ANALYZING THE RADIUS OF SPATIAL ANALYTICITY FOR SOLUTIONS OF NONLINEAR DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

Sileshi Mebrate Tegegn

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Sciences

Presented in Fulfillment of the Requirement for the Degree of Doctor of Philosophy in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

June, 2024

Adama, Ethiopia

ANALYZING THE RADIUS OF SPATIAL ANALYTICITY FOR SOLUTIONS OF NONLINEAR DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

Sileshi Mebrate Tegegn

Main supervisor: Dr. Achenef Tesfahun Temesgen

Cosupervisor: Dr. Tamirat Temesgen Dufera

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Sciences

Presented in Fulfillment of the Requirement for the Degree of Doctor of Philosophy in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

June, 2024

Adama, Ethiopia

APPROVAL PAGE

We hereby certify that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the dissertation entitled "**Analyzing radius of spatial analyticity for solutions of nonlinear dispersive partial differential equations**" by **Sileshi Mebrate Tegegn**.

____________________	____________	________________
Main supervisor	Signature	Date
____________________	____________	________________
Co-supervisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the dissertation open defense by Sileshi Mebrate Tegegn have read and evaluated the dissertation entitled "**Analyzing radius of spatial analyticity for solutions of nonlinear dispersive partial differential equations**" and examined the candidate during open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in Applied Mathematics.

____________________	____________	________________
Chair person	Signature	Date
____________________	____________	________________
Internal Examiner 1	Signature	Date
____________________	____________	________________
Internal Examiner 2	Signature	Date
____________________	____________	________________
External Examiner 1	Signature	Date
____________________	____________	________________
External Examiner 2	Signature	Date

Finally, approval and acceptance of the dissertation is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the candidate's Department Graduate Council (DGC) and School Graduate Committee (SGC).

____________________	_____________	_________________
Department head	Signature	Date
____________________	_____________	_________________
School Dean	Signature	Date
____________________	_____________	_________________
Office of Postgraduate Studies, Dean	Signature	Date

DECLARATION

I declare that this dissertation entitled "**Analyzing radius of spatial analyticity for solutions of nonlinear dispersive partial differential equations**" is my own work and has not been submitted to any university for similar purpose. The references used in this dissertation are duly recognized by proper citations.

______________________	______________	__________________
Name of Student	Signature	Date

I, the main supervisor/co-supervisor of this dissertation, hereby certify that I have closely supervised the student while developing this dissertation and read the draft entitled "**Analyzing radius of spatial analyticity for solutions of nonlinear dispersive partial differential equations**" prepared under my guidance by **Sileshi Mebrate Tegegn** submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics. Therefore, I recommend the submission of the dissertation to the department for further review and evaluation.

______________________	______________	__________________
Main supervisor	Signature	Date
______________________	______________	__________________
Co-supervisor	Signature	Date

ACKNOWLEDGEMENT

This work was successful with the help from the almighty God. Next to God my sincere thanks be expressed to my main supervisor, **Dr. Achenef Tesfahun**, for supporting me in all the way I walked. His patient guidance, encouragement, incredible support and immense knowledge on the subject has helped me a lot to accomplish my research works. I appreciate and thank him for what I learnt from him during our virtual communication what research in mathematics is. His meticulous edition gave life to my scratch works and helped me much in understanding what I wrote. His edition helped me to learn writting skills in mathematics. I thank him for his continued encouragement and enthusiasm throughout the research process, along with the personal and proffessional support offered by him throughout. Also I donot forget the sacrifies Dr. Achenef paid to me when he came and visited me in Adama.

I would like also to thank my co-supervisor **Dr. Tamirat Temesgen**, giving me the opportunity to join PhD program under Adama Science and Technology University and for motivation, encouragement and technical support he has provided me through out the study. I was always surprised for his patience to gothrough my work, check every thing and give me strong feedback. Also he taught me how to use Latex from the very beginning with great care. Dr. Tamirat thank you for your all rounded support.

My special thanks goes to the Department of Applied Mathematics, Adama Science and Technology University, for giving me the chance to join the PhD program and for its financial support. Also I would like to thank Adis Ababa university department of mathematics for inviting many guests where I meet my main supervisor for the first time. Here also my heart felt thank goes to Dr.Legese Lemecha who have strong motivation to help students by inviting guest professors from abroad.

My wife Rahel, kids Nathan and Heran you were the reason for my strength thank you. I am also thankful to my fellow PhD students for thier valuable advice and encouragements.

Finally, I am extremely grateful to thank my uncle Belete Adugna and his family for their encouragement and support throughout my life and particularly during the study.

Contents

List of Abrevations

BBM	Benjamin-Bona-Mahony
IVP	Initial value problem
KdV	Korteweg de-Vries
KdV-BBM	Korteweg de-Vries- Benjamin-Bona-Mahony
LHS	Left hand side
ODE	Ordinary differential equation
PDE	Partial differential equation
RHS	Right hand side
Supp$f$	Support of f

List of symbols

$\mathscr{F}$	Fourier transform of f
$\mathscr{F}^{-1}(x)$	Inverse Fourier transform of f
$\mathscr{F}_x(\xi)$	Fourier transform with respect to space
$\mathscr{F}_t(\tau)$	Fourier transform with respect to time
$\mathscr{F}_{x,t}(\xi,\tau)$	Fourier transform with respect to space and time
$\mathscr{S}(\mathbb{R})$	The space of Shwartz functions
$\mathscr{S}'(\mathbb{R})$	Distributions
$C_0^\infty(\mathbb{R})$	Space of continuous functions with compact support
$\overline{f(x)}$	Complex conjugate of f
Δ	Laplace operator
$p \lesssim q$	There exist constant $c > 0$ such that $p \leq cq$
$p \sim q$	$p \lesssim q$ and $q \lesssim p$
$\langle \xi \rangle$	$\sqrt{1+\lvert\xi\rvert^2}$
D	$-i\partial_x$
$\mathbb{N}$	Set of natural numbers

List of publications

Part of this dissertation has been published and accepted in the form of the following papers in peer reviewed indexed journals in web of science and scopes

- Tamirat T.Dufera, Sileshi Mebrate and Achenef Tesfahun(2022): On the persistence of spatial analyticity for the beam equation. Journal of Mathematical Analysis and Applications. Volume 509, Issue 2. https://doi.org/10.1016/j.jmaa.2022.126001
- Tamirat T.Dufera, Sileshi Mebrate and Achenef Tesfahun (2024): New lower bounds for the radius of analyticity of solutions to the fifth order KdV-BBM model. Bulletin of the Iranian Mathematical Society (Submitted on October 24, 2023 and accepted for publication on March 19, 2024)

ABSTRACT

Water waves are caused by wind that blows on the surface of water. In ocean and sea, water wave is one of natural happenings that causes considerable damage on ships, boats and other onshore activities. Thus, for human being it is natural to forecast the future disorder to reduce or overcome damages and exploit an opportunities. Different mathematical models were used for such forcasting of natural phenomenon. Therefore, in this dissertation we consider the beam and fifth order Kortwege-de-Vries- Benjamin-Bona-Mahony (KdV-BBM) equations in one dimension that models weak interaction of dispersive waves and the unidirectional water waves respectively. The concept of dispersive partial differential equation which depends on the dispersion relation was explained. A new hyperbolic weight function was introduced. Using this newly introduced weight function a new modified space, which is norm equivalent to the existing Gervey space was defined. In this newly modified space the persistence of spatial analyticity for the solution of the beam equation was analyzed. In particular, for a class of analytic initial data with uniform radius of analyticity $\sigma_0 > 0$*, we obtained an asymptotic lower bound* $\sigma(t) \leq c/\sqrt{t}$ *on the uniform radius of analyticity* $\sigma(t)$ *of solution as time* t *goes to infinity. Also for the fifth-order KdV-BBM model with analytic initial data and fixed radius* σ_0*, we proved that the radius of spatial analyticity can not decay faster than* $1/\sqrt{T}$ *for any large time* T*.*

Key words: Beam equation, KdV-BBM equation, Modified Gervey space, radius of spatial analyticity, dispersive PDEs.

Chapter 1

Introduction

In this chapter we introduce some theories about water waves. Also we consider the origin of two water wave models, namely beam and fifth order KdV-BBM equations.

1.1 Background of the study

The waves on the surface of water are one of the most common phenomena we observe in our day-to-day life as all shipping activities takes place on water surfaces. Such waves are formed by wind. In ocean, ocean waves are formed as wind blows across the surface of the ocean, creating small ripples (a small wave or series of waves on the surface of water, especially as caused by a slight breeze or an object dropped in to it), which eventually becomes waves with increasing time and distance (Benjamin et al., 1972; Russell, 1885).

Most of the time waves begin as a disturbance of some kind, and the energy of that disturbance gets propagated in the form of waves. We are most familiar with the kind of waves that break on shore, or rock a boat at sea, but there are many other types of waves that are important to oceneography. Some of these waves are internal waves, tidal waves, tsunamies, splash waves and atmospheric waves. When waves reach shallow water, they become unstable and begin to break and can impose large hydrodynamic forces on organisms living in these regions. Also it affects the activities on the water part. Due to its influence on marine activities, the theory of water wave attracted many mathematicians. A french mathematician and physist, Joseph Valentin Boussinesq, Diederik Kortwedge and Gustav de Vries were among the first.

More generally, water waves are formed as the displaced water under the influence of gravity attempts to regain its equilibrium position. These waves are surface waves, a mixture of longitudinal and transverse waves. For example, surface waves in oceanography are deformation of the sea water.

Mathematical descriptions of many existing physical problems including water waves lead to either ordinary differential equations (ODE) or partial differential equations (PDE).

The model of many physical problems which appears in reality are PDEs. Among those models of PDEs few are dispersive and non-dispersive, linear and non linear. In this dissertation we mainly focus on nonlinear dispersive water wave models. The notion of dispersive PDE will be discused in chapter three.

It is challenging to predict and understand large magnitude, nonlinear water waves due to lack of appropreate mathematical models for the analysis of the underlying physics (Müller et al., 2005). Such large magnitude water wave can cause the catastrophic impact on ocean engineering structures and naval operations. Findings suggest that an extreme form of such dynamical evolution which takes place under water is the cause of freak or rogue waves with wave height which can be as large as eight times the standard deviation of the surrounding wave field (Onorato et al., 2005). Waves of this magnitude have caused considerable damage to ships, oil rigs and human life. In addition, many naval operations, e.g. transfer of cargo between ships moored together in a sea base, landing on aircraft carriers, or path planning of high-speed surface vehicles, require short-term prediction of the surrounding wave field. To make such predictions, unusual high wave elevations must be predicted reliably.

The most common nonlinear dispersive PDE models which are helpful to make prediction about waves on water are Kortweg-de Varies (KdV), Benjamin-Bona-Mahony (BBM) and Beam equation (in the case of one dimension).

The Kortweg-de Varies (KdV) equation is a nonlinear dispersive partial differential equation and was first introduced by Joseph Valentin Boussinesq in 1877 and rediscovered by Diederic Kortweg and Gustav de Varies in 1895. This is a mathematical model of waves on shallow water surfaces where the governing equation takes the form

$$\eta_t + \eta_x + \eta_{xxx} + \eta\eta_x = 0 \quad x,t \in \mathbb{R}, \tag{1.1.1}$$

where $\eta = \eta(x,t)$ is the displacement. Due to the dispersive term (that is, of third order), waves decay while waves still steepen due to the non linear term. The discovery of solitons is connected to this problem, where a soliton is a self re-inforcing solitary wave that maintains its shape while it travels at a constant speed. The soliton was first observed by John Scott Russell in 1834, while he was conducting experiments to determine the most efficient design for canal boats. This young Scottish engineer wrote the event in (Russell, 1885): "He was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in the channel which it had put in motion stoped; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. He followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles he lost it in the winding of the channel."

The KdV equation has also come up in the theory of plasma and several other branches of physics. The remarkable stability of the soliton, discovered only in the 1960s by computer experimentation in (Zabusky and Kruskal, 1965) can be described as follows. If we start with two solitons, the faster one will overtake the slower one and, after a complicated nonlinear interaction, the two solitons will emerge unscathed as they move to the right, except for a slight delay. In fact, it was observed from the computer output that every solution of exact equation (1.1.1), with any initial function $\eta(x,0) = \eta_0(x)$, seems to decompose as $t \to \infty$ into a finite number of solitons (of various speeds) plus a dispersive tail which gradually disappears. This kind of behavior is expected for linear problems, but that it could happen for a nonlinear problem was a complete surprise at the time. This special behavior induced physicists to use the soliton as a mathematical model of a stable elementary particle.

The second model, Benjamin-Bona-Mahony (BBM) is another water wave model which is governed by

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0 \quad x,t \in \mathbb{R}, \tag{1.1.2}$$

and it describes approximately the unidirectional propagation of long water waves in certain nonlinear dispersive system. The BBM equation is a regularized long wave equation which is equivalent to KdV (Benjamin et al., 1972) .

We consider the combined KdV and BBM equations in this study. The fifth order KdV-BBM partial differential equation is mathematically given by,

$$\begin{aligned} \eta_t + \eta_x - \gamma_1\eta_{xxt} + \gamma_2\eta_{xxx} + \sigma_1\eta_{xxxxt} + \sigma_2\eta_{xxxxx} = \\ -\frac{3}{4}(\eta^2)_x - \gamma(\eta^2)_{xxx} + \frac{7}{48}(\eta_x^2)_x + \frac{1}{8}(\eta^3)_x, \end{aligned} \tag{1.1.3}$$

which describes the unidirectional propagation of water waves, and this was recently introduced in (Bona et al., 2018) using the second order approximation in the two way model, the so-called *abcd*-system derived in (Bona et al., 2002, 2004). Carvajal and Panthee (2019, 2020) studied the global well-posedness and radius of spatial analyticity for the KdV-BBM equation. Also, (Belayneh et al., 2022) studied about the lower bound on the radius of spatial analyticity for the solution of fifth order KdV-BBM equation and obtained $\sigma(t) \sim 1/t$.

The next model we consider in this dissertation is the beam equation in one dimension. In construction (the case for two or three dimensions) beams are horizontal structure elements that withstand vertical loads, shear forces and bending moments. They transfer loads that imposed along their length to their end points such as walls, columns, foundations and so on. Some examples of beams in construction are simply supported beam, fixed beam, cantilever beam, continuous beam, Re-inforced concrete beam, beam bridge and so on. Mathematically the nonlinear beam equation that we consider in this dissertation is the fourth-order partial differential equation given by,

$$\begin{cases} u_{tt} + \Delta^2 u + mu + |u|^{p-1}u = 0 \\ (u, u_t)(x, 0) = (u_0, u_1)(x). \end{cases} \tag{1.1.4}$$

where, $u : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, $p > 1$, $m > 0$ and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Equation (1.1.4) is also referred to as Bretherton's type equation or simply the beam equation. The original Bretherton equation, written down for $n = 1$ by (Bretherton, 1964), arises in the study of weak interactions of dispersive waves. A similar equation to Bretherton for $n = 2$ was proposed in (Love, 2013) for the motion of a clamped plate. (Levandosky, 1998a; Pausader, 2007; Pausader and Strauss, 2009; Pausader, 2010) studied well about the beam equations.

For the above water wave models, if initial data are given, which extends analytically to a strip about the real axis and satisfy some weak integrability conditions, then it can be shown that the solution can also be extended analytically to a possibly smaller strip as long as the solution exists. The width of this strip is usually called the radius of spatial analyticity as it is around the real axis. In this dissertation, we study the radius of spatial analyticity for the solution of beam equation (1.1.4) and fifth order KdV-BBM equation (1.1.3) with given initial data in a class of analytic functions.

1.2 Statment of the Problem

The analysis of water wave is crutial in a day to day life of human being as most of intercontinental trade activities in the world takes place on water part. Eventhogh cost of transportation and risk is minimum on sea, waves on water surfaces are among the most challenging things for such trade activities accomplished by boat and ship. Thus we need to guess the wave behaviour ahead of time. To forecast such wave nature, we have to know source of the wave, height, speed, radius and behaviour as time goes. The beam equation and fifth order KdV-BBM equations are among few water wave models. The analysis of water wave using these models play crucial role for the safety of ships and other onshore activities. Analyzing the radius of spatial analyticity for a holomorphic function (PDE models) plays a great role in forcasting the disaster caused by water waves and alarming onshore activities. Thus, it is reasonable to study the rate at which the radius of waves decreases as time goes.

Many authors (Ahn et al., 2021; Carvajal and Panthee, 2020; Levandosky, 1998b,a; Selberg and Da Silva, 2015; Selberg and Tesfahun, 2017) studied about the radius of spatial analyticity for different solutions of partial differential equations and established either an exponential or polynomial decay rate. The reason for the authors to achieve an exponential decay rate is due to consideration of initial data in the Sobolev space only. In particular, (Carvajal and Panthee, 2020) established an exponential radius of analyticity for the solution of fifth order KdV-BBM equation in the sobolev space and established an exponential decay rate. Few years latter, (Belayneh et al., 2022) improved the radius of analyticity for the solution of fifth order KdV-BBM equation to polynomial, where the authers considered a Gervey space. Thus, we raise a question ”why only the Sobolev space and Gervey space?” It is still possible to improve further this radius of spatial analyticity by introducing a new function space. This newly introduced space can also be implemented to the analysis of radius of spatial analyticity for the solution of beam equation.

The purpose of this research is to make analysis on the radius of spatial analyticity for the solution of water wave models inorder to know their behaviour. There are other water wave models, but we consider only the beam and fifth order KdV-BBM models.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this study is to investigate lower bound on the radius of spatial analyticity for the solution of the beam and fifth order KdV-BBM nonlinear dispersive PDEs.

1.3.2 Specific objectives

The specific objectives of the study are to:

- investigate lower bound on the radius of spatial analyticity for the solution of beam equation.
- improve the radius of spatial analyticity for the solution of fifth order KdV-BBM equation.

1.4 Significance of the study

Nonlinear dispersive PDE governs the behaviuor of waves in various physical systems such as fluid dynamics, nonlinear optics, plasma in physics and solid state physics (Tao, 2006). Understanding the radius of analyticity for solutions of the governing equation provides an insight in to the formation, propagation and interaction of waves in these systems. Since water waves cause significant damage on offshore activities like cargo exchanges and ship movements, individuals who work research on forcasting and minimizing such wave damages can use the result for further analysis. Thus, the results obtained so far in this dissertation are important for the mathematicians and engineers. The weight function we introduced here is new and hence many mathematicians may consider it to improve the lower bound on the radius of spatial analyticity for the solution of some nonelinear PDEs.

Inconclusion, studying the propagation of spatial analyticity for the solution of nonlinear dispersive PDEs is not only to advance understanding wave phenomena but also has broad implications for scientific research, engineering applications and innovations.

1.5 Scope of the study

This dissertation is limited to the study of radius of spatial analyticity for the solution of the beam equation and fifth order KdV-BBM equations due to time.

1.6 Organization of the dissertation

The next parts of this disseration are organized as follows. In chapter 2, we review some significant and related litratures on well-posedness of the beam and fifth order KdV-BBM equations. Chapter 3 explains the method we used to prove well-posedness of a problem in the given function space and some mathematical concepts that we use in the dissertation.

In chapter 4, the beam equation is introduced and the Paley-Wiener theorom is stated. We also introduced modified Gervey space $H^{\sigma,s}(\mathbb{R}^n)$ and show equivalence with the existing Gervey space $G^{\sigma,s}(\mathbb{R}^n)$. Local well-posedness and approximate conservation law for the beam equations using a modified Gervey space is stated and proved. Ofcourse consequative lemmas are introduced and proved inorder to prove approximate conservation law. Finally we state our main results and prove it using local well-posedness theorem and approximate conservation law successively.

In chapter 5, the above steps were repeated for the fifth order KdV-BBM equation and this completes the dissertation.

Chapter 2

Review of related litratures

In this chapter, we consider and rehears the existing well-posedness theories of the beam equation in the space $H^2 \times L^2$ and the fifth order KdV-BBM equation in H^s. After the early work by Isaac Newton, who attempt theory of water waves, the eighteenth and early nineteenth century French mathematicians Laplace, Lagrange, Poisson, and Cauchy made real theoretical advances in the linear theory of water waves (Craik, 2004). In Germany, Gerstner considered nonlinear waves. Later in Britain during 1837-1847, Rusell, Green, Kelland, Airy, and Earnshaw all made sustainable contributions. They set the scene for subsequent work by Stokes(1846) and others.

An overwhelmingly large portion of modeling of the world is done by posing a condition and solving differential equations, which leads to well-posedness of the problem. For several years, researchers studied the well-posedness of many physical problems in an appropreate spaces. Because of this well-posedness (both local and global) property further analysis of the problem can be conducted. Also, after (Foias and Temam, 1989) introduced the Gervey space $G^{\sigma,s}(\mathbb{R})$, many researches were conducted on the radius of spatial analyticity for the solution of different linear and nonlinear dispersive PDEs. In this chapter, we consider the well-posedness theories in litrature for the beam and fifth order KdV-BBM equations.

2.1 Well-posedness of beam equation

The Brethorton equation (1.1.4) (the beam equation in one dimension) was introduced by a Mathematician Francis Brethorton in 1964. The original equation studied by Bretherton has quadratic nonlinearity, $p = 2$. Well-posedness of the beam equation, which models weak interaction of water waves in one dimension was studied by different authors. For example, (Levandosky, 1998a) made an analysis of the stablity of traveling wave solutions of the beam equation (1.1.4) (fixing $m = 1$). Levandowsky show that the evolution equation admits solutions in the space $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ which exists locally in time for given initial data in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ provided $p < \frac{n+4}{n-4}$ for $n \geq 5, n \in \mathbb{N}$ and without any restriction on p for

$n < 5$. The author also showed that there exist solitary wave solution of equation (1.1.4) and prove criterion for their stability and instability.

Nine years latter, (Pausader, 2007) investigated the scattering theory in the energy space $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for the equation (1.1.4). The author observed that equation (1.1.4) is a formal fourth order extension of the classical Klein-Gordon equation, but it also inherits a Schrodinger structure because of the decomposition of differential operator given by,

$$\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta).$$

However, it can be noted that the equation satisfies neither finite speed propagation nor mass conservation which creates dificulties. Here scattering is to mean roughly that solutions of the equation can be approximated by solutions of a model equation, in the case of (1.1.4) the linear equation, when time becomes infinite.

Next, (Zhang, 2010) proved that the defocussing cubic equation (i.e, $p = 3$ in (1.1.4)) is globally well-posed in the energy space $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ given initial data $(u_0, u_1) \in H^s(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with $3 \leq n \leq 7$ provided that $\min\{\frac{n-2}{2}, \frac{n}{4}\} < s < 2$. The author remarks that the case for $n = 3$ for the result is similar to the case for $n = 4$. This is true because one can modify the ideas to obtain $s > \frac{n-2}{2}$. The index $\frac{n-2}{2}$ comes from the similar arguement of $n = 4$, since $2n = 4n - 8$ for $n = 4$. Also for $n = 7$, the proof of $n = 5, 6$ can be modified to gain an analogous result which holds for $s > 7/4$ and $n = 7$.

2.2 Well-posedness of the fifth order KdV-BBM equation

There are diffent water wave models and assumptions. In particular, it is understood that the KdV equations, appear to describe behaviour of unidirectional long water waves in nonlinear dispersive media (Zakharov, 1968). The derivation of such model equation in specific physical situations is presented in (Benjamin et al., 1972). Concerning the fifth order KdV-BBM equation, (Bona et al., 2018) established local well-posedness of the Cauchy problem (1.1.3) with initial data in the Sobolev space $H^s(\mathbb{R})$ for $s \geq 1$. When the parameters $\gamma_1, \sigma_1 > 0$ and $\gamma = 7/48$, the authors used the energy conservation to prove global well-posedness of (1.1.3) for data in $H^s(\mathbb{R})$, $s \geq 2$. Furthermore, the authors used the method of *high-low frequency splitting* to obtain global well-posedness for data with Sobolev regularity $H^s(\mathbb{R})$ where $3/2 \leq s < 2$.

The global well-posedness result of the KdV-BBM equation was further improved in (Carvajal and Panthee, 2019) for initial data with Sobolev regularity $s \geq 1$. The authors also proved an ill-posedness result by showing that the flow-map is not continuous if the given data has Sobolev regularity $s < 1$. Furthermore, (Carvajal and Panthee, 2020) studied the well-posedness of the fifth order KdV-BBM equation in the space of analytic functions the so called Gevrey class of functions, and evolution of radius of analyticity. Initially they found a solution $\eta(x,t)$ of the initial value problem (1.1.3) with real analytic initial data η_0 which admits extension as an analytic function to the complex strip $s_{\sigma_0} := \{x+iy : |y| < \sigma_0\}$ for some $\sigma_0 > 0$ at least for a short time (i.e., local in time). They established various multilinear estimates which were useful in the proof of the local well-posedness result. First they record the $G^{\sigma,s}(\mathbb{R})$ version of the sharp bilinear estimate obtained in (Bona and Tzvetkov, 2009) and then estimate the nonlinear terms. Finally they established an exponential radius of analyticity for the solution of fifth order KdV-BBM equation.

Few years latter, (Belayneh et al., 2022) improved the earlier results of Carvajal and Panthee. The authors showed the uniform radius of spatial analyticity $\sigma(t)$ of solution at time t for (1.1.3) with initial data cannot decay faster than $1/t$. First, the authors proved local in time well-posedness of (1.1.3) in the Gervey space. Next, they introduced an idea of approximate conservation law by multiplying the solution $\eta(x,t)$ with an exponential weight function $e^{\sigma|D|}$, where D is differential operator. The new energy (called modified energy) is computed and observed that for $\sigma = 0$ energy is conserved. However this conservation fails to hold for $\sigma > 0$, which leads the authors to prove the approximate conservation law. Finally after proving local well-posedness and approximate conservation law the authors constructed a solution on $[0,T]$ where T is arbitrarly large time. Thus, they established a decay rate of $1/T$ for the fifth order KdV-BBM equation.

The above flow of work showed that still it is open and possible to improve the lower bound on the radius of spatial analyticity for the solution of beam and fifth order KdV-BBM equations.

Chapter 3

Research Methodology and Material

3.1 Research methedology

Many authors followed different approches to get the maximum decay rate for dispersive PDEs. In this dissertation we introduce a new weight function (consequently a new space) to get the maximum decay rate. Multiplying the solution of beam and fifth order KdV-BBM equations by a weight function $e^{\sigma|\xi|}$ that already exist in litratures to introduce the idea of approximate conservation law can not help us to improve further the decay rate achieved earlier. Thus, we introduce a new weight function $\cosh(\sigma|\xi|)$ to improve earlier rsults as it gives more fredom than the previous exponential weight function.

We apply the Banach Fixed point theorem (contraction mapping) that can be stated below inorder to check whether a given equation has a unique solution or not.

3.1.1 Banach fixed point theorem

Banach's fixed point theorem (also called the contraction principle) is one of the most important tool used to solve nonlinear equations. This is for instance the main ingredient in the implicit function theorem and the inverse function theorem. It can also be used to prove the Picard-Lindelöf theorem on existence and uniqueness of solutions of the initial value problem for ordinary differential equations.

Definition 3.1.1. (Royden and Fitzpatrick, 1988) Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces. A map $T : X \to Y$ is called a contraction if there exist $\theta \in (0,1)$ such that

$$\|T(x) - T(y)\|_Y \leq \theta \|x - y\|_X, \quad \text{for all} \quad x, y \in X.$$

Theorem 3.1.1. *(Royden and Fitzpatrick, 1988)[Banach Fixed Point Theorem] Let $(X, \|.\|_X)$ be a Banach space. If $T : X \to X$ is a contraction map, then there exist a unique solution $x \in X$ of the fixed point equation $T(x) = x$.*

Proof. Assume x and y are two solutions of equation $T(x) = x$, i.e, $T(x) = x$ and $T(y) = y$. Then

$$\|x-y\|_X = \|T(x)-T(y)\|_X \leq \theta \|x-y\|_X,$$

which inturn implies

$$(1-\theta)\|x-y\|_X \leq 0.$$

But since $\theta \in (0,1)$, we have $\|x-y\|_X = 0$ where it follows $x = y$ and this proves uniqueness of solution.

To prove existence of solution let's pick arbitrary point $x_0 \in X$ and set $x_1 = T(x_0)$, $x_2 = T(x_1)$, $x_3 = T(x_2), \ldots$. Then from the assumption,

$$\begin{aligned}\|x_{n+1}-x_n\|_X &= \|T(x_n)-T(x_{n-1})\|_X \\ &\leq \theta \|x_n - x_{n-1}\|_X = \theta \|T(x_{n-2})-T(x_{n-2})\|_X \\ &\leq \theta \|x_{n-1}-x_{n-2}\|_X.\end{aligned}$$

Repeating the iteration n times we have

$$\|x_{n+1}-x_n\|_X \leq \theta^n \|x_1 - x_0\|_X.$$

Now for $m > n$,

$$\begin{aligned}\|x_m - x_n\|_X &= \|x_m - x_{m-1} + x_{m-1} + x_{m-2} - x_{m-2} + \cdots - x_n\|_X \\ &\leq \|x_m - x_{m-1}\|_X + \|x_{m-1} - x_{m-2}\|_X + \cdots + \|x_{n+1} - x_n\|_X \\ &\leq (\theta^m + \theta^{m-1} + \theta^{m-2} + \ldots)\|x_1 - x_0\|_X \\ &\leq \sum_{k=n}^{\infty} \theta^k \|x_1 - x_0\|_X \longrightarrow 0\end{aligned}$$

because $\sum_{k=n}^{\infty} \theta^k$ is a geometric series. Thus

$$\|x_m - x_n\|_X \longrightarrow 0 \quad \text{as} \quad n, m \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence and hence it has a limit say $x \in X$ as X is a Banach space. The fact that x is a solution of the fixed point problem follows by passing to the limit in the relation $x_{n+1} = T(x_n)$ and as $x_{n+1} \to x$ it follows that $T(x_n) \to T(x)$. □

Example 3.1.1. Consider an initial value problem (IVP) to the non-linear Ordinary differential (ODE) equation given by:

$$\begin{cases} u'(t) = f(t,u) \\ u(0) = u_0 \in \mathbb{R}^d \end{cases}$$

where f is a function such that $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and satisfies the following properties,

i) $\|f(t,x(t))\| \leq c\|x(t)\|$ which guarantees f is bounded .

ii) f is lipschitz, i.e,

$$\|f(t,x(t)) - f(t,y(t))\| \leq c\|x(t) - y(t)\|$$

for some constant $c > 0$ and all $x,y \in \mathbb{R}^d, t \in \mathbb{R}$.

Here $\|x\|$ is the usual norm in $\mathbb{R}^d$, i.e, $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$.

Now, the major question of interest is that does a solution to the above IVP exist? if so, is the solution unique? In order to answer the above questions let's transform the IVP in to an integral equation of the form

$$u(t) = u_0 + \int_0^t f(s,u(s))ds.$$

Define the mapping from u to $\Gamma(u)$ by

$$\Gamma(u(t)) := u_0 + \int_0^t f(s,u(s))ds$$

such that the problem of existence and uniqueness of solution reduces to finding a fixed point of the mapping $\Gamma : X \to X$ for some Banach space X. Now, as a contraction space choose the Banach space X such that

$$X = C\big([0,T], \mathbb{R}^d\big)$$

endowed with the norm

$$\|u\|_X = \max_{t \in [0,T]} |u(t)|.$$

Next to this, we have to show

a) Γ maps X in to X. i.e, Γ is well defined in X.

b) Γ is a contraction in X. i.e,

$$\|\Gamma(x)-\Gamma(y)\|_X \leq \theta\|x-y\|_X \text{ for } 0<\theta<1.$$

To show that Γ is well defined

$$\begin{aligned}
\|\Gamma(u(t))\| &= \left\| u_0 + \int_0^t f(s,u(s))ds \right\| \\
&\leq \|u_0\| + \int_0^t \|f(s,u(s))ds\| \\
&\leq \|u_0\| + \int_0^T c\|u(s)\|ds \\
&\leq \|u_0\| + Tc\|u(s)\|.
\end{aligned}$$

Thus, taking maximum over $s,t \in [0,T]$ on both sides of the above last equation we have

$$\|\Gamma(u(t))\|_X \leq \|u_0\|_X + Tc\|u(s)\|_X.$$

This shows that for any $u \in X$, $\Gamma(u) \in X$ because $\|\Gamma(u)\|_X < \infty$. Next, to show that Γ is a contraction mapping, consider

$$\Gamma(u(t)) - \Gamma(v(t)) = \int_0^t \Big(f(s,u(s)) - f(s,v(s)) \Big) ds,$$

and hence from Minkowiski's inequality for integral

$$\begin{aligned}
\|\Gamma(u(t)) - \Gamma(v(t))\| &\leq \int_0^T \|f(s,u(s)) - f(s,v(s))\| ds \\
&\leq \int_0^T c\|u(s)-v(s)\|ds.
\end{aligned}$$

Taking maximum over $s,t \in [0,T]$ it follows
$\|\Gamma(u)-\Gamma(v)\|_X \leq cT\|u-v\|_X$.
Now, if we choose $T < 1/c$, then Γ is contraction in X. Therefore by the Banach fixed point theorem there exist a unique solution $u \in X$ whenever $T < 1/c$.

3.2 Materials

In this section, we give emphasis to some existing concepts, definitions and remarks that we frequently use in the dissertation.

3.2.1 Dispersive Partial differential equations

To illustrate the notion of dispersive partial differential equations let's consider theory of waves. The first encounter with mathematical theory of wave is usually with cosine or sine waves of the form

$$u(x,t) = a\cos(kx - \omega t) \quad x,t \in \mathbb{R},$$

while the parameters a represent the amplitude of the wave, k represent the wave number and ω is the angular frequency. Also note that $u(x,t)$ is periodic both in space and time with periods $\lambda = 2\pi/k, \quad T = 2\pi/\omega$. The parameter λ is called the wave length and T is period. The wave has local maxima (crest) when $kx - \omega t$ is an even multiple of π and local minima (troughs) when it is an odd multiple of π. Now we can rewrite u as

$$u(x,t) = a\cos(k(x - ct)),$$

where $c = \omega/k$. As time changes the fixed shape is simply being translated at a constant speed c along the x-axis. The number c is called the wave speed or phase speed. Such waves move to the right or left depending on the sign of c. A wave of permanent shape moving at a constant speed is called traveling waves.

In general, one considers sinusoidal wave (periodic oscillations) of the form

$$u(x,t) = ae^{i(kx \pm \omega t)} = a\left[\cos(kx \pm \omega t) + i\sin(kx \pm \omega t)\right],$$

to test weather a wave is traveling at a constant speed or not.

Now, using this sinusoidal wave we introduce the notion of dispersion. In many situations and reality the assumption that wave speed becomes constant is unrealistic. Thus, if c is a non constant function of k, i.e, $c = c(k)$ one says that the wave (or rather the equation) is dispersive (Tao, 2006). The differential equation is dispersive means waves of different wave lengths travel with different speeds. An initially localized wave will disintegrate into separate components according wave length and disperse. The relation between ω and k is called the dispersion relation. For example, if we consider the Schrodinger equation

$$iu_t + u_{xx} = 0, \quad x,t \in \mathbb{R}.$$

and seek for a sinusoidal solution of the form

$$u(x,t) = a\big[\cos(kx - \omega t) + i\sin(kx - \omega t)\big],$$

then we have the dispersion relation $\omega : \omega(k) = k^2$ and the wave speed $c : c(k) = k$ for the Schrodinger equation. Since c is not constant, the Schrodinger equation is dispersive. On the contrary, the linear wave equation in one dimension given by,

$$u_{tt} - u_{xx} = 0,$$

is non-dispersive as the dispersive relation $\omega = |k|$ and the wave speed is $c = \pm 1$, which is to mean the wave travels to either right or left depending on the sign of c with a constant speed.

3.2.2 L^p function spaces

Definition 3.2.1. (Folland, 1999) A function $f : X \to Y$ is said to be measurable if for each $a \in Y$, $\{x \in X; f(x) > a\}$ is a measurable set.

Definition 3.2.2. (Folland, 1999) Suppose f is a measurable function such that $f : \mathbb{R}^d \to C$, then norm of f in L^p is defined as

$$\|f\|_{L^p(\mathbb{R}^d)} = \begin{cases} \big(\int_{\mathbb{R}^d} |f(x)|^p dx\big)^{\frac{1}{p}} & \text{for} \quad p \in [1, \infty), \\ ess\sup|f(x)| & \text{for} \quad p = \infty. \end{cases}$$

Shortly, we can write

$$L^p(\mathbb{R}^d) = \{f : \mathbb{R}^d \to C, \quad f \text{ is measurable such that} \quad \|f\|_{L^p(\mathbb{R}^d)} < \infty\}.$$

The following theorems, Minkowski's inequality, Minkowski's inequality for integral and Holder inequality are taken from (Folland, 1999) without proof for later use.

Theorem 3.2.1 (Minkowski's inequality)**.** *If $1 \leq p \leq \infty$ and two functions $f, g \in L^p(\mathbb{R}^d)$, then*

$$\|f + g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} + \|g\|_{L^p(\mathbb{R}^d)}.$$

This Minkowski's inequality is also called triangle inequality.

Minkowski's inequality shows that $\|.\|_{L^p(\mathbb{R}^d)}$ defines a semi-norm on L^p for $1 \le p \le \infty$. It is not a norm, however, since $\|f\|_{L^p(\mathbb{R}^d)} = 0$ when $f = 0$ almost everywhere (a.e). This is easily fixed by introducing an equivalence class on measurable functions by defining two functions in $L^p(\mathbb{R}^d)$ as equivalent if they are equal a.e. So an element of $L^p(\mathbb{R}^d)$ is not a function; it is an equivalence class of functions which are equal a.e. A great advantage of working with $L^p(\mathbb{R}^d)$ spaces is that they are complete normed space, i.e. it's a Banach space.

Theorem 3.2.2 (Minkowski's inequality for integral)**.** *Let* $1 \le p \le \infty$. *Suppose that* f *is measurable on* $\mathbb{R}^n \times \mathbb{R}^d$ *, that* $f(.,y) \in L^p(\mathbb{R}^n)$ *for almost every* $y \in \mathbb{R}^d$ *, and that the function* $y \to \|f(.,y)\|_{L^p(\mathbb{R}^n)}$ *belongs to* $L^1(\mathbb{R}^d)$. *Then the function* $x \to \int_{\mathbb{R}^d} f(x,y)dy$ *belongs to* $L^p(\mathbb{R}^d)$ *and*

$$\left\| \int_{\mathbb{R}^d} f(x,y)dy \right\|_{L^p(\mathbb{R}^n)} \le \int_{\mathbb{R}^d} \|f(.,y)\|_{L^p(\mathbb{R}^n)} dy.$$

Theorem 3.2.3 (Holder inequality)**.** *Suppose that* $1 \le p \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *(i.e.,* p,q *are conjugates). If* f *and* g *are measurable functions on* $\mathbb{R}^d$ *such that* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *then,* $fg \in L^1(\mathbb{R}^d)$ *and*

$$\|fg\|_{L^1(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

More generally, if $1 \le p,q \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *then for* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *we have* $fg \in L^1(\mathbb{R}^d)$ *and*

$$\|fg\|_{L^r(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

3.2.3 Convolution

Definition 3.2.3. (McLean and McLean, 2000) If f and g are measurable functions, the convolution of f and g is denoted by $f * g$ and defined by

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy, \quad x,y \in \mathbb{R}^d.$$

Note that convolution is commutative. i.e, $(f*g)(x) = (g*f)(x)$. Also this definition makes sense for example if f is in L^1 and g is bounded.

Corollary 3.2.4. *(McLean and McLean, 2000)[Young's inequality] Suppose that* $1 \le p,q,r \le \infty$ *and* $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ *and* $f \in L^p(\mathbb{R}^d)$ *and* $g \in L^q(\mathbb{R}^d)$, *then* $f*g \in L^r(\mathbb{R}^d)$ *with*

$$\|f*g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Definition 3.2.4. (McLean and McLean, 2000) Support of a function f is denoted by $Suppf$ and given by $Suppf = \{x \in \mathbb{R}^d : f(x) \neq 0\}$.

Example 3.2.1. Consider

$$f(x) = \begin{cases} 2x^2 + 1, & -1 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

$Suppf = [-1, 1]$.

3.2.4 Fourier Transform

A mathematical transform which decomposes functions depending on space or time in to functions depending on spatial frequency or temporal frequency is known as Fourier transform (Kammler (2007)). We define the Fourier transform of function in the Schwarz class $\mathscr{S}(\mathbb{R})$ that will be defined latter. The main reason for using $\mathscr{S}(\mathbb{R})$ instead of $C_0^\infty(\mathbb{R})$ (the space of continuous function with compact support) is that, the former class is invariant under the Fourier transform, while the second class may not be. We restrict our definition of Fourier transform to functons of one variable, which can be extended to $\mathbb{R}^d$ for $d \geq 2$.

Definition 3.2.5. (McLean and McLean, 2000)[Schwartz space] The Schwartz space on $\mathbb{R}$ denoted $\mathscr{S}(\mathbb{R})$, is the space of rapidly decreasing smooth functions,

$$\mathscr{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^j f^k(x)| < \infty \quad \text{for all} \quad j, k \in \mathbb{N}_0\}.$$

Now, we can see topology on the Schwartz space $\mathscr{S}(\mathbb{R})$

Definition 3.2.6. (McLean and McLean, 2000) A sequence $\varphi_n \in \mathscr{S}(\mathbb{R})$ converges to a function φ, written as $n \to \infty$

$$\varphi_n \to \varphi \in \mathscr{S}(\mathbb{R}),$$

if the following are satisfied

i) There exist a compact set $K \subset \mathbb{R}$ such that support of $\varphi_n \subset K$ for all n.

ii) There exist $\varphi \in \mathscr{S}(\mathbb{R})$ such that as $n \to \infty$, $\varphi_n^k \to \varphi^k$ uniformly for all $k \in \mathbb{N}_0$.

Example 3.2.2. Let $f(x) = e^{-x^2}$, then $f(x)$ is a Schwartz function, i.e, $f(x) \in \mathscr{S}(\mathbb{R})$. Also we notice that $f(x)$ is not in $C_0^\infty(\mathbb{R})$ (the space of continuous function with compact support) as the function has no compact support on $\mathbb{R}$.

3.2.5 Distributions

Distributions are generalizations of the classical concept of functions. One of the most useful aspects of the theory of distribution is that, in application, discontinuous functions can be handled as easily as continuous functions.

Definition 3.2.7. (McLean and McLean, 2000) Suppose Ω be a domain in $\mathbb{R}$ and $\mathscr{S}(\Omega)$ be a test function. The set of distributions (generalized functions) denoted by $\mathscr{S}'(\Omega)$ is the collection of all complex- valued linear functionals f over $\mathscr{S}(\Omega)$, i.e.,$f : \mathscr{S}(\Omega) \to \mathbb{C}$ where the value of f acting on a test function φ denoted by $f(\varphi) := f\varphi := \langle f, \varphi \rangle$, satisfies the following linearity and continuity criteria

i) For any $\alpha, \beta \in \mathbb{C}$, $\varphi_1, \varphi_2 \in \mathscr{S}(\Omega)$, $\langle f, \alpha\varphi_1 + \beta\varphi_2 \rangle = \alpha\langle f, \varphi_1 \rangle + \beta\langle f, \varphi_2 \rangle$.

ii) $\langle f, \varphi_n \rangle \to \langle f, \varphi \rangle$ for $n \to \infty$ in $\mathbb{C}$, whenever $\varphi_n \to \varphi$ in $\mathscr{S}(\Omega)$.

The simplest example of distribution is the functional generated by a locally integrable function, $f(x) \in L^1_{Loc}(\Omega)$ given by

$$\langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx \quad \text{for all} \quad \varphi \in \mathscr{S}(\Omega).$$

Example 3.2.3. The Dirac Delta function at a point (let the point be 0) denoted by δ_0 is defined as

$$\delta_0 : \mathscr{S} \to \mathbb{C}, \quad \text{such that} \quad \delta_0(\varphi) = \varphi(0)$$

is a distribution.

To show linearity, let α, β be any two constants in the set of complex numbers and φ be test function then,

$$\begin{aligned}\langle \delta_0, \alpha\varphi + \beta\varphi \rangle &= (\alpha\varphi + \beta\varphi)(0)\\ &= \alpha\varphi(0) + \beta\varphi(0)\\ &= \alpha\langle \delta, \varphi \rangle + \beta\langle \delta, \varphi \rangle.\end{aligned}$$

This shows the Dirac Delta function is linear. To show continuity let φ_n be a test function that converges to φ in $\mathscr{S}(\Omega)$ then,

$$\langle \delta_0, \varphi_n \rangle = \varphi_n(0) \to \varphi(0) = \langle \delta_0, \varphi \rangle.$$

Thus $\langle \delta_0, \varphi_n \rangle \to \langle \delta_0, \varphi \rangle$. This concludes that the Dirac Delta function is a distribution.

Note that Dirac Delta function is not a function as it measures what happens only at a single point (not about away from that point).

Definition 3.2.8. (McLean and McLean, 2000)[Fourier transform] The spatial Fourier transformation of a function $f(x) \in \mathscr{S}(\mathbb{R})$ is defined as

$$\mathscr{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \quad (\xi \in \mathbb{R}).$$

Usually we use the notation $\widehat{f}(\xi) = \mathscr{F}(f)(\xi)$ in this dissertation.

From the definition of Fourier transform, if $f \in \mathscr{S}(\mathbb{R})$, then $\widehat{f}$ is bounded function with

$$\left\| \widehat{f} \right\|_{L^\infty(\mathbb{R})} \leq c \|f\|_{L^1(\mathbb{R})}, \quad \text{where } c \text{ is a constant.}$$

i.e, by definition and

$$\begin{aligned} |\widehat{f}(\xi)| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |e^{-ix\xi} f(x) dx| = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x)| dx \\ &\leq c \|f\|_{L^1(\mathbb{R})}, \end{aligned}$$

where,

$$\begin{aligned} \left| e^{-ix\xi} \right| &= \left| \cos(x\xi) + i \sin(x\xi) \right| \\ &= \sqrt{\cos^2(x\xi) + \sin^2(x\xi)} \\ &= 1. \end{aligned}$$

Taking supremum on both sides,

$$\sup_{x\in\mathbb{R}} |\widehat{f}(\xi)| \leq c\|f\|_{L^1(\mathbb{R})}$$
$$\left\|\widehat{f}\right\|_{L^\infty(\mathbb{R})} \leq c\|f\|_{L^1(\mathbb{R})}.$$

Theorem 3.2.5 (Basic property of Fourier transform)**.** *(Stein and Shakarchi, 2003) Let* $(\partial_x^k f)(x) = f^k(x)$*, where* $f^k(x)$ *is the* k^{th} *derivative of* f*. If* $f \in \mathscr{S}(\mathbb{R})$*, then* $\widehat{f}(\xi) \in C^\infty(\mathbb{R})$ *and* $\mathscr{F}(\partial_x^k f)(\xi) = (i\xi)^k \widehat{f}(\xi)$*.*

Proof. By definition

$$\widehat{\partial_x^k f}(\xi) = \mathscr{F}(\partial_x^k f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \frac{\partial}{\partial_x}^k (f(x))dx.$$

Now we can use induction on k. For $k = 1$,

$$\widehat{\partial_x f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f'(x)dx.$$

Using integration by parts,
i.e., substituting $u = e^{-ix\xi}$, $dv = f'(x)dx$ such that $du = -i\xi e^{-ix\xi}dx$, $v = f(x)$ and the assumption that $f \in \mathscr{S}(\mathbb{R})$ (which means $f(x)e^{-ix\xi}|_{-\infty}^{\infty} = 0$) it follows

$$\begin{aligned}
\widehat{\partial_x f}(\xi) &= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} udv = uv|_{\mathbb{R}} - \int_{\mathbb{R}} vdu \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(e^{-ix\xi} f(x)|_{\mathbb{R}} - \int_{\mathbb{R}} (-i\xi)e^{-ix\xi} f(x)dx \right) \\
&= \frac{1}{\sqrt{2\pi}} i\xi \int_{\mathbb{R}} e^{-ix\xi} f(x)dx \\
&= (i\xi)\widehat{f}(\xi).
\end{aligned}$$

Assume the statment is true for $n = k \in \mathbb{N}$, i.e,

$$\widehat{\partial_x^k f}(\xi) = (i\xi)^k \widehat{f}(\xi),$$

then by definition and using integration by parts for $n = k+1$ we have

$$\begin{aligned}\widehat{\partial_x^{k+1} f}(\xi) &= \widehat{\partial_x^k(\partial_x f)}(\xi) \\ &= (i\xi)^k(\widehat{\partial_x f}) \\ &= (i\xi)^{k+1}\widehat{f}(\xi),\end{aligned}$$

and this completes the proof. □

Example 3.2.4. Let's see the spatial Fourier transform of $f(x) = e^{-x^2/2}$.

First note that $f(x) \in \mathscr{S}(R)$ and also $f(x)$ satisfies

$$f'(x) = -xf(x).$$

Applying Fourier transform on both sides of the above equalities, i.e.,

$$\widehat{f'(x)} = \widehat{xf(x)}.$$

From basic properties of Fourier transform it follows that

$$(i\xi)\widehat{f}(\xi) = -i\widehat{f}'(\xi),$$

and hence we get separable ordinary differential equation in $\widehat{f}(\xi)$ such that

$$\widehat{f}'(\xi) = -\xi\widehat{f}(\xi). \tag{3.2.1}$$

Solving the linear first order ode (3.2.1), we get $\widehat{f}(\xi) = ce^{-\xi^2/2}$ where c is an integration constant that we fix latter from initial condition. If we suppose $c = \widehat{f}(0)$, then by definition

$$\begin{aligned}\widehat{f}(0) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix.0} f(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}dx.\end{aligned}$$

But, since $f(x)$ is an even function $\int_{\mathbb{R}} e^{-x^2/2}dx = 2\int_0^{\infty} e^{-x^2/2}dx$. Hence

$$\begin{aligned}c^2 &= \frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{-x^2/2}dx\right)^2 = \frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{-x^2/2}dx\right)\left(\int_{\mathbb{R}} e^{-y^2/2}dy\right)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)/2}dxdy.\end{aligned}$$

Now letting $x = r\cos\theta$ and $y = r\sin\theta$ it follows,

$$\begin{aligned}c^2 &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}rdrd\theta\\ &= \int_0^{\infty} e^{-r^2/2}rdr\\ &= 1.\end{aligned}$$

Thus, we conclude that

$$\widehat{f}(\xi) = e^{-\xi^2/2}.$$

Theorem 3.2.6. *(McLean and McLean, 2000)[Fourier transform of convolution] Let $f, g \in \mathscr{S}(R)$. Then*

$$\widehat{f*g}(\xi) = \sqrt{2\pi}\widehat{f}(\xi)\widehat{g}(\xi).$$

Proof. By definition of convolution

$$f*g = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Thus,

$$\begin{aligned}\left(\widehat{f*g}\right)(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\left[\int_{\mathbb{R}} f(x-y)g(y)dy\right]dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(e^{-ix\xi}f(x-y)dx\right)g(y)dy.\end{aligned}$$

By Fubinni's theorem the above integral on the right hand side is equal to

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-ix\xi}f(x-y)dx\right)g(y)dy.$$

Therefore, set $z = x - y$ and it follows

$$
\begin{aligned}
(\widehat{f*g})(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-i(y+z)\xi} f(z)dz \right) g(y)dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-iy\xi} e^{-iz\xi} f(z)dz \right) g(y)dy \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} e^{-iy\xi} g(y)dy \int_{\mathbb{R}} e^{-iz\xi} f(z)dz \right) \\
&= \sqrt{2\pi} \widehat{f}(\xi) \widehat{g}(\xi).
\end{aligned}
$$

This completes the proof.

□

Definition 3.2.9. (McLean and McLean, 2000)[Inverse Fourier transform] The inverse Fourier transform of Schwartz function f is denoted by

$$\mathscr{F}^{-1}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) d\xi.$$

We use the notation $\check{f}(x) = \mathscr{F}^{-1}(f)(x)$ for the inverse Fourier transform. Since $\mathscr{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p \le \infty$, it is natural to try to extend the Fourier transform to $L^p(\mathbb{R})$ by continuity. For example, to extend the Fourier transform in $L^2(\mathbb{R})$, first recall the inner product of two $L^2(\mathbb{R})$ functions;

$$(f,g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{g(x)}dx,$$

and then the following theorem.

Theorem 3.2.7. *Let $f, g \in \mathscr{S}(\mathbb{R})$. Then $(\widehat{f}, g)_{L^2(\mathbb{R})} = (f, \check{g})_{L^2(\mathbb{R})}$.*

Proof. By definition of inner product,

$$(\widehat{f}, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \widehat{f}(x)\overline{g(x)}dx,$$

which inturn implies

$$\begin{aligned}
(\widehat{f},g)_{L^2(\mathbb{R})} &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-iyx} f(y)dy\right)\overline{g(x)}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \overline{e^{iyx} g(x)dx}\right) f(y)dy \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}\int_{\mathbb{R}} \overline{\breve{g}(y)} f(y)dy = \int_{\mathbb{R}} f(y)\overline{\breve{g}(y)}dy \\
&= (f,\breve{g}).
\end{aligned}$$

□

Corollary 3.2.8. *Let $f,g \in \mathscr{S}(\mathbb{R})$ then*

$$(\widehat{f},\widehat{g})_{L^2(\mathbb{R})} = (f,g)_{L^2(\mathbb{R})}.$$

In particular when $f = g$, $\left\|\widehat{f}\right\|_{L^2(\mathbb{R})} = \|\breve{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$.

Theorem 3.2.9 (Plancherel theorem)**.** *(Folland, 1999) The Fourier mapping $\mathscr{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ extends to a bounded linear operator*

$$\mathscr{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}),$$

and we have for all $f,g \in L^2(\mathbb{R})$,

$$(\widehat{f},\widehat{g})_{L^2(\mathbb{R})} = (f,g)_{L^2(\mathbb{R})}.$$

Inparticular

$$\left\|\widehat{f}\right\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

3.2.6 Sobolev space ($H^s(\mathbb{R})$)

A sobolev space is a Banach space that measures differentiability of functions in $L^2(\mathbb{R})$ spaces.

Definition 3.2.10. The Sobolev space denoted by $H^k(\mathbb{R})$ is the space of functions $f \in L^2(\mathbb{R})$ such that the derivatives $f^j, j = 0,1,2,\ldots,k$ are also in $L^2(\mathbb{R})$, where the derivatives are interpreted in the distributional sense (McLean and McLean, 2000).

That is, if $f \in L^2(\mathbb{R})$ then $f^j \in L^2(\mathbb{R})$ for $j = 1, 2, 3, ...k$, and the derivatives are in distributional sense

$$\langle f^j, \varphi \rangle = (-1)^j \langle f, \varphi^j \rangle$$

for any $\varphi \in \mathscr{S}(\mathbb{R})$. The norm of u in $H^k(\mathbb{R}^d)$ for $k \in \mathbb{R}$ is defined as

$$\|u\|_{H^k(\mathbb{R}^d)} = \left[\int_{\mathbb{R}^d} \langle \xi \rangle^{2k} |\widehat{u}(\xi)|^2 d\xi \right]^{\frac{1}{2}}.$$

For any $s > r$, the embedding $H^s \subset H^r$ holds true. This follows from the fact that

$$\langle \xi \rangle^r < \langle \xi \rangle^s,$$

where $\langle a \rangle = \sqrt{1 + a^2}$. Note that $H^k(\mathbb{R})$ is a member of a more general family $W^{k,p}(\mathbb{R})$, where

$$W^{k,p}(\mathbb{R}) = \{ f \in L^p(\mathbb{R}); f^j \in L^p(\mathbb{R}), j = 0, 1, 2, \ldots, k \}.$$

Thus, $H^k(\mathbb{R}) = W^{k,2}(\mathbb{R})$. The space $L^2(\mathbb{R})$ is a Hilbert space, and hence we can define an inner product on $H^k(\mathbb{R})$ by

$$(f, g)_{H^k(\mathbb{R})} = \sum_{j=0}^{k} \int_{\mathbb{R}} f^j(x) \overline{g^j(x)} dx = \sum_{j=0}^{k} (f^j, g^j)_{L^2(\mathbb{R})}.$$

In particular,

- for $k = 0$, $H^0(\mathbb{R}) = L^2(\mathbb{R})$ with inner product

$$(f, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

- for $k = 1$, $H^1(\mathbb{R})$ with inner product

$$(f, g)_{H^1(\mathbb{R})} = \int_{\mathbb{R}} f(x) \overline{g(x)} dx + \int_{\mathbb{R}} f'(x) \overline{g'(x)} dx.$$

Definition 3.2.11. $C^{\cdot k}(\mathbb{R})$ is the space of k times continuously differentiable function f :

$\mathbb{R} \longrightarrow C$ such that $\lim_{|x|\to\infty} |f^j(x)| = 0$ for $j = 0,1,2,\ldots,k$, endowed with norm

$$\|f\|_{C_b^k(\mathbb{R})} := \max_{0\le j\le k} \sup_{x\in\mathbb{R}} |f^j(x)|,$$

where $C_b^k(\mathbb{R})$ is the space of k times continuously differentiable and bounded functions.

Theorem 3.2.10 (Sobolev embedding theorem)**.** *(McLean and McLean, 2000) Let $k \geq 0$, the Sobolev space $H^s(\mathbb{R})$ is continuously embeded in $C^{\cdot k}(\mathbb{R})$ for $s > k + \frac{1}{2}$.*

Proof. As a first step let's start by proving the inequality

$$\left\|\varphi^j(x)\right\|_{C_b^k(\mathbb{R})} \leq c\|\varphi\|_{H^s(\mathbb{R})}, \tag{3.2.2}$$

for all $\varphi \in \mathscr{S}(\mathbb{R})$ and a constant c depending on k and s.
By Fourier inversion formula and Cauchy-schwartz inequality we have

$$\begin{aligned}
\varphi^j(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{\varphi^j}(\xi) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} (i\xi)^j \widehat{\varphi}(\xi) d\xi \\
\left|\varphi^j(x)\right| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^j |\widehat{\varphi}(\xi)| d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^j |\widehat{\varphi}(\xi)| \langle\xi\rangle^s \langle\xi\rangle^{-s} d\xi \\
&\leq \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\xi|^{2j} \langle\xi\rangle^{-2s} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \langle\xi\rangle^{2s} |\widehat{\varphi}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\
&\leq c\|\varphi\|_{H^s(\mathbb{R})}
\end{aligned}$$

where the constant c is given by

$$c^2 = \frac{1}{2\pi} \left(\int_{\mathbb{R}} |\xi|^{2j} \langle\xi\rangle^{-2s} d\xi \right)^{\frac{1}{2}}.$$

Since $|\xi|^{2j} \leq \langle\xi\rangle^{2j}$

$$\begin{aligned} c^2 &\leq \frac{1}{2\pi}\left(\int_{\mathbb{R}} \langle\xi\rangle^{2j}\langle\xi\rangle^{-2s} d\xi\right)^{\frac{1}{2}} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \langle\xi\rangle^{2(j-s)} d\xi \\ &< \infty, \end{aligned}$$

provided that $2(s-j) < -1$ and hence $c < \infty$ if and only if $s > j + 1/2$. This implies

$$\max_{0\leq j\leq k} \sup_{0x\in\mathbb{R}} |\varphi^j(x)| \leq c\|\varphi\|_{H^s(\mathbb{R})}$$

which is to mean

$$\|\varphi\|_{C_b^k(\mathbb{R})} \leq c\left\|\varphi_{H^s(\mathbb{R})}\right\| \quad \text{for } s > k + 1/2.$$

Next, pick a sequence $\{\varphi_n\} \in \mathscr{S}(\mathbb{R})$ with $\varphi_n \to f$ in the space $H^s(\mathbb{R})$ which means $f \in H^s(\mathbb{R})$. By (3.2.2)

$$\|\varphi_n - \varphi_m\|_{C_b^k(\mathbb{R})} \leq c\|\varphi_n - \varphi_m\|_{H^s(\mathbb{R})} \to 0$$

as $n, m \to \infty$. This shows that $\{\varphi_n\}$ is a cauchy sequence in $C^k(\mathbb{R})$. Since $C^k(\mathbb{R})$ is a Banach space

$$\{\varphi_n\} \to g \quad \text{in } C^k(\mathbb{R}),$$

but both f and g are in $\mathscr{S}(\mathbb{R})$. It follows that $f = g$ almost everywhere (a.e) so,

$$\|f\|_{C_b^k(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})},$$

for $f \in H^s(\mathbb{R})$ where $s > k + 1/2$. □

Theorem 3.2.11 ($H^s(\mathbb{R})$-algebra)**.** *The Sobolev space $H^s(\mathbb{R})$ is an algebra for $s > \frac{1}{2}$.*

This theorem states that if $s > \frac{1}{2}$ and $f, g \in H^s(\mathbb{R})$ then $fg \in H^s(\mathbb{R})$ and that there exist a positive constant $c = c(s)$ such that

$$\|fg\|_{H^s(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}\|g\|_{H^s(\mathbb{R})}$$

.

Chapter 4

Persistence of spatial analyticity for the beam equation

In this chapter, we introduce the newly modified Gervey space, state and prove local well-posedness and approximate conservation law of beam equation.

4.1 Introduction

In this chapter, we study on the persistence (continuing to exist or endure over a long period) of spatial analyticity for the solution of beam equation with initial data in a class of analytic functions. Recall that the beam equation is given by

$$\begin{cases} u_{tt} + \Delta^2 u + mu = \lambda |u|^{p-1} u \\ (u, u_t)(x, 0) = (u_0, u_1)(x), \end{cases} \tag{4.1.1}$$

where, $u : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, $p > 1$, and $m > 0$. Note that if $\lambda < 0$, then we call the beam equation is defocusing whereas for $\lambda > 0$ it is focussing. Throughout our study we consider the defocusing beam equation where $\lambda = 1$ is fixed.

A Paley–Wiener theorem is a theorem that relates decay properties of a function or distribution at infinity with analyticity of its Fourier transform. Thus, due to this theorem the radius of spatial analyticity of a function can be related to decay properties of its Fourier transform. It is therefore natural to take data for (1.1.4) in the Gevrey space $G^{\sigma,s}(\mathbb{R}^n)$, defined by the norm

$$\|f\|_{G^{\sigma,s}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} e^{2\sigma|\xi|} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}, \quad \sigma \geq 0,$$

i.e.,

$$\|f\|_{G^{\sigma,s}(\mathbb{R}^n)} = \left\| \exp(\sigma|\xi|) \langle \xi \rangle^s \widehat{f} \right\|_{L^2_\xi(\mathbb{R}^n)}, \quad \sigma \geq 0,$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. In particular for $\sigma = 0$, this space coincides with the Sobolev space $H^s(\mathbb{R}^n)$, with norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \left\|\langle\xi\rangle^s \widehat{f}\right\|_{L^2_\xi(\mathbb{R}^n)},$$

while for all $\sigma > 0$, any function in $G^{\sigma,s}(\mathbb{R}^n)$ has a radius of analyticity of at least σ at each point $x \in \mathbb{R}^n$. This fact is contained in the following theorem whose proof can be found in (Katznelson, 2004) in the case $s = 0$ and $n = 1$. The general case follows from a simple modification.

Theorem 4.1.1. ***(Paley-Wiener Theorem.)****(Katznelson, 2004; Yosida, 2012) Let $\sigma > 0$ and $s \in \mathbb{R}$. If $f \in G^{\sigma,s}(\mathbb{R}^n)$, then f is the restriction to $\mathbb{R}^n$ of a function F which is holomorphic in the strip*

$$S_\sigma = \{x+iy \in \mathbb{C}^n : \; |y| < \sigma\}.$$

Moreover, the function F satisfies the estimates

$$\sup_{|y|<\sigma} \|F(\cdot + iy)\|_{H^s} < \infty.$$

We remark that Gevrey spaces satisfy the embeddings

$$\|f\|_{G^{\sigma,s}} \lesssim \|f\|_{G^{\sigma',s'}}, \tag{4.1.2}$$

for any $s, s' \in \mathbb{R}$ and $\sigma \leq \sigma'$, from which we easily obtain

$$\|f\|_{H^s} \lesssim \|f\|_{G^{\sigma',s'}}, \tag{4.1.3}$$

from which we may also obtain

$$\|f\|_{L^p} \lesssim \|f\|_{G^{\sigma,s}} \tag{4.1.4}$$

for $2 \leq p \leq \infty$ by Sobolev embedding.

The space $G^{\sigma,s}(\mathbb{R}^n)$ were introduced by (Foias and Temam, 1989) (see also (Kato and Masuda, 1986)) in the study of spatial analyticity of solutions to Navier-Stokes equations, and various refinements of their method have been applied to prove lower bounds on the radius of spatial analyticity for a number of nonlinear evolution equations (Ferrari and Titi, 1998; Foias and Temam, 1989; Hannah et al., 2011; Himonas and Petronilho, 2012; Levermore and Oliver, 1997; Oliver and Titi, 2001; Panizzi, 2012; Selberg and Da Silva, 2015;

Selberg and Tesfahun, 2017, 2015; Tesfahun, 2019a,b). The method we used here for proofing lower bounds on the radius of analyticity was introduced in (Selberg and Tesfahun, 2015) in the context of $1D$ Dirac-Klein-Gordon equations. This method is based on an approximate conservation laws, and has been applied to prove an algebraic lower bound (decay rate) of order $t^{-1/\alpha}$ for some $\alpha \in (0,1]$ on the radius of spatial analyticity of solutions to a number of nonlinear dispersive and wave equations. See e.g., (Belayneh et al., 2022; Selberg and Da Silva, 2015; Selberg and Tesfahun, 2017; Tesfahun, 2019a,b; Zhang, 2010) . The optimal decay rate that can be obtained in this setting is $1/t$, which corresponds to $\alpha = 1$, see e.g.,(Belayneh et al., 2022; Tesfahun, 2019a; Zhang, 2010). This decay rate is related to the behaviour of the exponential weight , $\exp(\sigma|\xi|)$ that sits in the Gevrey norm. More specifically, it stems from the simple estimate

$$\exp(\sigma|\xi|) - 1 \leq (\sigma|\xi|)^{\alpha}.\exp(\sigma|\xi|)) \quad (0 < \alpha \leq 1),$$

which follows from an interpolation between $\exp r - 1 \leq \exp r$ and $\exp r - 1 \leq r \exp r$ for $r \geq 0$. Since $\exp r > 0$ for all r, it is true that $\exp r < 1 + \exp r$ and using the Taylor series expansion of $\exp r$, we show that $\exp r < 1 + r \exp r$.

In this disertation, in an attempt to improve the decay rate obtained so far for the beam equation (1.1.4) and in (Belayneh et al., 2022) for the KdV-BBM equation (1.1.3) we introduce a modified Gevrey space $H^{\sigma,s}(\mathbb{R}^n)$ with norm by

$$\|f\|_{H^{\sigma,s}(\mathbb{R}^n)} = \left\|\cosh(\sigma|\xi|)\langle\xi\rangle^s \widehat{f}\right\|_{L^2_\xi(\mathbb{R}^n)} \quad (\sigma \geq 0),$$

where the exponential weight $\exp(\sigma|\xi|)$ in the Gevrey norm is now replaced by a hyperbolic weight $\cosh(\sigma|\xi|)$. These two weights are equivalent in the sense that

$$\frac{1}{2}\exp(\sigma|\xi|) \leq \cosh(\sigma|\xi|) \leq \exp(\sigma|\xi|). \tag{4.1.5}$$

Thus, the associated $G^{\sigma,s}(\mathbb{R}^n)$ and $H^{\sigma,s}(\mathbb{R}^n)$ norms are equivalent, i.e.,

$$\|f\|_{H^{\sigma,s}(\mathbb{R}^n)} \sim \|f\|_{G^{\sigma,s}(\mathbb{R}^n)}, \tag{4.1.6}$$

and so the statment of Paley-Wiener theorem still holds for functions in $H^{\sigma,s}(\mathbb{R}^n)$. The space

$H^{\sigma,s}(\mathbb{R}^n)$, however, has an advantage since $\cosh(\sigma|\xi|)$ satisfies the estimate

$$\cosh(\sigma|\xi|)-1 \leq (\sigma|\xi|)^{2\alpha}\cosh(\sigma|\xi|) \quad (0<\alpha\leq 1). \tag{4.1.7}$$

Equation (4.1.7) follows from

$$\cosh r - 1 \leq \cosh r \quad \text{and} \quad \cosh r - 1 \leq r^2 \cosh r \quad (r\in\mathbb{R}).$$

Therefore, in view of (4.1.7), an application of our method in the $H^{\sigma,s}(\mathbb{R}^n)$ set up can yield a decay rate of order $t^{-1/2\alpha}$ for some $\alpha\in(0,1]$ provided that the nonlinear estimates in the derivation of an approximate conservation law can absorb the weight $|\xi|^{2\alpha}$. In this work, we managed to obtain the optimal decay rate of $t^{-1/2}$ (which corresponds to $\alpha=1$) for the Cauchy problems (1.1.4) and (1.1.3).

4.2 Energy of the beam equation

In order to get energy of the beam equation, we multiply (1.1.4) by u_t and integrating over $\mathbb{R}$, then we obtain

$$\int_{\mathbb{R}}(u_t u_{tt}+u_t\Delta^2 u+muu_t)dx=\int_{\mathbb{R}}|u|^{p-1}uu_t dx.$$

Note that, we can rewrite,

$$u_t u_{tt}=\frac{1}{2}(u_t^2)_t, \quad uu_t=\frac{1}{2}(u^2)_t, \quad |u|^{p-1}uu_t=\frac{1}{p+1}(|u|^{p+1})_t$$

and $u_t\Delta^2 u=\frac{1}{2}(\Delta u)_t^2$.

Assuming that the solution is smooth and vanishes as $x\to\pm\infty$, the energy for (1.1.4) is given by

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^n}\left(u_t^2+(\Delta u)^2+mu^2+\frac{2}{p+1}|u|^{p+1}\right)dx$$

and is conserved by the flow of (1.1.4). i.e., $E(t)=E(0)\ \forall t$.

4.3 Local well-posedness in $H^{\sigma,2}(\mathbb{R}^n) \times H^{\sigma,0}(\mathbb{R}^n)$

Well-posedness, blow-up in finite time, long time existence, and the existence of uniform bounds for global solutions of the beam equation (1.1.4) were adressed by several authors. For instance, local well-posedness, scattering, and stability in the energy space $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ was studied by (Levandosky, 1998a,b) and (Levandosky and Strauss, 2000), results which were extended by (Pausader, 2007; Pausader and Strauss, 2009). For the global problem, low-regularity global well-posedness was also shown by (Zhang, 2010) in dimensions $n \in \mathbb{N}$ such that $3 \leq n \leq 7$ in the cubic case for data in $H^s(\mathbb{R}^n) \times H^{s-2}(\mathbb{R}^n)$ satisfying

$$s > \min\left\{\frac{n-2}{2}, \frac{n}{4}\right\}.$$

We remark that as a consequence of the embedding

$$H^{\sigma,s}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \quad (\sigma \geq 0), \tag{4.3.1}$$

and the existing well posedness theory in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, one can conclude that the Cauchy problem (1.1.4), with $1 \leq n \leq 3$ and $p \geq 1$, has a unique, global-in-time solution, given initial data $(u_0, u_1) \in H^{\sigma_0,2}(\mathbb{R}^n) \times H^{\sigma_0,0}(\mathbb{R}^n)$ for some $\sigma_0 \geq 0$.

Theorem 4.3.1. *(Local well-posedness). Let $n \in \mathbb{N}$ such that $1 \leq n \leq 3$, $p \geq 1$ be an odd integer and $\sigma > 0$. Given $(u_0, u_1) \in H^{\sigma,2}(\mathbb{R}^n) \times H^{\sigma,0}(\mathbb{R}^n)$, then there exists a time $\delta > 0$ and a unique solution*

$$(u, u_t) \in C\left([0,\delta]; H^{\sigma,2}(\mathbb{R}^n)\right) \times C^1\left([0,\delta]; H^{\sigma,0}(\mathbb{R}^n)\right),$$

of the Cauchy problem (1.1.4) *on* $[0,\delta] \times \mathbb{R}^n$. *Moreover, the existence time is given by*

$$\delta = c_0(\|u_0\|_{H^{\sigma,2}} + \|u_1\|_{H^{\sigma,0}})^{-(p-1)}. \tag{4.3.2}$$

Proof. Theorem 4.3.1 can be proved using energy inequality, Sobolev embedding and a standard contraction arguement. Indeed, consider the Cauchy problem for linear beam equation

$$\begin{aligned} &u_{tt} + \left(m + \Delta^2\right) u = F(x,t), \\ &(u, u_t)|_{t=0} = (u_0, u_1). \end{aligned} \tag{4.3.3}$$

Note that to get solution of equation (4.3.3), we take Fourier transform of the Cauchy problem, reduce it to an ordinary differential equation and then solve it. Indeed, applying the Fourier transform with respect to space variable on the homogeneous part,

$$u_{tt} + \left(m + \Delta^2\right) u = 0,$$

we have an ordinary differential equation

$$\widehat{u}_{tt}(\xi) + (m + \xi^4)\widehat{u}(\xi) = 0.$$

Denote $m + \xi^4 = \langle \Delta \rangle_m$, then using the initial conditions and Duhamel's formula, the solution of (4.3.3) is given by

$$u(t) = \cos\left(t\langle\Delta\rangle_m\right) u_0 + \frac{\sin\left(t\langle\Delta\rangle_m\right)}{i\langle\Delta\rangle_m} u_1 - \int_0^t \frac{\sin\left((t-\tau)\langle\Delta\rangle_m\right)}{i\langle\Delta\rangle_m} F(\tau)\, d\tau. \tag{4.3.4}$$

or in short

$$u(t) = s'_m(t)u_0 + s_m(t)u_1 + \int_0^t s_m(t-t')F(t')dt', \tag{4.3.5}$$

where

$$s_m(t) = \frac{\sin(t\langle\Delta\rangle_m)}{\langle\Delta\rangle_m}.$$

Applying $\cosh(\sigma|D|)$ to (4.3.5) and taking the H^2 norm on both sides yields the energy inequality (where first we differentiate equation (4.3.5) with respect to time)

$$\sup_{0\le t\le\delta} \left[\|u\|_{H^{\sigma,2}} + \|u_t\|_{H^{\sigma,0}}\right] \lesssim \|u_0\|_{H^{\sigma,2}} + \|u_1\|_{H^{\sigma,0}} + \int_0^\delta \|F(\tau)\|_{H^{\sigma,0}}\, d\tau, \tag{4.3.6}$$

for some $\sigma > 0$.

Now, consider the integral formulation of (1.1.4),

$$u(t) = s'_m(t)u_0 + s_m(t)u_1 + \int_0^t s_m(t-t')u^p(t')dt', \tag{4.3.7}$$

where we used the fact that $|u|^{p-1}u = u^p$ for odd p.

Then from the inequality (4.3.6) and a standard contraction arguement, Theorem 4.3.1

reduces to proving the nonlinear estimate

$$\|u^p\|_{H^{\sigma,0}} \lesssim \|u\|^p_{H^{\sigma,2}}, \tag{4.3.8}$$

which is also equivalent to

$$\|u^p\|_{G^{\sigma,0}} \lesssim \|u\|^p_{G^{\sigma,2}}.$$

Setting $U = \exp(\sigma|D|)u$, this estimate further reduces to

$$\|\exp(\sigma|D|)[(\exp(-\sigma|D|)U)^p]\|_{L^2_x} \lesssim \|U\|^p_{H^2}. \tag{4.3.9}$$

Applying Plancherel theorem gives,

$$\begin{aligned}
LHS(4.3.9) &= \left\|\mathscr{F}_x\{\exp(\sigma|D|)[(\exp(-\sigma|D|)U)^P]\}(\xi)\right\|_{L^2_\xi} \\
&= \left\| \int_{\xi=\sum_{j=1}^p \xi_j} \exp\left(\sigma\left[|\xi| - \sum_{j=1}^p |\xi_j|\right]\right) \prod_{i=1}^p \widehat{U}(\xi_j) d\xi_1 d\xi_2 ... d\xi_p \right\|_{L^2_\xi} \\
&\le \left\| \int_{\xi=\sum_{j=1}^p \xi_j} \prod_{i=1}^p |\widehat{U}(\xi_j)| d\xi_1 d\xi_2 ... d\xi_p \right\|_{L^2_\xi} \\
&= \|V\|^p_{L^2_x}
\end{aligned}$$

where $V = \mathscr{F}^{-1}[|\widehat{U}|]$. To obtain the third line we used the fact that $|\xi| \le \sum_{j=1}^p |\xi_j|$, which follows from triangle inequality.

Now, by Sobolev embedding theorem,

$$\|V^p\|_{L^2_x} = \|V\|^p_{L^{2p}_x} \lesssim \|V\|^p_{H^2} = \|U\|^p_{H^2},$$

for all $p \ge 1$. This concludes the proof of (4.3.9), and hence (4.3.8). □

4.4 Approximate conservation law for the beam equation

The second step to prove our main result is to state and prove an approximate conservation law for the norm of solution, that involves a small parameter $\delta > 0$ and which reduces to the exact energy conservation law in the limit as $\delta \to 0$. To derive this approximate conservation law, we set

$$v_\sigma(x,t) := \cosh(\sigma|D|)u(x,t),$$

where $u(x,t)$ is the solution of (1.1.4). Define a modified energy associated with v_σ by

$$E_\sigma(t) = \frac{1}{2}\int_{\mathbb{R}^n}\left((\partial_t v_\sigma)^2 + (\Delta v_\sigma)^2 + m v_\sigma^2 + \frac{2}{p+1}|v_\sigma|^{p+1}\right)dx.$$

We state and prove the following three Lemmas which are crutial in the proof of approximate conservation law.

Lemma 4.4.1. *For $a, b \in \mathbb{R}$, we have*

$$|\cosh b - \cosh a| \le \frac{1}{2}|b^2 - a^2|(\cosh b + \cosh a). \tag{4.4.1}$$

Proof. Note that $\cosh r$ is an increasing function for $r \ge 0$. Since $\cosh r$ is even, i.e., $\cosh r = \cosh|r|$, we may assume $a, b \ge 0$. By symmetry we may also assume $b \ge a$, and thus

$$\begin{aligned}\cosh b - \cosh a &= \int_a^b \int_0^s \cosh r\, dr ds \\ &= \int_a^b \sinh s ds.\end{aligned}$$

Therefore we have an inequality

$$\cosh b - \cosh a = \int_a^b \int_0^s \cosh r\, dr ds \le \cosh b \int_a^b \int_0^s dr ds = \frac{1}{2}\left(b^2 - a^2\right)\cosh b.$$

□

Lemma 4.4.2. *Let $\xi = \sum_{j=1}^p \xi_j$ for $\xi_j \in \mathbb{R}$, where $p \ge 1$ is an integer. Then*

$$|1 - \cosh|\xi| \prod_{j=1}^p sech|\xi_j|| \le 2^p \sum_{j \ne k = 1}^p |\xi_j||\xi_k|. \tag{4.4.2}$$

Proof. First observe that the product of cosine hyperbolic functions

$$\prod_{j=1}^p \cosh|\xi_j| = 2^{1-p} \sum_{s_2, s_3, \ldots, s_p} \cosh\left(|\xi_1| + \sum_{j=2}^p s_j|\xi_j|\right), \tag{4.4.3}$$

where $s_2, s_3, \ldots, s_p$ are independent signs (+ or -). Indeed, the case $p = 1$ is obvious, while the case $p = 2$ follows from the hyperbolic identity

$$2\cosh|\xi_1|\cosh|\xi_2| = \cosh(|\xi_1| - |\xi_2|) + \cosh(|\xi_1| + |\xi_2|),$$

which also implies equation (4.4.3) for $p = 3$. The general case follows by induction. It follows from equation (4.4.3) that

$$\cosh\left(|\xi_1| + \sum_{j=2}^{p} s_j|\xi_j|\right) + \cosh|\xi| \le 2^p \prod_{j=1}^{p} \cosh(|\xi_j|). \tag{4.4.4}$$

Observe that

$$\left|\left(|\xi_1| + \sum_{j=2}^{p} s_j|\xi_j|\right)^2 - |\xi|^2\right| \le 2\sum_{j\neq k=1}^{p} |\xi_j||\xi_k|. \tag{4.4.5}$$

By applying equation (4.4.3), and inequalities (4.4.1), (4.4.4), (4.4.5) we obtain

$$\begin{aligned}
\left|\prod_{j=1}^{p}\cosh|\xi_j| - \cosh|\xi|\right| &= |2^{1-p}\sum_{s_2,s_3,\ldots,s_p}\cosh\left(|\xi_1| + \sum_{j=2}^{p} s_j|\xi_j|\right) - \cosh|\xi|| \\
&= \left|2^{1-p}\sum_{s_2,s_3,\ldots,s_p}\left[\cosh\left(|\xi_1| + \sum_{j=2}^{p} s_j|\xi_j|\right) - \cosh|\xi|\right]\right| \\
&\le 2^{1-p}\sum_{s_2,s_3,\ldots,s_p}\frac{1}{2}\left|\left(|\xi_1| + \sum_{j=2}^{p} s_j|\xi_j|\right)^2 - |\xi|^2\right|\cosh\left(|\xi_1| + \sum_{j=2}^{p} s_j|\xi_j|\right) + \cosh|\xi| \\
&\le 2^{1-p}\sum_{s_2,s_3,\ldots,s_p}\left(\sum_{j\neq k=1}^{p}|\xi_j||\xi_k|\right).2^p\prod_{j=1}^{p}\cosh(|\xi_j|) \\
&= 2^p\left(\sum_{j\neq k=1}^{p}|\xi_j||\xi_k|\right)\prod_{j=1}^{p}\cosh(|\xi_j|).
\end{aligned}$$

Dividing by $\prod_{j=1}^{p}\cosh(|\xi_j|)$ yields the desired estimate (4.4.2). □

From the modified energy, using integration by parts and equation (1.1.4), we obtain

$$\begin{aligned}
E'_\sigma(t) &= \int_{\mathbb{R}^n} \partial_t v_\sigma \left[\partial_{tt} v_\sigma + \Delta^2 v_\sigma + m v_\sigma + v_\sigma^p\right] dx \\
&= \int_{\mathbb{R}^n} \partial_t v_\sigma \left[\cosh(\sigma|D|)\left(\partial_{tt} u + \Delta^2 u + mu\right) + v_\sigma^p\right] dx \\
&= \int_{\mathbb{R}^n} \partial_t v_\sigma \left[-\cosh(\sigma|D|)u^p + v_\sigma^p\right] dx \\
&= \int_{\mathbb{R}^n} \partial_t v_\sigma . N_p(v_\sigma) dx,
\end{aligned}$$

where

$$N_p(v_\sigma) = v_\sigma^p - \cosh(\sigma|D|)[sech(\sigma|D|)v_\sigma]^p.$$

Therefore, integrating both sides with respect to time yields

$$E_\sigma(t) = E_\sigma(0) + \int_0^t \int_{\mathbb{R}^n} \partial_t v_\sigma(x,s).N_p(v_\sigma(x,s))dxds. \tag{4.4.6}$$

Now, we can sate a key lemma as follows

Lemma 4.4.3. *For $\partial_t v_\sigma \in L_x^2$ and $v_\sigma \in H^2$, we have the estimate*

$$\Big|\int_{\mathbb{R}^n} \partial_t v_\sigma.N_p(v_\sigma)dx\Big| \le c\sigma^2 \|\partial_t v_\sigma\|_{L_x^2} \|v_\sigma\|_{H^2}^p, \tag{4.4.7}$$

for some constant $c > 0$.

Proof. Recall that

$$N_p(v_\sigma) = v_\sigma^p - \cosh(\sigma|D|)[sech(\sigma|D|)v_\sigma]^p.$$

By Cauchy-Schwartz inequality

$$\Big|\int_{\mathbb{R}^n} \partial_t v_\sigma.N_p(v_\sigma)dx\Big| \le \|\partial_t v_\sigma\|_{L_x^2} \big\|N_p(v_\sigma)\big\|_{L_x^2},$$

which reduces to prove

$$\big\|N_p(v_\sigma)\big\|_{L_x^2} \lesssim \sigma^2 \|v_\sigma\|_{H^2}^p. \tag{4.4.8}$$

Taking the Fourier transform with respect to the space variable we have,

$$\mathscr{F}_x[N_p(v_\sigma)](\xi) = \int_{\xi=\sum_{j=1}^p \xi_j} \left[1 - \cosh(\sigma|\xi|)\prod_{j=1}^p sech(\sigma|\xi_j|)\right] \prod_{j=1}^p \widehat{v_\sigma}(\xi_j) d\xi_1 d\xi_2 \dots d\xi_p. \tag{4.4.9}$$

By symmetry, we may assume $|\xi_1| \ge |\xi_2| \ge \dots \ge |\xi_p|$. Using Lemma (4.4.2), we have

$$\left|1 - \cosh(\sigma|\xi|)\prod_{j=1}^p sech(\sigma|\xi_j|)\right| \le 2^p \sum_{j\ne k=1}^p |\sigma\xi_j||\sigma\xi_k| \le c(p)\sigma^2|\xi_1||\xi_2|, \tag{4.4.10}$$

where $c(p) = p^2 2^p$. Now set

$$w_\sigma := \mathscr{F}_x^{-1}(|\widehat{v_\sigma}|).$$

Then applying inequality (4.4.10) to (4.4.9) gives

$$
\begin{aligned}
\left|\mathscr{F}_x[N_p(v_\sigma)](\xi)\right| &\leq c(p)\sigma^2 \int_{\xi=\sum_{\xi_j}^{j=1}\xi_j} |\xi_1||\xi_2||\widehat{v_\sigma}(\xi_1)||\widehat{v_\sigma}(\xi_2)|\prod_{j=3}^{p}|\widehat{v_\sigma}(\xi_j)|d\xi_1\xi_2\ldots d\xi_p \\
&= c(p)\sigma^2 \int_{\xi=\sum_{\xi_j}^{j=1}\xi_j} |\xi_1||\xi_2|\widehat{w_\sigma}(\xi_1)\widehat{w_\sigma}(\xi_2)\prod_{j=3}^{p}|\widehat{w_\sigma}(\xi_j)|d\xi_1\xi_2\ldots d\xi_p \\
&= c(p)\sigma^2\mathscr{F}_x\left((|D|w_\sigma)^2.w_\sigma^{p-2}\right)(\xi).
\end{aligned}
$$

Therefore, using Plancherel, Holder and Sobolev embedding it follows that

$$
\begin{aligned}
\|N_p(v_\sigma)\|_{L_x^2} &\leq c(p)\sigma^2\left\|(|D|w_\sigma)^2.w_\sigma^{p-2}\right\|_{L_x^2} \\
&\leq c(p)\sigma^2\||D|w_\sigma\|_{L_x^2}^2.\|w_\sigma\|_{L_x^\infty}^{p-2} \\
&\leq c(p)\sigma^2\|w_\sigma\|_{H^2}^2\|w_\sigma\|_{H^2}^{p-2} \\
&= c(p)\sigma^2\|v_\sigma\|_{H^2}^p
\end{aligned}
$$

as desired in equation (4.4.8). □

Theorem 4.4.4. *(Approximate conservation law). Let $n \in \mathbb{N}$ such that $1 \leq n \leq 3$, $p \geq 1$ be an odd integer and $\sigma_0 > 0$. Given $(u_0, u_1) \in H^{\sigma,2}(\mathbb{R}^n) \times H^{\sigma,0}(\mathbb{R}^n)$, let u be the solution of equation* (1.1.4) *on $\mathbb{R}^n \times [0,\delta]$ that is obtained in Theorem 4.3.1. Then*

$$
\sup_{0\leq t\leq\delta} E_\sigma(t) = E_\sigma(0) + \delta\sigma^2.\mathscr{O}\left((E_\sigma(0))^{\frac{p+1}{2}}\right). \tag{4.4.11}
$$

Observe that in the limit as $\delta \to 0$ the second term on the right hand side of equation (4.4.11) goes to zero and hence we recover the conservation $E_0(t) = E_0(0)$ for $0 \leq t \leq \delta$ (note that $v_0 = u$).

Proof. Let $n \in \mathbb{N}$ such that $1 \leq n \leq 3$, $p \geq 1$ is an odd integer, $\sigma > 0$, and $\delta > 0$ be the local existence time for the solution obtained in Theorem 4.3.1. Recall that $v_\sigma(x,t) = \cosh(\sigma|D|)u(x,t)$, where u is the solution to equation (1.1.4). Thus, $u(x,t) = sech(\sigma|D|)v_\sigma(x,t)$. Now we use equation (4.4.6) and inequality (4.4.7) to obtain the a prior energy estimate

$$
\sup_{0\leq t\leq\delta} E_\sigma(t) = E_\sigma(0) + \delta\sigma^2.\mathscr{O}\left(\|\partial_t v_\sigma\|_{L_\delta^\infty L_x^2}\|v_\sigma\|_{L_\delta^\infty H^2}^p\right), \tag{4.4.12}
$$

where we use the notation

$$L^\infty_\delta X := L^\infty_t X([0,\delta]\times\mathbb{R}^n),$$

with $X = L^2_x$ or H^2. As a consequence of Theorem 4.3.1, we get

$$\begin{aligned}\|v_\sigma\|_{L^\infty_\delta H^2} + \|\partial_t v_\sigma\|_{L^\infty_\delta L^2_x} &= \|u\|_{L^\infty_\delta H^{\sigma,2}} + \|u_t\|_{L^\infty_\delta H^{\sigma,0}} \\ &\leq C(\|u_0\|_{H^{\sigma,2}} + \|u_1\|_{H^{\sigma,0}}) \\ &= C\left(\|v_\sigma(.,0)\|_{H^2} + \|\partial_t v_\sigma(.,0)\|_{L^2_x}\right).\end{aligned}$$

Thus,

$$\|v_\sigma\|_{L^\infty_\delta H^2} + \|\partial_t v_\sigma\|_{L^\infty_\delta L^2_x} \leq C\left(\|v_\sigma(.,0)\|_{H^2} + \|\partial_t v_\sigma(.,0)\|_{L^2_x}\right). \tag{4.4.13}$$

Since

$$\begin{aligned}E_\sigma(0) &= \frac{1}{2}\int_{\mathbb{R}^n}\left([\partial_t v_\sigma(x,0)]^2 + (\Delta v_\sigma(x,0))^2 + m[v_\sigma(x,0)]^2 + \frac{2}{p+1}|v_\sigma(x,0)|^{p+1}\right)dx \\ &\geq C\left(\|v_\sigma(.,0)\|_{H^2} + \|\partial_t v_\sigma(.,0)\|_{L^2_x}\right)^2.\end{aligned}$$

It follows from estimate (4.4.13) that

$$\|v_\sigma\|_{L^\infty_\delta H^2} + \|\partial_t v_\sigma\|_{L^\infty_\delta L^2_x} \lesssim [E_\sigma(0)]^{\frac{1}{2}}. \tag{4.4.14}$$

Finally using estimate (4.4.14) in equation (4.4.12), we obtain the desired estimate, equation (4.4.11). □

4.5 Main result and its proof

Theorem 4.5.1. *(Lower bound for the radius of analyticity). Let $n \in \mathbb{N}$ such that $1 \leq n \leq 3$, $p \geq 1$ be an odd integer and $\sigma_0 > 0$. If $(u_0, u_1) \in H^{\sigma_0,2}(\mathbb{R}^n) \times H^{\sigma_0,0}(\mathbb{R}^n)$, then for any time $T > 0$ the solution of (1.1.4) satisfies*

$$(u, u_t) \in C\left([0,T]; H^{\sigma,2}(\mathbb{R}^n)\right) \times C^1\left([0,T]; H^{\sigma,0}(\mathbb{R}^n)\right),$$

with

$$\sigma := \sigma(T) = \min\{\sigma_0, cT^{-\frac{1}{2}}\},$$

where $c > 0$ is a constant depending on the initial data norm.

In view of the Paley-Wiener Theorem and (4.1.6), this result implies that the solution $u(.,t)$ has radius of analyticity at least $\sigma(t)$ for every $t > 0$. Applying the approximate conservation law and local well-posedness theorems repeatedly, and then by taking $\sigma > 0$ small enough we can recover any time interval $[0,T]$ and obtain the main result Theorem 4.5.1.

Proof. Suppose that $(u_0,u_1) \in H^{\sigma_0,2} \times H^{\sigma_0,0}$ for some $\sigma_0 > 0$. This implies

$$v_{\sigma_0}(.,0) = \cosh(\sigma_0|D|)u_0 \in H^2, \quad \partial_t v\sigma_0(.,0) = \cosh(\sigma_0|D|)u_1 \in L^2.$$

Then by Sobolev embedding,

$$E_{\sigma_0}(0) \lesssim \|v_{\sigma_0}(.,0)\|^2_{H^2} + \|\partial_t v_{\sigma_0}(.,0)\|^2_{L^2_x} + \|v_{\sigma_0}(.,0)\|^{p+1}_{H^2} < \infty.$$

Now following the arguement in (Selberg and Tesfahun, 2015) (see also Selberg and Da Silva (2015)) we can construct a solution on $[0,T]$ for arbitrarily large time T by applying the approximate conservation law (4.4.11), so as to repeat the local result in Theorem 4.4.4 on successive short time intervals of size δ to reach T by adjusting the strip width parameter $\sigma \in (0,\sigma_0)$ of the solution according to the size of T.

The goal is to prove that for a given parameter $\sigma \in (0,\sigma_0)$ and large $T > 0$,

$$\sup_{t\in[0,T]} E_\sigma(t) \le 2E_{\sigma_0}(0) \quad \text{for} \quad \sigma = c/\sqrt{T}, \tag{4.5.1}$$

where $c > 0$ depends only on the initial data norm, σ_0 and p. This would imply $E_\sigma(t) < \infty$ for all $t \in [0,T]$, and hence

$$(u,u_t)(.,t) \in H^{\sigma,2} \times H^{\sigma,0} \quad \text{for} \quad \sigma = c/\sqrt{T} \quad \text{and} \quad t \in [0,T].$$

Now to proceed to the proof of estimate (4.5.1) first observe that for a given parameter $\sigma \in [0,\sigma_0]$ and $0 < t \le \delta$, by Theorem 4.3.1 and 4.5.1, we have

$$\sup_{t\in[0,t_0]} E_\sigma(t) \le E_\sigma(0) + c\delta\sigma^2(E_\sigma(0))^{\frac{p+1}{2}} \le E_{\sigma_0}(0) + c\delta\sigma^2(E_{\sigma_0}(0))^{\frac{p+1}{2}}.$$

This holds true since $\cosh r$ is increasing for $r \geq 0$ and it follows $E_\sigma(0) \leq E_{\sigma_0}(0)$ for $\sigma \leq \sigma_0$. Thus,

$$\sup_{t\in[0,t_0]} E_\sigma(t) \leq 2E_{\sigma_0}(0), \tag{4.5.2}$$

provided that

$$c\delta\sigma^2(E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq E_{\sigma_0}(0). \tag{4.5.3}$$

Then, we can apply Theorem 4.3.1, with initial time $t = t_0$ and the time step δ as in (4.3.2) to extend the solution to $[t_0, t_0 + \delta]$. By Theorem 4.4.4, the approximate conservation law, and (4.5.2) we have

$$\sup_{t\in[t_0,t_0+\delta]} E_\sigma(t) \leq E_\sigma(t_0) + c\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}}. \tag{4.5.4}$$

In this way, we recover time intervals $[0,\delta],[\delta,2\delta],[2\delta,3\delta]$ etc, and obtain

$$\begin{aligned}
E_\sigma(\delta) &\leq E_\sigma(0) + c\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}},\\
E_\sigma(2\delta) &\leq E_\sigma(\delta) + c\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq E_\sigma(0) + c2\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}},\\
E_\sigma(3\delta) &\leq E_\sigma(\delta) + c\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq E_\sigma(0) + c3\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}}.
\end{aligned}$$

Continuing this we arrive at

$$E_\sigma(n\delta) \leq E_\sigma(0) + cn\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}},$$

which continues as long as

$$cn\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq E_{\sigma_0}(0) \tag{4.5.5}$$

and hence

$$E_\sigma(n\delta) \leq E_\sigma(0) + cn\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq 2E_{\sigma_0}(0).$$

Note also that inequality (4.5.3) follows from inequality (4.5.5). Thus, induction stops at the first integer n for which

$$cn\delta\sigma^2(2E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq E_{\sigma_0}(0),$$

and then we have reached the final time

$$T = n\delta,$$

when

$$cT\sigma^2(2E_{\sigma_0}(0))^{\frac{p-1}{2}} \geq 1.$$

Note that T will be arbitrarily large for $\sigma > 0$ small enough. Moreover,

$$\sigma^2 > C^{-1}(2E_{\sigma_0}(0))^{\frac{-p+1}{2}} T^{-1},$$

yields

$$\sigma \geq cT^{-\frac{1}{2}},$$

as claimed. □

In conclusion, the local well-posedness result of the fourth order beam equation in the space $H^{\sigma(t),2} \times H^{\sigma(t),0}$ can be extended to global well-posedness as far as the radius of analyticity $\sigma(t) \sim \frac{1}{\sqrt{T}}$ for any large time T.

Chapter 5

Radius of analyticity of solution for the fifth order KdV-BBM equation

5.1 Introduction

From day to day life it is necessary to study waves on water because many activities like shipping, fishing and so on took place there. Remember that the fifth order KdV-BBM type equation which describes the unidirectional propagation of water waves is given by

$$\begin{aligned}\eta_t+\eta_x-\gamma_1\eta_{xxt}+\gamma_2\eta_{xxx}+\sigma_1\eta_{xxxxt}+\sigma_2\eta_{xxxxx}\\=-\frac{3}{4}(\eta^2)_x-\gamma(\eta^2)_{xxx}+\frac{7}{48}(\eta_x^2)_x+\frac{1}{8}(\eta^3)_x,\end{aligned} \tag{5.1.1}$$

with initial data

$$\eta(x,0)=\eta_0(x). \tag{5.1.2}$$

The unknown function is

$$\eta:\mathbb{R}^{1+1}\to\mathbb{R}.$$

The parameters $\gamma,\gamma_1,\gamma_2,\sigma_1,\sigma_2$ are constants that satisfy certain constraints; see (Bona et al., 2018; Carvajal and Panthee, 2020) for more details.

Here, our main concern is to get new lower bound on the radius of spatial analyticity for the solution of Cauchy problem (5.1.1)-(5.1.2), given initial data in a class of analytic functions. This improves the existing results on the lower bound for the fifth order KdV-BBM equation.

5.2 Energy of the fifth order KdV-BBM equation

In the case $\gamma=\frac{7}{48}$, the energy

$$E(\eta(t)):=\frac{1}{2}\int_{\mathbb{R}}(\eta^2+\gamma_1\eta_x^2+\sigma_1\eta_{xx}^2)dx \tag{5.2.1}$$

is conserved by the flow of (5.1.1). i.e.,

$$E(\eta(t)) = E(\eta(0)) \quad \text{for all t.} \tag{5.2.2}$$

Indeed, multiply (5.1.1) by η and integrating with respect to space, we have

$$\int_{\mathbb{R}} \eta\left(\eta_t + \eta_x - \gamma_1 \eta_{xxt} + \gamma_2 \eta_{xxx} + \sigma_1 \eta_{xxxxt} + \sigma_2 \eta_{xxxxx}\right) dx$$
$$= \int_{\mathbb{R}} \eta\left(-\frac{3}{4}(\eta^2)_x - \gamma(\eta^2)_{xxx} + \frac{7}{48}(\eta_x^2)_x + \frac{1}{8}(\eta^3)_x\right) dx.$$

Using integration by parts, assuming the solution is sufficiently regular and using the following identities,

$$\eta\eta_x = \frac{1}{2}\left(\eta^2\right)_x, \qquad \eta\eta_{xxx} = (\eta\eta_{xx})_x - \frac{1}{2}\left(\eta_x^2\right)_x,$$
$$\eta\eta_{xxxxx} = (\eta\eta_{xxxx})_x - (\eta_x\eta_{xxx})_x + \frac{1}{2}\left(\eta_{xx}^2\right)_x,$$

and

$$\eta\left(\eta^2\right)_x = \frac{2}{3}\left(\eta^3\right)_x, \qquad \eta\left(\eta^3\right)_x = \frac{3}{4}\left(\eta^4\right)_x,$$
$$\eta(\eta^2)_{xxx} = 2\left(\eta^2\eta_{xx}\right)_x + \eta\left(\eta_x^2\right)_x,$$
$$\eta\eta_{txx} = (\eta\eta_{xt})_x - \frac{1}{2}(\eta_x^2)_t,$$

we have,

$$\int_{\mathbb{R}} \left(\eta\eta_t - \gamma_1\eta\eta_{xxt} + \sigma_1\eta\eta_{xxxxt}\right) dx = 0.$$

Thus, it follows that

$$\frac{d}{dt}\int_{\mathbb{R}} (\eta^2 + \gamma_1\eta_x^2 + \sigma_1\eta_{xx}^2) dx = 0,$$

and hence

$$E(\eta(t)) := \int_{\mathbb{R}} (\eta^2 + \gamma_1\eta_x^2 + \sigma_1\eta_{xx}^2 dx)$$

is constant. In order to prove our main result, Theorem 5.5.1, we have to prove the following local-in-time result, where the radius of analyticity remains constant as a first step.

5.3 Local well-posedness in $H^{\sigma,s}(\mathbb{R})$

Theorem 5.3.1. *(Local well posedness). Let $s \geq 1$, $\sigma_0 > 0$ and $\eta_0 \in H^{\sigma_0,s}(\mathbb{R})$. Then there exists a unique solution*

$$\eta \in C([0,\delta]; H^{\sigma_0,s}(\mathbb{R})),$$

of the Cauchy problem (5.1.1)-(5.1.2) *where the existence time is*

$$\delta \sim (\|\eta_0\|_{H^{\sigma_0,s}(\mathbb{R})} + \|\eta_0\|^2_{H^{\sigma_0,s}(\mathbb{R})})^{-1}.$$

Moreover, $\|\eta\|_{L^\infty_\delta H^{\sigma_0,s}(\mathbb{R})} \leq \|\eta_0\|_{H^{\sigma_0,s}(\mathbb{R})}$.

In order to prove Theorem 5.3.1 first we write the integral representation for the KdV-BBM equation ((5.1.1)-(5.1.2)) and then state a crucial lemma. Here we apply the contraction argument in the space $H^{\sigma_0,s}(\mathbb{R})$ to obtain local well-posedness result for the KdV-BBM equation with the initial data in $H^{\sigma_0,s}(\mathbb{R})$ for $s \geq 1, \sigma_0 > 0$. Applying the spatial Fourier transform to equation (5.1.1) it follows that,

$$\begin{aligned}\widehat{\eta}_t + i\xi\widehat{\eta} + \gamma_1\xi^2\widehat{\eta}_t - i\gamma_2\xi^3\widehat{\eta} + \sigma_1\xi^4\widehat{\eta}_t + i\sigma_2\xi^5\widehat{\eta} \\ = -\frac{3}{4}i\xi\widehat{\eta^2} + i\gamma\xi^3\widehat{\eta^2} + \frac{7}{48}i\xi\widehat{\eta_x^2} + \frac{1}{8}i\xi\widehat{\eta^3}.\end{aligned}$$

Re-arranging terms gives

$$\begin{aligned}\partial_t\left(1+\gamma_1\xi^2+\sigma_1\xi^4\right)\widehat{\eta} + i\xi\left(1-\gamma_2\xi^2+\sigma_2\xi^4\right)\widehat{\eta} \\ = \frac{1}{4}i\xi\left(-3+4\gamma\xi^2\right)\widehat{\eta^2} + \frac{7}{48}i\xi\widehat{\eta_x^2} + \frac{1}{8}i\xi\widehat{\eta^3}.\end{aligned}$$

Thus, it follows that

$$i\widehat{\eta}_t - \phi(\xi)\widehat{\eta} = \tau(\xi)\widehat{\eta^2} - \frac{1}{8}\psi(\xi)\widehat{\eta^3} - \frac{7}{48}\psi(\xi)\widehat{\eta_x^2}, \tag{5.3.1}$$

where

$$\phi(\xi) = \frac{\xi(1-\gamma_2\xi^2+\sigma_2\xi^4)}{\varphi(\xi)}, \quad \psi(\xi) = \frac{\xi}{\varphi(\xi)}, \quad \tau(\xi) = \frac{\xi(3-4\gamma\xi^2)}{4\varphi(\xi)},$$

and $\varphi(\xi) = 1+\gamma_1\xi^2+\sigma_1\xi^4$. Since $\gamma_1, \sigma_1 > 0$, a polynomial $\varphi(\xi)$ is always positive. Now, define the Fourier multiplier operators $\phi(D)$, $\psi(D)$ and $\tau(D)$ as follows

$$\mathscr{F}[\phi(D)f] = \phi(\xi)\widehat{f}(\xi), \quad \mathscr{F}[\psi(D)f] = \psi(\xi)\widehat{f}(\xi), \quad \mathscr{F}[\tau(D)f] = \tau(\xi)\widehat{f}(\xi).$$

Therefore, we can rewrite equation (5.3.1) in an operator form as

$$i\eta_t - \phi(D)\eta = F(\eta), \tag{5.3.2}$$

where

$$F(\eta) = \tau(D)\eta^2 - \frac{1}{8}\psi(D)\eta^3 - \frac{7}{48}\psi(D)\eta_x^2. \tag{5.3.3}$$

Thus, equations (5.3.2)-(5.3.3) together with the initial data (5.1.2) can be rewritten as

$$\begin{cases} i\eta_t = \phi(D)\eta + F(\eta), \\ \eta(x,0) = \eta_0(x). \end{cases}.$$

By Duhamel's formula the integral representation of equations (5.3.2)-(5.3.3) with initial data $\eta(x,0) = \eta_0(x)$ is given by

$$\eta(t) = S(t)\eta_0 - i\int_0^t S(t-t')F(\eta)(t')dt', \tag{5.3.4}$$

where $S(t) = e^{-it\phi(D)}$ is a unitary operator in $H^s(\mathbb{R})$ and hence also unitary in $H^{\sigma,s}(\mathbb{R})$. Next we estimate $F(\eta)$ in $H^{\sigma,s}(\mathbb{R})$.

Lemma 5.3.2. *Let $F(\eta)$ be defined as in equation* (5.3.3)*. Then for $s \geq 1$, $\sigma > 0$, we have nonlinear estimate*

$$\|F(\eta)\|_{H^{\sigma,s}(\mathbb{R})} \lesssim [1 + \|\eta\|_{H^{\sigma,s}(\mathbb{R})}]\|\eta\|^2_{H^{\sigma,s}(\mathbb{R})} \quad \textit{for all} \quad \eta \in H^{\sigma,s}(\mathbb{R}). \tag{5.3.5}$$

Proof. In (Carvajal and Panthee, 2020, Lemma 2.2-2.4) it was proved that

$$\|F(\eta)\|_{G^{\sigma,s}(\mathbb{R})} \lesssim [1 + \|\eta\|_{G^{\sigma,s}(\mathbb{R})}]\|\eta\|^2_{G^{\sigma,s}(\mathbb{R})}. \tag{5.3.6}$$

Then, estimate (5.3.5) follows from estimates (5.3.6) and (4.1.6). □

Now, we can prove Theorem 5.3.1 as follows. Define a mapping Γ by

$$\Gamma(\eta(t)) := S(t)\eta_0 - i\int_0^t S(t-t')F(\eta)(t')dt'.$$

Here, we want to show that the mapping Γ is a contraction for some local time on a closed ball with radius $r > 0$ and center at the origin in $C([0,\delta] : H^{\sigma,s}(\mathbb{R}))$. We choose the contraction space

$$X = C([0,\delta] : H^{\sigma,s}(\mathbb{R})),$$

with norm

$$\|\eta\|_X = \sup_{0\le t\le\delta} \|\eta\|_{H^{\sigma,s}(\mathbb{R})}.$$

Also consider a closed ball of radius r in X where

$$X_r = \{\eta \in X : \|\eta\|_X \le r\}.$$

Then, X_r is a closed subspace of X and hence it is a Banach space as X is a Banach space. We show Γ maps X_r in to X_r and Γ is a contraction mapping.

i) Γ maps X_r in to X_r. i.e., for all $\eta \in X_r, \Gamma(\eta(t)) \in X_r$.

$$\begin{aligned}\|\Gamma(\eta(t))\|_{H^{\sigma,s}(\mathbb{R})} &= \left\| S(t)\eta_0 - i\int_0^t S(t-t')F(\eta)(t')dt' \right\|_{H^{\sigma,s}(\mathbb{R})} \\ &\le \|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \int_0^t \left\|F(\eta)(t')\right\|_{H^{\sigma,s}(\mathbb{R})} dt'\end{aligned}$$

Now, taking the supremum over $[0,\delta]$ on both sides, (Carvajal and Panthee, 2020, Lemma 2.2-2.4) and equation (5.3.3), it follows that

$$
\begin{aligned}
\|\Gamma(\eta(t))\|_X &\leq \|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \int_0^\delta \left\|F(\eta)(t')\right\|_{H^{\sigma,s}(\mathbb{R})} dt' \\
&\leq \|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \int_0^\delta \left[c_1\|\eta\|^2_{H^{\sigma,s}} + c_2\|\eta\|^3_{H^{\sigma,s}} + c_3\|\eta\|^2_{H^{\sigma,s}}\right] dt \\
&= \|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \delta\left[c_4\|\eta\|^2_{H^{\sigma,s}} + c_2\|\eta\|^3_{H^{\sigma,s}}\right] \quad \text{where} \quad c_4 = c_1 + c_3 \\
&\leq \|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \delta\left[c_4 \sup_{0\leq t\leq\delta}\|\eta\|^2_{H^{\sigma,s}} + c_2 \sup_{0\leq t\leq\delta}\|\eta\|^3_{H^{\sigma,s}}\right] \\
&= \|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \delta\left[c_4\|\eta\|^2_X + c_2\|\eta\|^3_X\right] \\
&\leq \|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \delta[c_4 r^2 + c_2 r^3].
\end{aligned}
$$

Choose $r = 2\|\eta_0\|_{H^{\sigma,s}(\mathbb{R})}$, then for any $\delta \leq \frac{1}{2(c_4 r + c_2 r^2)}$, we have

$$
\|\Gamma(\eta(t))\|_X \leq r.
$$

Here it follows that the existing time is given by

$$
\delta \sim (\|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \|\eta_0\|^2_{H^{\sigma,s}(\mathbb{R})})^{-1}. \tag{5.3.7}
$$

ii) To prove the mapping is contraction, we need to show for all $\eta_1, \eta_2 \in X_r$

$$
\|\Gamma(\eta_1) - \Gamma(\eta_2)\|_X \leq \theta\|\eta_1 - \eta_2\|_X, \quad \text{where} \quad 0 < \theta < 1.
$$

For $c_4 = c_1 + c_3$ and $F(\eta)$ as in equation (5.3.3) it follows that

$$
\begin{aligned}
\|F(\eta_1) - F(\eta_2)\|_{H^{\sigma,s}(\mathbb{R})} &\leq c_4\left\|\eta_1^2 - \eta_2^2\right\|_{H^{\sigma,s}(\mathbb{R})} + c_2\left\|\eta_1^3 - \eta_2^3\right\|_{H^{\sigma,s}(\mathbb{R})} \\
&= c_4\|(\eta_1 - \eta_2)(\eta_1 + \eta_2)\|_{H^{\sigma,s}(\mathbb{R})} \\
&\quad + c_2\left\|(\eta_1 - \eta_2)(\eta_1^2 + \eta_1\eta_2 + \eta_2^2)\right\|_{H^{\sigma,s}(\mathbb{R})}
\end{aligned}
$$

Since $s > 1/2$, by (4.1.6), $H^{\sigma,s}(\mathbb{R})$ is an algebra and it follows

$$\begin{aligned}\|F(\eta_1) - F(\eta_2)\|_{H^{\sigma,s}(\mathbb{R})} &\leq c_4\|\eta_1 - \eta_2\|_{H^{\sigma,s}(\mathbb{R})}\|\eta_1 + \eta_2\|_{H^{\sigma,s}(\mathbb{R})} \\ &+ c_2\|\eta_1 - \eta_2\|_{H^{\sigma,s}(\mathbb{R})} c_2 \left\|\eta_1^2 + \eta_1\eta_2 + \eta_2^2\right\|_{H^{\sigma,s}(\mathbb{R})} \\ &\leq \|\eta_1 - \eta_2\|_{H^{\sigma,s}(\mathbb{R})}\left[c_4 2r + c_2 3r^2\right]\end{aligned}$$

Thus,

$$\begin{aligned}\|\Gamma(\eta_1) - \Gamma(\eta_2)\|_X &\leq \int_0^\delta \|F(\eta_1) - F(\eta_2)\|_{H^{\sigma,s}} dt \\ &\leq \delta\|\eta_1 - \eta_2\|_{H^{\sigma,s}(\mathbb{R})}\left[c_4 2r + c_2 3r^2\right]\end{aligned}$$

Now, if we choose $\delta = \frac{1}{2(2c_4 r + c_2 3r^2)}$ it follows

$$\|\Gamma(\eta_1) - \Gamma(\eta_2)\|_X \leq \frac{1}{2}\|\eta_1 - \eta_2\|_X.$$

We conclude that Γ is contraction mapping and hence there exists a unique solution to Cauchy problem (5.1.1)-(5.1.2) by Banach fixed point theorem. In (Carvajal and Panthee, 2020) uniqueness and continuous dependence of solution on initial data was proved. Thus, due to equivalence of the Gervey space and the modified Gervey space, it follows that the solution of the fifth order KdV-BBM equation exists and continuously dependent on the initial data in the modified Gervey space.

5.4 Approximate conservation law of the KdV-BBM equation

Next, we need to prove an approximate conservation law for the norm of the solution, that involves a small parameter $\sigma > 0$ and which reduces to the exact energy conservation law in the limit as $\sigma \to 0$. In order to derive this approximate conservation law, set

$$v_\sigma(x,t) := \cosh(\sigma|D|)\eta(x,t),$$

where $D=-i\partial_x$ and $\eta(x,t)$ is solution of the KdV-BBM equation. The modified energy associated with $v_\sigma(t)$ is given by

$$E_\sigma(t)=\frac{1}{2}\int_{\mathbb{R}}(v_\sigma^2+\gamma_1(\partial_x v_\sigma)^2+\sigma_1(\partial_x^2 v_\sigma)^2)dx. \tag{5.4.1}$$

Observe that if $\sigma=0$, we have $v_\sigma=\eta$ and therefore the energy equation (5.4.1) is conserved. i.e., $E_0(t)=E_0(0)$ for all t. However, this fails to hold for $\sigma>0$. And hence we prove the approximate conservation

$$\sup_{0\le t\le\delta} E_\sigma(t)=E_\sigma(0)+\sigma^2.\mathscr{O}\left([1+(E_\sigma(0))]^{\frac{1}{2}}(E_\sigma(0))^{\frac{3}{2}}\right),$$

for δ as stated in the above Theorem 5.3.1. Thus as $\sigma\to 0$, we recover the conservation

$$E_\sigma(t)=E_\sigma(0).$$

Fix $\gamma_1,\sigma_2>0$. Letting $\gamma=\frac{7}{48}$ and applying the operator $v_\sigma:=\cosh(\sigma|D|)\eta$ to (5.1.1) it follows that

$$\begin{aligned}\partial_t v_\sigma+&\partial_x v_\sigma-\gamma_1\partial_t\partial_x^2 v_\sigma+\gamma_2\partial_x^3 v_\sigma+\sigma_1\partial_t\partial_x^4 v_\sigma+\sigma_2\partial_x^5 v_\sigma\\ &=-\frac{3}{4}\partial_x(v_\sigma^2)-\gamma\partial_x^3(v_\sigma^2)+\gamma\partial_x(\partial_x v_\sigma)^2+\frac{1}{8}\partial_x(v_\sigma^3)+N(v_\sigma),\end{aligned} \tag{5.4.2}$$

where

$$N(v_\sigma)=\left(\frac{3}{4}+\gamma\partial_x^2\right)\partial_x N_1(v_\sigma)-\gamma\partial_x N_2(v_\sigma)-\frac{1}{8}\partial_x N_3(v_\sigma), \tag{5.4.3}$$

with

$$\begin{aligned}N_1(v_\sigma)&=v_\sigma^2-\cosh(\sigma|D|)(sech^2(\sigma|D|)v_\sigma),\\ N_2(v_\sigma)&=(\partial_x v_\sigma)^2-\cosh(\sigma|D|)(\eta_x^2),\\ N_3(v_\sigma)&=v_\sigma^3-\cosh(\sigma|D|)(sech^3(\sigma|D|)v_\sigma).\end{aligned} \tag{5.4.4}$$

Assuming that the solution is vanishing at infinity, using integration by parts and (5.4.2)-

(5.4.4) it follows that

$$
\begin{aligned}
E'_\sigma(t) &= \int_{\mathbb{R}} v_\sigma \partial_t v_\sigma + \gamma_1 \partial_x v_\sigma \partial_t(\partial_x v_\sigma) + \sigma_1 \partial_{xx} v_\sigma \partial_t(\partial_{xx} v_\sigma) dx \\
&= \int_{\mathbb{R}} v_\sigma [\partial_t v_\sigma - \gamma_1 \partial_t \partial_x^2 v_\sigma + \sigma_1 \partial_t \partial_x^4 v_\sigma] dx \\
&= \int_{\mathbb{R}} v_\sigma \left[\partial_x v_\sigma + \gamma_2 \partial_x^3 v_\sigma + \sigma_2 \partial_x^5 v_\sigma + \frac{3}{4}\partial_x(v_\sigma^2) + \gamma \partial_x^3(v_\sigma^2) - \gamma \partial_x(\partial_x v_\sigma)^2 - \frac{1}{8}\partial_x(v_\sigma)^3 \right] dx \\
&+ \int_{\mathbb{R}} v_\sigma N(v_\sigma) dx.
\end{aligned}
$$

The integral on the third line is zero (see (Belayneh et al., 2022)). Therefore,

$$
E'_\sigma(t) = \int_{\mathbb{R}} v_\sigma(x,t) N(v_\sigma(x,t)) dx.
$$

Consequently, integrating both sides with respect to time gives

$$
E_\sigma(t) = E_\sigma(0) + \int_0^t \int_{\mathbb{R}} v_\sigma(x,t') N(v_\sigma(x,t')) dx dt'. \tag{5.4.5}
$$

We now state and prove a consecutive Lemmas that will help us to prove the approximate conservation law.

Lemma 5.4.1. *For $N(v_\sigma)$ as in equations* (5.4.3)-(5.4.4)*, we have the inequality*

$$
\left| \int_{\mathbb{R}} v_\sigma N(v_\sigma) dx \right| \leq c\sigma^2 \left[1 + \|v_\sigma\|_{H^2(\mathbb{R})} \right] \|v_\sigma\|^3_{H^2(\mathbb{R})}.
$$

We aim to show

$$
\left| \int_{\mathbb{R}} v_\sigma N(v_\sigma) dx \right| \leq c\sigma^2 \left[1 + \|v_\sigma\|_{H^2(\mathbb{R})} \right] \|v_\sigma\|^3_{H^2(\mathbb{R})},
$$

where $N(v_\sigma)$ is given as in (5.4.3)-(5.4.4).

Using equation (5.4.3) and Plancherel Theorem, we get

$$
\begin{aligned}
\int_{\mathbb{R}} v_\sigma N(v_\sigma) dx &= \int_{\mathbb{R}} \left[v_\sigma (\frac{3}{4} + \gamma \partial_x^2) \partial_x N_1(v_\sigma) - \gamma v_\sigma \partial_x N_2(v_\sigma) - \frac{1}{8} v_\sigma \partial_x N_3(v_\sigma) \right] dx \\
&= I_1 + I_2 + I_3
\end{aligned}
$$

where,

$$I_1 = \int_{\mathbb{R}} \left(\frac{3}{4} + \gamma\partial_x^2\right) v_\sigma \partial_x N_1(v_\sigma)dx,$$
$$I_2 = \gamma \int_{\mathbb{R}} \partial_x v_\sigma N_2(v_\sigma)dx,$$
$$I_3 = \frac{1}{8} \int_{\mathbb{R}} \partial_x v_\sigma N_3(v_\sigma)dx.$$

Now, we need to show that

$$|I_j| \lesssim \sigma^2 \|v_\sigma\|^3_{H^2(\mathbb{R})}, j = 1,2, \tag{5.4.6}$$

and

$$|I_3| \lesssim \sigma^2 \|v_\sigma\|^4_{H^2(\mathbb{R})}. \tag{5.4.7}$$

In order to prove estimates (5.4.6)-(5.4.7), we need the following lemma.

Lemma 5.4.2. *Let* $\xi = \sum_{j=1}^{p} \xi_j$*,* $\xi_j \in \mathbb{R}$ *and* $p \in \mathbb{N}$*, then*

a) For $p = 2$*,*

$$\left|1 - \frac{\cosh(\sigma|\xi|)}{\prod_{j=1}^{2}\cosh(\sigma|\xi_j|)}\right| \leq c\sigma^2|\xi_1||\xi_2|.$$

b) For $p = 3$*,*

$$\left|1 - \frac{\cosh(\sigma|\xi|)}{\prod_{j=1}^{3}\cosh(\sigma|\xi_j|)}\right| \leq c\sigma^2\xi_{\max}^2$$

where $\xi_{\max}^2 = \max\{\xi_1^2, \xi_2^2, \xi_3^2\}$ *and c is a positive constant.*

Proof. The proof was given in Lemma 4.4.2 for the more general case. However, for the reader's convinence we include the proof of (a) and (b) below.

(a) The hyperbolic identity

$$\prod_{j=1}^{2}\cosh(\sigma|\xi_j|) = \frac{1}{2}\Big[\cosh\left(\sigma(|\xi_1| - |\xi_2|)\right) + \cosh(\sigma(|\xi_1| + |\xi_2|))\Big], \tag{5.4.8}$$

can be rewritten as

$$\prod_{j=1}^{2}\cosh(\sigma|\xi_j|) = 2^{-1}\sum_{s_2}\cosh(\sigma\left(|\xi_1| + s_2|\xi_2|\right)), \tag{5.4.9}$$

where s_2 is the independent signs ($+$ or $-$). It follows from equation (5.4.9) that

$$\cosh\Big(\sigma(|\xi_1|+s_2|\xi_2|)\Big)+\cosh(\sigma|\xi|)\le 4\prod_{j=1}^{2}\cosh(\sigma|\xi_j|). \tag{5.4.10}$$

Also observe that

$$\Big|(\sigma(|\xi_1|+s_2|\xi_2|))^2-(\sigma|\xi|)^2\Big|\le 2\sigma^2\sum_{j\neq k=1}^{2}|\xi_j||\xi_k|. \tag{5.4.11}$$

Thus,

$$\Big|\prod_{j=l}^{2}\cosh(\sigma|\xi_j|)-\cosh(\sigma|\xi|)\Big|=\Big|2^{-1}\sum_{s_2}\Big[\cosh\big(\sigma(|\xi_1|+s_2|\xi_2|)\big)-\cosh(\sigma|\xi|)\Big]\Big|. \tag{5.4.12}$$

From Lemma 4.4.1, inequality (5.4.10) and (5.4.11), it follows that

$$\begin{aligned}
&\text{RHS of (5.4.12)}\\
&\le 2^{-1}\sum_{s_2}\left[\frac{1}{2}\Big|\sigma^2\Big(|\xi_1|+s_2|\xi_2|\Big)^2-(\sigma|\xi|)^2\Big|\right]\Big(\cosh\Big(\sigma(|\xi_1|+s_2|\xi_2|)\Big)+\cosh(\sigma|\xi|)\Big)\\
&\le 2^{-1}\sum_{s_2}\Big(\sigma^2\sum_{j\neq k=1}^{2}|\xi_j||\xi_k|\Big)2\prod_{j=1}^{2}\cosh(\sigma|\xi_j|)\\
&=4\sigma^2\Big(\sum_{j\neq k=1}^{2}|\xi_j||\xi_k|\Big)\prod_{j=1}^{2}\cosh(\sigma|\xi_j|).
\end{aligned}$$

Dividing by $\prod_{j=1}^{2}\cosh(\sigma|\xi_j|)$, we arrive at the desired inequality.

(b) Multiplying equation (5.4.8) by $\cosh(\sigma|\xi_3|)$ leads to

$$4\prod_{j=1}^{3}\cosh(\sigma|\xi_j|)=\sum_{j=1}^{4}\cosh(\sigma(k_j)), \tag{5.4.13}$$

where $k_1=|\xi_1|-|\xi_2|-|\xi_3|$, $k_2=|\xi_1|-|\xi_2|+|\xi_3|$, $k_3=|\xi_1|+|\xi_2|-|\xi_3|$ and $k_4=|\xi_1|+|\xi_2|+|\xi_3|$.

Then (5.4.13) can be rewritten as

$$\prod_{j=1}^{3}\cosh(\sigma|\xi_j|)=2^{-2}\sum_{s_2,s_3}\cosh\Big(\sigma(|\xi|+\sum_{j=2}^{3}s_j|\xi_j|)\Big), \tag{5.4.14}$$

where s_2, s_3 are independent signs ($+$ or $-$). From equation (5.4.14) we get

$$\cosh\Big(\sigma(|\xi_1|+\sum_{j=2}^{3} s_j|\xi_j|)\Big)+\cosh(\sigma|\xi|)) \leq 8\prod_{j=1}^{3}\cosh(\sigma|\xi_j|). \tag{5.4.15}$$

Also observe that for an independent signs s_2, s_3, we have

$$\Big|\sigma^2\Big(|\xi_1|+\sum_{j=2}^{3} s_j|\xi_j|\Big)^2-(\sigma|\xi|)^2\Big| \leq 2\sigma^2\sum_{j\neq k=1}^{3}|\xi_j||\xi_k|. \tag{5.4.16}$$

Applying (5.4.14) and(5.4.15) we have

$$\Big|\prod_{j=1}^{3}\cosh(\sigma|\xi_j|)-\cosh(\sigma|\xi|)\Big| = \Big|2^{-2}\sum_{s_2,s_3}\Big[\cosh\Big(\sigma(|\xi_1|+\sum_{j=2}^{3} s_j|\xi_j|)\Big)-\cosh(\sigma|\xi|)\Big]\Big|. \tag{5.4.17}$$

Next, from Lemma 4.4.1 and inequality (5.4.16), we have

RHS of equation (5.4.17)

$$\begin{aligned}
&\leq 2^{-2}\sum_{s_2,s_3}\frac{1}{2}\Big|\sigma^2\Big(|\xi_1|+\sum_{j=2}^{3} s_j|\xi_j|\Big)^2-(\sigma|\xi|)^2\Big|\Big(\cosh(\sigma(|\xi_1|+\sum_{j=2}^{3} s_j|\xi_j|))+\cosh(\sigma|\xi|))\Big)\\
&\leq 2^{-2}\sum_{s_2,s_3}\Big(\sigma^2\sum_{j\neq k=1}^{3}|\xi_j||\xi_k|\Big)4\prod_{j=1}^{3}\cosh(\sigma|\xi_j|)\\
&= 8\sigma^2\Big(\sum_{j\neq k=1}^{3}|\xi_j||\xi_k|\Big)\prod_{j=1}^{3}\cosh(\sigma|\xi_j|).
\end{aligned}$$

Dividing by $\prod_{j=1}^{3}\cosh(\sigma|\xi_j|)$ leads to the desired result.

□

Proof of estimate (5.4.6): For $j=1$, using Hölder inequality,

$$\begin{aligned}
|I_1| = \Big|\int_{\mathbb{R}}(\frac{3}{4}+\gamma\partial_x^2)v_\sigma.\partial_x N_1(v_\sigma)dx\Big| &\leq \Big\|(\frac{3}{4}+\gamma\partial_x^2)v_\sigma\Big\|_{L_x^2(\mathbb{R})}\|\partial_x N_1(v_\sigma)\|_{L_x^2(\mathbb{R})}\\
&\lesssim \|v_\sigma\|_{H^2(\mathbb{R})}\|\partial_x N_1(v_\sigma)\|_{L_x^2(\mathbb{R})}.
\end{aligned}$$

Next, we show

$$\|\partial_x N_1(v_\sigma)\|_{L_x^2(\mathbb{R})} \lesssim \sigma^2\|v_\sigma\|_{H^2(\mathbb{R})}^2.$$

Taking the spatial Fourier Transform of $\partial_x N_1(v_\sigma)$ and using Lemma 5.4.2 (a), we obtain

$$\begin{aligned}\left|\mathscr{F}_x(\partial_x N_1(v_\sigma)(\xi))\right| &= \int_{\xi=\xi_1+\xi_2} |\xi|\left|\prod_{j=1}^{2}\cosh(\sigma|\xi_j|)-\cosh(\sigma|\xi|)\right||\widehat{\eta}(\xi_1)||\widehat{\eta}(\xi_2)|d\xi_1 d\xi_2 \\ &= \int_{\xi=\xi_1+\xi_2} |\xi|\left|1-\frac{\cosh(\sigma|\xi|)}{\prod_{j=1}^{2}\cosh(\sigma|\xi_j|)}\right||\widehat{v_\sigma}(\xi_1)||\widehat{v_\sigma}(\xi_2)|d\xi_1 d\xi_2 \\ &\le 4\sigma^2\int_{\xi=\xi_1+\xi_2}|\xi_1+\xi_2||\xi_1||\xi_2||\widehat{v_\sigma}(\xi_1)||\widehat{v_\sigma}(\xi_2)|d\xi_1 d\xi_2.\end{aligned}$$

Assume that $|\xi_1|\le|\xi_2|$, then

$$\left|\mathscr{F}_x(\partial_x N_1(v)(\xi))\right| \le 4\sigma^2\int_{\xi=\xi_1+\xi_2} 2|\xi_1||\widehat{v_\sigma}(\xi_1)||\xi_2|^2|\widehat{v_\sigma}(\xi_2)|d\xi_1 d\xi_2.$$

Set $V=\mathscr{F}_x^{-1}(|\widehat{v_\sigma}|)$, the it follows that

$$\begin{aligned}\left|\mathscr{F}_x(\partial_x N_1(v_\sigma)(\xi))\right| &\le 8\sigma^2\int_{\xi=\xi_1+\xi_2}|\xi_1\widehat{V}(\xi_1)|\xi_2|^2\widehat{V}(\xi_2)|d\xi_1 d\xi_2 \\ &= 8\sigma^2\mathscr{F}_x[|D|V.|D|^2V](\xi).\end{aligned}$$

Taking an L^2 norm, using Plancherel Theorem, applying Hölder inequality and Sobolev embedding yields

$$\begin{aligned}\|\partial_x N_1(v)(\xi)\|_{L^2_x(\mathbb{R})} &\le c\sigma^2\||D|V\|_{L^\infty_x(\mathbb{R})}\left\||D|^2V\right\|_{L^2_x(\mathbb{R})} \\ &\lesssim \sigma^2\|V\|^2_{H^2(\mathbb{R})} \\ &\lesssim \sigma^2\|v_\sigma\|^2_{H^2(\mathbb{R})}.\end{aligned}$$

Thus estimate (5.4.6) holds true for $j=1$.

To prove the case $j=2$, first claim that

$$\left\|\langle D\rangle^{-1}N_2(v_\sigma)\right\|_{L^2_x(\mathbb{R})} \lesssim \sigma^2\|v_\sigma\|^2_{H^2(\mathbb{R})}.$$

Taking the Fourier Transform of $N_2(v_\sigma)$, using Lemma 5.4.2(a) and $\widehat{\eta}_x(\xi)=i\xi\widehat{\eta}(\xi)$, we obtain

$$
\begin{aligned}
\left|\mathscr{F}_x(N_2(v_\sigma)(\xi))\right| &= \int_{\xi=\xi_1+\xi_2}\left|\prod_{j=1}^{2}\cosh(\sigma|\xi_j|)-\cosh(\sigma|\xi|)\right||\xi_1||\xi_2||\widehat{\eta}(\xi_1)||\widehat{\eta}(\xi_2)|d\xi_1 d\xi_2 \\
&\leq 4\sigma^2\int_{\xi=\xi_1+\xi_2}|\xi_1||\xi_2||\xi_1||\xi_2||\widehat{v_\sigma}(\xi_1)||\widehat{v_\sigma}(\xi_2)|d\xi_1 d\xi_2 \\
&= 4\sigma^2\int_{\xi=\xi_1+\xi_2}|\xi_1|^2||\widehat{v_\sigma}(\xi_1)|\xi_2|^2|\widehat{v_\sigma}(\xi_2)|d\xi_1 d\xi_2 \\
&= 4\sigma^2\mathscr{F}_x[|D|^2V.|D|^2V](\xi).
\end{aligned}
$$

By Plancherel Theorem, we have

$$
\|N_2(v_\sigma)\|_{L^2_x}\leq 4\sigma^2\left\||D|^2V.|D|^2V\right\|_{L^2_x}. \tag{5.4.18}
$$

In the case of one dimension, from (Tao, 2006, the inhomogeneous Sobolev embedding A.12) we have

$$
\|f\|_{L^q_x(\mathbb{R})}\lesssim\|f\|_{W^{s,p}_x(\mathbb{R})}\quad\text{provided that}\quad\frac{1}{p}=\frac{1}{q}+s.
$$

Thus,

$$
\|f\|_{L^q(\mathbb{R})}\lesssim\|\langle D\rangle^s f\|_{L^p(\mathbb{R})}\iff\left\|\langle D\rangle^{-1/2}f\right\|_{L^2(\mathbb{R})}\lesssim\|f\|_{L^1(\mathbb{R})},
$$

and it follows that

$$
\left\|\langle D\rangle^{-1}f\right\|_{L^2_x(\mathbb{R})}\lesssim\|f\|_{L^1_x(\mathbb{R})}.
$$

Using estimate (5.4.18), Hölder inequality and the above inequality, we have

$$
\begin{aligned}
\left\|\langle D\rangle^{-1}N_2(v_\sigma)\right\|_{L^2_x(\mathbb{R})} &\leq 4\sigma^2\left\|\langle D\rangle^{-1}(|D|^2V.|D|^2V)\right\|_{L^2_x(\mathbb{R})} \\
&\lesssim\sigma^2\left\||D|^2V.|D|^2V\right\|_{L^1_x(\mathbb{R})} \\
&\lesssim\sigma^2\left\||D|^2V\right\|_{L^2_x(\mathbb{R})}\left\||D|^2V\right\|_{L^2_x(\mathbb{R})} \\
&\lesssim\sigma^2\|V\|_{H^2}.\|V\|_{H^2(\mathbb{R})} \\
&\sim\sigma^2\|v_\sigma\|^2_{H^2(\mathbb{R})}.
\end{aligned}
$$

Hence the claim. Also note that by Plancherel Theorem

$$
\int_{\mathbb{R}}\partial_x v_\sigma.N_2(v_\sigma)dx=\int_{\mathbb{R}}\langle D\rangle\partial_x v_\sigma.\langle D\rangle^{-1}N_2(v_\sigma)dx.
$$

Next, by Hölder inequality, we have

$$|I_2| = \left| \int_{\mathbb{R}} \partial_x v_\sigma . N_2(v_\sigma) dx \right| \leq \|\langle D \rangle \partial_x v_\sigma\|_{L^2_x(\mathbb{R})} \left\| \langle D \rangle^{-1} N_2(v_\sigma) \right\|_{L^2_x(\mathbb{R})}$$
$$\lesssim \|v_\sigma\|_{H^2(\mathbb{R})} \left\| \langle D \rangle^{-1} N_2(v_\sigma) \right\|_{L^2_x(\mathbb{R})}.$$

Thus, it follows that

$$|I_2| \lesssim \sigma^2 \|v_\sigma\|^3_{H^2(\mathbb{R})}.$$

Proof of estimate (5.4.7): By Hölder inequality, we have

$$|I_3| = \frac{1}{8} \left| \int_{\mathbb{R}} \partial_x v_\sigma N_3(v_\sigma) dx \right|$$
$$\lesssim \|\partial_x v_\sigma\|_{L^2_x(\mathbb{R})} \|N_3(v_\sigma)\|_{L^2_x(\mathbb{R})}$$
$$\lesssim \|v_\sigma\|_{H^1(\mathbb{R})} \|N_3(v_\sigma)\|_{L^2_x(\mathbb{R})}.$$

But, using Fourier Transform of $N_3(v_\sigma)$, we get

$$\left| \mathscr{F}_x(N_3(v_\sigma))(\xi) \right|$$
$$= \int_{\xi=\xi_1+\xi_2+\xi_3} \left| \prod_{j=1}^{3} \cosh(\sigma|\xi_j|) - \cosh(\sigma|\xi|) \right| |\widehat{\eta}(\xi_1)||\widehat{\eta}(\xi_2)||\widehat{\eta}(\xi_3)| d\xi_1 d\xi_2 d\xi_3.$$

By symmetry, we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3|$, then by Lemma 5.4.2 (b), we have

$$\left| \mathscr{F}_x(N_3(v_\sigma))(\xi) \right| \leq C\sigma^2 \int_{\xi=\xi_1+\xi_2+\xi_3} |\widehat{v_\sigma}(\xi_1)||\widehat{v_\sigma}(\xi_2)||\xi_3|^2 |\widehat{v_\sigma}(\xi_3)| d\xi_1 d\xi_2 d\xi_3.$$

Similarly, setting $V = \mathscr{F}_x^{-1}(|\widehat{v_\sigma}|)$ gives

$$|\mathscr{F}_x(N_3(v_\sigma))(\xi)| \leq C\sigma^2 \mathscr{F}_x(V.V.|D|^2 V)(\xi).$$

Next, by taking an L^2 norm, using Plancherel Theorem and applying Hölder inequality, we get

$$
\begin{aligned}
\|\mathscr{F}_x(N_3(v_\sigma))(\xi)\|_{L_x^2(\mathbb{R})} &\lesssim \sigma^2 \left\|V^2.|D|^2 V\right\|_{L_x^2(\mathbb{R})} \\
&\lesssim \sigma^2 \|V\|_{L_x^\infty(\mathbb{R})}^2 \left\||D|^2 V\right\|_{L_x^2(\mathbb{R})} \\
&\lesssim \sigma^2 \|V\|_{H^2(\mathbb{R})}^2 \|V\|_{H^2(\mathbb{R})} \\
&\lesssim \sigma^2 \|V\|_{H^2(\mathbb{R})}^3 \\
&\lesssim \sigma^2 \|v_\sigma\|_{H^2(\mathbb{R})}^3,
\end{aligned}
$$

which concludes estimate (5.4.7). This completes proof of Lemma 5.4.1.

Theorem 5.4.3. *(Almost conservation law). Let $\eta_0 \in H^{\sigma,2}(\mathbb{R})$, suppose that $\eta \in C([0,\delta]; H^{\sigma,2}(\mathbb{R}))$ is the local in time solution to the Cauchy problem* (5.1.1)-(5.1.2) *that is constructed in Theorem 5.3.1. Then*

$$
\sup_{0\le t\le\delta} E_\sigma(t) = E_\sigma(0) + \sigma^2 \delta . \mathscr{O}\left([1 + (E_\sigma(0))^{\frac{1}{2}}](E_\sigma(0))^{\frac{3}{2}}\right). \tag{5.4.19}
$$

Proof. In view of equation (5.4.5) and Lemma 5.4.1, we have the a prior energy estimate

$$
\sup_{0\le t\le\delta} E_\sigma(t) = E_\sigma(0) + \sigma^2 . \mathscr{O}\left([1 + \|v_\sigma\|_{L_\delta^\infty H^2}]\|v_\sigma\|_{L_\delta^\infty H^2}^3\right), \tag{5.4.20}
$$

where $L_\delta^\infty H^2 := L_t^\infty H^2([0,\delta]\times\mathbb{R})$.

By Theorem 5.3.1, we have the bound,

$$
\|v_\sigma\|_{L_\delta^\infty H^2(\mathbb{R})} = \|\eta\|_{L_\delta^\infty H^{\sigma,2}(\mathbb{R})} \le c\|\eta_0\|_{H^{\sigma,2}(\mathbb{R})} = c\|v_0\|_{H^2(\mathbb{R})}, \tag{5.4.21}
$$

where δ is as in Theorem 5.3.1. Also from the modified energy we have

$$
\begin{aligned}
E_0(t) &= \frac{1}{2}\int_{\mathbb{R}} [v_\sigma(x,0)]^2 + \gamma_1 [\partial_x v_\sigma(x,0)]^2 + \sigma_1 \left[\partial_x^2 v_\sigma(x,0)\right]^2 dx \\
&\sim \|v_\sigma(.,0)\|_{H^2(\mathbb{R})}^2
\end{aligned}
$$

From a prior energy estimate equation (5.4.20), we get

$$\sup_{0\le t\le \delta} E_\sigma(t) = E_\sigma(0) + c\sigma^2\delta.\mathscr{O}\left([1+\|v\|_{L^\infty_\delta H^2(\mathbb{R})}]\|v\|^3_{L^\infty_\delta H^2(\mathbb{R})}\right)$$
$$\lesssim E_\sigma(0) + c\sigma^2\delta.\mathscr{O}\left([1+(E_\sigma(0))^{1/2}](E_\sigma(0))^{3/2}\right).$$

This completes proof of Theorem 5.4.3. □

5.5 Main result and its proof

In the study of lower bound for the radius of spatial analyticity of equation (5.1.1), we use the idea introduced in (Dufera et al., 2021) where an exponential weight, $\exp(\sigma|\xi|)$ in the Gevrey space is replaced by hyperbolic weight, $\cosh(\sigma|\xi|)$ to improve the result significantly showing that $\sigma(t)$ can not decay faster than $1/\sqrt{t}$ for large time t.

Previously we state that by the Paley-Wiener Theorem, the radius of analyticity of a function can be related to decay properties of its Fourier transform. Thus, in this study a class of analytic function spaces suitable to study analyticity of solution of the KdV-BBM equation is the modified-Gevrey class. We are considering initial data in the space $G^{\sigma,s}(\mathbb{R})$ (also in $H^{\sigma,s}(\mathbb{R})$ due to norm equivalence (4.1.6)) because of the analyticity properties of Gevrey functions, which are detailed in the Paley-Weiner theorem.

Observe that the statment of Paley-Wiener Theorem still holds for function in $H^{\sigma,s}(\mathbb{R})$ because of norm equivalence (4.1.6). Particularly if $\sigma = 0$, it follows that $H^{0,s}(\mathbb{R}) = G^{0,s}(\mathbb{R}) = H^s(\mathbb{R})$. We note that because of the embedding,

$$H^{\sigma,s}(\mathbb{R}) \subset H^s(\mathbb{R}) \quad (\sigma \ge 0), \tag{5.5.1}$$

and the existing well-posedness theory in $H^s(\mathbb{R})$(see Bona et al. (2018)), we conclude that the KdV-BBM equation (with $\gamma_1, \sigma_1 > 0$ and $\gamma = 7/48$) has a unique, smooth solution for all time, given initial data $\eta_0 \in H^{\sigma_0,s}(\mathbb{R})$ for all $\sigma_0 > 0$ and $s \in \mathbb{R}$. Now, we can state our main result in this dissertation as follows.

Theorem 5.5.1. *Suppose the parameters $\gamma_1, \sigma_1 > 0$ and $\gamma = \frac{7}{48}$. Let η be the global solution of* (5.1.1)-(5.1.2) *with $\eta_0 \in H^{\sigma_0,2}(\mathbb{R})$ for $\sigma_0 > 0$, then*

$$\eta(t) \in H^{\sigma(t),2}(\mathbb{R}) \quad \textit{for all} \quad t > 0,$$

with the radius of analyticity $\sigma(t)$ *satisfying the lower bound*

$$\sigma(t) \sim c/\sqrt{t} \quad as \quad t \to \infty,$$

where $c > 0$ *is constant depending on the initial data norm* $\|\eta_0\|_{H^{\sigma_0,2}(\mathbb{R})}$.

Thus, the solution $\eta(x,t)$ at any time t is analytic in the strip $s_{\sigma(t)}$, which implies that the solution has radius of analyticity at least $\sigma(t)$ for all time t due to equivalence (4.1.6) and Paley-Wiener Theorem.

Proof. Suppose that the initial data $\eta(x,0) = \eta_0(x) \in H^{\sigma_0,2}(\mathbb{R})$ for some $\sigma_0 > 0$. From the local well-posedness theory there is a unique solution $\eta \in C([0,\delta];H^{\sigma_0,2}(\mathbb{R}))$ of the KdV-BBM equation constructed in Theorem 5.3.1 with existence time

$$\delta \sim (\|\eta_0\|_{H^{\sigma,s}(\mathbb{R})} + \|\eta_0\|^2_{H^{\sigma,s}(\mathbb{R})})^{-1}.$$

Note that

$$v_{\sigma_0}(.,0) := \cosh(\sigma_0|D|)\eta_0 \in H^2(\mathbb{R}).$$

Then

$$E_{\sigma_0}(0) \sim \|v_{\sigma_0}(.,0)\|^2_{H^2(\mathbb{R})} < \infty.$$

Next, we follow the argument in (Selberg and Da Silva, 2015; Selberg and Tesfahun, 2015) to construct a solution on the interval $[0,T]$ for arbitrarily large time T where we apply the approximate conservation law in Theorem 5.4.3, equation (5.4.19) so as to repeat the above local result on successive short time intervals of size δ to reach the arbitrary large time T by adjusting the strip width parameter $\delta \in (0,\sigma_0]$ according to the size of T.

The main goal is to prove that for a given parameter $\delta \in (0,\sigma_0]$ and large time $T > 0$,

$$\sup_{0\le t\le T} E_\sigma(t) \le 2E_{\sigma_0}(0) \tag{5.5.2}$$

for $\sigma(t) \ge c/\sqrt{T}$, where the constant c depends only on the initial data norm. Thus, $E_\sigma(t) < \infty$ for $t \in [0,T]$, which in turn implies

$$\eta(t) \in H^{\sigma(t),2}(\mathbb{R}) \quad \text{for all} \quad t \in [0,T].$$

To prove (5.5.2), first observe that for a given parameter $0 < \delta < \sigma_0$ and $0 < t_0 \le \delta$, by Theorem 5.3.1 and 5.4.3, we have

$$\sup_{0\le t\le t_0} E_\sigma(t) \le E_\sigma(0) + c\sigma^2\delta[1+(E_\sigma(0))^{1/2}](E_\sigma(0))^{3/2})$$
$$\le E_{\sigma_0}(0) + c\sigma^2\delta[1+(E_{\sigma_0}(0))^{1/2}](E_{\sigma_0}(0))^{3/2}).$$

This is true because for $\sigma \in (0,\sigma_0)$, $E_\sigma(0) \le E_{\sigma_0}(0)$. Thus,

$$\sup_{0\le t\le t_0} E_\sigma(t) \le 2E_{\sigma_0}(0),$$

provided that

$$c\sigma^2\delta[1+(E_{\sigma_0}(0))^{1/2}](E_{\sigma_0}(0))^{3/2} \le E_{\sigma_0}(0). \tag{5.5.3}$$

Next, we apply Theorem 5.3.1 with initial time $t = t_0$ and the time with step size δ as in (5.3.7) to extend the solution from $[0,t_0]$ to $[t_0,t_0+\delta]$. From Theorem 5.4.3 and assumption (5.5.3), we have

$$\sup_{t_0\le t\le t_0+\delta} E_\sigma(t) \le E_\sigma(t_0) + c\sigma^2\delta\left[1+(2E_{\sigma_0}(0))^{1/2}\right]\left[(2E_{\sigma_0}(0))^{3/2}\right]. \tag{5.5.4}$$

In this way, we cover all time intervals $[0,\delta],[\delta,2\delta],[2\delta,3\delta]$ etc of size δ and obtain

$$E_\sigma(\delta) \le E_\sigma(0) + c\sigma^2\delta[1+(2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2}$$
$$E_\sigma(2\delta) \le E_\sigma(\delta) + c\sigma^2\delta[1+(2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2}$$
$$\le E_\sigma(0) + c\sigma^2 2\delta[1+(2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2}$$
$$E_\sigma(3\delta) \le E_\sigma(0) + c\sigma^2 3\delta[1+(2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2}.$$

In a similar way we have,

$$E_\sigma(n\delta) \le E_\sigma(0) + c\sigma^2 n\delta[1+(2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2}.$$

We repeatedly continue this process as long as

$$c\sigma^2 n\delta[1+(2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2} \le E_{\sigma_0}(0). \tag{5.5.5}$$

Thus, it follows that $E_\sigma(n\delta) \leq E_\sigma(0) + c\sigma^2 n\delta[1 + (2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2} \leq 2E_{\sigma_0}(0)$.
The induction stops at the first integer n for which

$$c\sigma^2 n\delta[1 + (2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{3/2} \geq E_{\sigma_0}(0),$$

and then we have reached the finite time

$$T = n\delta,$$

when $c\sigma^2 T[1 + (2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{1/2} \geq 1$.
Note that T is arbitrary large for $\sigma > 0$ small enough. Moreover,

$$\sigma^2 > c^{-1}\left([1 + (2E_{\sigma_0}(0))^{1/2}](2E_{\sigma_0}(0))^{1/2}\right)^{-1} .(T)^{-1},$$

and this proves $\sigma(t) \geq c/\sqrt{T}$. □

From the result obtained, we conclude that using local well-posedness and approximate conservation low in the newly modified Gervey space can improve the existing radius of spatial analyticity for the solution of fifth order KdV-BBM equation. We treat this problem only in one dimension and hence the higher dimension is still an open problem.

Chapter 6

Conclusion and Recommendation

6.1 Conclusion

In this dissertation we consider beam equation, which arises in the study of weak interaction of dispersive waves in one dimension and fifth order KdV-BBM equations describing the unidirectional propagation of water waves. In order to forcast and reduce the damage caused by water waves it is necessary to study about the radius of spatial analyticity for the solution of the governing equation. Accordingly, we introduced a new weight function $\cosh(\sigma|D|)$ instead of the earlier function $\exp(\sigma|D|)$ and hence define new space. Then we define and show equivalence of modified Gervey space $H^{\sigma,s}(\mathbb{R}^n)$ and the existing Gervey space $G^{\sigma,s}(\mathbb{R}^n)$.

From the modified Gervey space and the existing Paley-Wiener theorem, the solution of both beam and fifth order KdV-BBM equations has radius of analyticity at least $\sigma(t)$ for every time $t > 0$. Following the work of (Selberg and Tesfahun, 2015) on the Dirac Klein Gordon equations, we find the energy, show local well-possednes in the new space and finally state and prove an approximate conservation laws for the norm of solutions. The approximate conseravtion law involves a small parameter $\sigma > 0$ that reduces to the exact energy conservation law in the limit as $\sigma \to 0$. Applying the local well-posedness theorem and approximate conservation law repeatdly, and then by taking $\sigma > 0$ small enough we can recover any time interval $[0,T]$ for large T and obtain the lower bound $\sigma(t) \lesssim \frac{1}{\sqrt{T}}$. This improves the previous works on lower bounds of the beam and fifth order KdV-BBM equations.

6.2 Recommendation

In mathematics, there is no one fixed way to study about nature of problems, anlysis of solutions and so on. For example, when we come specifically to this study, in order to improve the radius of spatial analyticity for the solution of many dispersive PDEs one way can be introducing a new space and define appropreate norm. Thus, others may consider another spaces. The last, but not the least is to check equivalence of the new space and the existing Gervey space $G^{\sigma,s}$. For the fifth order KdV-BBM equation the higher dimension $n > 1$ is still an open problem. Since we consider only the beam and fifth order KdV-BBM equations, one can check if this newly introduced space is applicable in the analysis of lower bound on the radius of spatial analyticity for other non linear dispersive partial differential equations and extend the idea.

6.3 Future work

For $n \in \mathbb{N}$ we can extend our work to $n = 2, 3, \ldots$ dimensions for the fifth order KdV-BBM equation. Using this new space we can further apply it on different water wave models like Kawahara equation and Airy equation. More over we can also try to introduce another new space to improve the erlier results we achieved.

Bibliography

Ahn, J., Kim, J., and Seo, I. (2021). Lower bounds on the radius of spatial analyticity for the kawahara equation. *Analysis and Mathematical Physics*, 11(1):1–22.

Belayneh, B., Tegegn, E., and Tesfahun, A. (2022). Lower bound on the radius of analyticity of solution for fifth order kdv–bbm equation. *Nonlinear Differential Equations and Applications NoDEA*, 29(1):1–12.

Benjamin, T. B., Bona, J. L., and Mahony, J. J. (1972). Model equations for long waves in nonlinear dispersive systems. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 272(1220):47–78.

Bona, J. and Tzvetkov, N. (2009). Sharp well-posedness results for the bbm equation. *Discrete & Continuous Dynamical Systems*, 23(4):1241.

Bona, J. L., Carvajal, X., Panthee, M., and Scialom, M. (2018). Higher-order hamiltonian model for unidirectional water waves. *Journal of Nonlinear Science*, 28(2):543–577.

Bona, J. L., Chen, M., and Saut, J.-C. (2002). Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. i: Derivation and linear theory. *Journal of Nonlinear Science*, 12(4).

Bona, J. L., Chen, M., and Saut, J.-C. (2004). Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: Ii. the nonlinear theory. *Nonlinearity*, 17(3):925.

Bretherton, F. P. (1964). Resonant interactions between waves. the case of discrete oscillations. *Journal of Fluid Mechanics*, 20(3):457–479.

Carvajal, X. and Panthee, M. (2019). On sharp global well-posedness and ill-posedness for a fifth-order kdv-bbm type equation. *Journal of Mathematical Analysis and Applications*, 479(1):688–702.

Carvajal, X. and Panthee, M. (2020). On the radius of analyticity for the solution of the fifth order kdv-bbm model. *arXiv preprint arXiv:2009.09328*.

Craik, A. D. (2004). The origins of water wave theory. *Annual Reviiews*.

Dufera, T., Sileshi, T., and Achenef, T. (2021). On the prsistence of spatial analyticity for the beam equation. *Journal of Mathematical Applications and Analysis, 2022*.

Ferrari, A. B. and Titi, E. S. (1998). Gevrey regularity for nonlinear analytic parabolic equations. *Communications in Partial Differential Equations*, 23(1-2):424–448.

Foias, C. and Temam, R. (1989). Gevrey class regularity for the solutions of the navier-stokes equations. *Journal of Functional Analysis*, 87(2):359–369.

Folland, G. B. (1999). *Real analysis: modern techniques and their applications*, volume 40. John Wiley & Sons.

Hannah, H., Himonas, A. A., and Petronilho, G. (2011). Gevrey regularity of the periodic gkdv equation. *Journal of Differential Equations*, 250(5):2581–2600.

Himonas, A. A. and Petronilho, G. (2012). Analytic well-posedness of periodic gkdv. *Journal of Differential Equations*, 253(11):3101–3112.

Kammler, D. W. (2007). *A first course in Fourier analysis*. Cambridge University Press.

Kato, T. and Masuda, K. (1986). Nonlinear evolution equations and analyticity. i. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 3, pages 455–467. Elsevier.

Katznelson, Y. (2004). *An introduction to harmonic analysis*. Cambridge University Press.

Levandosky, S. (1998a). Stability and instability of fourth-order solitary waves. *Journal of Dynamics and Differential Equations*, 10(1):151–188.

Levandosky, S. P. (1998b). Decay estimates for fourth order wave equations. *journal of differential equations*, 143(2):360–413.

Levandosky, S. P. and Strauss, W. A. (2000). Time decay for the nonlinear beam equation. *Methods and Applications of Analysis*, 7(3):479–488.

Levermore, C. D. and Oliver, M. (1997). Analyticity of solutions for a generalized euler equation. *Journal of differential equations*, 133(2):321–339.

Love, A. E. H. (2013). *A treatise on the mathematical theory of elasticity*. Cambridge university press.

McLean, W. and McLean, W. C. H. (2000). *Strongly elliptic systems and boundary integral equations*. Cambridge university press.

Müller, P., Garrett, C., and Osborne, A. (2005). Rogue waves. *Oceanography*, 18(3):66.

Oliver, M. and Titi, E. S. (2001). On the domain of analyticity for solutions of second order analytic nonlinear differential equations. *Journal of Differential Equations*, 174(1):55–74.

Onorato, M., Osborne, A. R., Serio, M., and Cavaleri, L. (2005). Modulational instability and non-gaussian statistics in experimental random water-wave trains. *Physics of Fluids*, 17(7):078101.

Panizzi, S. (2012). On the domain of analyticity of solutions to semilinear klein–gordon equations. *Nonlinear Analysis: Theory, Methods & Applications*, 75(5):2841–2850.

Pausader, B. (2007). Scattering and the levandosky–strauss conjecture for fourth-order nonlinear wave equations. *Journal of Differential Equations*, 241(2):237–278.

Pausader, B. (2010). Scattering for the defocusing beam equation in low dimensions. *Indiana University mathematics journal*, pages 791–822.

Pausader, B. and Strauss, W. (2009). Analyticity of the scattering operator for the beam equation. *Discrete Contin. Dyn. Syst*, 25:617–626.

Royden, H. and Fitzpatrick (1988). Real analysis, volume 32. *Macmillan New York*.

Russell, J. S. (1885). *The wave of translation in the oceans of water, air, and ether*. Trübner & Company.

Selberg, S. and Da Silva, D. O. (2015). Lower bounds on the radius of spatial analyticity for the kdv equation. *arXiv preprint arXiv:1508.06116*.

Selberg, S. and Tesfahun, A. (2015). On the radius of spatial analyticity for the 1d dirac–klein–gordon equations. *Journal of Differential Equations*, 259(9):4732–4744.

Selberg, S. and Tesfahun, A. (2017). On the radius of spatial analyticity for the quartic generalized kdv equation. In *Annales Henri Poincaré*, volume 18, pages 3553–3564. Springer.

Stein, E. M. and Shakarchi, R. (2003). Fourier analysis. an introduction. princeton lectures in analysis, 1. *Princeton, NJ: Princeton Univ. Press. MR1970295*, 44:45.

Tao, T. (2006). *Nonlinear dispersive equations: local and global analysis*. Number 106. American Mathematical Soc.

Tesfahun, A. (2019a). Asymptotic lower bound for the radius of spatial analyticity to solutions of kdv equation. *Communications in Contemporary Mathematics*, 21(08):1850061.

Tesfahun, A. (2019b). Remark on the persistence of spatial analyticity for cubic nonlinear schrödinger equation on the circle. *NODEA-NONLINEAR DIFFERENTIAL EQUATIONS AND APPLICATIONS*, 26(2).

Yosida, K. (2012). *Functional analysis*, volume 123. Springer Science & Business Media.

Zabusky, N. J. and Kruskal, M. D. (1965). Interaction of" solitons" in a collisionless plasma and the recurrence of initial states. *Physical review letters*, 15(6):240.

Zakharov, V. E. (1968). Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194.

Zhang, J. (2010). Global existence for rough solutions of a fourth-order nonlinear wave equation. *Journal of mathematical analysis and applications*, 369(2):635–644.