HIGHER ORDER COMPACT FINITE DIFFERENCE METHOD FOR FOURTH-ORDER EULER-BERNOULLI BEAM PARTIAL DIFFERENTIAL EQUATION

Temesgen Alemayehu Kebede

A Thesis Submitted to the department of Applied Mathematics
School of Applied Natural Science

Presented in Partial Fulfillment of the Required for the Degree of Master's in Applied Mathematics (Numerical Analysis)

Office of Graduate Studies
Adama Science and Technology University

June, 2024
Adama, Ethiopia

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Temesgen Alemayehu Kebede

Major-Advisor: Dr. Tekle Gemechu Dinka
Co-Advisor: Dr. Mesfin Mekuria Woldaregay

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Declaration

I hereby declare that this Master Thesis entitled ***" Higher Order Compact Finite Difference Method for Fourth-Order Euler-Bernoulli Beam Partial Differential Equation"*** is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

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Recommendation of Advisors

We, the advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"Higher Order Compact Finite Difference Method for Fourth-Order Euler-Bernoulli Beam Partial Differential Equation"*** prepared under our guidance by **Temesgen Alemayehu Kebede** submitted in partial fulfillment of the requirements for the degree of Master's of science in **Applied Mathematics**. Therefore, we recommend the submission of revised version the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

we, the advisors of the thesis entitled ***"Higher Order Compact Finite Difference Method for Fourth-Order Euler-Bernoulli Beam Partial Differential Equation"*** and developed by **Temesgen Alemayehu Kebede**, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval of Board of Examiners

We, the undersigned, members of the Board of Examiners of the thesis by **Temesgen Alemayehu Kebede** have read and evaluated the thesis entitled ***"Higher Order Compact Finite Difference Method for Fourth-Order Euler-Bernoulli Beam Partial Differential Equation"*** and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of science in **Applied Mathematics**.

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Final approval and acceptance of the thesis is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate Council (DGC) and School Graduate Committee (SGC).

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School Dean	Signature	Date
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Table of contents

List of Tables

List of Figures

List of Acronyms and Abbreviations

ADI	Alternating Direction Implicit.
ADM	Adomian Decomposition Method.
AGE	Alternating Group Explicit.
CFD	Compact Finite Difference.
CFDM	Compact Finite Difference Method.
FEM	Finite Element Method.
MAE	Maximum Absolute Error.
1D	One Dimensional.
PDE	Partial Differential Equation.
PDEs	Partial Differential Equations.
3D	Three Dimensional.
2D	Two Dimensional.
VIM	Variational Iteration Method.

Abstract

In this thesis, we develop a higher order compact finite difference method to solve numerically a fourth-order partial differential equation that governs the transverse displacement of the Euler–Bernoulli beam equations. We develop an eight-order compact finite difference approximation for both the spatial derivative and the time derivative. An advantage of this method is that it completely satisfies the boundary conditions, eliminating the need for further approximations at the boundaries and it is easy to implement. Moreover, the proposed method is eight-order accurate and is based upon a single compact stencil. The discretizing procedure transforms the initial boundary value problem of the partial differential equation into a linear system of algebraic equations that can be solved by matrix inversion method. The stability, consistency and convergence were analyzed. Error analysis of compact finite difference method is based on Taylor's theorem. The test example confirms the eight-order compact finite difference method has small maximum absolute error when compared with those already available in the literature. The results indicate that the eight-order compact finite difference method is an applicable technique and approximates the exact solution very well.

Keywords: Euler-Bernoulli beam partial differential equation, Higher-order compact finite difference method, Stability, Consistency, Convergence analysis

CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

A partial differential equation is a differential equation that involves two or more independent variables, an unknown function and partial derivatives of the unknown function with respect to the independent variables. The order of a partial differential equation is the order of the highest derivative involved in the equation. A solution to a partial differential equation is a function that solves the equation or in other words turns it into an identity when substituted into the equation. A solution is called general if it contains all particular solutions of the equation concerned.

Structural mechanics is a long-standing discipline that was originally developed to address construction needs, primarily in the field of Civil Engineering. It relies on simplified models for swift structural analysis, particularly for slender structures. These structures also known as beams or arches depending on whether their mean line is straight or not, if their mean line is straight they are called beams, but if it is curve they are called arches. They are characterized by a length that significantly exceeds their other two dimensions. Today, these types of structures are prevalent in various industrial sectors including aeronautics, automotive, rail, shipbuilding and civil engineering (Aouragh, Khali, Khallouq, & Segaoui, 2024).

A beam is a structural constituent that has an ability to withstand load fundamentally by resisting against bending. Beams are characterised by their length, cross-section area and nature of material. Depending on length, they called Short beams when their length is small compared to their cross-sectional dimensions and long beams, when their length is significant compared to their cross-sectional dimensions. The length of a beam affects its deflection and the distribution of stresses along its length. When we consider their cross-sectional area, they have different cross-sectional shapes and sizes, such as rectangular, circular, I-shaped or T-shaped. Beams which have larger cross-sectional areas have higher load-carrying capacities and the shape of the cross-section also affects the distribution of stresses within the beam. Beams can be also depend on materials made, including steel, concrete, wood or composite materials. The choice of material depends on factors such as strength requirements, cost, availability, and environmental considerations.

Beams can be horizontal, vertical or inclined at an angle, depending on their orientation, horizontal beams are positioned parallel to the ground or any other reference plane, vertical beams are positioned perpendicular to the ground or any other reference plane and inclined beams are positioned at an angle between horizontal and vertical. Beams generally hold vertical gravitational forces but can also be made to hold horizontal loads. The loads carried by a beam are channelled to columns, walls or girders, which then channel the force to the adjacent structural compression members. Compression parts are structural elements subjected only to axial compressive forces. When both shear strains and effects of inertia are considered negligible, the dynamics of a long, thin and extensible beam can be described by the Euler–Bernoulli beam equation which is a fourth-order parabolic differential equation (Gorman, 1975).

The bending effect is the single most significant factor in a transversely vibrating beam (Mwabora, Sigey, Okelo, & Giterere, 2019). The Euler-Bernoulli beam model includes the bending behavior, transverse displacement and it assumes small deformations, linear elasticity, neglects shear deformation and rotary inertia effects. In real life applications such as rail tracks, bridges, pavements, underground pipelines, foundation beams and even animal vertebra columns have been modeled as beams resting on elastic foundations. To investigate the dynamics of such applications, the vibration behavior of these models need to be accurately determined. The finite element method, transfer matrix method, Rayleigh-Ritz method, differential quadrature element method, Galerkin procedure and perturbation techniques are some of the numerical methods used to investigate the vibration behavior of different types of linear or non-linear beams resting on linear or non-linear foundations.

Fourth-order partial differential equations (PDEs) are widely used to describe the mathematical models in a variety of physical phenomenon and dynamic processes in physics, engineering and geological science such as fluids in lungs, physical flows include ice formation, episodic vibration of a uniform elastic beams and plate deviation theory (Royster & Conte, 1956). Several methods have been used to find the analytic solution of linear and non-linear fourth-order PDEs, for instance variation iteration method, homotopy perturbation method, adomian decomposition method and homotopy analysis method. But for most fourth-order PDEs, the analytic solutions are not available. Hence, there is a need to develop an efficient numerical method to solve these PDEs, because the solutions of fourth-order PDEs are of great importance to scientists and engineers.

The compact finite difference method is one of a numerical method to compute finite difference approximations. Such approximations tend to be more accurate for their stencil sizes and for hyperbolic problems, have favorable dispersive error and dissipative error properties when compared to explicit methods. The compact methods are implicit and require to solve a diagonal matrix system for the evaluation of derivatives at all grid points (Lele, 1992). Due to their excellent stability properties, compact methods are a popular choice for use in higher-order numerical solvers for boundary value problems of partial differential equations.

The fourth order Euler-Bernoulli beam partial differential equation is a mathematical equation that describes the behavior of a beam under certain condition. It is a specific type of partial differential equation that takes into account the bending stiffness and inertia of the beam. This equation is commonly used in engineering and physics to analyze the deflection and vibration characteristics of beams subjected to lateral loads.

The Euler-Bernoulli beam theory is a mathematical model used to calculate the load-carrying and deflection characteristics of beams (Timoshenko, 1983). It is a simplification of the linear theory of elasticity and is widely used in structural and mechanical engineering ,which describes the relationship between the beam's deflection and the applied load. It's physically given by:

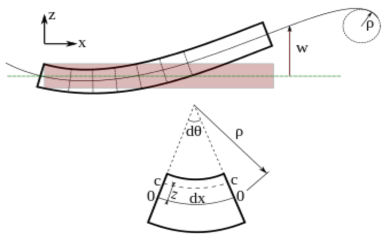

Figure 1.1: Shows the relationship between external load and deflection.

Assumption of the Euler-Bernoulli beam equation

The fundamental assumptions of the Euler-Bernoulli beam equations are:

1. The beam section is infinitely rigid in its own plane. According to this assumption, there is no deformation in the plane of the cross-section.
2. The cross-section of the beam remains plane to the deformed axis of the beam.
3. The cross-section of the beam remains normal to the deformed axis of the beam.

Derivation of Euler-Bernoulli beam equation

The Euler-Bernoulli beam equation is derived by combining four segments of beam theory (Bauchau & Craig, 2009). Those are kinematics, constitutive, force result and equilibrium equations. These four equations govern the behavior of beams and when they are combined they give the Euler-Bernoulli beam equation.
Kinematics $\rightarrow$ Constitutive $\rightarrow$ Force resultant $\rightarrow$ equilibrium = Beam equation

$$\text{Kinematics } \aleph = -\theta = -\frac{du}{dx} \tag{1.1}$$

$$\text{Constitutive } \quad \sigma(x,y) = E\varepsilon(x,y) \tag{1.2}$$

$$\text{Resultant } \quad M(x) = \iint y\sigma(x,y)dydz \tag{1.3}$$

$$\text{Equilibrium } \quad \frac{dM}{dx} = V \qquad \frac{dV}{dx} = -f \tag{1.4}$$

$$\text{where } \quad V(x) = \iint \sigma_{xy} dydz,$$

θ represents the neutral surface rotation

$\sigma(x,y)$ is stress in the beam at a given point (x,y)

E represents Young's modulus of the beam

$\varepsilon(x,y)$ represents the corresponding strain

V is the shear force and

M is the bending moment of the beam

To derive the formula for the resultant in the Euler-Bernoulli beam equation from the expression of the resultant moment, we can start with the following equation:

Resultant Moment $M(x) = \iint y\sigma(x,y)dydz$.

where $M(x)$ represents the moment at a given position x along the beam, y is the perpendicular distance from the neutral axis of the beam and $\sigma(x, y)$ is the stress distribution in the beam. To derive the formula for the resultant, we need to integrate the stress distribution over the cross-sectional area of the beam. This can be done by considering the stress as a function of the moment arm, which is the distance from the neutral axis to the point where the stress is acting.

Let's assume that the stress distribution can be represented by a function $\sigma(x, y) = f(y)g(x)$, where $f(y)$ represents the stress distribution along the y-axis and $g(x)$ represents the variation of stress along the x-axis.

Using this assumption, we can rewrite the equation for the resultant moment as:

$$M(x) = \iint yf(y)g(x)dydz.$$

To simplify the integration, we can separate the variables and integrate each term separately. Since $f(y)$ depends only on y and $g(x)$ depends only on x, we can rewrite the equation as:

$$M(x) = g(x) \int yf(y)dy \int dz.$$

The first integral represents the moment arm, which is the distance from the neutral axis to the point where the stress is acting. This can be denoted as $\hat{y}$, the centroid of the cross-sectional area. Therefore, the first integral simplifies to:

$$\int yf(y)dy = \hat{y} \int f(y)dy.$$

The second integral represents the cross-sectional area of the beam, denoted as A. Therefore, the second integral simplifies to: $\int dz = A$.

Substituting these simplifications back into the equation, we get:

$$M(x) = g(x)\hat{y} \int f(y)dyA.$$

Finally, we can define the resultant R as the product of the stress distribution function $f(y)$ and the cross-sectional area A. Therefore, we have:

Resultant $(R) = \int f(y)dyA$.

Substituting this into the previous equation, we obtain the formula for the resultant in the Euler-Bernoulli beam equation:

Resultant $(R) = \frac{M(x)}{(g(x)\hat{y})}$.

This formula relates the resultant to the moment at a given position x along the beam, the moment arm $\hat{y}$, and the variation of stress along the x-axis represented by $g(x)$.

By combining the two equilibrium equation (1.4), it gives

$$\frac{d^2M}{dx^2} = -f \tag{1.5}$$

Then, in equation (1.5), moment M is replaced by equation (1.3) given by

$$\frac{d^2}{dx^2}\left(\iint y\sigma dydz\right) = -f \tag{1.6}$$

To eliminate σ from equation (1.6), we use constitutive relation (1.2) and then use kinematics (1.1) to replace ε in the normal displacement (u), thus we obtained:

$$\frac{d^2}{dx^2}\left(E\left(\iint y\varepsilon dydz\right)\right) = -f \qquad \frac{d^2}{dx^2}\left(E\frac{d\aleph}{dx}\left(\iint y^2dydz\right)\right) = -f$$

$$\frac{d^2}{dx^2}\left(E\frac{d^2u}{dx^2}\iint y^2dydz\right) = -f \tag{1.7}$$

putting, moment of inertia $I = \iint y^2dydz$ in equation (1.7), obtain Euler-Bernoulli equation which describes the relationship between the beam's deflection and the applied load (Gere & Timoshenko, 1997)

$$\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right) = f \tag{1.8}$$

Here, EI represents the bending stiffness which is constant, then equation (1.8) becomes

$$EI\frac{d^4u}{dx^4} = f$$

Thus, the above equation indicates only the position part of Euler-Bernoulli beam equation, but our aim is arriving to the Euler-Bernoulli beam PDE which have position and time, therefore to get that PDE we follow the successive derivatives of the deflection u which have important physical meanings, because $\frac{du}{dx}$ is the slope of the beam, which is the anti-clockwise angle of rotation about the y-axis in the limit of small displacements. Then after simplifications we get:

$$M = -\, EI\frac{d^2u}{dx^2} \quad \text{is the bending moment in the beam and} \tag{1.9}$$

$$V = -\, \frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right) \quad \text{is the shear force in the beam} \tag{1.10}$$

The stresses in a beam can be calculated from the above expressions after the deflection due to a given load has been determined. Thus, the dynamic beam equation is the Euler-Lagrange equation for the following action.

$$S = \int_{t_1}^{t_2}\int_0^L \left[\frac{1}{2}\mu\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + f(x)u(x,t)\right] dxdt. \tag{1.11}$$

The first term represents the kinetic energy where μ is the mass per unit length, the second term represents the potential energy due to internal forces (when considered with a negative sign) and the third term represents the potential energy due to the external load $f(x)$. The Euler–Lagrange equation is used to determine the function that minimizes the functional S. For a dynamic Euler–Bernoulli beam equation, the Euler–Lagrange equation is written by:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) = -\mu\frac{\partial^2 u}{\partial t^2} + f(x). \tag{1.12}$$

But, the one dimensional non-homogeneous fourth-order Euler-Bernoulli beam partial differential equation that we considered is of the form:

$$EI\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = f(x,t) \tag{1.13}$$

With the following initial and boundary conditions.

$$u(0,t) = u(l,t) = 0 \qquad\qquad 0 < t \le T \tag{1.14}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0 \tag{1.15}$$

$$u(x,0) = f(x),\ \ \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 \le x \le l \tag{1.16}$$

where $u(x,t)$ represents the transverse displacement of the beam at position x and time t. E is the elastic modulus. I is the area moment of inertia, since both E and I are constants. $f(x,t)$ represents external force or load applied to the given Euler-Bernoulli beam at a given position x and time t.

This equation relates the fourth derivative of the transverse displacement $u(x,t)$ with respect to the position x and the second derivative with respect to time t. The left-hand side of the equation represents the bending stiffness of the beam, while the right-hand side is $f(x,t)$, indicating that there is external forces acting on the beam.

Solving the fourth-order Euler-Bernoulli beam partial differential equation involves finding the function $u(x,t)$ that satisfies the equation and the initial boundary conditions of the specific problem. The boundary conditions typically include constraints on the displacement and its derivatives at the ends of the beam.

The outline of this thesis is as follows. In "High-Order Compact Finite Difference Method" section, we present the process of developing compact finite difference method for (1.13). Furthermore, we approximate the fourth-order spatial derivative and second-order time derivative using eight-order compact finite difference method, it gives a 3-point compact stencil approximation. In " Stability, Consistency and Convergence Analysis" section, we discuss the stability of the proposed method by using the Von Neumann stability analysis method with Routh-Hurwitz criteria for stability, which is shown to be unconditionally stable. In "Result and Discussion" section, we present numerical results for the example and compare errors with those from existing works in the literature. Finally, in "Conclusion" section, we mention the conclusion of the implementation and the superiority of the proposed method based on numerical results from the problems.

1.2 Statement of the problem

In recent years, there has been considerable interest in the exploration of higher-order compact finite difference methods for the resolution of partial differential equations (PDEs). Specifically, for fourth-order Euler-Bernoulli beam PDEs, researchers have conducted several investigations utilizing fourth and sixth-order compact finite difference methods. However, a gap exists in the current literature pertaining to the utilization of an eight-order compact finite difference method for solving these PDEs.

The objective of this thesis is to fill this gap by examining and developing a higher-order compact finite difference method, specifically an eight-order compact finite difference method for effectively solving fourth-order Euler-Bernoulli beam PDEs. The main aim is to investigate the stability, consistency and convergence analysis of the proposed method and comparison with the existing compact finite difference approaches in the literature. This study addresses to the following basic questions:

1. How can we construct compact finite difference method to approximate the Euler-Bernoulli beam PDEs?

2. How can we find the convergency of compact finite difference method?

3. How the order of compact finite difference method affects the accuracy of the solution?

1.3 Objectives

1.3.1 General Objective

The general objective of this thesis is constructing eight-order compact finite difference method for solving fourth-order Euler-Bernoulli beam partial differential equations.

1.3.2 Specific Objectives

The specific objectives of this study are to:

- apply compact finite difference methods for approximate solution of fourth-order Euler-Bernoulli beam equations.

- investigate the stability, consistency and convergence analysis of the compact finite difference methods.

- test the validation of the method by taking an example and write Matlab code for the problem.

1.4 Significance of the study

This study enhances numerical accuracy using an eight-order compact finite difference method by benefiting theory and practice. It captures complex details in fourth-order Euler-Bernoulli beam PDEs while maintaining a comparison between exact and numerical efficiency. It addresses numerical stability and reduces computational costs for complex systems. By using its concept, we can apply it to other fourth-order partial differential equations (PDEs) in various fields. Its practical applications involve improved understanding of beam behavior, response prediction and optimized designs for enhanced structural integrity and safety. In addition to that, the result obtained from this study was:

- used as reference material for scholars who works on this research area.
- offer a numerical method for solving fourth-order Euler-Bernoulli beam equations.

1.5 Limitation of the study

When using the eight-order compact finite difference method to solve fourth-order Euler-Bernoulli beam PDEs include domain restrictions for irregular domains, difficulties with singularities and discontinuities, high computational demands for large-scale problems, sensitivity to parameter choices, limited validation and comparison with other methods and potential limitations in handling nonlinear problems or material properties and also the study only concerning the one dimension beam equation, anybody will study two or more dimension beam equation.

CHAPTER TWO
LITERATURE REVIEW

Partial differential equations (PDEs) are vital in science and engineering, with the fourth-order Euler-Bernoulli beam equation being a fundamental model for slender beams under bending forces. Non-homogeneous versions of this equation pose challenges in solving and requiring advanced mathematical techniques. Accurate determination of deflection and bending moment distribution is crucial for optimizing design. This literature review aims to provide an overview of state of the art approaches for solving non-homogeneous fourth-order Euler-Bernoulli beam PDEs, exploring mathematical techniques, computational algorithms and validation methodologies. Identifying research gaps and suggesting future directions will enhance our understanding and improve methods for solving these complex equations.

Several methods have been developed to derive analytical solutions of fourth-order PDEs, including the Adomian Decomposition Method (ADM) by (Wazwaz, 1995, 2001) and the Variational Iteration Method (VIM) (Biazar & Ghazvini, 2007). (Dehghan & Manafian, 2009) employed the Homotopy Perturbation Method to address boundary value problems related to fourth-order parabolic PDEs with variable coefficients. Recently, (Khan, Ara, Afzal, & Khan, 2010) used the Homotopy Analysis Method for the same problem.

Over the years, many researchers have numerically analysed the beam vibration on excited structures such as bridges, buildings, trains which results from natural frequencies and high speed movements on those structures. The vibration analysis of exponential cross-section beam using Galerkin's method and FEM utilised to compare the results (Achawakorn & Jearsiripongkul, 2012). The analytical method and the FEM give the conformable results, with slightly different value of the natural frequencies of the beam. The dynamic analysis of non-uniform beams on elastic foundations by transforming the fourth-order differential equation of beam vibration under appropriate boundary conditions to the Bessel equation by factorization (Abohadima & Taha, 2009). Modes of shapes and damped natural frequencies of the beam are obtained for broad range of beam-foundation system characteristics. The effect of foundation stiffness is more pronounced on frequencies of lower modes. Analyzing non-uniform beams on elastic foundations making use of the recursive differentiation method (Abohadima, Taha, Abdeen, et al., 2015). Foundation stiffness influence was detectable on both the critical load and natural frequency in the case of thinner beams. Effect of the end conditions reduces as the thinness parameter of the beam increases. Neither the critical load nor the natural frequency corresponding to the first mode is always the smallest one in the case of beams on the elastic foundations

The study by (Twizell & Khaliq, 1983) developed a finite difference method to solve numerically a fourth-order parabolic partial differential equation in one and two space variables. This method is derived from a multi-derivative method for second-order ordinary differential equation. (Andrade & McKee, 1977) proposed a high accuracy alternating direction implicit (ADI) methods are derived for solving fourth-order parabolic equations. The approach by (Evans & Yousif, 1991) that utilized the Alternating Group Explicit (AGE) method was achieved a higher accuracy level. A novel space semi discretized numerical scheme based on the finite volume method was proposed in (Liu & Guo, 2019). Discrete Galerkin methods were formulated for approximating a two-dimensional linear fourth-order parabolic partial differential equation in (Fairweather, 1972).

The method of lines is generalized for the numerical solution of the one-dimensional forced beam vibration problems having more than one dependent variable or degree of freedom (Sarker & Chakravarty, 2020). Its objective is to demonstrate the implementation of the generalized method of lines and its effectiveness by several realistic case studies with limited/no availability of the closed-form solutions.

Higher-order compact finite difference schemes have been used to solve unsteady convection diffusion equations. (Noye & Tan, 1988) established a five-point higher-order compact finite difference scheme with a large stability region. The truncation error of their scheme was $O(k^2 + h^3)$. A higher-order compact finite difference scheme by (Kalita, Dalal, & Dass, 2002) applied for the two variables convection equation, which is fourth-order in space and not more than second-order in time according to weighted discretization. In addition, for the 2D problem, (Karaa & Zhang, 2004) proposed a fourth-order alternating direction implicit (ADI) scheme, which produces an efficient solver by using 1D tridiagonal algorithm and the unconditional stability was proved by discrete Fourier analysis. (Z. Tian & Ge, 2007) derived a compact ADI scheme by using a spatial discrete exponential fourth-order compact finite difference method and the Crank–Nicolson method for the time discretization. (Z. F. Tian, 2011) also proposed another unconditionally stable rational compact ADI difference method. Their method was unconditionally stable too and compared with (Karaa & Zhang, 2004), it has a smaller dissipation error and better resolution properties, while both schemes have the same order. (Li, Jiang, & Yin, 2018) formulated a fourth-order compact scheme of the 2D equation to solve groundwater pollution problems, which is also unconditionally stable. (Sun & Li, 2014) proposed a six-order scheme by using a combined compact difference scheme for the spatial discretization and the Crank–Nicolson scheme for the temporal discretization. Although the scheme is sixth-order in space and it is only second-order in time, to match the sixth-order accuracy in space, a very small time step-length must be adopted in the calculation.

Among the above mentioned numerical methods to solve a fourth-order parabolic PDE, the finite difference method is attractive because of its relative ease of implementation, flexibility and accuracy in the solution values (Aouragh et al., 2024). The finite difference method splits the area of interest into a grid system over which the partial derivatives can be expressed as finite differences. The problem is reduced to finding the solution at the grid points by solving a system of linear algebraic equations. High-order compact finite difference methods have been developed to solve several partial differential equations.

The aim of our thesis is to develop an eight-order compact finite difference method to solve the non-homogeneous fourth-order parabolic problem (1.13 - 1.16). This method employs a compact finite difference (CFD) for both the spatial derivative and the temporal derivative using a 3-point compact stencil approximation. We also discuss the stability, consistency and convergence analysis of this method, which is unconditionally stable, with eight-order accuracy. We consider the non-homogeneous problem example studied by (Ghafoor, Haq, Hussain, Abdeljawad, & Alqudah, 2022) and present its numerical results to illustrate the accuracy of the proposed method. The test problems confirm that the computed solution not only align well with the exact solutions but are also competitive with the solutions derived in earlier research studies.

CHAPTER THREE
RESEARCH METHODOLOGY

This section consists of methods and materials used to undertake the study. Those are study design and Procedures of the study.

3.1 Study Design

This study applied both the documentation review and numerical experimentation.

3.2 Procedures of the study

To attain the objective of the study, the following procedures were undertaken:

- Domain discretiziation of the given partial differential equation.
- Construction of compact finite difference methods to the problem.
- Obtaining system of equations.
- Assembling the obtained system of equations in a penta-diagonal form.
- Writing a code for the problem solving by using Matlab software.
- Validating the methods using numerical examples.
- Discussing about the methods from the graphs.

CHAPTER FOUR

COMPACT FINITE DIFFERENCE METHOD FOR EULER-BERNOULLI BEAM PARTIAL DIFFERENTIAL EQUATIONS

The one-dimensional non-homogeneous fourth-order Euler-Bernoulli beam PDE is given by:

$$EI\frac{\partial^4 u}{\partial x^4}(x,t) + \frac{\partial^2 u}{\partial t^2}(x,t) = f(x,t), \qquad 0 < x < l, \tag{4.1}$$

with boundary conditions

$$u(0,t) = u(l,t) = 0, \qquad 0 < t \leq T,$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0, \tag{4.2}$$

and initial conditions

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 \leq x \leq l, \tag{4.3}$$

where $u(x,t)$ represents the displacement with position x and time t. E and I are constants that represents the elastic modulus and the area moment of inertia respectively. In order for this equation to be solvable the initial conditions equation (4.3) as well as the boundary conditions equation (4.2) should be provided (Duressa, Bullo, & Kiltu, 2016). To describe the scheme, we divide interval $[0,l]$ and $[0,T]$ in m and n equal sub intervals of mesh length h and k respectively.

Let $0 = x_0 < x_1 < ... < x_{m-1} < x_m = l$ and $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ be the mesh points with $x_i = x_0 + ih$ and $t_j = t_0 + jk$ for $i = 1,2,...,m$ and $j = 1,2,...,n$. We take the assumptions that $u(x,t)$ has continuous higher order partial derivatives on the region $[0,l]\times$ $[0,T]$. We use u_i^j in place of $u(x_i,t_j)$ and also we use $(\frac{\partial u}{\partial x})_i^j$ in place of $\frac{\partial u}{\partial x}(i,j)$ or $(\frac{\partial u}{\partial t})_i^j$ in place of $\frac{\partial u}{\partial t}(i,j)$ for derivatives. Thus by using Taylor series expansion, we have:

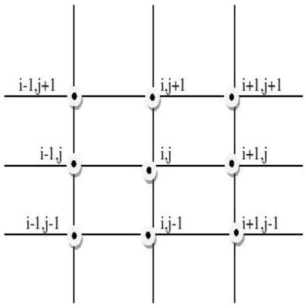

Figure 4.1: Discretization of the finite difference method.

$$u_{i+1}^{j} = u_i^j + h(\frac{\partial u}{\partial x})_i^j + \frac{h^2}{2!}(\frac{\partial^2 u}{\partial x^2})_i^j + \frac{h^3}{3!}(\frac{\partial^3 u}{\partial x^3})_i^j + \frac{h^4}{4!}(\frac{\partial^4 u}{\partial x^4})_i^j + \frac{h^5}{5!}(\frac{\partial^5 u}{\partial x^5})_i^j + \frac{h^6}{6!}(\frac{\partial^6 u}{\partial x^6})_i^j$$

$$+\frac{h^7}{7!}(\frac{\partial^7 u}{\partial x^7})_i^j + \frac{h^8}{8!}(\frac{\partial^8 u}{\partial x^8})_i^j + \frac{h^9}{9!}(\frac{\partial^9 u}{\partial x^9})_i^j + \frac{h^{10}}{10!}(\frac{\partial^{10} u}{\partial x^{10}})_i^j + ... \tag{4.4}$$

$$u_{i-1}^{j} = u_i^j - h(\frac{\partial u}{\partial x})_i^j + \frac{h^2}{2!}(\frac{\partial^2 u}{\partial x^2})_i^j - \frac{h^3}{3!}(\frac{\partial^3 u}{\partial x^3})_i^j + \frac{h^4}{4!}(\frac{\partial^4 u}{\partial x^4})_i^j - \frac{h^5}{5!}(\frac{\partial^5 u}{\partial x^5})_i^j + \frac{h^6}{6!}(\frac{\partial^6 u}{\partial x^6})_i^j$$

$$-\frac{h^7}{7!}(\frac{\partial^7 u}{\partial x^7})_i^j + \frac{h^8}{8!}(\frac{\partial^8 u}{\partial x^8})_i^j - \frac{h^9}{9!}(\frac{\partial^9 u}{\partial x^9})_i^j + \frac{h^{10}}{10!}(\frac{\partial^{10} u}{\partial x^{10}})_i^j + ... \tag{4.5}$$

$$u_{i}^{j+1} = u_i^j + k(\frac{\partial u}{\partial t})_i^j + \frac{k^2}{2!}(\frac{\partial^2 u}{\partial t^2})_i^j + \frac{k^3}{3!}(\frac{\partial^3 u}{\partial t^3})_i^j + \frac{k^4}{4!}(\frac{\partial^4 u}{\partial t^4})_i^j + \frac{k^5}{5!}(\frac{\partial^5 u}{\partial t^5})_i^j + \frac{k^6}{6!}(\frac{\partial^6 u}{\partial t^6})_i^j$$

$$+\frac{k^7}{7!}(\frac{\partial^7 u}{\partial t^7})_i^j + \frac{k^8}{8!}(\frac{\partial^8 u}{\partial t^8})_i^j + \frac{k^9}{9!}(\frac{\partial^9 u}{\partial t^9})_i^j + \frac{k^{10}}{10!}(\frac{\partial^{10} u}{\partial t^{10}})_i^j + ... \tag{4.6}$$

$$u_{i}^{j-1} = u_i^j - k(\frac{\partial u}{\partial t})_i^j + \frac{k^2}{2!}(\frac{\partial^2 u}{\partial t^2})_i^j - \frac{k^3}{3!}(\frac{\partial^3 u}{\partial t^3})_i^j + \frac{k^4}{4!}(\frac{\partial^4 u}{\partial t^4})_i^j - \frac{k^5}{5!}(\frac{\partial^5 u}{\partial t^5})_i^j + \frac{k^6}{6!}(\frac{\partial^6 u}{\partial t^6})_i^j$$

$$-\frac{k^7}{7!}(\frac{\partial^7 u}{\partial t^7})_i^j + \frac{k^8}{8!}(\frac{\partial^8 u}{\partial t^8})_i^j - \frac{k^9}{9!}(\frac{\partial^9 u}{\partial t^9})_i^j + \frac{k^{10}}{10!}(\frac{\partial^{10} u}{\partial t^{10}})_i^j + ... \tag{4.7}$$

$$u_{i+2}^{j} = u_i^j + 2h(\frac{\partial u}{\partial x})_i^j + \frac{(2h)^2}{2!}(\frac{\partial^2 u}{\partial x^2})_i^j + \frac{(2h)^3}{3!}(\frac{\partial^3 u}{\partial x^3})_i^j + \frac{(2h)^4}{4!}(\frac{\partial^4 u}{\partial x^4})_i^j + \frac{(2h)^5}{5!}(\frac{\partial^5 u}{\partial x^5})_i^j$$

$$+\frac{(2h)^6}{6!}(\frac{\partial^6 u}{\partial x^6})_i^j + \frac{(2h)^7}{7!}(\frac{\partial^7 u}{\partial x^7})_i^j + \frac{(2h)^8}{8!}(\frac{\partial^8 u}{\partial x^8})_i^j + \frac{(2h)^9}{9!}(\frac{\partial^9 u}{\partial x^9})_i^j + \frac{(2h)^{10}}{10!}(\frac{\partial^{10} u}{\partial x^{10}})_i^j + ... \tag{4.8}$$

$$u_{i-2}^{j} = u_i^j - 2h(\frac{\partial u}{\partial x})_i^j + \frac{(2h)^2}{2!}(\frac{\partial^2 u}{\partial x^2})_i^j - \frac{(2h)^3}{3!}(\frac{\partial^3 u}{\partial x^3})_i^j + \frac{(2h)^4}{4!}(\frac{\partial^4 u}{\partial x^4})_i^j - \frac{(2h)^5}{5!}(\frac{\partial^5 u}{\partial x^5})_i^j$$

$$+\frac{(2h)^6}{6!}(\frac{\partial^6 u}{\partial x^6})_i^j - \frac{(2h)^7}{7!}(\frac{\partial^7 u}{\partial x^7})_i^j + \frac{(2h)^8}{8!}(\frac{\partial^8 u}{\partial x^8})_i^j - \frac{(2h)^9}{9!}(\frac{\partial^9 u}{\partial x^9})_i^j + \frac{(2h)^{10}}{10!}(\frac{\partial^{10} u}{\partial x^{10}})_i^j + ... \quad (4.9)$$

Subtracting equation (4.5) from equation (4.4) and equation (4.7) from equation (4.6), we obtain the second-order central $\delta_c^1 u_i^j$ difference for the first derivative of u_i^j:

$$\delta_{cx}^1 u_i^j = \frac{u_{i+1}^j - u_{i-1}^j}{2h} + \tau_1 \text{ and } \delta_{ct}^1 u_i^j = \frac{u_i^{j+1} - u_i^{j-1}}{2k} + \tau_2, \quad (4.10)$$

where $\quad \tau_1 = -\frac{h^2}{6}(\frac{\partial^3 u}{\partial x^3})_i^j$ and $\tau_2 = -\frac{k^2}{6}(\frac{\partial^3 u}{\partial t^3})_i^j$.

Similarly, adding equation (4.6) with equation (4.7), we obtain the second-order central difference $\delta_c^2 u_i^j$ for the second derivative of u_i^j:

$$\delta_{ct}^2 u_i^j = \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \tau_3, \quad (4.11)$$

where $\quad \tau_3 = -\frac{k^2}{12}(\frac{\partial^4 u}{\partial t^4})_i^j$.

And also adding equation (4.8) with equation (4.9), we obtain the fourth-order central difference $\delta_{cx}^4 u_i^j$ for fourth derivatives of u_i^j.

$$\delta_{cx}^4 u_i^j = \frac{u_{i+2}^j - 4u_{i+1}^j + 6u_i^j - 4u_{i-1}^j + u_{i-2}^j}{h^4} + \tau_4, \quad (4.12)$$

where $\quad \tau_4 =-\frac{h^2}{6}(\frac{\partial^6 u}{\partial x^6})_i^j$.

Substituting equation (4.4 - 4.9) into equation (4.11) and (4.12) gives:

$$\delta_{ct}^2 u_i^j = (\frac{\partial^2 u}{\partial t^2})_i^j + \frac{k^2}{12}(\frac{\partial^4 u}{\partial t^4})_i^j + \frac{k^4}{360}(\frac{\partial^6 u}{\partial t^6})_i^j + \frac{k^6}{20160}(\frac{\partial^8 u}{\partial t^8})_i^j + \tau_5, \quad (4.13)$$

where $\quad \tau_5=\frac{k^8}{1814400}(\frac{\partial^{10} u}{\partial t^{10}})_i^j$.

$$\delta_{cx}^4 u_i^j = (\frac{\partial^4 u}{\partial x^4})_i^j + \frac{h^2}{6}(\frac{\partial^6 u}{\partial x^6})_i^j + \frac{h^4}{80}(\frac{\partial^8 u}{\partial x^8})_i^j + \tau_6, \quad (4.14)$$

where $\quad \tau_6=\frac{h^6}{1779}(\frac{\partial^{10} u}{\partial x^{10}})_i^j$.

Using from equation (4.1) and successive differentiation and also using a^2 in place of EI gives:

$$(\frac{\partial^6 u}{\partial x^6})_i^j = \frac{1}{a^2}(\frac{\partial^2 f}{\partial x^2})_i^j - \frac{1}{a^2}\frac{\partial^2}{\partial x^2}(\frac{\partial^2 u}{\partial t^2})_i^j = \frac{1}{a^2}\delta_{cx}^2 f_i^j - \frac{1}{a^2}\delta_{cx}^2(\delta_{ct}^2 u_i^j). \tag{4.15}$$

$$(\frac{\partial^8 u}{\partial x^8})_i^j = \frac{1}{a^2}\frac{\partial^2}{\partial x^2}(\frac{\partial^2 f}{\partial x^2})_i^j - \frac{1}{a^2}\frac{\partial^2}{\partial x^2}(\frac{\partial^2}{\partial x^2}(\frac{\partial^2 u}{\partial t^2})_i^j) = \frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2 f_i^j)$$

$$-\frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u_i^j)). \tag{4.16}$$

$$(\frac{\partial^4 u}{\partial t^4})_i^j = (\frac{\partial^2 f}{\partial t^2})_i^j - a^2\frac{\partial^2}{\partial t^2}(\frac{\partial^4 u}{\partial x^4})_i^j = \delta_{ct}^2 f_i^j - a^2\delta_{ct}^2(\delta_{cx}^4 u_i^j). \tag{4.17}$$

$$(\frac{\partial^6 u}{\partial t^6})_i^j = \frac{\partial^2}{\partial t^2}(\frac{\partial^2 f}{\partial t^2})_i^j - a^2\frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial t^2}(\frac{\partial^4 u}{\partial x^4}))_i^j = \delta_{ct}^2(\delta_{ct}^2 f_i^j) - a^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j)). \tag{4.18}$$

$$(\frac{\partial^8 u}{\partial t^8})_i^j = \frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial t^2}(\frac{\partial^2 f}{\partial t^2}))_i^j - a^2\frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial t^2}(\frac{\partial^4 u}{\partial x^4})))_i^j = \delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2 f_i^j)) - a^2\delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j)))$$

But,

$$\delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j))) = \delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 f_i^j)) - \delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j))$$

Thus,

$$(\frac{\partial^8 u}{\partial t^8})_i^j = \delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2 f_i^j)) - a^2\left(\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 f_i^j)) - \delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j))\right). \tag{4.19}$$

Substituting equation (4.15 - 4.19) into equation (4.13) and (4.14) respectively gives:

$$\delta_{ct}^2 u_i^j = (\frac{\partial^2 u}{\partial t^2})_i^j + \frac{k^2}{12}(\delta_{ct}^2 f_i^j - a^2\delta_{ct}^2(\delta_{cx}^4 u_i^j)) + \frac{k^4}{360}(\delta_{ct}^2(\delta_{ct}^2 f_i^j) - a^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j)))$$

$$+\frac{k^6}{20160}\left(\delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2 f_i^j)) - a^2\left(\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 f_i^j)) - \delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j))\right)\right) + \tau_5$$

$$\delta_{cx}^4 u_i^j = (\frac{\partial^4 u}{\partial x^4})_i^j + \frac{h^2}{6}(\frac{1}{a^2}\delta_{cx}^2 f_i^j - \frac{1}{a^2}\delta_{cx}^2(\delta_{ct}^2 u_i^j)) + \frac{h^4}{80}(\frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2 f_i^j) - \frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u_i^j))) + \tau_6$$

$$(\frac{\partial^2 u}{\partial t^2})_i^j = \delta_{ct}^2 u_i^j - \frac{k^2}{12}(\delta_{ct}^2 f_i^j - a^2\delta_{ct}^2(\delta_{cx}^4 u_i^j)) - \frac{k^4}{360}(\delta_{ct}^2(\delta_{ct}^2 f_i^j) - a^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j)))$$

$$-\frac{k^6}{20160}\left(\delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2 f_i^j)) - a^2\left(\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 f_i^j)) - \delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j))\right)\right) - \tau_5. \tag{4.20}$$

$$(\frac{\partial^4 u}{\partial x^4})_i^j = \delta_{cx}^4 u_i^j - \frac{h^2}{6}(\frac{1}{a^2}\delta_{cx}^2 f_i^j - \frac{1}{a^2}\delta_{cx}^2(\delta_{ct}^2 u_i^j)) - \frac{h^4}{80}(\frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2 f_i^j)$$

$$-\frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u_i^j))) - \tau_6. \tag{4.21}$$

Again, substituting equation (4.20) and (4.21) into equation (4.1), we obtain:

$$\delta_{ct}^2 u_i^j - \frac{k^2}{12}(\delta_{ct}^2 f_i^j - a^2\delta_{ct}^2(\delta_{cx}^4 u_i^j)) - \frac{k^4}{360}(\delta_{ct}^2(\delta_{ct}^2 f_i^j) - a^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j))) - \frac{k^6}{20160}(\delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2 f_i^j))$$

$$-a^2(\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 f_i^j)) - \delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j)))) - \tau_5 = f_i^j - a^2[\delta_{cx}^4 u_i^j - \frac{h^2}{6}(\frac{1}{a^2}\delta_{cx}^2 f_i^j - \frac{1}{a^2}\delta_{cx}^2(\delta_{ct}^2 u_i^j))$$

$$-\frac{h^4}{80}(\frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2 f_i^j) - \frac{1}{a^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u_i^j))) - \tau_6]$$

$$\delta_{ct}^2 u_i^j - \frac{k^2}{12}(\delta_{ct}^2 f_i^j - a^2\delta_{ct}^2(\delta_{cx}^4 u_i^j)) - \frac{k^4}{360}(\delta_{ct}^2(\delta_{ct}^2 f_i^j) - a^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j))) - \frac{k^6}{20160}(\delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2 f_i^j))$$

$$-a^2(\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 f_i^j)) - \delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j)))) + a^2\delta_{cx}^4 u_i^j - \frac{h^2}{6}(\delta_{cx}^2 f_i^j - \delta_{cx}^2(\delta_{ct}^2 u_i^j)) - \frac{h^4}{80}(\delta_{cx}^2(\delta_{cx}^2 f_i^j)$$

$$-\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u_i^j))) - f_i^j = \tau_5 + a^2\tau_6. \tag{4.22}$$

And also,

$$\delta_{cx}^2(\delta_{cx}^2 f_i^j) = \frac{1}{h^4}[f_{i+2}^j - 4f_{i+1}^j + 6f_i^j - 4f_{i-1}^j + f_{i-2}^j]. \tag{4.23}$$

$$\delta_{ct}^2(\delta_{ct}^2 f_i^j) = \frac{1}{k^4}[f_i^{j+2} - 4f_i^{j+1} + 6f_i^j - 4f_i^{j-1} + f_i^{j-2}]. \tag{4.24}$$

$$\delta_{cx}^2(\delta_{ct}^2 u_i^j) = \frac{1}{h^2k^2}[u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} - 2u_{i+1}^j + 4u_i^j - 2u_{i-1}^j + u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}]. \tag{4.25}$$

$$\delta_{ct}^2(\delta_{cx}^4 u_i^j) = \frac{1}{k^2h^4}[u_{i+2}^{j+1} - 2u_{i+2}^j + u_{i+2}^{j-1} - 4u_{i+1}^{j+1} + 8u_{i+1}^j - 4u_{i+1}^{j-1} + 6u_i^{j+1} - 12u_i^j$$

$$+6u_i^{j-1} - 4u_{i-1}^{j+1} + 8u_{i-1}^j - 4u_{i-1}^{j-1} + u_{i-2}^{j+1} - 2u_{i-2}^j + u_{i-2}^{j-1}]. \tag{4.26}$$

$$\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u_i^j)) = \frac{1}{h^4k^2}[u_{i+2}^{j+1} - 4u_{i+1}^{j+1} + 6u_i^{j+1} - 4u_{i-1}^{j+1} + u_{i-2}^{j+1} - 2u_{i+2}^j + 8u_{i+1}^j - 12u_i^j$$

$$+8u_{i-1}^j - 2u_{i-2}^j + u_{i+2}^{j-1} - 4u_{i+1}^{j-1} + 6u_i^{j-1} - 4u_{i-1}^{j-1} + u_{i-2}^{j-1}]. \tag{4.27}$$

$$\delta_{ct}^2(\delta_{ct}^2(\delta_{ct}^2 f_i^j)) = \frac{1}{k^6}[f_i^{j+3} - 6f_i^{j+2} + 15f_i^{j+1} - 20f_i^j + 15f_i^{j-1} - 6f_i^{j-2} + f_i^{j-3}]. \tag{4.28}$$

$$\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 u_i^j)) = \frac{1}{k^4h^4}[u_{i+2}^{j+2} - 4u_{i+2}^{j+1} + 6u_{i+2}^j - 4u_{i+2}^{j-1} + u_{i+2}^{j-2} - 4u_{i+1}^{j+2} + 16u_{i+1}^{j+1} - 24u_{i+1}^j$$

$$+16u_{i+1}^{j-1} - 4u_{i+1}^{j-2} + 6u_i^{j+2} - 24u_i^{j+1} + 36u_i^j - 24u_i^{j-1} + 6u_i^{j-2} - 4u_{i-1}^{j+2} + 16u_{i-1}^{j+1} - 24u_{i-1}^j$$

$$+16u_{i-1}^{j-1} - 4u_{i-1}^{j-2} + u_{i-2}^{j+2} - 4u_{i-2}^{j+1} + 6u_{i-2}^j - 4u_{i-2}^{j-1} + u_{i-2}^{j-2}]. \tag{4.29}$$

$$\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^4 f_i^j)) = \frac{1}{k^4h^4}[f_{i+2}^{j+2} - 4f_{i+2}^{j+1} + 6f_{i+2}^j - 4f_{i+2}^{j-1} + f_{i+2}^{j-2} - 4f_{i+1}^{j+2} + 16f_{i+1}^{j+1} - 24f_{i+1}^j$$

$$+16f_{i+1}^{j-1} - 4f_{i+1}^{j-2} + 6f_i^{j+2} - 24f_i^{j+1} + 36f_i^j - 24f_i^{j-1} + 6f_i^{j-2} - 4f_{i-1}^{j+2} + 16f_{i-1}^{j+1} - 24f_{i-1}^j$$

$$+16f_{i-1}^{j-1} - 4f_{i-1}^{j-2} + f_{i-2}^{j+2} - 4f_{i-2}^{j+1} + 6f_{i-2}^j - 4f_{i-2}^{j-1} + f_{i-2}^{j-2}]. \tag{4.30}$$

Substituting equation (4.11), (4.12) and equation (4.23 - 4.30) into equation (4.22) gives:

$$\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{k^2}+\tau_3-\frac{1}{12}([f_i^{j+1}-2f_i^j+f_i^{j-1}]-a^2(\frac{1}{h^4}[u_{i+2}^{j+1}-2u_{i+2}^j+u_{i+2}^{j-1}-4u_{i+1}^{j+1}$$

$$+8u_{i+1}^j-4u_{i+1}^{j-1}+6u_i^{j+1}-12u_i^j+6u_i^{j-1}-4u_{i-1}^{j+1}+8u_{i-1}^j-4u_{i-1}^{j-1}+u_{i-2}^{j+1}-2u_{i-2}^j+u_{i-2}^{j-1}]))$$

$$-\frac{1}{360}([f_i^{j+2}-4f_i^{j+1}+6f_i^j-4f_i^{j-1}+f_i^{j-2}]-a^2(\frac{1}{h^4}([u_{i+2}^{j+2}-4u_{i+2}^{j+1}+6u_{i+2}^j-4u_{i+2}^{j-1}$$

$$+u_{i+2}^{j-2}-4u_{i+1}^{j+2}+16u_{i+1}^{j+1}-24u_{i+1}^j+16u_{i+1}^{j-1}-4u_{i+1}^{j-2}+6u_i^{j+2}-24u_i^{j+1}+36u_i^j-24u_i^{j-1}$$

$$+6u_i^{j-2}-4u_{i-1}^{j+2}+16u_{i-1}^{j+1}-24u_{i-1}^j+16u_{i-1}^{j-1}-4u_{i-1}^{j-2}+u_{i-2}^{j+2}-4u_{i-2}^{j+1}+6u_{i-2}^j-4u_{i-2}^{j-1}$$

$$+u_{i-2}^{j-2}])))-\frac{1}{20160}([f_i^{j+3}-6f_i^{j+2}+15f_i^{j+1}-20f_i^j+15f_i^{j-1}-6f_i^{j-2}+f_i^{j-3}]-a^2(\frac{k^2}{h^4}$$

$$([f_{i+2}^{j+2}-4f_{i+2}^{j+1}+6f_{i+2}^j-4f_{i+2}^{j-1}+f_{i+2}^{j-2}-4f_{i+1}^{j+2}+16f_{i+1}^{j+1}-24f_{i+1}^j+16f_{i+1}^{j-1}-4f_{i+1}^{j-2}$$

$$+6f_i^{j+2}-24f_i^{j+1}+36f_i^j-24f_i^{j-1}+6f_i^{j-2}-4f_{i-1}^{j+2}+16f_{i-1}^{j+1}-24f_{i-1}^j+16f_{i-1}^{j-1}-4f_{i-1}^{j-2}$$

$$+f_{i-2}^{j+2}-4f_{i-2}^{j+1}+6f_{i-2}^j-4f_{i-2}^{j-1}+f_{i-2}^{j-2}]-[u_{i+2}^{j+2}-4u_{i+2}^{j+1}+6u_{i+2}^j-4u_{i+2}^{j-1}+u_{i+2}^{j-2}-4u_{i+1}^{j+2}$$

$$+16u_{i+1}^{j+1}-24u_{i+1}^j+16u_{i+1}^{j-1}-4u_{i+1}^{j-2}+6u_i^{j+2}-24u_i^{j+1}+36u_i^j-24u_i^{j-1}+6u_i^{j-2}-4u_{i-1}^{j+2}$$

$$+16u_{i-1}^{j+1}-24u_{i-1}^j+16u_{i-1}^{j-1}-4u_{i-1}^{j-2}+u_{i-2}^{j+2}-4u_{i-2}^{j+1}+6u_{i-2}^j-4u_{i-2}^{j-1}+u_{i-2}^{j-2}])))$$

$$+a^2(\frac{u_{i+2}^j-4u_{i+1}^j+6u_i^j-4u_{i-1}^j+u_{i-2}^j}{h^4}+\tau_4)-\frac{1}{6}([f_{i+1}^j-2f_i^j+f_{i-1}^j]-\frac{1}{k^2}[u_{i+1}^{j+1}-2u_i^{j+1}$$

$$+u_{i-1}^{j+1}-2u_{i+1}^j+4u_i^j-2u_{i-1}^j+u_{i+1}^{j-1}-2u_i^{j-1}+u_{i-1}^{j-1}])-\frac{1}{80}([f_{i+2}^j-4f_{i+1}^j+6f_i^j-4f_{i-1}^j$$

$$+f_{i-2}^{j}] - \frac{1}{k^2}[u_{i+2}^{j+1} - 4u_{i+1}^{j+1} + 6u_{i}^{j+1} - 4u_{i-1}^{j+1} + u_{i-2}^{j+1} - 2u_{i+2}^{j} + 8u_{i+1}^{j} - 12u_{i}^{j} + 8u_{i-1}^{j} - 2u_{i-2}^{j}$$

$$+u_{i+2}^{j-1} - 4u_{i+1}^{j-1} + 6u_{i}^{j-1} - 4u_{i-1}^{j-1} + u_{i-2}^{j-1}]) - f_{i}^{j} = \tau_5 + a^2\tau_6. \quad (4.31)$$

And after simplification, we obtain:

$$[16968h^4 - 58464a^2k^2 + 24a^2k^4](u_{i+1}^{j} + u_{i}^{j-1} + u_{i}^{j+1} + u_{i-1}^{j}) + [56a^2k^2 - a^2k^4](u_{i+2}^{j+2}$$

$$+u_{i-2}^{j-2} + u_{i+2}^{j-2} + u_{i-2}^{j+2}) + [2352h^4 - 5824a^2k^2 - 16a^2k^4](u_{i+1}^{j-1} + u_{i-1}^{j-1} + u_{i+1}^{j+1} + u_{i-1}^{j+1})$$

$$-[29904h^4 - 102816a^2k^2 + 36a^2k^4](u_{i}^{j}) + [252h^4 + 1456a^2k^2 + 4a^2k^4](u_{i-1}^{j-2} + u_{i+1}^{j-2}$$

$$+u_{i+1}^{j+2} + u_{i-1}^{j+2} + u_{i+2}^{j+1} + u_{i-2}^{j+1} + u_{i+2}^{j-1} + u_{i-2}^{j-1}) - [504h^4 - 17136a^2k^2 + 6a^2k^4](u_{i}^{j-2} + u_{i}^{j+2}$$

$$+u_{i+2}^{j} + u_{i-2}^{j}) + [a^2k^4](f_{i+2}^{j+2} + f_{i-2}^{j-2} + f_{i+2}^{j-2} + f_{i-2}^{j+2}) + [16a^2k^4](f_{i+1}^{j-1} + f_{i-1}^{j-1} + f_{i+1}^{j+1} + f_{i-1}^{j+1})$$

$$-[4a^2k^4](f_{i-1}^{j-2} + f_{i+1}^{j-2} + f_{i+1}^{j+2} + f_{i-1}^{j+2} + f_{i+2}^{j+1} + f_{i-2}^{j+1} + f_{i+2}^{j-1} + f_{i-2}^{j-1}) - [3823k^2h^4 + 24a^2k^4]$$

$$(f_{i+1}^{j} + f_{i-1}^{j} + f_{i}^{j+1} + f_{i}^{j-1}) - [11908k^2h^4 - 36a^2k^4](f_{i}^{j}) - [302k^2h^4 - 6a^2k^4](f_{i-2}^{j} + f_{i}^{j+2}$$

$$+f_{i}^{j-2} + f_{i+2}^{j}) - [k^2h^4](f_{i}^{j-3} + f_{i}^{j+3}) - \tau_7 = 0, \quad (4.32)$$

where $\tau_7 = 20160k^2h^4(\tau_5 + a^2\tau_6 - a^2\tau_4 - \tau_3)$ is a local truncation error.

In order to satisfy the initial boundary conditions and also to reduce the algorithm complexity we can use the following Taylor series estimates.

$$u_{i+2}^{j+2} + u_{i-2}^{j-2} + u_{i+2}^{j-2} + u_{i-2}^{j+2} = 16u_{i+1}^{j+1} - 24u_{i}^{j+1} + 16u_{i-1}^{j+1} - 24u_{i+1}^{j} + 36u_{i}^{j} - 24u_{i-1}^{j}$$

$$+16u_{i+1}^{j-1} - 24u_{i}^{j-1} + 16u_{i-1}^{j-1}. \quad (4.33)$$

$$u_{i-1}^{j-2}+u_{i+1}^{j-2}+u_{i+1}^{j+2}+u_{i-1}^{j+2}+u_{i+2}^{j+1}+u_{i-2}^{j+1}+u_{i+2}^{j-1}+u_{i-2}^{j-1}=8u_{i+1}^{j+1}+8u_{i+1}^{j-1}-6u_{i+1}^{j}-6u_{i}^{j+1}$$

$$-6u_{i}^{j-1}-6u_{i-1}^{j}+8u_{i-1}^{j+1}+8u_{i-1}^{j-1}. \tag{4.34}$$

$$u_{i}^{j-2}+u_{i}^{j+2}+u_{i+2}^{j}+u_{i-2}^{j}=4u_{i}^{j+1}-12u_{i}^{j}+4u_{i}^{j-1}+4u_{i+1}^{j}+4u_{i-1}^{j}. \tag{4.35}$$

$$f_{i+2}^{j+2}+f_{i-2}^{j-2}+f_{i+2}^{j-2}+f_{i-2}^{j+2}=16f_{i+1}^{j+1}-24f_{i}^{j+1}+16f_{i-1}^{j+1}-24f_{i+1}^{j}+36f_{i}^{j}-24f_{i-1}^{j}$$

$$+16f_{i+1}^{j-1}-24f_{i}^{j-1}+16f_{i-1}^{j-1}. \tag{4.36}$$

$$f_{i-1}^{j-2}+f_{i+1}^{j-2}+f_{i+1}^{j+2}+f_{i-1}^{j+2}+f_{i+2}^{j+1}+f_{i-2}^{j+1}+f_{i+2}^{j-1}+f_{i-2}^{j-1}=8f_{i+1}^{j+1}+8f_{i+1}^{j-1}-6f_{i+1}^{j}-6f_{i}^{j+1}$$

$$-6f_{i}^{j-1}-6f_{i-1}^{j}+8f_{i-1}^{j+1}+8f_{i-1}^{j-1}. \tag{4.37}$$

$$f_{i}^{j-2}+f_{i}^{j+2}+f_{i+2}^{j}+f_{i-2}^{j}=4f_{i}^{j+1}-12f_{i}^{j}+4f_{i}^{j-1}+4f_{i+1}^{j}+4f_{i-1}^{j}. \tag{4.38}$$

$$f_{i}^{j+3}+f_{i}^{j-3}=9f_{i+1}^{j}-16f_{i}^{j}+9f_{i}^{j-1}. \tag{4.39}$$

By substituting equations (4.33 - 4.39) into equation (4.32), we get the following:

$$[16968h^4-58464a^2k^2+24a^2k^4](u_{i+1}^{j}+u_{i}^{j-1}+u_{i}^{j+1}+u_{i-1}^{j})+[56a^2k^2-a^2k^4](16u_{i+1}^{j+1}$$

$$-24u_{i}^{j+1}+16u_{i-1}^{j+1}-24u_{i+1}^{j}+36u_{i}^{j}-24u_{i-1}^{j}+16u_{i+1}^{j-1}-24u_{i}^{j-1}+16u_{i-1}^{j-1})+[2352h^4$$

$$-5824a^2k^2-16a^2k^4](u_{i+1}^{j-1}+u_{i-1}^{j-1}+u_{i+1}^{j+1}+u_{i-1}^{j+1})-[29904h^4-102816a^2k^2+36a^2k^4](u_{i}^{j})$$

$$+[252h^4+1456a^2k^2+4a^2k^4](8u_{i+1}^{j+1}+8u_{i+1}^{j-1}-6u_{i+1}^{j}-6u_{i}^{j+1}-6u_{i}^{j-1}-6u_{i-1}^{j}+8u_{i-1}^{j+1}$$

$$+8u_{i-1}^{j-1})-[504h^4-17136a^2k^2+6a^2k^4](4u_{i}^{j+1}-12u_{i}^{j}+4u_{i}^{j-1}+4u_{i+1}^{j}+4u_{i-1}^{j})=-[a^2k^4]$$

$$(16f_{i+1}^{j+1}-24f_i^{j+1}+16f_{i-1}^{j+1}-24f_{i+1}^{j}+36f_i^{j}-24f_{i-1}^{j}+16f_{i+1}^{j-1}-24f_i^{j-1}+16f_{i-1}^{j-1})$$

$$-[16a^2k^4](f_{i+1}^{j-1}+f_{i-1}^{j-1}+f_{i+1}^{j+1}+f_{i-1}^{j+1})+[4a^2k^4](8f_{i+1}^{j+1}+8f_{i+1}^{j-1}-6f_{i+1}^{j}-6f_i^{j+1}-6f_i^{j-1}$$

$$-6f_{i-1}^{j}+8f_{i-1}^{j+1}+8f_{i-1}^{j-1})+[3823k^2h^4+24a^2k^4](f_{i+1}^{j}+f_{i-1}^{j}+f_i^{j+1}+f_i^{j-1})+[11908k^2h^4$$

$$-36a^2k^4](f_i^{j})+[302k^2h^4-6a^2k^4](4f_i^{j+1}-12f_i^{j}+4f_i^{j-1}+4f_{i+1}^{j}+4f_{i-1}^{j})+[k^2h^4](9f_{i+1}^{j}$$

$$-16f_i^{j}+9f_i^{j-1}). \tag{4.40}$$

After rearranging the above terms we get the following more simplified formula for our problem.

$$A(u_{i-1}^{j}+u_{i+1}^{j}+u_i^{j-1}+u_i^{j+1})+B(u_{i-1}^{j-1}+u_{i-1}^{j+1}+u_{i+1}^{j-1}+u_{i+1}^{j+1})+C(u_i^{j})=D(f_i^{j})$$

$$+E(f_{i+1}^{j}+f_{i-1}^{j}+f_i^{j+1}+f_i^{j-1}), \tag{4.41}$$

where $A = 13440h^4$, $B = 4368h^4 + 6720a^2k^2$, $C = -23856h^4 - 100800a^2k^2$, $D = 8268k^2h^4$ and $E = 5040k^2h^4$.

And also, equation (4.41) can be written as:

$$\alpha u_{i-1}^{j+1}+\beta u_i^{j+1}+\alpha u_{i+1}^{j+1}=\gamma u_{i-1}^{j}+\eta u_i^{j}+\gamma u_{i+1}^{j}-\alpha u_{i-1}^{j-1}-\beta u_i^{j-1}-\alpha u_{i+1}^{j-1}+K_i^{j}, \tag{4.42}$$

where

$$\begin{aligned}
\alpha &= 4368h^4+6720a^2k^2,\\
\beta &= 13440h^4,\\
\eta &= 23856h^4+100800a^2k^2,\\
\gamma &= -13440h^4,\\
K_i^{j} &= (8268k^2h^4(f_i^{j})+5040k^2h^4(f_{i+1}^{j}+f_{i-1}^{j}+f_i^{j+1}+f_i^{j-1})),
\end{aligned}$$

For $i = 1, 2, 3, ..., m-1$ and $j = 1, 2, 3, ..., n-1$.

4.1 Stability, Consistency and Convergence Analysis

Stability: A method is said to be stable if it produces a bounded solution which estimates the exact solution otherwise it is said to be unstable. The error caused by a small perturbation in the numerical method remains bounded, when the difference between the computed numerical solution and the exact solution of the discrete equation remains bounded as time grows (Duressa et al., 2016).

Consistence: The numerical scheme is called consistent with the differential equation, if the truncation error goes to zero when the numerical parameters are made arbitrarily small (Duressa et al., 2016).

Convergence: A numerical scheme is convergent when the solution to the numerical scheme approaches the exact solution of the PDE when the mesh is refined. The numerical scheme formulation must have the same set of initial conditions and boundary conditions (Duressa et al., 2016).

Von Neumann stability analysis is a procedure used to check the stability of finite difference schemes as applied to linear partial differential equations (Isaacson & Keller, 1994). It is based on decomposing the numerical solution into Fourier harmonics on the spatial grid. Although this method does not capture the influence of the boundary conditions, it is quite easy to formulate and usually accurate enough to provide practical stability criteria. We can decompose the solution into Fourier modes on the mesh as cited by (Duressa et al., 2016), assume that the solution of equation (4.42) at the grid point (ih, jk) is:

$$u_i^j = \sum_m u_m(t_j)e^{iq\theta}, \tag{4.43}$$

$$u_i^j = \lambda^j e^{iq\theta}, \tag{4.44}$$

where $q = \sqrt{-1}$, θ is a real number and $|\lambda^j|$ is the amplitude of j^{th} component. By linearity, it is enough to examine the behavior of a single Fourier component, so we replace u_i^j by $\lambda^j e^{iq\theta}$ and rearrange to get:

$$\lambda = \frac{\lambda^{j+1}}{\lambda^j} \leq 1.$$

This is the Von Neumann condition for stability, λ is the amplification factor. Substituting equation (4.44) into equation (4.42) gives:

$$\alpha\lambda^{j+1}e^{(i-1)q\theta} + \beta\lambda^{j+1}e^{iq\theta} + \alpha\lambda^{j+1}e^{(i+1)q\theta} = \gamma\lambda^{j}e^{(i-1)q\theta} + \eta\lambda^{j}e^{iq\theta} + \gamma\lambda^{j}e^{(i+1)q\theta}$$
$$-\alpha\lambda^{j-1}e^{(i-1)q\theta} - \beta\lambda^{j-1}e^{iq\theta} - \alpha\lambda^{j-1}e^{(i+1)q\theta} + K_i^j$$
$$\implies \lambda^j e^{iq\theta}(\alpha\lambda e^{-q\theta} + \beta\lambda + \alpha\lambda e^{q\theta}) = \lambda^j e^{iq\theta}(\gamma e^{-q\theta} + \eta + \gamma e^{q\theta} - \alpha\lambda^{-1}e^{-q\theta} - \beta\lambda^{-1} - \alpha\lambda^{-1}e^{q\theta}) + K_i^j$$

Then multiplying both sides of the equation by $\lambda e^{q\theta}$ to remove the minus superscripts.

$$\implies \alpha\lambda^2 + \beta\lambda^2 e^{q\theta} + \alpha\lambda^2 e^{2q\theta} = \lambda\gamma + \eta\lambda e^{q\theta} + \gamma\lambda e^{2q\theta} - \alpha - \beta e^{q\theta} - \alpha e^{2q\theta} + K_i^j$$
$$\implies (\alpha e^{2q\theta} + \beta e^{q\theta} + \alpha)\lambda^2 = (\gamma e^{2q\theta} + \eta e^{q\theta} + \gamma)\lambda - (\alpha e^{2q\theta} + \beta e^{q\theta} + \alpha) + K_i^j$$
$$\implies (\alpha e^{2q\theta} + \beta e^{q\theta} + \alpha)\lambda^2 - (\gamma e^{2q\theta} + \eta e^{q\theta} + \gamma)\lambda + (\alpha e^{2q\theta} + \beta e^{q\theta} + \alpha) + K_i^j = 0$$

$$x\lambda^2 + y\lambda + t = 0, \tag{4.45}$$

where

$$x = \alpha e^{2q\theta} + \beta e^{q\theta} + \alpha,$$
$$y = -(\gamma e^{2q\theta} + \eta e^{q\theta} + \gamma),$$
$$t = \alpha e^{2q\theta} + \beta e^{q\theta} + \alpha + K_i^j.$$

Theorem 4.1 *Routh-Hurwitz stability theorem: The system is stable if and only if all coefficients in the first column of a complete Routh Array are of the same sign. The number of sign changes indicates the number of unstable poles.*

Definition 1 *Routh-Hurwitz Stability Criteria "For a system to be stable, it is necessary and sufficient that each term of first column of Routh Array formed of its characteristic equation be positive if $a_0 > 0$*

The Routh-Hurwitz stability criteria is an algorithm used to determine whether the zeros of a polynomial lie in the left half of the complex plane, which is often referred to as being "Hurwitz." A polynomial satisfying the Hurwitz condition is a crucial requirement for a linear continuous time-invariant system to be stable, meaning that all bounded inputs will produce bounded outputs. By using Routh-Hurwitz criteria and using the transformation (Duressa et al., 2016), $\lambda = \frac{1+z}{1-z}$ substitute into equation (4.45), we have:

$$(x - y + t)\lambda^2 + 2(x - t)\lambda + (x + y + t) = 0. \tag{4.46}$$

If $|\lambda| \leq 1$, then the difference scheme of equation (4.42) is stable. It is sufficient to show that:

$$y < 0 \text{ and } x + t > 0.$$

From equation (4.45), we have:

$$\begin{aligned}
y &= -(e^{q\theta}(23856h^4 + 100800a^2k^2) - e^{2q\theta}(13440h^4) - 13440h^4) \\
&= -h^4(23856e^{q\theta} - 13440(e^{2q\theta} + 1)) - k^2(100800a^2e^{q\theta})
\end{aligned}$$

$$\begin{aligned}
x + t &= \alpha e^{2q\theta} + \beta e^{q\theta} + \alpha + \alpha e^{2q\theta} + \beta e^{q\theta} + \alpha + K_i^j \\
&= 2\alpha e^{2q\theta} + 2\beta e^{q\theta} + 2\alpha + K_i^j \\
&= 2(4368h^4 + 6720a^2k^2)e^{2q\theta} + 2(13440h^4)e^{q\theta} + 2(4368h^4 + 6720a^2k^2) + (8268k^2h^4(f_i^j) \\
&+ 5040k^2h^4(f_{i+1}^j + f_{i-1}^j + f_i^{j+1} + f_i^{j-1})) \\
&= 2h^4(13440e^{q\theta} + 4368(e^{2q\theta} + 1)) + 2k^2(6720a^2(e^{2q\theta} + 1) + 4134h^4(f_i^j) \\
&+ 2520h^4(f_{i+1}^j + f_{i-1}^j + f_i^{j+1} + f_i^{j-1}))
\end{aligned}$$

Clearly, for sufficiently small k, $y < 0$ and $x + t > 0$. Thus, the difference scheme given in equation (4.42) is unconditionally stable for Euler-Bernoulli beam PDE. Now, expand equation (4.32) in Taylor series and replace the derivatives involving x and t for the relation.

$$(\frac{\partial^2 u}{\partial t^2})_i^j + a^2(\frac{\partial^4 u}{\partial x^4})_i^j = f_i^j \tag{4.47}$$

and then we drive the local truncation error. The principal part of the local truncation error of the proposed method using equation (4.11), (4.12),(4.13),(4.14) and (4.32) for Euler-Bernoulli beam PDE is:

$$\begin{aligned}
T_i^j =& 20160k^2h^4(\tau_5 + a^2\tau_6 - a^2\tau_4 - \tau_3) \\
=& 20160k^2h^4(\frac{k^8}{1814400}(\frac{\partial^{10}u}{\partial t^{10}})_i^j + a^2\frac{h^6}{1779}(\frac{\partial^{10}u}{\partial x^{10}})_i^j + a^2\frac{h^2}{6}(\frac{\partial^6 u}{\partial x^6})_i^j + \frac{k^2}{12}(\frac{\partial^4 u}{\partial t^4})_i^j) \\
=& 168h^4k^4(\frac{\partial^4 u}{\partial t^4})_i^j + 3360a^2h^6k^2(\frac{\partial^6 u}{\partial x^6})_i^j + O(h^{10}k^2 + h^4k^{10})
\end{aligned}$$

Thus, $T \to 0$ as h and $k \to 0$.

So that, the scheme is consistent with the order of $O(h^4k^4 + h^6k^2)$.

Theorem 4.2 *Lax-equivalence theorem on convergence: States that for a consistent finite difference method for a well-posed linear initial value problem, the method is convergent if and only if it is stable (Strikwerda, 2004).*

The importance of the theorem is that while the convergence of the solution of the finite difference method to the solution of the partial differential equation is what is desired, it is ordinarily difficult to establish because the numerical method is defined by a recurrence relation while the differential equation involves a differentiable function.

However, consistency is the requirement that the finite difference method approximates the exact partial differential equation is straightforward to verify and stability is typically much easier to show than convergence (and would be needed in any event to show that round-off error will not destroy the computation).

Therefore, based on the theorem discussed above, the proposed method exhibits convergence when applied to the Euler-Bernoulli beam partial differential equation.

CHAPTER FIVE

RESULT AND DISCUSSION

In this section the Matlab version of the Compact finite difference method of 1D non-homogeneous Euler-Bernoulli beam partial differential equation is presented and demonstrated.

5.1 Test problem

Consider the initial boundary value problem (Ghafoor et al., 2022).

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = (\pi^4 - 1)sin(\pi x)cos(t) \tag{5.1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0 \qquad\qquad 0 \leq t \leq \frac{1}{100} \tag{5.2}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0 \qquad 0 \leq x \leq 1 \tag{5.3}$$

and initial conditions

$$u(x,0) = sin(\pi x), \quad \frac{\partial u}{\partial t}(x,0) = 0 \qquad 0 \leq x \leq 1 \tag{5.4}$$

The exact analytical solution of the above beam partial differential equation is:

$$u(x,t) = sin(\pi x)cos(t).$$

Solve this Problem by using eight-order compact finite difference method.

To solve the above problem, we have to construct the matrix for the same mesh grids and step by step combine each matrix in one place to insert it in Matlab code as we wanted.

Let $h = \frac{1}{4}$ and $k = \frac{1}{400}$.

From the difference equation, we have to approximate for $(\frac{\partial u}{\partial t})_i^0$

$$g_i = (\frac{\partial u}{\partial t})_i^0 = \frac{u_i^{j+1} - u_i^{j-1}}{2k} = 0$$

$$\implies u_i^{j+1} = u_i^{j-1} \tag{5.5}$$

From equation (4.41), we have

$$A(u_{i-1}^j + u_{i+1}^j + u_i^{j-1} + u_i^{j+1}) + B(u_{i-1}^{j-1} + u_{i-1}^{j+1} + u_{i+1}^{j-1} + u_{i+1}^{j+1}) + C(u_i^j) = D(f_i^j)$$

$$+E(f_{i+1}^j + f_{i-1}^j + f_i^{j+1} + f_i^{j-1}) \tag{5.6}$$

Where,
$A = 13440h^4$, $B = 4368h^4 + 6720a^2k^2$, $C = -23856h^4 - 100800a^2k^2$,
$D = 8268k^2h^4$ and $E = 5040k^2h^4$
Now construct the matrix to solve the given example above with the same mesh grids. Working on the first column of the inner grid points gives us for
i=1,j=1

$$A(u_0^1 + u_2^1 + u_1^0 + u_1^2) + B(u_0^0 + u_0^2 + u_2^0 + u_2^2) + C(u_1^1) = D(f_1^1) + E(f_2^1 + f_0^1 + f_1^2 + f_1^0)$$

i=2,j=1

$$A(u_1^1 + u_3^1 + u_2^0 + u_2^2) + B(u_1^0 + u_1^2 + u_3^0 + u_3^2) + C(u_2^1) = D(f_2^1) + E(f_3^1 + f_1^1 + f_2^2 + f_2^0)$$

i=3,j=1

$$A(u_2^1 + u_4^1 + u_3^0 + u_3^2) + B(u_2^0 + u_2^2 + u_4^0 + u_4^2) + C(u_3^1) = D(f_3^1) + E(f_4^1 + f_2^1 + f_3^2 + f_3^0)$$

By arranging in matrix-vector form gives:

$$\begin{bmatrix} C & A & 0 \\ A & C & A \\ 0 & A & C \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} + \begin{bmatrix} A & B & 0 \\ B & A & B \\ 0 & B & A \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} = \begin{bmatrix} D(f_1^1) + E(f_2^1 + f_0^1 + f_1^2 + f_1^0) \\ D(f_2^1) + E(f_3^1 + f_1^1 + f_2^2 + f_2^0) \\ D(f_3^1) + E(f_4^1 + f_2^1 + f_3^2 + f_3^0) \end{bmatrix}$$

$$- \begin{bmatrix} A(u_0^1 + u_1^0) + B(u_0^0 + u_0^2 + u_2^0) \\ A(u_2^0) + B(u_1^0 + u_3^0) \\ A(u_4^1 + u_3^0) + B(u_2^0 + u_4^0 + u_4^2) \end{bmatrix}$$

The second column of the inner grid point gives us for

i=1,j=2

$$A(u_0^2 + u_2^2 + u_1^1 + u_1^3) + B(u_0^1 + u_0^3 + u_2^1 + u_2^3) + C(u_1^2) = D(f_1^2) + E(f_2^2 + f_0^2 + f_1^3 + f_1^1)$$

i=2,j=2

$$A(u_1^2 + u_3^2 + u_2^1 + u_2^3) + B(u_1^1 + u_1^3 + u_3^1 + u_3^3) + C(u_2^2) = D(f_2^2) + E(f_3^2 + f_1^2 + f_2^3 + f_2^1)$$

i=3,j=2

$$A(u_2^2 + u_4^2 + u_3^1 + u_3^3) + B(u_2^1 + u_2^3 + u_4^1 + u_4^3) + C(u_3^2) = D(f_3^2) + E(f_4^2 + f_2^2 + f_3^3 + f_3^1)$$

By arranging in matrix-vector form gives:

$$\begin{bmatrix} C & A & 0 \\ A & C & A \\ 0 & A & C \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} + \begin{bmatrix} A & B & 0 \\ B & A & B \\ 0 & B & A \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} + \begin{bmatrix} A & B & 0 \\ B & A & B \\ 0 & B & A \end{bmatrix} \begin{bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{bmatrix}$$

$$= \begin{bmatrix} D(f_1^2) + E(f_2^2 + f_0^2 + f_1^3 + f_1^1) \\ D(f_2^2) + E(f_3^2 + f_1^2 + f_2^3 + f_2^1) \\ D(f_3^2) + E(f_4^2 + f_2^2 + f_3^3 + f_3^1) \end{bmatrix} - \begin{bmatrix} A(u_0^2) + B(u_0^1 + u_0^3) \\ 0 \\ A(u_4^2) + B(u_4^1 + u_4^3) \end{bmatrix}$$

The third column of the inner grid point gives us for
i=1,j=3

$$A(u_0^3 + u_2^3 + u_1^2 + u_1^4) + B(u_0^2 + u_0^4 + u_2^2 + u_2^4) + C(u_1^3) = D(f_1^3) + E(f_2^3 + f_0^3 + f_1^4 + f_1^2)$$

i=2,j=3

$$A(u_1^3 + u_3^3 + u_2^2 + u_2^4) + B(u_1^2 + u_1^4 + u_3^2 + u_3^4) + C(u_2^3) = D(f_2^3) + E(f_3^3 + f_1^3 + f_2^4 + f_2^2)$$

i=3,j=3

$$A(u_2^3 + u_4^3 + u_3^2 + u_3^4) + B(u_2^2 + u_2^4 + u_4^2 + u_4^4) + C(u_3^3) = D(f_3^3) + E(f_4^3 + f_2^3 + f_3^4 + f_3^2)$$

By arranging in matrix-vector form gives:

$$\begin{bmatrix} A & B & 0 \\ B & A & B \\ 0 & B & A \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} + \begin{bmatrix} C & A & 0 \\ A & C & A \\ 0 & A & C \end{bmatrix} \begin{bmatrix} u_1^3 \\ u_2^3 \\ u_3^3 \end{bmatrix} = \begin{bmatrix} D(f_1^3) + E(f_2^3 + f_0^3 + f_1^4 + f_1^2) \\ D(f_2^3) + E(f_3^3 + f_1^3 + f_2^4 + f_2^2) \\ D(f_3^3) + E(f_4^3 + f_2^3 + f_3^4 + f_3^2) \end{bmatrix}$$

$$- \begin{bmatrix} A(u_0^3 + u_1^4) + B(u_0^2 + u_0^4 + u_2^4) \\ A(u_2^4) + B(u_1^4 + u_3^4) \\ A(u_4^3 + u_3^4) + B(u_2^4 + u_4^2 + u_4^4) \end{bmatrix}$$

After rearranging the above matrix it gives us:

$$\begin{bmatrix} C & A & 0 & A & B & 0 & 0 & 0 & 0 \\ A & C & A & B & A & B & 0 & 0 & 0 \\ 0 & A & C & 0 & B & A & 0 & 0 & 0 \\ A & B & 0 & C & A & 0 & A & B & 0 \\ B & A & B & A & C & A & B & A & B \\ 0 & B & A & 0 & A & C & 0 & B & A \\ 0 & 0 & 0 & A & B & 0 & C & A & 0 \\ 0 & 0 & 0 & B & A & B & A & C & A \\ 0 & 0 & 0 & 0 & B & A & 0 & A & C \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_1^2 \\ u_2^2 \\ u_3^2 \\ u_1^3 \\ u_2^3 \\ u_3^3 \end{bmatrix} = \begin{bmatrix} D(f_1^1) + E(f_2^1 + f_0^1 + f_1^2 + f_1^0) \\ D(f_2^1) + E(f_3^1 + f_1^1 + f_2^2 + f_2^0) \\ D(f_3^1) + E(f_4^1 + f_2^1 + f_3^2 + f_3^0) \\ D(f_1^2) + E(f_2^2 + f_0^2 + f_1^3 + f_1^1) \\ D(f_2^2) + E(f_3^2 + f_1^2 + f_2^3 + f_2^1) \\ D(f_3^2) + E(f_4^2 + f_2^2 + f_3^3 + f_3^1) \\ D(f_1^3) + E(f_2^3 + f_0^3 + f_1^4 + f_1^2) \\ D(f_2^3) + E(f_3^3 + f_1^3 + f_2^4 + f_2^2) \\ D(f_3^3) + E(f_4^3 + f_2^3 + f_3^4 + f_3^2) \end{bmatrix}$$

$$- \begin{bmatrix} A(u_0^1 + u_1^0) + B(u_0^0 + u_0^2 + u_2^0) \\ A(u_2^0) + B(u_1^0 + u_3^0) \\ A(u_4^1 + u_3^0) + B(u_2^0 + u_4^0 + u_4^2) \\ A(u_0^2) + B(u_0^1 + u_0^3) \\ 0 \\ A(u_4^2) + B(u_4^1 + u_4^3) \\ A(u_0^3 + u_1^4) + B(u_0^2 + u_0^4 + u_2^4) \\ A(u_2^4) + B(u_1^4 + u_3^4) \\ A(u_4^3 + u_3^4) + B(u_2^4 + u_4^2 + u_4^4) \end{bmatrix}$$

Table 5.1: Shows the comparison of exact solution with in the numerical solution solved by using eight-order compact finite difference method using (h=0.1 and k=0.01)

Grid point	Exact Solution	Numerical Solution	Absolute Error
u_1^1	0.30897478281168	0.30901699437495	0.00004221156327
u_1^2	0.58770496112786	0.58778525229247	0.00008029116462
u_1^3	0.80890648306760	0.80901699437495	0.00011051130735
u_2^1	0.95092660246181	0.95105651629515	0.00012991383334
u_2^2	0.99986340051183	1.00000000000000	0.00013659948817
u_2^3	0.95092660246181	0.95105651629515	0.00012991383334
u_3^1	0.80890648306760	0.80901699437495	0.00011051130735
u_3^2	0.58770496112786	0.58778525229247	0.00008029116462
u_3^3	0.30897478281168	0.30901699437495	0.00004221156327
Maximum Absolute Error			0.00013659948817

Table 5.2: Shows the comparison of the maximum absolute error of our method with in the maximum absolute error of the same example worked before by CFDM for different values of h and $k = h^2$

Mesh	N	h	MAE(Aouragh et al., 2024)	MAE(Our method)
$Mesh_1$	32	0.0312	$1.1969e^{-04}$	$1.8927e^{-06}$
$Mesh_2$	64	0.0156	$7.5135e^{-06}$	$1.1841e^{-07}$
$Mesh_3$	128	0.0078	$4.6973e^{-07}$	$7.4024e^{-09}$
$Mesh_4$	256	0.0039	$2.9359e^{-08}$	$4.6268e^{-10}$
$Mesh_5$	512	0.0020	$1.8348e^{-09}$	$3.2000e^{-11}$

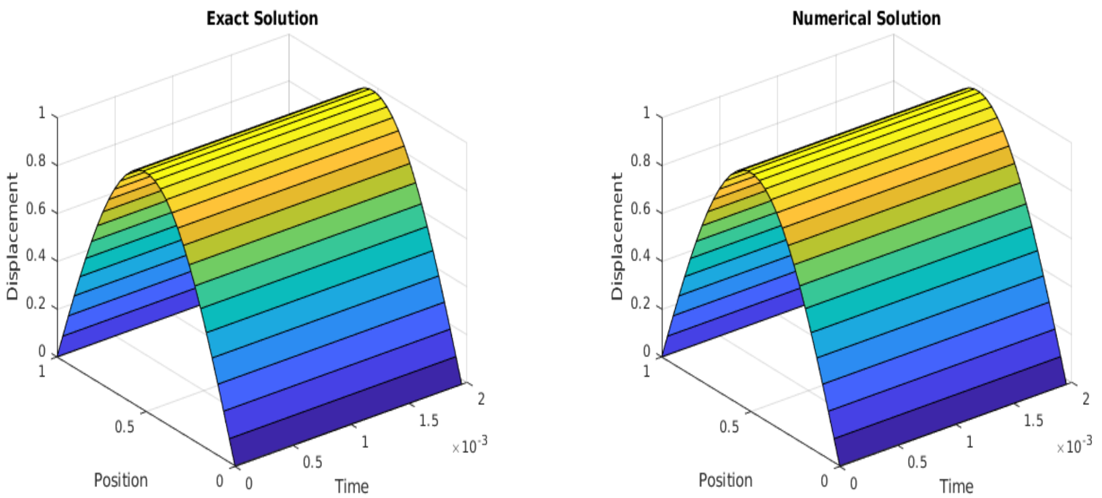

Figure 5.1: Graph of exact solution and numerical(CFDM) solution for $h = 0.0312$ and $k = h^2$

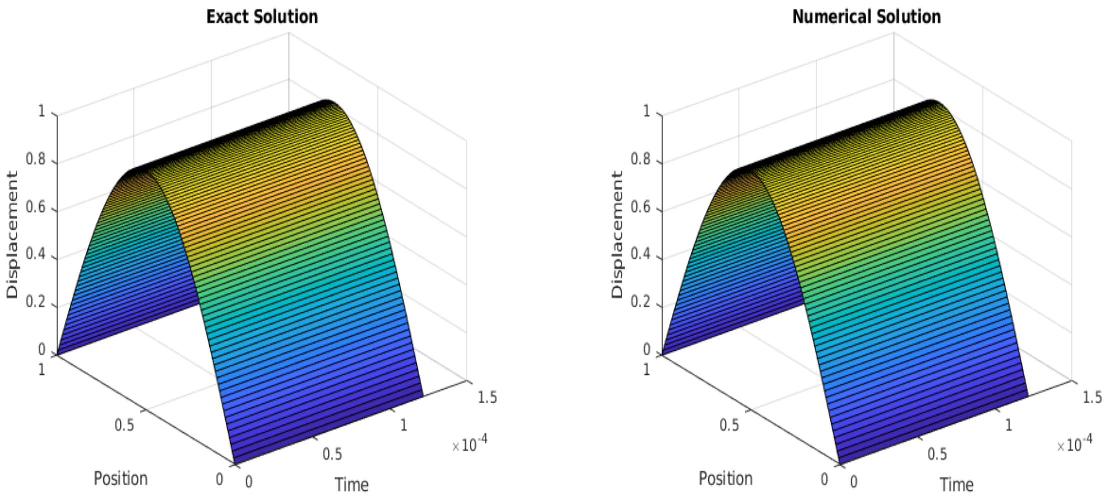

Figure 5.2: Graph of exact solution and numerical(CFDM) solution for $h = 0.0078$ and $k = h^2$

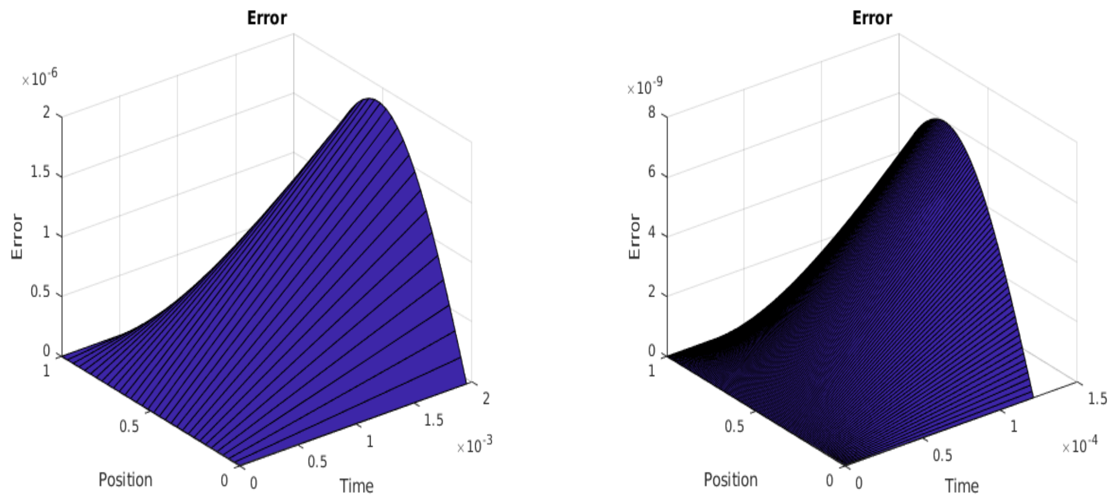

Figure 5.3: Error graphs for $h = 0.0312$ and $k = h^2$(left) and also for $h = 0.0078$ and $k = h^2$(right)

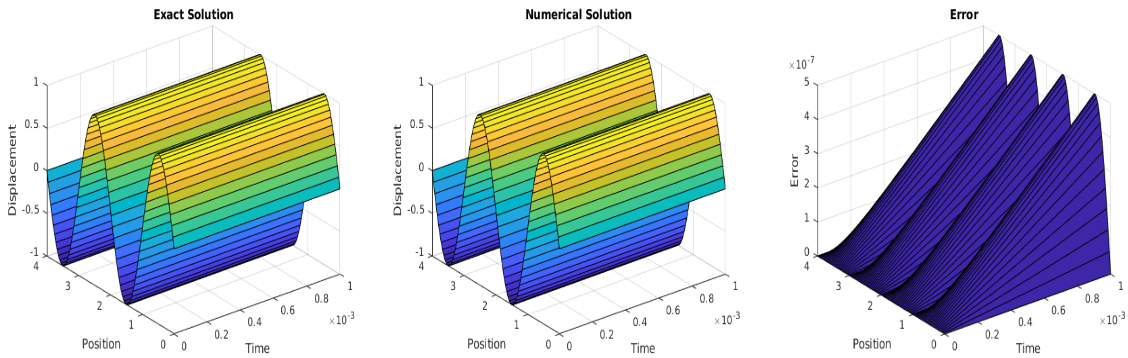

Figure 5.4: The graph of exact solution, numerical(CFDM) solution and its error for $h = 0.04$, $k = 0.0005$ and length of the beam $L = 4$

5.2 Discussion

In table (5.1), we have presented the results of our study where we compared the exact solutions with the solutions obtained using CFDM (Numerical Solution) for our test problem. The test problem was solved using a grid size of $h = 0.1$ and a time step size of $k = 0.01$. The results indicate that there is a strong agreement between the exact solutions and the numerical results obtained using CFDM. Furthermore, in table (5.2), we conducted a comparison of the maximum absolute errors obtained by CFDM for different values of h, where $k = h^2$ and also compared our method with previous approaches that were used on the same example. The results indicate that our method gives higher accuracy compared to the previous approaches.

In figures (5.1) and (5.2), the exact solution and the eight-order CFDM (Numerical Solution) with different step sizes are compared using 3D plots to visualize the beam's displacement. The result displays that reducing the time step size improves the accuracy of the compact finite difference method. Smaller step sizes result in more precise approximations of the beam's displacement. Comparing plots with different step sizes provides insights into the method's accuracy. Figure (5.3) presents a 3D plot showing the error between the exact solution and the CFDM method. It allows for an analysis of how varying step sizes affect the method's accuracy. Decreasing the values of h and k, indicating smaller step sizes, leads to a decrease in error. This suggests that using smaller step sizes yields more accurate approximations of the beam's displacement.

From figure (5.4),when the length of the beam is increased, the computational complexity of solving the problem also increases. This is because a larger beam requires more calculations and memory to accurately solve the PDE. Additionally, the accuracy of the solution also affected as errors and inaccuracies are more likely to occur with a longer beam. This is due to the increased number of grid points and the potential for numerical instability. Furthermore, the computational time needed to solve the problem is also increase as the algorithm has to perform more calculations, resulting in longer processing times.

CONCLUSION AND RECOMMENDATION

Conclusion

In this thesis, we developed an eight-order compact finite difference method for solving one dimensional non-homogeneous Euler-Bernoulli beam PDE. This method utilizes an eight-order compact finite difference approach to approximate both the spatial and time derivatives. It effectively incorporates the boundary conditions by utilizing only three spatial grid points within a single compact stencil. This eliminates the need for additional nodes or special schemes near the boundary, reducing the complexity often associated with central finite difference methods. The method is known for its ease of implementation, conciseness, accuracy and computational efficiency. Extensive demonstrations confirm the unconditional stability of the proposed method. Through example, the effectiveness of the method is showcased, providing results comparable to existing methods in the literature for one-dimensional Euler-Bernoulli beam PDEs. Consequently, the proposed method offers improved accuracy compared to existing approaches.

Recommendation

The results obtained in this thesis focused on comparing eight-order CFDM with existing methods for solving one dimensional non-homogeneous fourth-order Euler-Bernoulli beam PDEs. The analysis was conducted to evaluate stability, consistency, and convergence in 1D. Furthermore, the thesis suggests that the possibility of extending the research for 2D.

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References

Abohadima, S., & Taha, M. (2009). Dynamic analysis of nonuniform beams on elastic foundations. *The Open Applied Mathematics Journal*, *3*(1).

Abohadima, S., Taha, M., Abdeen, M., et al. (2015). General analysis of timoshenko beams on elastic foundation. *Mathematical Problems in Engineering*, *2015*.

Achawakorn, K., & Jearsiripongkul, T. (2012). Vibration analysis of exponential cross-section beam using galerkin's method. *International Journal of Applied Science and Technology*, *2*(6), 7–13.

Andrade, C., & McKee, S. (1977). High accuracy adi methods for fourth order parabolic equations with variable coefficients. *Journal of Computational and Applied Mathematics*, *3*(1), 11–14.

Aouragh, M. D., Khali, Y., Khallouq, S., & Segaoui, M. (2024). Compact finite difference scheme for euler–bernoulli beam equation with a simply supported boundary conditions. *International Journal of Applied and Computational Mathematics*, *10*(1), 17.

Bauchau, O. A., & Craig, J. I. (2009). Euler-bernoulli beam theory. In *Structural analysis* (pp. 173–221). Springer.

Biazar, J., & Ghazvini, H. (2007). He's variational iteration method for fourth-order parabolic equations. *Computers & Mathematics with Applications*, *54*(7-8), 1047–1054.

Dehghan, M., & Manafian, J. (2009). The solution of the variable coefficients fourth-order parabolic partial differential equations by the homotopy perturbation method. *Zeitschrift für Naturforschung A*, *64*(7-8), 420–430.

Duressa, G. F., Bullo, T. A., & Kiltu, G. G. (2016). Fourth order compact finite difference method for solving one dimensional wave equation. *International Journal of Engineering and Applied Sciences*, *8*(4), 30–39.

Evans, D. J., & Yousif, W. (1991). A note on solving the fourth order parabolic equation by the age method. *International journal of computer mathematics*, *40*(1-2), 93–97.

Fairweather, G. (1972). Galerkin methods for vibration problems in two space variables. *SIAM Journal on Numerical Analysis*, *9*(4), 702–714.

Gere, J. M., & Timoshenko, S. (1997). Mechanics of materials. ed. *Boston, MA: PWS*.

Ghafoor, A., Haq, S., Hussain, M., Abdeljawad, T., & Alqudah, M. A. (2022). Numerical solutions of variable coefficient higher-order partial differential equations arising in beam models. *Entropy*, *24*(4), 567.

Gorman, D. J. (1975). Free vibration analysis of beams and shafts. *(No Title)*.

Isaacson, E., & Keller, H. B. (1994). *Analysis of numerical methods.* Courier Corporation.

Kalita, J. C., Dalal, D., & Dass, A. K. (2002). A class of higher order compact schemes for the unsteady two-dimensional convection–diffusion equation with variable convection coefficients. *International Journal for Numerical Methods in Fluids*, *38*(12), 1111–1131.

Karaa, S., & Zhang, J. (2004). High order adi method for solving unsteady convection–diffusion problems. *Journal of Computational Physics*, *198*(1), 1–9.

Khan, N. A., Ara, A., Afzal, M., & Khan, A. (2010). Analytical aspect of fourth-order parabolic partial differential equations with variable coefficients. *Mathematical and Computational Applications*, *15*(3), 481–489.

Lele, S. K. (1992). Compact finite difference schemes with spectral-like resolution. *Journal of computational physics*, *103*(1), 16–42.

Li, L., Jiang, Z., & Yin, Z. (2018). Fourth-order compact finite difference method for solving two-dimensional convection–diffusion equation. *Advances in Difference Equations*, *2018*(1), 1–24.

Liu, J., & Guo, B.-Z. (2019). A novel semi-discrete scheme preserving uniformly exponential stability for an euler–bernoulli beam. *Systems & Control Letters*, *134*, 104518.

Mwabora, K., Sigey, J., Okelo, J., & Giterere, K. (2019). A numerical study on transverse vibration of euler-bernoulli beam. *Int. J. Eng. Sci. Innov. Technol.*, *8*(3), 31–40.

Noye, B., & Tan, H. (1988). A third-order semi-implicit finite difference method for solving the one-dimensional convection-diffusion equation. *International journal for numerical methods in engineering*, *26*(7), 1615–1629.

Royster, W., & Conte, S. (1956). Convergence of finite difference solutions to a solution of the equation of the vibrating rod. *Proceedings of the American Mathematical Society*, *7*(4), 742–749.

Sarker, P., & Chakravarty, U. K. (2020). A generalization of the method of lines for the numerical solution of coupled, forced vibration of beams. *Mathematics and Computers in Simulation*, *170*, 115–142.

Strikwerda, J. C. (2004). *Finite difference schemes and partial differential equations.* SIAM.

Sun, H.-W., & Li, L. Z. (2014). A ccd–adi method for unsteady convection–diffusion equations. *Computer Physics Communications*, *185*(3), 790–797.

Tian, Z., & Ge, Y. (2007). A fourth-order compact adi method for solving two-dimensional unsteady convection–diffusion problems. *Journal of Computational* *and Applied Mathematics*, *198*(1), 268–286.

Tian, Z. F. (2011). A rational high-order compact adi method for unsteady convection–diffusion equations. *Computer Physics Communications*, *182*(3), 649–662.

Timoshenko, S. (1983). *History of strength of materials: with a brief account of the history of theory of elasticity and theory of structures*. Courier Corporation.

Twizell, E., & Khaliq, A. (1983). A difference scheme with high accuracy in time for fourth-order parabolic equations. *Computer methods in applied mechanics and engineering*, *41*(1), 91–104.

Wazwaz, A.-M. (1995). On the solution of the fourth order parabolic equation by the decomposition method. *International journal of computer mathematics*, *57*(3-4), 213–217.

Wazwaz, A.-M. (2001). Analytic treatment for variable coefficient fourth-order parabolic partial differential equations. *Applied Mathematics and Computation*, *123*(2), 219–227.