=1}^{J-1} U^{*(i,j)}(x, y) \tag{4.2}$$

Let's define the local interpolating polynomial $U^{*(i,j)}(x, y)$ as a linear bivariate polynomial. Element (i,j) is illustrated in Figure 4.1(c). Let's use a local coordinate system, where node i,j is at (0.0), node i+1,j is at ($\triangle x$,0), etc. Denote the grid points as 1,2,3 and 4. The linear interpolating polynomial, $U^{*(i,j)}(x, y)$, corresponding to element(i,j) is given by:

$$U^{*(i,j)}(x, y) = U_1^* N_1(x, y) + U_2^* N_2(x, y) + U_3^* N_3(x, y) + U_4^* N_4(x, y) \tag{4.3}$$

where U_1^* , U_2^*, U_3^*, U_4^* are the values of $U^*(x, y)$ at nodes 1, 2, 3, 4 respectively, and $N_1(x, y), N_2(x, y), N_3(x, y), N_4(x, y)$ are linear interpolating polynomials within element(i,j). The interpolating polynomials $, N_1(x, y), N_2(x, y), N_3(x, y),$ $N_4(x, y)$ are called **shape functions**. The subscripts of the shape functions denote the node at which the corresponding shape function is equal to unity. The shape function is defined to be zero at the other three nodes and zero every where outside of the element. Since the element approach is being used, only one element is involved. Thus, the superscript (i,j) identifying the element will be omitted for clarity. Next,let's develop the expressions for the shape functions $N_1(x, y), N_2(x, y)$, etc. First, consider $N_1(x, y)$:

$$N_1(x, y) = a_0 + a_1 x^* + a_2 y^* + a_3 x^* y^* \tag{4.4}$$

where x^* and y^* are normalized values of x and y, respectively, that is, $x^* = x/\triangle x$ and $y^* = y/\triangle y$. Introducing the values of $N_1(x, y)$ at the four nodes into Eq. (4.4) gives:

$$N_1(0, 0) = 1.0 = a_0 + a_1(0) + a_2(0) + a_3(0)(0) = a_0$$

$$N_1(1, 0) = 0.0 = a_0 + a_1(1) + a_2(0) + a_3(1)(0) = a_0 + a_1$$

$$N_1(0,1) = 0.0 = a_0 + a_1(0) + a_2(1) + a_3(0)(1) = a_0 + a_2$$

$$N_1(1,1) = 0.0 = a_0 + a_1(1) + a_2(1) + a_3(1)(1) = a_0 + a_1 + a_2 + a_3$$

Let $a^T = [a_0, a_1, a_2, a_3]$. Solving by Gauss elimination yields $a^T =$ [1 -1 -1 0] . Thus, $N_1(x, y)$ is given by

$$N_1(x,y) = 1 - x^* - y^* + x^*y^* \tag{4.5}$$

In a similar manner, it is found that

$$N_2(x,y) = x^* - x^*y^*$$

$$N_3(x,y) = x^*y^*$$

$$N_4(x,y) = y^* - x^*y^*$$

Then Eq.(4.3) becomes

$$U^{*(i,j)}(x,y) = U_1^*(1 - x^* - y^* + x^*y^*) + U_2^*(x^* - x^*y^*) + U_3^*(x^*y^*) + U_4^*(y^* - x^*y^*) \tag{4.6}$$

Equation(4.6) is the interpolating polynomial for a rectangular element.

4.1.2 The Galerkin Weighted Residual Approach

The Galerkin weighted residual approach is applied in this study to develop a finite element approximation of the 2D Poisson equation, Eq.(4.1)[28]

$$U_{xx} + U_{yy} = f(x,y) \tag{4.7}$$

Let's approximate the exact solution U(x,y) by the approximate solution $U^*(x, y)$ given Eq. (4.2). Substituting U(x,y) into Eq. (4.7)gives the residual R(x,y):

$$R(x,y) = U_{xx}^* + U_{yy}^* - f \tag{4.8}$$

The residual R(x,y) is multiplied by a set of weighting functions $W_k(x, y)$ (k =1, 2, ...) and integrated over the global solution domain Ω(x, y) to obtain the weighted residual integral $I(U^*(x, y))$, which is equated to zero. Consider the general weighting function W(x,y). Then

$$I(U^*(x,y)) = \int\int_{\Omega} W(U^*_{xx} + U^*_{yy} - f)dxdy \tag{4.9}$$

The first two terms in Eq. (4.9) can be integrated by parts. Thus,

$$WU^*_{xx} = W(U^*_x)_x = (WU^*_x)_x - W_x U^*_x \tag{4.10}$$

$$WU^*_{yy} = W(U^*_y)_y = (WU^*_y)_y - W_y U^*_y \tag{4.11}$$

Substituting Eq. (4.10) and (4.11) into Eq. (4.9) gives :

$$I(U^*(x,y)) = \int\int_{\Omega} [(WU^*_x)_x + (WU^*_y)_y - W_y U^*_y - W_x U^*_x - Wf]dxdy \tag{4.12}$$

But,

$$\int\int_{\Omega} (WU^*_x)_x dxdy = \oint_B WU^*_x n_x ds$$
$$\int\int_{\Omega} (WU^*_y)_y dxdy = \oint_B WU^*_x n_x ds$$

where the line integrals are evaluated around the outer boundary B of the global solution domain Ω(x,y) and n_x and n_y are the components of the unit normal vector to the outer boundary of **n**. Note that the flux of $U^*(x,y)$ crossing the outer boundary B of the global solution domain Ω(x, y) is given by

$$q_n = n. \triangle U^* \quad = \quad n_x U^*_x + n_y U^*_y \tag{4.13}$$

then eq.(4.12) becomes:

$$I(U^*(x,y)) = -\int\int_{\Omega} [(W_x U^*_x + W_y U^*_y + Wf]dxdy + \oint_B Wq_n ds \tag{4.14}$$

The line integral in Eq.(4.14) specifies the flux q_n normal to the outer boundary of the global solution domain Ω(x,y). For all interior elements which do not coincide with a portion of the outer boundary, these fluxes cancel out when all the interior elements are assembled. For any element which has a side coincident with a portion of the outer boundary, the line integral expresses the boundary condition on that side. For Dirichlet boundary conditions, $U^*(x,y)$ is specified on the boundary, and the line integral is not needed. In terms of the global approximate solution $U^*(x,y)$ and the discretized global solution domain illustrated in Figure. 4.1(b), Eq. (4.14) can be written as follows [29]:

$$I(U^*(x,y)) = I^{(1,1)}(U^*(x,y))+...+I^{(i,j)}(U^*(x,y))+...+I^{(I-1,J-1)}(U^*(x,y))+\oint_B Wq_n ds \tag{4.15}$$

where $I^{(i,j)}(U^*(x,y))$ is given by

$$I^{(i,j)}(U^*(x,y)) = -\int\int_\Omega [W_x U_x^* + W_y U_y^* + Wf]dxdy \tag{4.16}$$

where $U^*(x,y)$ is the approximate solution given by Eq. (4.6) W(x,y) is an unspecified weighting function. The evaluation of the weighed residual integral, Eq. (4.16), requires the function $U^*(x,y)$ and its partial derivatives with respect to x and y. From Eqs. (4.6)

$$U^{*(i,j)}(x,y) = U_1^*(1-x^*-y^*+x^*y^*)+U_2^*(x^*-x^*y^*)+U_3^*(x^*y^*)+U_4^*(y^*-x^*y^*) \tag{4.17}$$

Differentiating Eq. (4.17) with respect to x and y gives

$$U_x^* = U_1^*(-\frac{1}{\triangle x}+\frac{y^*}{\triangle x}) + U_2^*(\frac{1}{\triangle x}-\frac{y^*}{\triangle x}) + U_3^*(\frac{y^*}{\triangle x}) + U_4^*(-\frac{y^*}{\triangle x}) \tag{4.18}$$

$$U_y^* = U_1^*(-\frac{1}{\triangle y}+\frac{x^*}{\triangle y}) + U_2^*(-\frac{x^*}{\triangle y}) + U_3^*(\frac{x^*}{\triangle y}) + U_4^*(\frac{1}{\triangle y}-\frac{x^*}{\triangle y}) \tag{4.19}$$

Substituting Eq. (4.18) and Eq. (4.19) into Eq. (4.16) yields

$$I^{(i,j)}(U^*(x,y)) = \int\int_\Omega [W_x\frac{1}{\triangle x}[U_1^*(-1+y^*)+U_2^*(1-y^*)+U_3^*(y^*)+U_1^*(-y^*)]dxdy$$

$$-\int\int_\Omega [W_y\frac{1}{\triangle y}[U_1^*(-1+x^*)+U_2^*(-x^*)+U_3^*(x^*)+U_1^*(1-x^*)]dxdy$$

$$-\int\int_\Omega Wf]dxdy = 0 \tag{4.20}$$

where the superscript (i,j) has been dropped from I for simplicity. In the Galerkin weighed residual approach, the weighting factors $W_k(x,y)$ (k =1,2,...) are chosen to be the shape functions $N_1(x,y), N_2(x,y)$, etc., specified by Eq. (4.5). Let's evaluate Eq. (4.20) W(x,y) = $N_1(x,y)$. From Eq. (4.5),

$$W_1(x,y) = N_1(x,y) = 1 - x^* - y^* + x^*y^* \tag{4.21}$$

Differentiating Eq. (4.21) with respect to x and y gives

$$(W_1)_x = (-\frac{1}{\triangle x} + \frac{y^*}{\triangle x}) \qquad and \qquad (W_1)_y = (-\frac{1}{\triangle y} + \frac{x^*}{\triangle y}) \tag{4.22}$$

Substituting Eqs. (4.21) and (4.22) into Eq. (4.20) yields

$$I^{(i,j)}(U^*(x,y)) = -\frac{1}{\triangle x^2}\int\int_\Omega (-1+y^*)[U_1^*(-1+y^*)+U_2^*(1-y^*)+U_3^*(y^*)+U_1^*(-y^*)]dxdy$$

$$-\frac{1}{\triangle y^2}\int\int_\Omega (-1+x)[U_1^*(-1+x^*)+U_2^*(-x^*)+U_3^*(x^*)+U_1^*(1-x^*)]dxdy$$

$$-\int\int_\Omega (1-x^*-y^*+x^*y^*)f]dxdy = 0 \tag{4.23}$$

Recall that $x^* = x/\triangle x$, and $y^* = y/\triangle y$. Thus, $dx = \triangle x dx^*$ and $dy = \triangle y dy^*$. Thus, Eq.(4.23) can be written as

$$I^{(i,j)}(U^*(x,y)) = \int_0^1 [\int_0^1 (...) \triangle x dx^*] \triangle y dy^* \tag{4.24}$$

where the integrand of the inner integral, denoted as (...), is obtained from Eq. (4.23). Evaluating the inner integral in Eq. (4.24) gives

$$(...) = -\triangle x \int_0^1 \frac{1}{\triangle x^2}[U_1^*(1-2y^*+y^{*2}) + U_2^*(-1+2y^*-y^{*2})]$$

$$+[U_3^*(-y^*+y^{*2}) + U_4^*(y^*-y^{*2})]dx^*$$

$$-\triangle x \int_0^1 \frac{1}{\triangle y^2}[U_1^*(1-2x^*+x^{*2}) + U_2^*(x^*-x^{*2})]$$

$$+[U_3^*(-x^*+x^{*2}) + U_4^*(-1+2x^*-x^{*2})]dx^*$$

$$-\triangle x \int_0^1 [(1-x^*-y^*+x^*y^*)f dx \tag{4.25}$$

Let $f^*(x,y)$ denote the average value of f(x,y) in the element. Integrating Eq. (4.25) gives

$$(...) = -\frac{\triangle x}{\triangle x^2}[U_1^*(1-2y^*+y^{*2})+U_2^*(-1+2y^*-y^{*2})]+[U_3^*(-y^*+y^{*2})+U_4^*(y^*-y^{*2})]x^*|_0^1$$

$$-\frac{\triangle x}{\triangle y^2}[U_1^*(x^*-x^{*2}+\frac{x^{*3}}{3})+U_2^*(\frac{x^{*2}}{2}-\frac{x^{*3}}{3})]+[U_3^*(\frac{-x^{*2}}{2}+\frac{x^{*3}}{3})+U_4^*(-x^*+x^{*2}-\frac{x^{*3}}{3})]|_0^1$$
$$-\triangle x f^*[(x^*-\frac{x^{*2}}{2}-y^*x^*+\frac{x^{*2}}{2}y^*)]|_0^1 \tag{4.26}$$

Evaluating Eq. (4.26) yields

$$(...)=-\frac{\triangle x}{\triangle x^2}[U_1^*(1-2y^*+y^{*2})+U_2^*(-1+2y^*-y^{*2})]+[U_3^*(-y^*+y^{*2})+U_4^*(y^*-y^{*2})]$$
$$-\frac{\triangle x}{\triangle y^2}[\frac{1}{3}U_1^*+\frac{1}{6}U_2^*-\frac{1}{6}U_3^*-\frac{1}{3}U_4^*]-\frac{\triangle x^* f^*}{2}(1-y^*) \tag{4.27}$$

Substituting Eq. (4.27) into Eq. (4.24) gives

$$I(U^*(x,y))=-\frac{\triangle x\triangle y}{\triangle x^2}\int_0^1[U_1^*(1-2y^*+y^{*2})+U_2^*(-1+2y^*-y^{*2})+$$
$$[U_3^*(-y^*+y^{*2})+U_4^*(y^*-y^{*2})]dy^*-\frac{\triangle x\triangle y}{\triangle y^2}\int_0^1[\frac{1}{3}U_1^*+\frac{1}{6}U_2^*-\frac{1}{6}U_3^*-\frac{1}{3}U_4^*]dy^*$$
$$-\triangle x\triangle y\int_0^1\frac{f^*}{2}(1-y^*)dy^* \tag{4.28}$$

Integrating Eq. (4.28) gives

$$I(U^*(x,y))=-\frac{\triangle x\triangle y}{\triangle y^2}[U_1^*(y^*-y^{*2}+\frac{y^{*3}}{3})+U_2^*(-y^*+y^{*2}-\frac{y^{*3}}{3})$$
$$+U_3^*(\frac{-y^{*2}}{2}+\frac{y^{*3}}{3})+U_4^*(\frac{y^{*2}}{2}\frac{-y^{*3}}{3})]|_0^1$$
$$-\frac{\triangle x\triangle y}{\triangle y^2}[\frac{1}{3}U_1^*+\frac{1}{6}U_2^*-\frac{1}{6}U_3^*-\frac{1}{3}U_4^*]y^*|_0^1$$
$$\frac{-\triangle x\triangle y f^*}{2}(y^*-\frac{y^{*2}}{2})|_0^1 \tag{4.29}$$

Evaluating Eq. (4.29) and collecting terms yields

$$\frac{-1}{3}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_1^*+\frac{1}{6}(\frac{2\triangle y}{\triangle x}-\frac{\triangle x}{\triangle y})U_2^*$$

$$+\frac{1}{6}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_3^*+\frac{1}{6}(\frac{-\triangle y}{\triangle x}+\frac{2\triangle x}{\triangle y})U_4^*-\frac{\triangle x\triangle y f^*}{4}=0 \tag{4.30}$$

Repeating the steps in Eqs. (4.20) to (4.30) for $W_2=N_2, W_3=N_3, and W_4=N_4$ yields the following results:

$$\frac{1}{6}(2\frac{\triangle y}{\triangle x}-\frac{\triangle x}{\triangle y})U_1^*-\frac{1}{3}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_2^*$$

$$+\frac{1}{6}(-\frac{\triangle y}{\triangle x}+2\frac{\triangle x}{\triangle y})U_3^*+\frac{1}{6}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_4^*-\frac{\triangle x\triangle yf^*}{4}=0 \quad (4.31)$$

$$\frac{1}{6}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_1^*+\frac{1}{6}(-\frac{\triangle y}{\triangle x}+\frac{2\triangle x}{\triangle y})U_2^*$$
$$\frac{-1}{3}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_3^*+\frac{1}{6}(\frac{2\triangle y}{\triangle x}-\frac{\triangle x}{\triangle y})U_4^*-\frac{\triangle x\triangle yf^*}{4}=0 \quad (4.32)$$

$$\frac{1}{6}(\frac{-\triangle y}{\triangle x}+\frac{2\triangle x}{\triangle y})U_1^*+\frac{1}{6}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_2^*$$
$$+\frac{1}{6}(2\frac{\triangle y}{\triangle x}-\frac{\triangle x}{\triangle y})U_3^*-\frac{1}{3}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_4^*-\frac{\triangle x\triangle yf^*}{4}=0 \quad (4.33)$$

Equations(4.30) to (4.33) are the element equations for element (i,j) for $\triangle x \neq \triangle y$. The next step is to assemble the element equations, Eqs. (4.30) to (4.33), to obtain the nodal equation for node i,j. This process is considerably more complicated for two and three-dimensional problems than for one-dimensional problems because each node belongs to several elements. Consider the portion of the discretized global solution domain which surrounds node i,j, which is illustrated in Figure 4.1. Note that $\triangle x_- = (x_i - x_{i-1}) \neq \triangle x_+ = (x_{i+l} - x_i), and \triangle y_- = (y_i - y_{i-1}) \neq \triangle y_+ = (y_{i+1} - y_i)$. These differences in the grid increments must be accounted for while assembling the equations for four different elements using the element equations derived for a single element. Also note that the average value of f(x,y), denoted by $f^*(x,y)$, can be different in the four elements surrounding node i,j. Local node 0 is surrounded by local elements(1), (2), (3), and (4). The assembled nodal equation for node 0 is obtained by combining all of the element equations for elements(1), (2), (3), and (4), respectively, which correspond to the shape functions associated with node 0. Figure 4.2 illustrates the process. Figure 4.3(a) illustrates the basic element used to derive Eqs. 4.3(a) to 4.3(d). Figures 4.3(b) to 4.3(e) illustrate elements(1) to (4), respectively, from Figure 4.2. Consider element (1) illustrated in Figure 4.3(b) [29].

Node 0 in element (1) corresponds to node 3 in the basic element. Thus, the element equation corresponding to node 3, Eq. (4.32), is part of the nodal equation for node

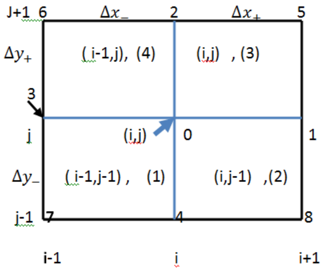
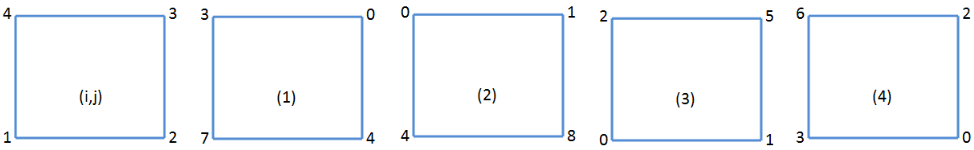


Figure 4.2: Portion of global grid surrounding node i,j.



(a) (b) (c) (d) (e)

Figure 4.3: Element correspondence

Renumbering the function values in Eq.(4.32) to correspond to the nodes in Figure 4.3(b) yields:

Element (1):Equation (4.32) with $\triangle x = \triangle x_-$ and $\triangle y = \triangle y_-$

$$\frac{1}{6}\left(\frac{\triangle y_-}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_-}\right)U_7^* + \frac{1}{6}\left(-\frac{\triangle y_-}{\triangle x_-} + \frac{2\triangle x_-}{\triangle y_-}\right)U_4^*$$

$$\frac{-1}{3}\left(\frac{\triangle y_-}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_-}\right)U_0^* + \frac{1}{6}\left(\frac{2\triangle y_-}{\triangle x_-} - \frac{\triangle x_-}{\triangle y_-}\right)U_3^* - \frac{\triangle x_- \triangle y_- f^{*(1)}}{4} = 0 \qquad (4.34)$$

Repeating the process for elements (2) to (4) show that Eq. (4.33) corresponds element(2), Eq. (4.30) corresponds to element (3), and Eq. (4.31) corresponds element (4). Renumbering the function values in those equations to agree with the node numbers in Figure 4.3 yields the remaining three element equations corresponding to node 0.

Element (2): Equation (4.33) with $\triangle x = \triangle x_+$ and $\triangle y = \triangle y_-$

$$\frac{1}{6}\left(\frac{-\triangle y_-}{\triangle x_+} + \frac{2\triangle x_+}{\triangle y_-}\right)U_4^* + \frac{1}{6}\left(\frac{\triangle y_-}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_-}\right)U_8^*$$

$$+\frac{1}{6}\left(2\frac{\triangle y_-}{\triangle x_+} - \frac{\triangle x_+}{\triangle y_-}\right)U_1^* - \frac{1}{3}\left(\frac{\triangle y_-}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_-}\right)U_0^* - \frac{\triangle x_+ \triangle y_- f^{*(2)}}{4} = 0 \qquad (4.35)$$

Element (3):Equation (4.30) with $\triangle x = \triangle x_+$ and $\triangle y = \triangle y_+$

$$\frac{-1}{3}(\frac{\triangle y_+}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_+})U_0^* + \frac{1}{6}(\frac{2\triangle y_+}{\triangle x_+} - \frac{\triangle x_+}{\triangle y_+})U_1^*$$

$$+\frac{1}{6}(\frac{\triangle y_+}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_+})U_5^* + \frac{1}{6}(\frac{-\triangle y_+}{\triangle x_+} + \frac{2\triangle x_+}{\triangle y_+})U_2^* - \frac{\triangle x_+ \triangle y_+ f^{*(3)}}{4} = 0 \qquad (4.36)$$

Element(4):Equation (4.31) with $\triangle x = \triangle x_-$ and $\triangle y = \triangle y_+$

$$\frac{1}{6}(2\frac{\triangle y_+}{\triangle x_-} - \frac{\triangle x_-}{\triangle y_+})U_3^* - \frac{1}{3}(\frac{\triangle y_+}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_+})U_0^*$$

$$+\frac{1}{6}(-\frac{\triangle y_+}{\triangle x_-} + 2\frac{\triangle x_-}{\triangle y_+})U_2^* + \frac{1}{6}(\frac{\triangle y_+}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_+})U_6^* - \frac{\triangle x_- \triangle y_+ f^{*(4)}}{4} = 0 \qquad (4.37)$$

Summing Eqs. (4.34) to (4.37) yields the nodal equation for node 0:

$$\begin{aligned}
U_0^*[&-\frac{1}{3}(\frac{\triangle y_-}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_-}) - \frac{1}{3}(\frac{\triangle y_-}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_-}) - \frac{1}{3}(\frac{\triangle y_+}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_+}) - \frac{1}{3}(\frac{\triangle y_+}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_+})] \\
&+U_1^*[\frac{1}{6}(\frac{2\triangle y_-}{\triangle x_+} - \frac{\triangle x_+}{\triangle y_-}) + \frac{1}{6}(\frac{2\triangle y_+}{\triangle x_+} - \frac{\triangle x_+}{\triangle y_+})] \\
&+U_2^*[\frac{1}{6}(\frac{-\triangle y_+}{\triangle x_+} + \frac{2\triangle x_+}{\triangle y_+}) + \frac{1}{6}(\frac{-\triangle y_+}{\triangle x_-} + \frac{2\triangle x_-}{\triangle y_+})] \\
&+U_3^*[\frac{1}{6}(\frac{2\triangle y_-}{\triangle x_-} - \frac{\triangle x_-}{\triangle y_-}) + \frac{1}{6}(\frac{2\triangle y_+}{\triangle x_-} - \frac{\triangle x_-}{\triangle y_+})] \\
&+U_4^*[\frac{1}{6}(\frac{-\triangle y_-}{\triangle x_-} + \frac{2\triangle x_-}{\triangle y_-}) + \frac{1}{6}(\frac{-\triangle y_-}{\triangle x_+} + \frac{2\triangle x_+}{\triangle y_-})] \\
&+U_5^*[\frac{1}{6}(\frac{\triangle y_+}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_+})] + U_6^*[\frac{1}{6}(\frac{\triangle y_+}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_+})] \\
&+U_7^*[\frac{1}{6}(\frac{\triangle y_-}{\triangle x_-} + \frac{\triangle x_-}{\triangle y_-})] + U_8^*[\frac{1}{6}(\frac{\triangle y_-}{\triangle x_+} + \frac{\triangle x_+}{\triangle y_-})] - \frac{1}{4}[\triangle x_- \triangle y_- f^{*(1)} \\
&+\triangle x_+ \triangle y_- f^{*(2)} + \triangle x_+ \triangle y_+ f^{*(3)} + \triangle x_- \triangle y_+ f^{*(4)}] = 0 \qquad (4.38)
\end{aligned}$$

Let $\triangle x = \triangle x_+ = \triangle x_-$ and $\triangle y = \triangle y_- = \triangle y_+$, multiply through by 3, and collect terms.

$$-4(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})U_0^*+(\frac{2\triangle y}{\triangle x}-\frac{\triangle x}{\triangle y})U_1^*+(2\frac{\triangle x}{\triangle y}-\frac{\triangle y}{\triangle x})U_2^*+(2\frac{\triangle y}{\triangle x}-\frac{\triangle x}{\triangle y})U_3^*+(2\frac{\triangle x}{\triangle y}-\frac{\triangle y}{\triangle x})U_4^*$$

$$+\frac{1}{2}(\frac{\triangle y}{\triangle x}+\frac{\triangle x}{\triangle y})(U_5^*+U_6^*+U_7^*+U_8^*)-\frac{\triangle x \triangle y(f^{*(1)}+f^{*(2)}+f^{*(3)}+f^{*(4)})}{4} = 0 \quad (4.39)$$

For $\triangle x = \triangle y = \triangle L$,Eq.(4.39) yields

$$-8U_0^* + (U_1^* + U_2^* + U_3^* + U_4^* + U_5^* + U_6^* + U_7^* + U_8^*) - \triangle L^2 f = 0 \tag{4.40}$$

In terms of the i,j notation, Eq.(4.40) becomes[29]

$$-8U_{ij}^* + (U_{i+1j+1}^* + U_{i+1j}^* + U_{i+1j-1}^* + U_{ij+1}^* + U_{ij-1}^* + U_{i-1j+1}^* + U_{i-1j}^* + U_{i-1j-1}^*) - \triangle L^2 f_{ij} = 0 \tag{4.41}$$

The Eq.(4.41) is the assembled elements , which gives system of linear equation and we solve it by Thomas algorithms.

4.2 Error Analysis for Finite Element Method

Consider the following 2D Poisson equation with Dirichlet boundary condition: [30]

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x,y) \quad (x,y) \in \Omega \tag{4.42}$$

We start with the variational formulation for the problem (4.42), through multiplying the equation by a test function, integrating over Ω and using the Green's formula: Find a solution U(x,y) such that U(x,y) = 0 on Γ = ∂Ω an

$$(VF) : \int_\Omega \bigtriangledown U \bigtriangledown v dxdy = \int_\Omega v f dxdy, \ \forall v \ such \quad V = 0 \ on \ \Gamma \tag{4.43}$$

We prepare for a finite element method where we shall approximate the exact solution U(x,y) by a suitable discrete solution $U^*(x,y)$. To this approach let T = K : ⋃K =Ω be a rectangular of the domain Ω and φ_j, j = 1,2, ...,n be the corresponding basis functions, such that $\varphi_j(x)$ is continuous, linear in x on each K and

$$\varphi_j(N_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{4.44}$$

where $N_1, N_2, ..., N_n$ are the inner nodes in the rectangular. Now we set the approximate solution $U^*(x,y)$ to be a linear combination of the basis functions φ_j,j=1,...,n:

$$U^*(x,y) = U_1^*\varphi_1(x,y) + U_2^*\varphi_2(x,y) + U_3^*\varphi_3(x,y) + ... + U_n^*\varphi_n(x,y) \tag{4.45}$$

and seek the coefficients $U_j^* = U^*(N_j)$, i.e., the nodal values of $U^*(x, y)$, at the nodes N_j, $1 \leq i \leq n$, so that

$$(FEM) : \int_\Omega \triangledown U^* \triangledown \varphi_i dxdy = \int_\Omega \varphi_i.f dxdy, \quad i = 1, 2, ...n, \tag{4.46}$$

or equivalently

$$(V_h^0) : \int_\Omega \triangledown U^* \triangledown v dxdy = \int_\Omega v.f dxdy, \quad \forall v \in V_h^0 \tag{4.47}$$

Where V_h^0={v(x):v is continuous, piecewise linear(on τ), and v=0 on $\Gamma = \partial\Omega$}. Note that every $v \in V_h^0$ can be represented by [30]

$$v(x, y) = v(N_1)\varphi_1(x, y) + v(N_2)\varphi_2(x, y) + v(N_3)\varphi_3(x, y) + ... + v(N_n)\varphi_n(x, y) \tag{4.48}$$

Theorem 1:(A priori error estimate for the solution $e = U - U^*$)

For a general mesh we have the following a priori error estimate for the solution of the Poisson equation (4.42):[30]

$$\|e\| = \|(U - U^*)\| \leq C^2 C_\Omega (\max_\Omega h)\|hD^2U\| \tag{4.49}$$

Proof: Let

$$\| \triangledown e\| = \| \triangledown (U - U^*)\| \leq C\|hD^2U\| \tag{4.50}$$

which is indicating that the error is small if h(x) is sufficiently small depending on D^2u and let φ be the solution of the dual problem

$$\begin{cases} - \triangle \varphi = e & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{4.51}$$

Then we have Green's formula

$$\|e\|^2 = \int_\Omega e.(- \triangledown \varphi) dxdy = \int_\Omega \triangledown e. \triangledown \varphi dxdy \tag{4.52}$$

$$\int_\Omega \triangledown e. \triangledown (\varphi - v) dxdy \leq \| \triangledown e\|.\| \triangledown (\varphi - v)\| \;\; \forall v \in V_h^0,$$

where in the last equality we have used the Galerkin orthogonality. We now choose v such that

$$\| \bigtriangledown (\varphi - v)e\| \leq C\|h.D^2\varphi C\| \leq C(\max_{\Omega} h)\|h.D^2\varphi C\| \tag{4.53}$$

Assume that

$$\|D^2\varphi\| \leq C_\Omega\| \bigtriangleup \varphi\| = C_\Omega\|e\| \tag{4.54}$$

Now (4.50)-(4.54) implies that

$$\|e\|^2 \leq \| \bigtriangledown e\|.\| \bigtriangledown (\varphi - v)\| \leq \| \bigtriangledown e\|.C(\max_{\Omega} h)\|D^2\varphi\| \tag{4.55}$$

$$\leq \| \bigtriangledown e\|.C(\max_{\Omega} h)C_\Omega\|e\| \leq C^2C_\Omega \max_{\Omega} h\|e\|.\|hD^2U\|$$

Thus we have obtained the desired result: a priori error estimate:

$$\|e\| = \|(U - U^*)\| \leq C^2C_\Omega(\max_{\Omega} h)\|hD^2U\| \tag{4.56}$$

4.3 Higher Order Compact Finite Difference Method for 2D Poisson equation

In the high-order compact difference methods, the spatial derivatives in the governing PDEs are not approximated directly by some finite differences. They are evaluated by some compact difference schemes. The compact finite difference method (CFDM) is one of the methods used to increase the accuracy of the numerical solutions of the partial differential equation. Therefore, we can use many methods to derive the compact finite difference scheme for any problem. In [10] use the Taylor series to drive a fourth and sixth compact finite difference for 2D and 3D Poisson equation. In [3] derive a fourth order compact finite difference for 2D Poisson equation by using compact operators method.

In this thesis we use the Taylor series expansion to derive the eight order compact finite difference for 2D Poisson equation.

Consider Poisson equation of the form:

$$U_{xx} + U_{yy} = f(x,y) \qquad (x,y) \in \Omega \tag{4.57}$$

Where Ω is a rectangular domain, or a union of rectangular domains, with suitable boundary condition defined on $\partial\Omega$. The solution U(x,y) and the forcing function f(x,y) are assumed to be sufficiently smooth and have required continuous partial derivatives. For convenience, let us consider a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$. Here subscripts are obviously not derivatives. We discretize Ω with uniform mesh sizes $\triangle x$ and $\triangle y$ respectively in the x and y coordinate directions. Denote $N_x = L_x / \triangle x$ and $N_y = L_y / \triangle y$ the numbers of uniform intervals along the x and y coordinate directions, respectively. The mesh point are (x_i, y_j) with $x_i = i \triangle x$ and $y_j = j \triangle y$, $0 \le i \le N_x, 0 \le j \le N_y$. In the sequel, we may also use the index pair (i,j) to represent the mesh point (x_i, y_j). In this thesis we take $\triangle x = \triangle y = h$ .also $N_x = N_y = N$ and consider the figure 4.1(b).

By using Taylor series expression, we have:[22]

$$U_{i+1j} = Uij + h\frac{\partial U_{ij}}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 U_{ij}}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 U_{ij}}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 U_{ij}}{\partial x^4} + \frac{h^5}{5!}\frac{\partial^5 U_{ij}}{\partial x^5} + \frac{h^6}{6!}\frac{\partial^6 U_{ij}}{\partial x^6} + \frac{h^7}{7!}\frac{\partial^7 U_{ij}}{\partial x^7} + \ldots \tag{4.58}$$

$$U_{i-1j} = Uij - h\frac{\partial U_{ij}}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 U_{ij}}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 U_{ij}}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 U_{ij}}{\partial x^4} - \frac{h^5}{5!}\frac{\partial^5 U_{ij}}{\partial x^5} + \frac{h^6}{6!}\frac{\partial^6 U_{ij}}{\partial x^6} - \frac{h^7}{7!}\frac{\partial^7 U_{ij}}{\partial x^7} + \ldots \tag{4.59}$$

$$U_{ij+1} = Uij + h\frac{\partial U_{ij}}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 U_{ij}}{\partial y^2} + \frac{h^3}{3!}\frac{\partial^3 U_{ij}}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 U_{ij}}{\partial y^4} + \frac{h^5}{5!}\frac{\partial^5 U_{ij}}{\partial y^5} + \frac{h^6}{6!}\frac{\partial^6 U_{ij}}{\partial y^6} + \frac{h^7}{7!}\frac{\partial^7 U_{ij}}{\partial y^7} + \ldots \tag{4.60}$$

$$U_{ij-1} = Uij - h\frac{\partial U_{ij}}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 U_{ij}}{\partial y^2} - \frac{h^3}{3!}\frac{\partial^3 U_{ij}}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 U_{ij}}{\partial y^4} - \frac{h^5}{5!}\frac{\partial^5 U_{ij}}{\partial y^5} + \frac{h^6}{6!}\frac{\partial^6 U_{ij}}{\partial y^6} - \frac{h^7}{7!}\frac{\partial^7 U_{ij}}{\partial y^7} + \ldots \tag{4.61}$$

Subtracting Eq. (4.59) from Eq. (4.58) and Eq. (4.61) from Eq. (4.60), we obtain the second order finite difference $(\delta^1 U(ij))$ for the first derivative of U_{ij} :

$$\left.\begin{aligned} \delta_x^1 U_{ij} &= \frac{U_{i+1j} - U_{i-1j}}{2h} + O(h^2) \\ \delta_y^1 U_{ij} &= \frac{U_{ij+1} - U_{ij-1}}{2h} + O(h^2) \end{aligned}\right\} \qquad 4.61a$$

Similarly, adding Eq. (4.58) with Eq. (4.59) and Eq. (4.60) with Eq. (4.61), we obtain the second order finite difference $(\delta^2 U_{ij})$ for the first derivative of U_{ij} :

$$\left.\begin{aligned}\delta_x^2 U_{ij} &= \frac{U_{i+1j} - 2U_{ij} + U_{i-1j}}{h^2} + O(h^2)\\ \delta_y^2 U_{ij} &= \frac{U_{ij+1} - 2U_{ij} + U_{ij-1}}{h^2} + O(h^2)\end{aligned}\right\} \tag{4.62}$$

Using from Eq.(4.58)-Eq.(4.61) we can approximated to second order accuracy as

$$\left.\begin{aligned}(\frac{\partial^2 U}{\partial x^2})_{ij} &= \delta_x^2 U_{ij} - \frac{h^2}{12}\frac{\partial^4 U}{\partial x^4} - \frac{h^4}{360}\frac{\partial^6 U}{\partial x^6} - \frac{h^6}{20160}\frac{\partial^8 U}{\partial x^8} + O(h^8)\\ (\frac{\partial^2 U}{\partial y^2})_{ij} &= \delta_y^2 U_{ij} - \frac{h^2}{12}\frac{\partial^4 U}{\partial y^4} - \frac{h^4}{360}\frac{\partial^6 U}{\partial y^6} - \frac{h^6}{20160}\frac{\partial^8 U}{\partial x^8} + O(h^8)\end{aligned}\right\} \tag{4.63}$$

Using the finite difference operators in (4.62) and (4.63), Eq.(4.57) can be discretized at a given grid point (x_i, y_i) as

$$\delta_x^2 Uij + \delta_y^2 Uij - \tau_{ij} = f_{ij} \tag{4.64}$$

Where the truncation error is

$$\tau_{ij} = \frac{h^2}{12}(\frac{\partial^4 U}{\partial x^4} + \frac{\partial^4 U}{\partial y^4})_{ij} + \frac{h^4}{360}(\frac{\partial^6 U}{\partial x^6} + \frac{\partial^6 U}{\partial y^6})_{ij} + \frac{h^6}{20160}(\frac{\partial^8 U}{\partial x^8} + \frac{\partial^8 U}{\partial y^8})_{ij} + O(h^8) \tag{4.65}$$

Equation (4.64) is called a high Central Difference Scheme (CDS). We have include both $O(h^2)$ and $O(h^4)$ term in Eq.(4.65) because we wish to approximate all of them in order to construct an $O(h^6)$ scheme[22].

To obtain compact approximation to the $O(h^2)$ terms in Eq.(4.65), we simply take the appropriate derivatives of Eq.(4.57),

$$\frac{\partial^4 U}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} - \frac{\partial^4 U}{\partial x^2 \partial y^2} \tag{4.66}$$

$$\frac{\partial^4 U}{\partial y^4} = \frac{\partial^2 f}{\partial y^2} - \frac{\partial^4 U}{\partial x^2 \partial y^2} \tag{4.67}$$

Substituting Eq.(4.66) and Eq.(4.67) into Eq.(4.65) yields

$$\tau_{ij} = \frac{h^2}{12}(\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^2 f}{\partial y^2})_{ij} + \frac{h^4}{360}(\frac{\partial^6 U}{\partial x^6} + \frac{\partial^6 U}{\partial y^6})_{ij} + \frac{h^6}{20160}(\frac{\partial^8 U}{\partial x^8} + \frac{\partial^8 U}{\partial y^8})_{ij} + O(h^8) \tag{4.68}$$

Note that the term on the right hand side of Eq.(4.68) have compact $O(h^2)$ approximations at noted i j, and the approximation of these terms has the following form:

$$(\frac{\partial^4 U}{\partial x^2 \partial y^2})_{ij} = \delta_x^2 \delta_y^2 U_{ij} - \frac{h^2}{12}(\frac{\partial^6 U}{\partial x^4 \partial y^2} + \frac{\partial^6 U}{\partial x^2 \partial y^4})_{ij} - \frac{h^4}{360}(\frac{\partial^8 U}{\partial x^6 \partial y^2} + \frac{\partial^8 U}{\partial x^2 \partial y^6})_{ij} + O(h^6) \tag{4.69}$$

Also,

$$\delta_x^2\delta_y^2 U_{ij} = \frac{1}{h^4}[4U_{ij} - 2(U_{i-1j} + U_{i+1j} + U_{ij-1} + U_{ij+1}) + U_{i-1j-1} + U_{i-1j+1} + U_{i+1j-1} + U_{i+1j+1}] \tag{4.70}$$

We can easily get an $O(h^4)$ method in the same way by substituting difference expressions for the $O(h^2)$ term in Eq.(4.68) and including these in the finite difference approximation (4.64). The resulting higher-order scheme follows from

$$\delta_x^2 U_{ij} + \delta_y^2 U_{ij} + \frac{h^2}{6}\delta_x^2\delta_y^2 U_{ij} - \tau_{ij} = f_{ij} + \frac{h^2}{12}(\delta_x^2 f_{ij} + \delta_y^2 f_{ij}) \tag{4.71}$$

Where $\tau_{ij} = O(h^4)$.

By using Eq.(4.62)and Eq.(4.70) we simplify Eq(4.71) as follows

$$\begin{aligned} 4(U_{i-1j} + U_{i+1j} + U_{ij-1} + U_{ij+1}) + U_{i-1j-1} + U_{i-1j+1} + U_{i+1j-1} + U_{i+1j+1} - 20U_{ij} \\ = \frac{h^2}{2}(8f_{ij} + f_{i+1j} + f_{i-1j} + f_{ij+1} + f_{ij-1}) \end{aligned} \tag{4.72}$$

Equation (4.72) is called **a high order compact difference scheme of order 4** (HOC-4). We can, however, obtain an $O(h^6)$ scheme for this governing equation. Note that the approximation to the cross derivative of U in Eq.(4.68) introduces additional $O(h^4)$ term that we must be careful to include in our derivation of an $O(h^6)$ scheme. More specifically, substituting the finite difference expression for the cross derivative of U and its truncation error terms into Eq.(4.68) give us

$$\begin{aligned} \tau_{ij} = \frac{h^2}{12}(\frac{\partial^2 f}{\partial x^2} - 2\delta_x^2\delta_y^2 U + \frac{\partial^2 f}{\partial y^2})_{ij} + \frac{h^4}{360}(\frac{\partial^6 U}{\partial x^6} + 5(\frac{\partial^6 U}{\partial x^4 \partial y^2} + \frac{\partial^6 U}{\partial x^2 \partial y^4}) + \frac{\partial^6 U}{\partial x^6}) \\ + \frac{h^6}{20160}(\frac{\partial^8 U}{\partial x^8} + \frac{\partial^8 U}{\partial y^8})_{ij} + O(h^8) \end{aligned} \tag{4.73}$$

Clearly, to get a compact $O(h^6)$ approximation, we require compact expressions for the four derivative of order 6 in Eq.(4.73) . This can actually be done by further differentiating (4.57). The required expressions are

$$\frac{\partial^6 U}{\partial x^6} = \frac{\partial^4 f}{\partial x^4} - \frac{\partial^6 U}{\partial x^4 \partial y^2} \tag{4.74}$$

$$\frac{\partial^6 U}{\partial y^6} = \frac{\partial^4 f}{\partial y^4} - \frac{\partial^6 U}{\partial x^2 \partial y^4} \tag{4.75}$$

And

$$\frac{\partial^6 U}{\partial x^4 \partial y^2} + \frac{\partial^6 U}{\partial x^2 \partial y^4} = \frac{\partial^4 f}{\partial x^2 \partial y^2} \tag{4.76}$$

The key here is that we can use Eqs. (4.74) and (4.75) to algebraically eliminate all the derivatives of U. The two dimensions, $O(h^6)$ compact approximation to (4.57) is therefore:[22]

$$\delta_x^2 U_{ij} + \delta_y^2 U_{ij} + \frac{h^2}{6}\delta_x^2\delta_y^2 U_{ij} - \tau_{ij} = f_{ij} + \frac{h^2}{12}(\delta_x^2 f_{ij} + \delta_y^2 f_{ij}) + \frac{h^4}{360}(\delta_x^4 f_{ij} + \delta_y^4 f_{ij}) + \frac{h^4}{90}\delta_x^2\delta_y^2 f_{ij} \tag{4.77}$$

Where $\tau_{ij} = O(h^6)$

By simplifying eq.(4.77) we obtain :

$$\begin{aligned} &60(U_{i-1j} + U_{i+1j} + U_{ij-1} + U_{ij+1}) + 15(U_{i-1j-1} + U_{i-1j+1} + U_{i+1j-1} + U_{i+1j+1}) - 300U_{ij} \\ &= \frac{h^2}{4}(262 f_{ij} + 18(f_{i+1j} + f_{i-1j} + f_{ij+1} + f_{ij-1}) + 4(f_{i-1j-1} + f_{i-1j+1} + f_{i+1j-1} + f_{i+1j+1}) \\ &\qquad + (f_{i-2j-1} + f_{i-1j+2} + f_{i+1j-2} + f_{i+2j+1})) \end{aligned} \tag{4.78}$$

From Eq(4.78) we can approximate $(f_{i-2j-1} + f_{i-1j+2} + f_{i+1j-2} + f_{i+2j+1})$ as follow

$$f_{i-2j-1} = f_{ij-1} - 2h(f_x)_{ij-1} + 2h^2(f_{xx})_{ij-1} + O(h^2) \tag{4.79}$$

$$f_{i+2j+1} = f_{ij+1} + 2h(f_x)_{ij+1} + 2h^2(f_{xx})_{ij+1} + O(h^2) \tag{4.80}$$

$$f_{i-1j+2} = f_{i-1j} + 2h(f_y)_{i-1j} + 2h^2(f_{yy})_{i-1j} + O(h^2) \tag{4.81}$$

$$f_{i+1j+2} = f_{i+1j} - 2h(f_y)_{i+1j} + 2h^2(f_{yy})_{i+1j} + O(h^2) \tag{4.82}$$

By using the central difference operator we can approximate $(f_{xx})_{ij+1}, (f_{xx})_{ij-1}, (f_{yy})_{i+1j}$ and $(f_{yy})_{i-1j}$

Therefore

$$\begin{aligned} (f_{i-2j-1} + f_{i-1j+2} + f_{i+1j-2} + f_{i+2j+1}) &= -3(f_{i+1j} + f_{i-1j} + f_{ij+1} + f_{ij-1}) \\ &+2(f_{i-1j-1} + f_{i-1j+1} + f_{i+1j-1} + f_{i+1j+1}) \end{aligned} \tag{4.83}$$

Equation (4.78) becomes

$$60(U_{i-1j}+U_{i+1j}+U_{ij-1}+U_{ij+1})+15(U_{i-1j-1}+U_{i-1j+1}+U_{i+1j-1}+U_{i+1j+1})-300U_{ij}$$
$$=\frac{h^2}{4}(268f_{ij}+15(f_{i+1j}+f_{i-1j}+f_{ij+1}+f_{ij-1})+6(f_{i-1j-1}+f_{i-1j+1}+f_{i+1j-1}+f_{i+1j+1})) \tag{4.84}$$

Equation (4.84) is called **a high order compact difference scheme of order 6** (HOC-6)

4.3.1 Eight order compact finite difference method

To get a compact $O(h^8)$ approximation, we require compact expressions for the four derivative of order 8 in Eq.(4.73) . This can actually be done by further differentiating (4.57). The required expressions are

$$\frac{\partial^8 U_{ij}}{\partial x^8}=\frac{\partial^6 f_{ij}}{\partial x^6}-\frac{\partial^8 U_{ij}}{\partial x^6\partial y^2} \tag{4.85}$$

$$\frac{\partial^8 U_{ij}}{\partial y^8}=\frac{\partial^6 f_{ij}}{\partial y^6}-\frac{\partial^8 U_{ij}}{\partial x^2\partial y^6} \tag{4.86}$$

The key here is that we can use Eqs. (4.85) and (4.86) to algebraically eliminate all the derivatives of U .

Also

$$\frac{\partial^8 U_{ij}}{\partial x^6\partial y^2}=\frac{\partial^6 f_{ij}}{\partial x^4\partial y^2}-\frac{\partial^8 U_{ij}}{\partial x^4\partial y^4} \tag{4.87}$$

$$\frac{\partial^8 U_{ij}}{\partial x^2\partial y^6}=\frac{\partial^6 f_{ij}}{\partial x^2\partial y^4}-\frac{\partial^8 U_{ij}}{\partial x^4\partial y^4} \tag{4.88}$$

By adding Eq(4.87) and Eq(4.88)

$$\left[\frac{\partial^8 U}{\partial x^2\partial y^6}+\frac{\partial^8 U}{\partial x^6\partial y^2}\right]_{ij}=\left[\frac{\partial^6 f}{\partial x^2\partial y^4}+\frac{\partial^6 f}{\partial x^4\partial y^2}-2\{\frac{\partial^8 U}{\partial x^4\partial y^4}\}\right]_{ij} \tag{4.89}$$

Substituting Eqs.(4.74) to (4.88) into Eq.(4.73) we obtain

$$\tau_{ij}=\frac{h^2}{12}\left(\frac{\partial^2 f}{\partial x^2}-2\delta_x^2\delta_y^2U+\frac{\partial^2 f}{\partial y^2}\right)_{ij}+\frac{h^4}{360}\left(\frac{\partial^4 f}{\partial x^4}+\frac{\partial^4 f}{\partial y^4}+4\frac{\partial^4 f}{\partial x^2\partial y^2}\right)_{ij}$$
$$+\frac{h^6}{181440}\left[9(\frac{\partial^6 f}{\partial x^6}+\frac{\partial^6 f}{\partial y^6})+75(\frac{\partial^8 U}{\partial x^2\partial y^6}+\frac{\partial^8 U}{\partial x^6\partial y^2})\right]_{ij}+O(h^8) \tag{4.90}$$

using of equations (4.76) and (4.89), into equation (4.90) yields

$$\tau_{ij} = \frac{h^2}{12}\left(\frac{\partial^2 f}{\partial x^2} - 2\delta_x^2\delta_y^2 U + \frac{\partial^2 f}{\partial y^2}\right)_{ij} + \frac{h^4}{360}\left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4} + 4\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)_{ij}$$

$$+\frac{h^6}{181440}\left[9(\frac{\partial^6 f}{\partial x^6} + \frac{\partial^6 f}{\partial y^6})\right] + \frac{h^6}{181440}\left[-150(\frac{\partial^8 U}{\partial x^4 \partial y^4}) - 75(\frac{\partial^6 f}{\partial x^2 \partial y^4} + \frac{\partial^6 f}{\partial x^4 \partial y^2})\right]_{ij} + O(h^8) \tag{4.91}$$

Finally, the eighth-order compact approximation for two dimensional Poisson's equation leads to

$$\delta_x^2 U_{ij} + \delta_y^2 U_{ij} + \frac{h^2}{6}\delta_x^2\delta_y^2 U_{ij} + \frac{150h^6}{181440}\delta_x^4\delta_y^4 U_{ij} - \tau_{ij} = f_{ij} + \frac{h^2 15120}{181440}(\delta_x^2 f_{ij} + \delta_y^2 f_{ij}) + \frac{h^4 504}{181440}(\delta_x^4 f_{ij} + \delta_y^4 f_{ij})$$

$$+\frac{h^2 2016}{181440}\delta_x^2\delta_y^2 f_{ij} + \frac{h^6}{20160}(\delta_x^6 f_{ij} + \delta_y^6 f_{ij}) + \frac{h^6 5}{12096}(\delta_x^4\delta_y^2 f + \delta_x^2\delta_y^4 f)_{ij} \tag{4.92}$$

Where $\tau_{ij} = O(h^8)$

By simplifying Eq.(4.92) using the definition of central difference operator Eq.(4.62) we obtain

$$\frac{117360}{181440}(U_{i+1j} + U_{ij-1} + U_{ij+1} + U_{i-1j}) + \frac{5}{6048}(U_{i+2j+2} + U_{i-2j-2} + U_{i+2j-2} + U_{i-2j+2})$$

$$+\frac{32640}{181440}(U_{i+1j-1} + U_{i-1j-1} + U_{i+1j+1} + U_{i-1j+1}) - \frac{599400}{181440}(U_{ij})$$

$$-\frac{600}{181440}(U_{i-1j-2} + U_{i+1j-2} + U_{i+1j+2} + U_{i-1j+2} + U_{i+2j+1} + U_{i-2j+1} + U_{i+2j-1} + U_{i-2j-1})$$

$$+\frac{900}{181440}(U_{ij-2} + U_{ij+2} + U_{i+2j} + U_{i-2j})$$

$$= \frac{132912h^2}{181440}f_{ij} + \frac{h^2 10257}{181440}(f_{i+1j} + f_{i-1j} + f_{ij+1} + f_{ij-1}) + \frac{h^2 1416}{181440}(f_{i-1j-1} + f_{i-1j+1}$$

$$+f_{i+1j-1} + f_{i+1j+1}) + \frac{h^2 5}{1206}(f_{i-1j-2} + f_{i+1j-2} + f_{i+1j+2} + f_{i-1j+2} + f_{1+2j+1}$$

$$+f_{i-2j+1} + f_{i+2j-1} + f_{i-2j-1}) + \frac{435h^2}{181440}(f_{i-2j} + f_{ij+2} + f_{ij-2} + f_{i+2j})$$

$$+\frac{h^2}{20160}(f_{i-3j} + f_{ij+3} + f_{ij-3} + f_{i+3j}) \tag{4.93}$$

It has 25-point stencils given in equation (4.93) and we can approximate the points which have i+2, i-2, j+2, j-2 subscripts as follow:

$$U_{i-2j-2} = U_{ij-2} - 2h(U_x)_{ij-2} + 2h^2(U_{xx})_{ij-2} + O(h^2) \tag{4.94}$$

$$U_{i+2j+2} = U_{ij+2} + 2h(U_x)_{ij+2} + 2h^2(U_{xx})_{ij+2} + O(h^2) \tag{4.95}$$

$$U_{i-2j+2} = U_{ij+2} + 2h(U_x)_{ij+2} + 2h^2(U_{xx})_{ij+2} + O(h^2) \tag{4.96}$$

$$U_{i+2j-2} = U_{ij-2} - 2h(U_x)_{ij-2} + 2h^2(U_{xx})_{ij-2} + O(h^2) \tag{4.97}$$

By approximating the $(U_{yy})_{ij-2}$ and $(U_{yy})_{ij-2}$ using central difference operator from (4.62), we obtain:

$$\begin{aligned}U_{i+2j+2}+U_{i+2j-2}+U_{i-2j-2}+U_{i-2j+2} &= 36U_{ij}+16(U_{i+1j+1}+U_{i+1j-1}+U_{i-1j-1}+U_{i-1j+1})\\ &\quad - 24(U_{i+1j} + U_{i-1j} + U_{ij-1} + U_{ij+1})\end{aligned} \tag{4.98}$$

In similar way

$$U_{i+2j} + U_{i-2j} + U_{ij-2} + U_{ij+2} = -12U_{ij} + 4(U_{i+1j} + U_{i-1j} + U_{ij-1} + U_{ij+1}) \tag{4.99}$$

$$\begin{aligned}&U_{i+2j+1} + U_{i+2j-1} + U_{i-2j-1} + U_{i-2j+1} + U_{i+1j-2} + U_{i+1j+2} + U_{i-1j-2} + U_{i-1j+2} =\\ &- 6(U_{i+1j} + U_{i-1j} + U_{ij-1} + U_{ij+1}) + 8(U_{i+1j} + U_{i-1j} + U_{ij-1} + U_{ij+1})\end{aligned} \tag{4.100}$$

Therefore equation (4.93) is reduced to nine-point stencils given Below.

$$\begin{aligned}&\frac{12096}{18144}(U_{i-1j}+U_{i+1j}+U_{ij-1}+U_{ij+1})+\frac{3024}{18144}(U_{i-1j-1}+U_{i-1j+1}+U_{i+1j-1}+U_{i+1j+1})-\frac{60480}{18144}U_{ij}\\ &= h^2(\frac{127404}{181440}f_{ij} + \frac{11628}{181440}(f_{i+1j} + f_{i-1j} + f_{ij+1} + f_{ij-1})\\ &\quad + \frac{2016}{181440}(f_{i-1j-1} + f_{i-1j+1} + f_{i+1j-1} + f_{i+1j+1}))\end{aligned} \tag{4.101}$$

Equation (4.101) is called **a high order compact difference scheme of order 8** (HOC-8),which makes system of linear equations to be solved by Mat-lab. Note that Eq.(4.93) corresponds to a 25-point stencil, encompassing all the adjacent nodes of the mesh located on the two grid planes that intersect the node ij.

4.3.2 Error analysis for Compact finite difference

We wish now to examine a different measure of the error, for two reasons. First, successively halving the mesh size in 2D quickly produces extremely large problems which we wish to avoid. Second, we wish to demonstrate that a global measure of the error also exhibits high-order convergence for HOC methods. Accordingly, first note that the numerical schemes in this work generate a class of linear systems that can be expressed in exact form as:[10]

$$\mathbf{M}_m\phi = \mathbf{f}_m - \sum_{p=m}^{\infty} \mathbf{c}_p\mathbf{h}^p \tag{4.102}$$

where m is the order of accuracy of the approximation scheme, $\mathbf{M}_m$ is the coefficient matrix,ϕ is the vector of nodal unknowns, $\mathbf{c}_p$ are truncation error coefficient vectors, and $\mathbf{f}_m$ is the right-hand-side vector. Of course, we do not solve (4.102), but rather the approximate equation,

$$\mathbf{M}_m\phi_h = \mathbf{f}_m \tag{4.103}$$

For computations in exact arithmetic, the error then satisfies

$$\mathbf{e} = \phi_h - \phi = \sum_{p=m}^{\infty} \mathbf{M}_m^{-1}\mathbf{c}_p\mathbf{h}^p \tag{4.104}$$

Dropping the higher-order terms, we see that

$$\mathbf{e} = \mathbf{M}_m^{-1}\mathbf{c}^m\mathbf{h}^m \tag{4.105}$$

and we expect the asymptotic behavior of the error at interior node ij to be $\mathbf{e}_{ij} \sim O(h^m)$ in the limit as $h \to 0$, as has been demonstrated here in 1D and 2D.

If, however, we desire an estimate of a global measure of the error, we can take some appropriate norm of (4.105),

$$E = \|\mathbf{e}\| \approx \|M_m^{-1}c_m\|h^m,$$

and provided $M_m^{-1}c_m$ is independent of h, the convergence rate of the global error in this norm is then also $O(h^m)$. Finding such a norm is not trivial, however, since we are comparing measures of error among different-sized vector spaces. Consider the simple distance norm ($\|.\|_d$) for the error on a uniform 2D mesh of N total grid points,

$$\|e\|_d = (\sum_{ij} e_{ij}^2)^{1/2} = [N \times O(h^{2m})]^{1/2}$$

We therefore choose the discrete L_2 norm

$$\|e\| = (\frac{\sum_{ij} e_{ij}^2}{N})^{1/2}$$

as the definition of our norm for the purposes of convergence analysis.

4.4 Result and Discussion

In this section, we performed a numerical experiment to solve a two dimensional Poisson equation (1.1) using finite element method and higher order compact finite difference method on the unit square domain $[0,1] \times [0,1]$. In that example, Dirichlet boundary conditions are prescribed on all sides of the unit square.

In order to compare the two methods (FEM and CFDM) we have to compare numerical solution U_{ij}^* to the exact solution U_{ij} , we used L_2 norm of the error vector e, define as

$$\|e\| = (\frac{1}{N}\sum_{i,j=0}^{N} e_{ij}^2)^{1/2}$$

where $e_{ij} = U_{ij} - U_{ij}^*$ and N is the number of nodes.

Example 1 : Consider the following 2D Poisson's equations

$$-\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = 2\pi^2 sin(\pi x)sin(\pi y)$$

For $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with Dirichlet boundary condition on all sides of f unit square that is U(0,y)=U(x,0)=U(x,1)=U(1,y)=0 and let $h = k = \frac{1}{80}$.

The exact solution is $U(x,y) = sin(\pi x)sin(\pi y)$.

To solve this 2D Poisson equation by using

i. Finite element method Eq.(4.41)

ii. Six order compact finite difference method Eq.(4.78)

iii. Eight order compact finite difference method Eq.(4.88)

Method 1:(Finite element method)

We want to approximate U(x,y) by using finite element method Eq.(4.41).

Let $h_x = 0.0125$,but $h_x = \frac{b-a}{n} \Longrightarrow n = \frac{b-a}{h_x} = \frac{1-0}{0.0125} = 80$ and $h_x = \frac{b-a}{n} \Longrightarrow$

$m = \dfrac{d-c}{h_y} = \dfrac{1-0}{0.0125} = 80$

On x-axis, the interval [0,1] is divided into n=80 equal parts of width $h_x = 1/80$ also the interval [0,1] is divided,On y-axis, the interval [0,1] is divided into m=80 equal parts of width $h_x = 1/80$.

Divide the horizontal grid lines as [29]

$x_i = a + ih_x, \qquad i = 0, 1, 2, 3, ...n = 80$

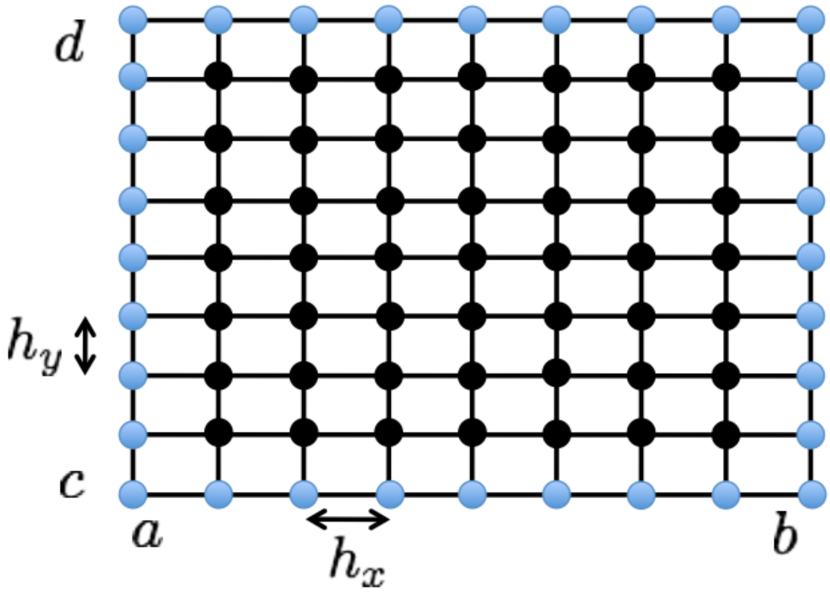


Figure 4.4: The discretization of Domain

for i=0: $x_0 = 0 + 0(1/80) = 0$; for i=1 $x_1 = 0 + 1(1/80) = 0.0125$

for i=2 $x_2 = 0 + 2(1/80) = 0.025$

for i=3: $x_3 = 0 + 3(1/80) = 0.0375$ ...; for i=80: $x_{80} = 0 + 80(1/80) = 1$

$y_j = a + jh_y, \qquad j = 0, 1, 2, 3, ...m = 4$

In the same manner ,divide the vertical grid lines as

for j=0: $y_0 = 0 + 0(1/80) = 0$; for j=1 $y_1 = 0 + 1(1/80) = 0.0125$

for j=2 $y_2 = 0 + 2(1/80) = 0.025$

for j=3: $y_3 = 0 + 3(1/4) = 0.0375$ ...;for j=80: $y_{80} = 0 + 80(1/80) = 1$

consider the figure (4.4)

The assembled rectangular finite element method from equation (4.41)

$$-8U^*_{ij}+(U^*_{i+1j+1}+U^*_{i+1j}+U^*_{i+1j-1}+U^*_{ij+1}+U^*_{ij-1}+U^*_{i-1j+1}+U^*_{i-1j}+U^*_{i-1j-1})-\triangle L^2 f_{ij}=0 \tag{4.106}$$

Working on the first column of the inner grid points gives us

i=1,j=1: $8U_{11}-U_{12}-U_{21}-U_{22}-U_{00}-U_{10}-U_{01}-U_{20}-U_{02}=h^2 f_{11}$

i=2,j=1: $8U_{21}-U_{22}-U_{31}-U_{32}-U_{10}-U_{20}-U_{11}-U_{30}-U_{12}=h^2 f_{21}$

i=3,j=1: $8U_{31}-U_{32}-U_{41}-U_{42}-U_{20}-U_{30}-U_{21}-U_{40}-U_{22}=h^2 f_{31}$

Arranging in matrix-vector form yields

$$\begin{pmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 8 \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix}+\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix}=h^2\begin{pmatrix} f_{11} \\ f_{21} \\ f_{31} \end{pmatrix}$$
$$+\begin{pmatrix} U_{00}+U_{10}+U_{01}+U_{20}+U_{02} \\ U_{10}+U_{20}+U_{30} \\ U_{20}+U_{30}+U_{40}+U_{41}+U_{42} \end{pmatrix}$$

The second column of the inner grid points gives

i=1,j=2: $8U_{12}-U_{13}-U_{22}-U_{23}-U_{01}-U_{11}-U_{02}-U_{21}-U_{03}=h^2 f_{12}$

i=2,j=2: $8U_{22}-U_{23}-U_{32}-U_{33}-U_{11}-U_{21}-U_{12}-U_{31}-U_{13}=h^2 f_{22}$

i=3,j=2: $8U_{32}-U_{33}-U_{42}-U_{43}-U_{21}-U_{31}-U_{22}-U_{41}-U_{23}=h^2 f_{32}$

Arranging in matrix-vector form yields:

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix}+\begin{pmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 8 \end{pmatrix}\begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix}+\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$
$$=h^2\begin{pmatrix} f_{12} \\ f_{22} \\ f_{32} \end{pmatrix}+\begin{pmatrix} U_{01}+U_{02}+U_{03} \\ 0 \\ U_{43}+U_{41}+U_{42} \end{pmatrix}$$

The third column gives

i=1,j=3: $8U_{13}-U_{14}-U_{23}-U_{24}-U_{02}-U_{12}-U_{03}-U_{22}-U_{04}=h^2 f_{13}$

i=2,j=3: $8U_{23} - U_{24} - U_{33} - U_{34} - U_{12} - U_{22} - U_{13} - U_{32} - U_{14} = h^2 f_{23}$

i=3,j=3: $8U_{33} - U_{34} - U_{43} - U_{44} - U_{22} - U_{32} - U_{23} - U_{42} - U_{24} = h^2 f_{33}$

Arranging in matrix-vector form yields:

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} + \begin{pmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 8 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = h^2 \begin{pmatrix} f_{13} \\ f_{23} \\ f_{33} \end{pmatrix} + \begin{pmatrix} U_{14} + U_{02} + U_{03} + U_{24} + U_{04} \\ U_{14} + U_{34} + U_{24} \\ U_{43} + U_{44} + U_{42} + U_{24} + U_{34} \end{pmatrix} \tag{4.107}$$

By rearranging the above matrix it forms Penta diagonal matrix as follows

$$\begin{pmatrix} 8 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 8 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 8 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 8 & -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 8 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & -1 & 8 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 8 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & 8 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 8 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ U_{22} \\ U_{32} \\ U_{13} \\ U_{23} \\ U_{33} \end{pmatrix} = h^2 \begin{pmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{pmatrix} + \begin{pmatrix} U_{00} + U_{10} + U_{01} + U_{20} + U_{02} \\ U_{10} + U_{20} + U_{30} \\ U_{20} + U_{30} + U_{40} + U_{41} + U_{42} \\ U_{01} + U_{02} + U_{03} \\ 0 \\ U_{43} + U_{41} + U_{42} \\ U_{14} + U_{02} + U_{03} + U_{24} + U_{04} \\ U_{14} + U_{34} + U_{24} \\ U_{43} + U_{44} + U_{42} + U_{24} + U_{34} \end{pmatrix}$$

The result which obtained from MATLAB code is given blow:

Table 4.1: Shows the comparison of Exact Solution with the Numerical Solution Solved By FEM(Nine Point)

Position	Exact Solution	Numerical solution	Absolute Error
U_{11}	0.00158058049265208	0.000527068502314871	0.0010535119903372
U_{21}	0.00315866176266422	0.00105330360099108	0.00210535816167315
U_{31}	0.00473174853918122	0.00157787321017241	0.0031538753290088
U_{12}	0.00315866176266422	0.00105330360099108	0.00210535816167315
U_{22}	0.00631232903183329	0.00210494171248728	0.00420738731934601
U_{32}	0.00945601521133286	0.00315325147847497	0.00630276373285789
U_{13}	0.00473174853918122	0.00157787321017241	0.00315387532900881
U_{23}	0.00945601521133286	0.00315325147847497	0.00630276373285789
U_{33}	0.014165329979795	0.00472364380805373	0.00944168617174126

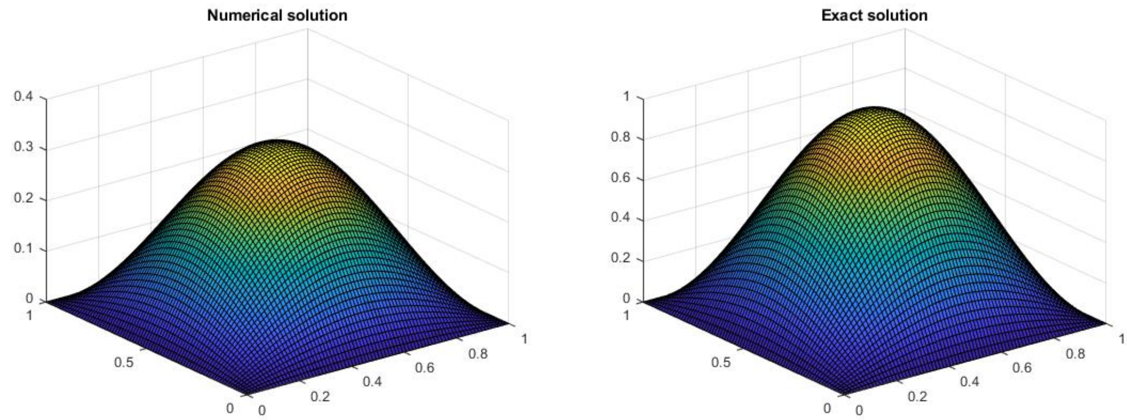


Figure 4.5: The relationship between Exact Solution and Numerical Solution which is solved by FEM

Method 2:(Sixth Order CFDM)

This method is also used to approximate the solution of 2D Poisson equation. Which was given as

$$60(U_{i-1j} + U_{i+1j} + U_{ij-1} + U_{ij+1}) + 15(U_{i-1j-1} + U_{i-1j+1} + U_{i+1j-1} + U_{i+1j+1}) - 300U_{ij}$$

$$= \frac{h^2}{4}(268f_{ij} + 15(f_{i+1j} + f_{i-1j} + f_{ij+1} + f_{ij-1}) + 6(f_{i-1j-1} + f_{i-1j+1} + f_{i+1j-1} + f_{i+1j+1})) \tag{4.108}$$

Consider the figure 4.4 above, we construct the matrix by using the given example above with the same mesh grid. Working on the first column of the inner grid points gives us

i=1,j=1: $300U_{11} - 60(U_{12} + U_{21} + U_{01} + U_{10}) - 15(U_{00} + U_{02} + U_{20} + U_{22}) =$

$$\frac{h^2}{4}[262f_{11} + 15(f_{12} + f_{21} + f_{01} + f_{10}) + 6(f_{00} + f_{02} + f_{20} + f_{22})]$$

i=2,j=1: $300U_{21} - 60(U_{22} + U_{31} + U_{11} + U_{20}) - 15(U_{10} + U_{12} + U_{30} + U_{32}) =$

$$\frac{h^2}{4}[262f_{21} + 15(f_{22} + f_{31} + f_{11} + f_{20}) + 6(f_{10} + f_{12} + f_{30} + f_{32})]$$

i=3,j=1: $300U_{31} - 60(U_{32} + U_{41} + U_{21} + U_{30}) - 15(U_{20} + U_{22} + U_{40} + U_{42}) =$

$$\frac{h^2}{4}[262f_{31} + 15(f_{32} + f_{41} + f_{21} + f_{30}) + 6(f_{20} + f_{22} + f_{40} + f_{42})]$$

Arranging in matrix-vector form yields

$$\begin{pmatrix} 300 & -60 & 0 \\ -60 & 300 & -60 \\ 0 & -60 & 300 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} + \begin{pmatrix} -60 & -15 & 0 \\ -15 & -60 & -15 \\ 0 & -15 & -60 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix}$$

$$= \frac{h^2}{4} \begin{pmatrix} 262f_{11} + 15(f_{12} + f_{21} + f_{01} + f_{10}) + 6(f_{00} + f_{02} + f_{20} + f_{22}) \\ 262f_{21} + 15(f_{22} + f_{31} + f_{11} + f_{20}) + 6(f_{10} + f_{12} + f_{30} + f_{32}) \\ 262f_{31} + 15(f_{32} + f_{41} + f_{21} + f_{30}) + 6(f_{20} + f_{22} + f_{40} + f_{42}) \end{pmatrix}$$

$$+ \begin{pmatrix} 60(U_{01} + U_{10}) + 15(U_{00} + U_{20} + U_{02}) \\ 60U_{20} + 15(U_{10} + U_{30}) \\ 60(U_{41} + U_{30}) + 15(U_{40} + U_{20} + U_{42}) \end{pmatrix}$$

The second column of the inner grid points gives

i=1,j=2: $300U_{12} - 60(U_{13} + U_{22} + U_{02} + U_{11}) - 15(U_{01} + U_{03} + U_{21} + U_{23}) =$

$$\frac{h^2}{4}[262f_{12} + 15(f_{13} + f_{22} + f_{02} + f_{11}) + 6(f_{01} + f_{03} + f_{21} + f_{23})]$$

i=2,j=2: $300U_{22} - 60(U_{23} + U_{32} + U_{12} + U_{21}) - 15(U_{11} + U_{13} + U_{31} + U_{33}) =$

$$\frac{h^2}{4}[262f_{22} + 15(f_{23} + f_{32} + f_{12} + f_{21}) + 6(f_{11} + f_{13} + f_{31} + f_{33})]$$

i=3,j=2: $300U_{32} - 60(U_{33} + U_{42} + U_{22} + U_{31}) - 15(U_{21} + U_{23} + U_{41} + U_{43}) =$

$$\frac{h^2}{4}[262f_{32} + 15(f_{33} + f_{42} + f_{22} + f_{31}) + 6(f_{21} + f_{23} + f_{41} + f_{43})]$$

Arranging in matrix-vector form yields

$$\begin{pmatrix} -60 & -15 & 0 \\ -15 & -60 & -15 \\ 0 & -15 & -60 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} + \begin{pmatrix} 300 & -60 & 0 \\ -60 & 300 & -60 \\ 0 & -60 & 300 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} + \begin{pmatrix} -60 & -15 & 0 \\ -15 & -60 & -15 \\ 0 & -15 & -60 \end{pmatrix} \begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

$$= \frac{h^2}{4} \begin{pmatrix} 262f_{12} + 15(f_{13} + f_{22} + f_{02} + f_{11}) + 6(f_{01} + f_{03} + f_{21} + f_{23}) \\ 262f_{22} + 15(f_{23} + f_{32} + f_{12} + f_{21}) + 6(f_{11} + f_{13} + f_{31} + f_{33}) \\ 262f_{32} + 15(f_{33} + f_{42} + f_{22} + f_{31}) + 6(f_{21} + f_{23} + f_{41} + f_{43}) \end{pmatrix}$$

$$+ \begin{pmatrix} 60(U_{02}) + 15(U_{01} + U_{03}) \\ 0 \\ 60(U_{42}) + 15(U_{41} + U_{43}) \end{pmatrix}$$

The third column gives

i=1,j=3: $300U_{13} - 60(U_{14} + U_{23} + U_{03} + U_{12}) - 15(U_{02} + U_{04} + U_{22} + U_{24}) =$

$$\frac{h^2}{4}[262f_{13} + 15(f_{14} + f_{23} + f_{03} + f_{12}) + 6(f_{02} + f_{04} + f_{22} + f_{24})]$$

i=2,j=3: $300U_{23} - 60(U_{24} + U_{33} + U_{13} + U_{22}) - 15(U_{12} + U_{14} + U_{32} + U_{34}) =$

$$\frac{h^2}{4}[262f_{23} + 15(f_{24} + f_{33} + f_{13} + f_{22}) + 6(f_{12} + f_{14} + f_{32} + f_{34})]$$

i=3,j=3: $300U_{33} - 60(U_{34} + U_{43} + U_{23} + U_{32}) - 15(U_{22} + U_{24} + U_{42} + U_{44}) =$

$$\frac{h^2}{4}[262f_{33} + 15(f_{34} + f_{43} + f_{23} + f_{32}) + 6(f_{22} + f_{24} + f_{42} + f_{44})]$$

Arranging in matrix-vector form yields

$$\begin{pmatrix} -60 & -15 & 0 \\ -15 & -60 & -15 \\ 0 & -15 & -60 \end{pmatrix}\begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix} + \begin{pmatrix} 300 & -60 & 0 \\ -60 & 300 & -60 \\ 0 & -60 & 300 \end{pmatrix}\begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

$$= \frac{h^2}{4}\begin{pmatrix} 262f_{13} + 15(f_{14} + f_{23} + f_{03} + f_{12}) + 6(f_{02} + f_{04} + f_{22} + f_{24}) \\ 262f_{23} + 15(f_{24} + f_{33} + f_{13} + f_{22}) + 6(f_{12} + f_{14} + f_{32} + f_{34}) \\ 262f_{33} + 15(f_{34} + f_{43} + f_{23} + f_{32}) + 6(f_{22} + f_{24} + f_{42} + f_{44}) \end{pmatrix}$$

$$+ \begin{pmatrix} 60(U_{14} + U_{03}) + 15(U_{02} + U_{04} + U_{24}) \\ 60(U_{24}) + 15(U_{14} + U_{34}) \\ 60(U_{43} + U_{34}) + 15(U_{24} + U_{42} + U_{44}) \end{pmatrix}$$

By rearranging the above matrix it forms Penta diagonal matrix as follows

$$\begin{pmatrix} 300 & -60 & 0 & -60 & -15 & 0 & 0 & 0 & 0 \\ -60 & 300 & -60 & -15 & -60 & -15 & 0 & 0 & 0 \\ 0 & -60 & 300 & 0 & -15 & -60 & 0 & 0 & 0 \\ -60 & -15 & 0 & 300 & -60 & 0 & -60 & -15 & 0 \\ -15 & -60 & -15 & -60 & 300 & -60 & -15 & -60 & -15 \\ 0 & -15 & -60 & 0 & -60 & 300 & 0 & -15 & -60 \\ 0 & 0 & 0 & -60 & -15 & 0 & 300 & -60 & 0 \\ 0 & 0 & 0 & -15 & -60 & -15 & -60 & 300 & -60 \\ 0 & 0 & 0 & 0 & -15 & -60 & 0 & -60 & 300 \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ U_{22} \\ U_{32} \\ U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

$$= \frac{h^2}{4}\begin{pmatrix} 262f_{11} + 15(f_{12} + f_{21} + f_{01} + f_{10}) + 6(f_{00} + f_{02} + f_{20} + f_{22}) \\ 262f_{21} + 15(f_{22} + f_{31} + f_{11} + f_{20}) + 6(f_{10} + f_{12} + f_{30} + f_{32}) \\ 262f_{31} + 15(f_{32} + f_{41} + f_{21} + f_{30}) + 6(f_{20} + f_{22} + f_{40} + f_{42}) \\ 262f_{12} + 15(f_{13} + f_{22} + f_{02} + f_{11}) + 6(f_{01} + f_{03} + f_{21} + f_{23}) \\ 262f_{22} + 15(f_{23} + f_{32} + f_{12} + f_{21}) + 6(f_{11} + f_{13} + f_{31} + f_{33}) \\ 262f_{32} + 15(f_{33} + f_{42} + f_{22} + f_{31}) + 6(f_{21} + f_{23} + f_{41} + f_{43}) \\ 262f_{13} + 15(f_{14} + f_{23} + f_{03} + f_{12}) + 6(f_{02} + f_{04} + f_{22} + f_{24}) \\ 262f_{23} + 15(f_{24} + f_{33} + f_{13} + f_{22}) + 6(f_{12} + f_{14} + f_{32} + f_{34}) \\ 262f_{33} + 15(f_{34} + f_{43} + f_{23} + f_{32}) + 6(f_{22} + f_{24} + f_{42} + f_{44}) \end{pmatrix}$$

$$+\begin{pmatrix} 60(U_{01}+U_{10})+15(U_{00}+U_{20}+U_{02}) \\ 60U_{20}+15(U_{10}+U_{30}) \\ 60(U_{41}+U_{30})+15(U_{40}+U_{20}+U_{42}) \\ 60(U_{02})+15(U_{01}+U_{03}) \\ 0 \\ 60(U_{42})+15(U_{41}+U_{43}) \\ 60(U_{14}+U_{03})+15(U_{02}+U_{04}+U_{24}) \\ 60(U_{24})+15(U_{14}+U_{34}) \\ 60(U_{43}+U_{34})+15(U_{24}+U_{42}+U_{44}) \end{pmatrix}$$

By solving the stiffness matrix above using MATLAB code we Obtain:

Table 4.2: Shows the comparison of Exact Solution with the Numerical Solution Solved By six order Compact finite difference Method

Position	Exact Solution	Numerical solution	Absolute Error
U_{11}	0.00158058049265208	0.00155244817175781	2.81323208942688e-05
U_{21}	0.00315866176266422	0.00310205691652636	5.6604846137866e-05
U_{31}	0.00473174853918122	0.00464641812077701	8.53304184042043e-05
U_{12}	0.00315866176266422	0.00310220315933827	5.64586033259492e-05
U_{22}	0.00631232903183329	0.00619877531590907	0.000113553715924227
U_{32}	0.00945601521133286	0.00928487384088197	0.000171141370450887
U_{13}	0.00473174853918122	0.00464685034534463	8.48981938365847e-05
U_{23}	0.00945601521133286	0.00928529980273972	0.000170715408593139
U_{33}	0.014165329979795	0.0139080768104968	0.000257253169298198

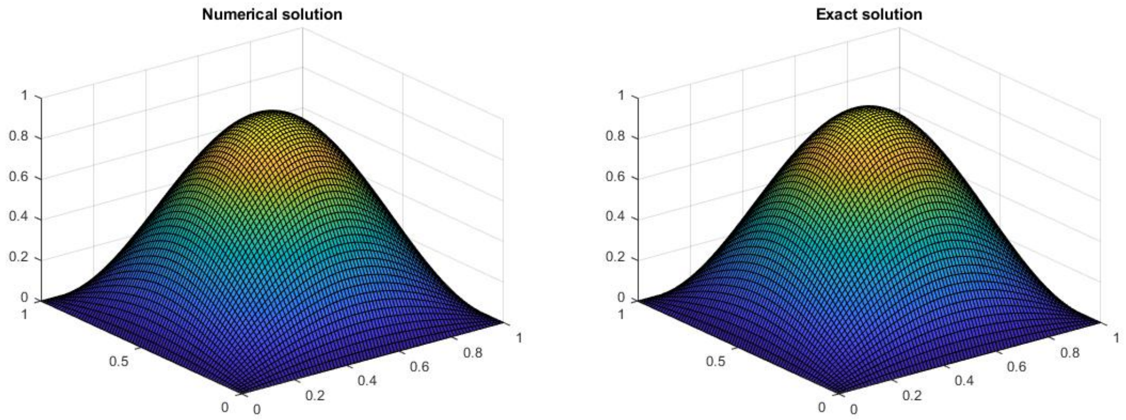


Figure 4.6: The relationship between Exact Solution and Numerical Solution of six order CFDM

Method 3:(Eighth Order CFDM)

Eight order of compact finite difference method is also used to approximate 2D Poisson equation. Which was given as

$$\frac{12096}{18144}(U_{i-1j}+U_{i+1j}+U_{ij-1}+U_{ij+1})+\frac{3024}{18144}(U_{i-1j-1}+U_{i-1j+1}+U_{i+1j-1}+U_{i+1j+1})-\frac{60480}{18144}U_{ij}$$
$$= h^2(\frac{127404}{181440}f_{ij} + \frac{11628}{181440}(f_{i+1j} + f_{i-1j} + f_{ij+1} + f_{ij-1})$$
$$+ \frac{2016}{181440}(f_{i-1j-1} + f_{i-1j+1} + f_{i+1j-1} + f_{i+1j+1})) \tag{4.109}$$

Now construct the matrix in the figure 4.4 to solve given example above with the same mesh grids.

Working on the first column of the inner grid points gives us

i=1,j=1:

$$\frac{60480}{18144}U_{11} - \frac{12096}{18144}(U_{12} + U_{21} + U_{01} + U_{10}) - \frac{3024}{18144}(U_{00} + U_{02} + U_{20} + U_{22}) =$$
$$h^2[\frac{127404}{181440}f_{11} + \frac{11628}{181440}(f_{12} + f_{21} + f_{01} + f_{10}) + \frac{2016}{181440}(f_{00} + f_{02} + f_{20} + f_{22})]$$

i=2,j=1:

$$\frac{60480}{18144}U_{21} - \frac{12096}{18144}(U_{22} + U_{31} + U_{11} + U_{20}) - \frac{3024}{18144}(U_{10} + U_{12} + U_{30} + U_{32}) =$$
$$h^2[\frac{127404}{181440}f_{21} + \frac{11628}{181440}(f_{22} + f_{31} + f_{11} + f_{20}) + \frac{2016}{181440}(f_{10} + f_{12} + f_{30} + f_{32})]$$

i=3,j=1:

$$\frac{60480}{18144}U_{31}-\frac{12096}{18144}(U_{32}+U_{41}+U_{21}+U_{30})-\frac{3024}{18144}(U_{20}+U_{22}+U_{40}+U_{42})=$$

$$h^2[\frac{127404}{181440}f_{31}+\frac{11628}{181440}(f_{32}+f_{41}+f_{21}+f_{30})+\frac{2016}{181440}(f_{20}+f_{22}+f_{40}+f_{42})]$$

Arranging in matrix-vector form yields

$$\begin{pmatrix}\frac{60480}{18144} & -\frac{12096}{18144} & 0\\ -\frac{12096}{18144} & \frac{60480}{18144} & -\frac{12096}{18144}\\ 0 & -\frac{12096}{18144} & \frac{60480}{18144}\end{pmatrix}\begin{pmatrix}U_{11}\\ U_{21}\\ U_{31}\end{pmatrix}+\begin{pmatrix}-\frac{12096}{18144} & -\frac{3024}{18144} & 0\\ -\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144}\\ 0 & -\frac{3024}{18144} & -\frac{12096}{18144}\end{pmatrix}\begin{pmatrix}U_{12}\\ U_{22}\\ U_{32}\end{pmatrix}$$

$$=h^2\begin{pmatrix}\frac{127404}{181440}f_{11}+\frac{11628}{181440}(f_{12}+f_{21}+f_{01}+f_{10})+\frac{2016}{181440}(f_{00}+f_{02}+f_{20}+f_{22})\\ \frac{127404}{181440}f_{21}+\frac{11628}{181440}(f_{22}+f_{31}+f_{11}+f_{20})+\frac{2016}{181440}(f_{10}+f_{12}+f_{30}+f_{32})\\ \frac{127404}{181440}f_{31}+\frac{11628}{181440}(f_{32}+f_{41}+f_{21}+f_{30})+\frac{2016}{181440}(f_{20}+f_{22}+f_{40}+f_{42})\end{pmatrix}$$

$$+\begin{pmatrix}\frac{12096}{18144}(U_{01}+U_{10})+\frac{3024}{18144}(U_{00}+U_{20}+U_{02})\\ \frac{12096}{18144}U_{20}+\frac{3024}{18144}(U_{10}+U_{30})\\ \frac{12096}{18144}(U_{41}+U_{30})+\frac{3024}{18144}(U_{40}+U_{20}+U_{42})\end{pmatrix}$$

The second column of the inner grid points gives

i=1,j=2:

$$\frac{60480}{18144}U_{12}-\frac{12096}{18144}(U_{13}+U_{22}+U_{02}+U_{11})-\frac{3024}{18144}(U_{01}+U_{03}+U_{21}+U_{23})=$$

$$h^2[\frac{127404}{181440}f_{12}+\frac{11628}{181440}(f_{13}+f_{22}+f_{02}+f_{11})+\frac{2016}{181440}(f_{01}+f_{03}+f_{21}+f_{23})]$$

i=2,j=2:

$$\frac{60480}{18144}U_{22}-\frac{12096}{18144}(U_{23}+U_{32}+U_{12}+U_{21})-\frac{3024}{18144}(U_{11}+U_{13}+U_{31}+U_{33})=$$
$$h^2[\frac{127404}{181440}f_{22}+\frac{11628}{181440}(f_{23}+f_{32}+f_{12}+f_{21})+\frac{2016}{181440}(f_{11}+f_{13}+f_{31}+f_{33})]$$

i=3,j=2:
$$\frac{60480}{18144}U_{32}-\frac{12096}{18144}(U_{33}+U_{42}+U_{22}+U_{31})-\frac{3024}{18144}(U_{21}+U_{23}+U_{41}+U_{43})=$$
$$h^2[\frac{127404}{181440}f_{32}+\frac{11628}{181440}(f_{33}+f_{42}+f_{22}+f_{31})+\frac{2016}{181440}(f_{21}+f_{23}+f_{41}+f_{43})]$$

Arranging in matrix-vector form yields

$$\begin{pmatrix}-\frac{12096}{18144} & -\frac{3024}{18144} & 0\\ -\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144}\\ 0 & -\frac{3024}{18144} & -\frac{12096}{18144}\end{pmatrix}\begin{pmatrix}U_{12}\\ U_{22}\\ U_{32}\end{pmatrix}+\begin{pmatrix}\frac{60480}{18144} & -\frac{12096}{18144} & 0\\ -\frac{12096}{18144} & \frac{60480}{18144} & -\frac{12096}{18144}\\ 0 & -\frac{12096}{18144} & \frac{60480}{18144}\end{pmatrix}\begin{pmatrix}U_{11}\\ U_{21}\\ U_{31}\end{pmatrix}$$
$$+\begin{pmatrix}-\frac{12096}{18144} & -\frac{3024}{18144} & 0\\ -\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144}\\ 0 & -\frac{3024}{18144} & -\frac{12096}{18144}\end{pmatrix}\begin{pmatrix}U_{12}\\ U_{22}\\ U_{32}\end{pmatrix}$$
$$=h^2\begin{pmatrix}\frac{127404}{181440}f_{12}+\frac{11628}{181440}(f_{13}+f_{22}+f_{02}+f_{11})+\frac{2016}{181440}(f_{01}+f_{03}+f_{21}+f_{23})\\ \frac{127404}{181440}f_{22}+\frac{11628}{181440}(f_{23}+f_{32}+f_{12}+f_{21})+\frac{2016}{181440}(f_{11}+f_{13}+f_{31}+f_{33})\\ \frac{127404}{181440}f_{32}+\frac{11628}{181440}(f_{33}+f_{42}+f_{22}+f_{31})+\frac{2016}{181440}(f_{21}+f_{23}+f_{41}+f_{43})\end{pmatrix}$$
$$+\begin{pmatrix}\frac{12096}{18144}(U_{02})+\frac{3024}{18144}(U_{01}+U_{03})\\ 0\\ \frac{12096}{18144}(U_{42})+15(U_{41}+U_{43})\end{pmatrix}$$

The third column gives

i=1,j=3:

$$\frac{60480}{18144}U_{13}-\frac{12096}{18144}(U_{14}+U_{23}+U_{03}+U_{12})-\frac{3024}{18144}(U_{02}+U_{04}+U_{22}+U_{24})=$$
$$h^2[\frac{127404}{181440}f_{13}+\frac{11628}{181440}(f_{14}+f_{23}+f_{03}+f_{12})+\frac{2016}{181440}(f_{02}+f_{04}+f_{22}+f_{24})]$$

i=2,j=3:

$$\frac{60480}{18144}U_{23}-\frac{12096}{18144}(U_{24}+U_{33}+U_{13}+U_{22})-\frac{3024}{18144}(U_{12}+U_{14}+U_{32}+U_{34})=$$
$$h^2[\frac{127404}{181440}f_{23}+\frac{11628}{181440}(f_{24}+f_{33}+f_{13}+f_{22})+\frac{2016}{181440}(f_{12}+f_{14}+f_{32}+f_{34})]$$

i=3,j=3:

$$\frac{60480}{18144}U_{33}-\frac{12096}{18144}(U_{34}+U_{43}+U_{23}+U_{32})-\frac{3024}{18144}(U_{22}+U_{24}+U_{42}+U_{44})=$$
$$h^2[\frac{127404}{181440}f_{33}+\frac{11628}{181440}(f_{34}+f_{43}+f_{23}+f_{32})+\frac{2016}{181440}(f_{22}+f_{24}+f_{42}+f_{44})]$$

Arranging in matrix-vector form yields

$$\begin{pmatrix} -\frac{12096}{18144} & -\frac{3024}{18144} & 0 \\ -\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144} \\ 0 & -\frac{3024}{18144} & -\frac{12096}{18144} \end{pmatrix}\begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix}+\begin{pmatrix} \frac{60480}{18144} & -\frac{12096}{18144} & 0 \\ -\frac{12096}{18144} & \frac{60480}{18144} & -\frac{12096}{18144} \\ 0 & -\frac{12096}{18144} & \frac{60480}{18144} \end{pmatrix}\begin{pmatrix} U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}$$

$$=h^2\begin{pmatrix} \frac{127404}{181440}f_{13}+\frac{11628}{181440}(f_{14}+f_{23}+f_{03}+f_{12})+\frac{2016}{181440}(f_{02}+f_{04}+f_{22}+f_{24}) \\ \frac{127404}{181440}f_{23}+\frac{11628}{181440}(f_{24}+f_{33}+f_{13}+f_{22})+\frac{2016}{181440}(f_{12}+f_{14}+f_{32}+f_{34}) \\ \frac{127404}{181440}f_{33}+\frac{11628}{181440}(f_{34}+f_{43}+f_{23}+f_{32})+\frac{2016}{181440}(f_{22}+f_{24}+f_{42}+f_{44}) \end{pmatrix}$$

$$+\begin{pmatrix} \frac{12096}{18144}(U_{14}+U_{03})+\frac{3024}{18144}(U_{02}+U_{04}+U_{24}) \\ \frac{12096}{18144}(U_{24})+\frac{3024}{18144}(U_{14}+U_{34}) \\ \frac{12096}{18144}(U_{43}+U_{34})+\frac{3024}{18144}(U_{24}+U_{42}+U_{44}) \end{pmatrix}$$

By rearranging the above matrix it forms Penta-diagonal matrix as follows

$$\begin{pmatrix}
\frac{60480}{18144} & -\frac{12096}{18144} & 0 & -\frac{12096}{18144} & -\frac{3024}{18144} & 0 & 0 & 0 & 0 \\
-\frac{12096}{18144} & \frac{60480}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144} & 0 & 0 & 0 \\
0 & -\frac{12096}{18144} & \frac{60480}{18144} & 0 & -\frac{3024}{18144} & -\frac{12096}{18144} & 0 & 0 & 0 \\
-\frac{12096}{18144} & -\frac{3024}{18144} & 0 & \frac{60480}{18144} & -\frac{12096}{18144} & 0 & -\frac{12096}{18144} & -\frac{3024}{18144} & 0 \\
-\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144} & -\frac{12096}{18144} & \frac{60480}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144} \\
0 & -\frac{3024}{18144} & -\frac{12096}{18144} & 0 & -\frac{12096}{18144} & \frac{60480}{18144} & 0 & -\frac{3024}{18144} & -\frac{12096}{18144} \\
0 & 0 & 0 & -\frac{12096}{18144} & -\frac{3024}{18144} & 0 & \frac{60480}{18144} & -\frac{12096}{18144} & 0 \\
0 & 0 & 0 & -\frac{3024}{18144} & -\frac{12096}{18144} & -\frac{3024}{18144} & -\frac{12096}{18144} & \frac{60480}{18144} & -60 \\
0 & 0 & 0 & 0 & -\frac{3024}{18144} & -\frac{12096}{18144} & 0 & -\frac{12096}{18144} & \frac{60480}{18144}
\end{pmatrix}$$

$$\begin{pmatrix} U_{11}\\ U_{21}\\ U_{31}\\ U_{12}\\ U_{22}\\ U_{32}\\ U_{13}\\ U_{23}\\ U_{33}\end{pmatrix} = h^2 \begin{pmatrix} \frac{127404}{181440}f_{11} + \frac{11628}{181440}(f_{12}+f_{21}+f_{01}+f_{10}) + \frac{2016}{181440}(f_{00}+f_{02}+f_{20}+f_{22}) \\ \frac{127404}{181440}f_{21} + \frac{11628}{181440}(f_{22}+f_{31}+f_{11}+f_{20}) + \frac{2016}{181440}(f_{10}+f_{12}+f_{30}+f_{32}) \\ \frac{127404}{181440}f_{31} + \frac{11628}{181440}(f_{32}+f_{41}+f_{21}+f_{30}) + \frac{2016}{181440}(f_{20}+f_{22}+f_{40}+f_{42}) \\ \frac{127404}{181440}f_{12} + \frac{11628}{181440}(f_{13}+f_{22}+f_{02}+f_{11}) + \frac{2016}{181440}(f_{01}+f_{03}+f_{21}+f_{23}) \\ \frac{127404}{181440}f_{22} + \frac{11628}{181440}(f_{23}+f_{32}+f_{12}+f_{21}) + \frac{2016}{181440}(f_{11}+f_{13}+f_{31}+f_{33}) \\ \frac{127404}{181440}f_{32} + \frac{11628}{181440}(f_{33}+f_{42}+f_{22}+f_{31}) + \frac{2016}{181440}(f_{21}+f_{23}+f_{41}+f_{43}) \\ \frac{127404}{181440}f_{13} + \frac{11628}{181440}(f_{14}+f_{23}+f_{03}+f_{12}) + \frac{2016}{181440}(f_{02}+f_{04}+f_{22}+f_{24}) \\ \frac{127404}{181440}f_{23} + \frac{11628}{181440}(f_{24}+f_{33}+f_{13}+f_{22}) + \frac{2016}{181440}(f_{12}+f_{14}+f_{32}+f_{34}) \\ \frac{127404}{181440}f_{33} + \frac{11628}{181440}(f_{34}+f_{43}+f_{23}+f_{32}) + \frac{2016}{181440}(f_{22}+f_{24}+f_{42}+f_{44}) \end{pmatrix}$$

$$+\begin{pmatrix} 60(U_{01}+U_{10})+15(U_{00}+U_{20}+U_{02}) \\ 60U_{20}+15(U_{10}+U_{30}) \\ 60(U_{41}+U_{30})+15(U_{40}+U_{20}+U_{42}) \\ 60(U_{02})+15(U_{01}+U_{03}) \\ 0 \\ 60(U_{42})+15(U_{41}+U_{43}) \\ 60(U_{14}+U_{03})+15(U_{02}+U_{04}+U_{24}) \\ 60(U_{24})+15(U_{14}+U_{34}) \\ 60(U_{43}+U_{34})+15(U_{24}+U_{42}+U_{44}) \end{pmatrix}$$

Table 4.3: Shows the comparison of Exact Solution with the Numerical Solution Solved By Eight order Compact finite difference Method

Position	Exact Solution	Numerical solution	Absolute Error
U_{11}	0.00158058049265208	0.00157717231141191	3.40818124016664e-06
U_{21}	0.00315866176266422	0.00315197178118063	6.68998148359735e-06
U_{31}	0.00473174853918122	0.00472190313807976	9.84540110146159e-06
U_{12}	0.00315866176266422	0.00315185078921897	6.81097344525702e-06
U_{22}	0.00631232903183329	0.00629895964710192	1.33693847313753e-05
U_{32}	0.00945601521133286	0.00943633997673275	1.96752346001087e-05
U_{13}	0.00473174853918122	0.00472154554307985	1.02029961013663e-05
U_{23}	0.00945601521133286	0.00943598756309898	2.00276482338786e-05
U_{33}	0.014165329979795	0.0141358560222863	2.94739575086977e-05

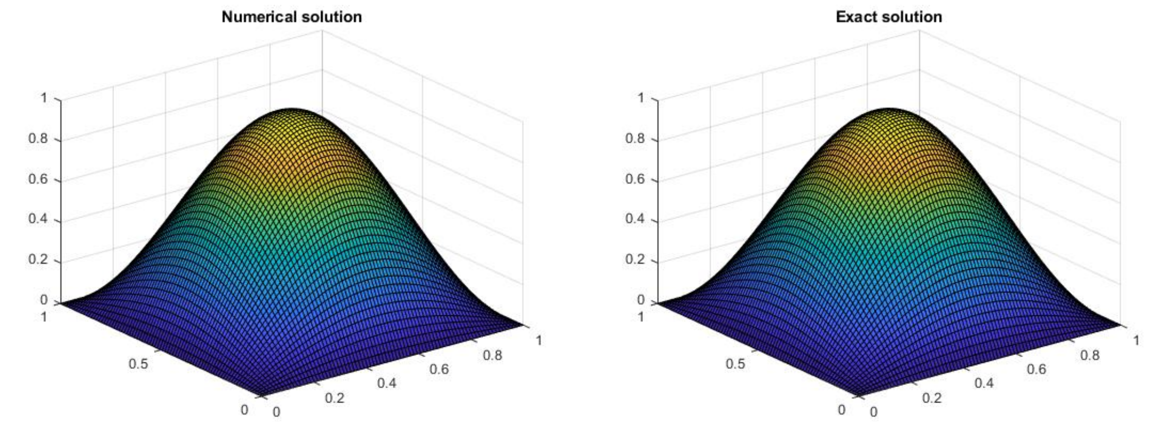


Figure 4.7: The relationship between exact solution and numerical solution of eight order CFDM

Generally we can conclude the error in the following table :

Table 4.4: Shows the comparison of FEM, six order CFDM and eight order CFDM concern with absolute error

Node No.	FEM error	6^{th} order CFDM error	8^{th} order CFDM error
N_1	0.0010535119903372	2.81323208942688e-05	3.40818124016664e-06
N_2	0.00210535816167315	5.6604846137866e-05	6.68998148359735e-06
N_3	0.0031538753290088	8.53304184042043e-05	9.84540110146159e-06
N_4	0.00210535816167315	5.64586033259492e-05	6.81097344525702e-06
N_5	0.00420738731934601	0.000113553715924227	1.33693847313753e-05
N_6	0.00630276373285789	0.000171141370450887	1.96752346001087e-05
N_7	0.00315387532900881	8.48981938365847e-05	1.02029961013663e-05
N_8	0.00630276373285789	0.000170715408593139	2.00276482338786e-05
N_9	0.00944168617174126	0.000257253169298198	2.94739575086977e-05

4.5 Discussion

In this thesis , we considered rectangular finite element method (9-point FDM) , sixth order CFDM and eighth order CFDM for solving 2D Poisson equation. The above two-dimensional Poisson's equation was solved on uniform grid to validate the rectangular finite element method (9-point FDM), six order CFDM and eight order CFDM. Table (4.1),table (4.2) and table (4.3) shows that absolute errors, Exact solution and Numerical solution solved by FEM six order CFDM and eight order CFDM. The numerical solutions obtained by FEM have lower accuracy than that of six order compact finite difference scheme as we observed from table (4.1) and table (4.2). The results indicate that the FEM based on a stencil of nine points can obtain the numerical solutions with second order accuracy, while the numerical solutions obtained by the six order CFDM has six order accuracy based on the uniform grid. The eight order CFDM which have a stencil of twenty five points, that is more accurate than FEM and six order CFDM as we observed from the table (4.3). The values of maximum absolute errors from the

FEM, six order compact finite difference method and six order CFDM are equal to $9.441686172 \times 10^{-3}$, $1.171141370 \times 10^{-4}$ and $2.947395751 \times 10^{-5}$, respectively. Figure (4.1) shows that the graph of numerical solution is not more related to the graph of exact solution. The figure (4.2) shows that the numerical solution is more related to the graph of exact solution when we compare with Figure (4.1), but in figure (4.3) the numerical solution is more related or quit equal to the exact solution. Therefore eight order CFDM is higher accuracy than FEM and six order CFDM for solving 2D-Poisson's equation with Dirichlet boundary condition on uniform grids.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

High order compact finite difference and rectangular finite element method are developed for solving 2D-elliptic equation. The high order compact finite difference schemes are derived to solve the Poisson equation subject to Dirichlet boundary condition on a bounded two dimension region. We use Taylor series expansion to drive the high order compact finite difference method. The proposed method is developed on a stencil of 25 points and it can approach to the eight order accuracy for solving 2-dimensional Poisson's equation. This thesis presents the basic understanding of six order compact finite difference method , eight order compact finite difference method and Finite Element Method. We use an example to test the accuracy of these schemes. Generally the results indicate that, the accuracy of the sixth order schemes is very high compare with the finite element scheme (9-point finite difference scheme); but the accuracy of the eight order schemes is very high compare with the both finite element scheme and six order schemes.

5.2 Recommendation

The result obtained in this study is on the solution of two dimensional Poisson equations using finite element approximation, six order CFDM and eight order CFDM with uniform mesh size for compeering the results.

This study can be extended further.

One may make analysis of this method for higher dimension with uniform mesh and can make comparison with other methods.

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