Numerical Solution of Heat Transfer Boundary Value Problem for Non-homogeneous Isotropic Material in a Plane

BY
ABDISA DEBEBE

A Thesis Submitted to Applied Mathematics Department School of Applied Natural Science
Presented in partial fulfillment of the requirement for the degree of master's in Applied Mathematics(Specialization in Numerical Analysis)

Office of Graduate Studies

Adama Science and Technology University

ADAMA,ETHIOPIA

March, 2019

Numerical Solution of Heat Transfer Boundary Value Problem for Non-homogeneous Isotropic Material in a Plane

BY

ABDISA DEBEBE

Advisor:

LEMI GUTA(PhD)

Co-advisor:

YADETA CHIMDESA(Msc)

A Thesis Submitted to Applied Mathematics Department School of Applied Natural Science

Presented in partial fulfillment of the requirement for degree of master's in Applied Mathematics(Specialization in Numerical Analaysis)

Office of Graduate Studies

Adama Science and Technology University

ADAMA, ETHIOPIA

March, 2019

Dedication

This thesis is dedicated to my beloved families, especially to my brather Tesfaye Debebe and to my Mother Ilfinesh Ejeta, whose advice, support, prayers and love helped me all along the right way and made me who I am today. They will always be in my heart forever!.

Table of Contents

List of Figures

List of Tables

List of Abbreviations

BDIE :	Boundary Domain Integral Equation
BDIDE :	Boundary Domain Integro-Differential Equation
BEM :	Boundary Element Method
BIE :	Boundary Integral Equation
FEM :	Finite Element Method
FDM:	Finite Diffrence Method
BVP :	Boundary-Value Problem
PDE :	Partial Differential Equation
2D :	Two-Dimensional
3D :	Three-Dimensional
$\bigtriangledown$:	Gradient operator
$\bigtriangledown^2$:	Laplace operator
L^2 :	square integrable functions
$\Gamma(x)$:	The gamma function
$\omega(x)$:	Localized domain with respect to collocation point y
$\overline{\Omega}$:	$\Omega \cup \partial\Omega$
$\partial\Omega$:	boundary of the domain
Ω :	domain of the problem
$\partial_D\Omega$:	Drichlet boundary

Abstract

In this thesis we studied the numerical solution of the heat transfer boundary value problem for non-homogeneous isotropic material in a plane. We focused on one of the numerical method in solving BVPs, i.e., Boundary Element Method (BEM). The formulations of the boundary-domain integral equation (BDIE) for heat transfer problems with variable coefficients are presented using a parametrix (Levi function), which is usually available. The mesh-based discretisation of BDIE with trangular domain elements leads to a system of linear algebric equations. Then, we solved the system by LU decomposition. Numerical examples are presented for several simple problems, for which exact solutions are available, to demonstrate the efficiency of the proposed approaches. convergence of the method was discussed.

- Key Words: Partial differential equation, Boundary element method, variable coefficient, boundary-domain integral equation, heat transfer.

Acknowledgements

First of all, I would like to thank the almighty God for His assistance from the beginning to now, enable me to be successful in different directions.

Socandly,I would take this opportunity to thank and my sincere gratitude to my adviser Dr.Lemi Guta, for his guidance, assistance throughout my work. This work would never have been completed without him help and guidance,and for that, I am deeply indebted to him.

I would like to acknowledge the financial support I received from the Adama Science and Technology University for my research.

Last, but not the least, I would like to express my deepest gratitude to my brather, Tasfye Debebe, for always being near me when I need someone to talk and for sharing all the ups and downs of the life with me. I am thankful for her continuous support, motivation and understanding.

Finally, I would like to express my sincere appreciation to all who has helped me in one way or another, but whose names are not mentioned

Chapter 1

Introduction

1.1 Background of the study

Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modeling of inhomogeneous media in solid mechanics, electromagnetics, thermoconductivity, fluid flows through porous media, and other areas of physics and engineering. Mathematical description of many physical processes leads to linear differential and integral equations, or even to integro-differential second order equations.

A reasonably broad class of physical and engineering problems may be reduced to linear second order differential equations. Through centuries, different method of solving such problems have been developed. In any method, it is crucial to investigate the well-posdness (uniqueness, existence and continuous dependence of initial data) of solution. In a rare case, one can find anlaytical solution for a given boundary value problem.

In most engineering and science problems, it is impossible to find the analytic solution for a boundary value problem. There are various methods for solving BVPs for PDE analytically e.g. methods of separation of variables, Fourier and Laplace transforms, integral transforms and variation of parameters. However, many problems encountered in applications cannot be solved using analytical methods. Therefore, it is necessary to resort approximate solution methods. Many methods have been developed for the numerical solution of partial differential equations and amongst the commonly used are finite difference method and finite element method. These methods are called domain discretization methods since they require the discretization of the domain of the problem under consideration into a number of elements or cells.

Although the method employs a comparatively straightforward internal discretization scheme and it is computationally economical, main difficulties of the technique lie in the consideration of curved geometries and the insertion of boundary conditions in this case. In that cases, one has to implement a suitable numerical method to be obtain the approximate solution. There are different kinds of transformations that are efficient in analyzing and obtaining solution. One of the classical method to investigate the existence of solution of various boundary value problems(BVP) comprises of reducing the solution of integral equations. The method of solving a boundary value problem by transforming to an integral equation or a systems of integral equations is called *boundary integral equation*(BIE) method. The boundary integral equation method has been developed over recent decades both theory and in engineering applications. The method of boundary integral equations has always had two important applications in the theory of boundary value problems for partial differential equations: as a theoretical tool for proving the existence of solutions and as a practical tool for the construction of solutions (M. Costabel. 2007). The solution of PDEs with variable coefficients is important in many practical engineering problems(such as, elastostatics for homogeneous and nonhomogeneous bodies, elastodynamics, thermoelasticity, and plate bending problem) and there is an effort to develop BEM formulations to treat these problems.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of Boundary Value Problems (BVPs) for such PDEs to explicit boundary integral equations to be effectively solved numerically. The boundary is discretized by elements, where the continuous function and its normal derivative along the boundary are approximated using interpolation functions. Many types of interpolation functions (constant, linear and quadratic) have been implemented to arrive at a system of linear algebraic equations.

This thesis presents how BEM is successfully applied to heat transfer problems. We investigate mathematical formulations leading to a boundary-domain integral equations (BDIE) based on the use of a parametrix (Levi function), which is usually available for heat transfer problems. The implementation of the BDIE formulations for stationary heat transfer in isotropic materials with variable coecients associated with Dirichlet boundary conditions is presented.

However, BDIEs do not enjoy the privilege of having problem dimensionality reduced by one like held for BIEs since they do not only consist of the boundary integral but also the domain integrals. For numerical solving the BDIEs, one should discretize not only the domain boundary but also the domain itself. Moreover, a parametrix is usually highly non-local like a fundamental solution. This leads the discretized BDIEs to systems of equations of the same size as obtained from Finite Element Method (FEM) but unlike FEM, the systems of equations are fully populated matrices. In addition to that, a parametrix like fundamental solution is singular that implies more expensive computationally in comparison with the FEM.

1.2 Statement of the problem

The BIE method has limitation in the sense that a fundamental solution is generally not available, or no applicable method is known to constract it for partial differential equation with variable coefficients. The construction of explicit fundamental solution is possible only for linear differential equation with constant coefficients except in same very spacial cases. In this cases, we can use a parmetrix(Levi function) instead of the fundamental solution in the greens formula. Here, the boundary value problem is transformed to boundary-domain integral equation(BDIE) or boundary-domain integro-differential equation (BDIDE), see e.g., (O. Chkadua, S. E. Mikhailov, and D. Natroshvili.(2013)). and the reference therein, for the corresponding two and three dimensional boundary value problems. Among variety of physical problems that can be solved using method of integral equation, the heat transfer problem is typical. It is derived from the energy conservation principle and from thermodynamics. Its general form is given as (Paris F., Canas J. 1997)

$$- \nabla \cdot q(x,t) + g(x,t) = \frac{\partial(cu\rho)}{\partial(t)} \tag{1.2.1}$$

where $x = (x_1, x_2) \in \Omega$, t time, q heat flux, g rate of interval heat generation due to heat sources, ρ material density, c specific heat, u(x,t) temperature, and Ω domain occupied by the body. In the steady-state heat transfer, that is when the thermal depend anymore on time and equation (1.2.1) simplified to

$$-\nabla \cdot q(x) + g(x) = 0 \tag{1.2.2}$$

For two dimensional problems, the heat may flow in the direction of the x_1x_2- plane. This flows described by the flux vector q. According to the generalized Fourier's law, the thermal flux density depends linearly on the gradient of the temperature field and is expressed as

$$q = -D\nabla u \tag{1.2.3}$$

where

$$D = \begin{bmatrix} a_{x_1x_1} & a_{x_1x_2} \\ a_{x_2x_1} & a_{x_2x_2} \end{bmatrix}$$

is a matrix that provides information about the heat transfer in any direction and it is referred to as the conductivity matrix.

The negative sign in equation (1.2.3) is due to the fact heat flows from higher to lower temperature regions, while the gradient ∇u is directed towards regions of higher temperature. In general D is not a symmetric matrix. However for the simplicity of the expressions it is assumed here to be symmetric, thus $a_{x_1x_2} = a_{x_2x_1}$. If the material is orthotropic, it will be $a_{x_1x_2} = a_{x_2x_1} = 0$. Moreover, we consider the case $a_{x_1x_1} = a_{x_2x_2} = k(x)$. Corresponding to an isotropic material and consequently, the equation (1.2.2) can be written as

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(k(x)\frac{\partial u}{\partial x_j}\right) + g(x) = 0. \tag{1.2.4}$$

The PDE (1.2.4) needs to be solved subject to appropriate prescribed boundary conditions. The numerical implementation of boundary integral equation which is called boundary element method is well developed recently. The numerical implementation of boundary domain integral equation for the PDE (1.2.4) in 3D has been studied. We consider the PDE of the form (1.2.4) in bounded two dimensional domain where the right hand side function is from $L_2(\Omega)$ with Dirichlet boundary condition, where the temperature on the boundary is prescribed.

We will transform the boundary value problem to an equivalent boundary-Domain integral equation. Then we will implement numerical technique on the boundary domain integral equation to simulate result. So, this study is intended to answer the following basic questions:

- Can equivalent BDIEs be found for the BVP in the appropriate soboleve space?
- What is the type of algebric system of equations that resulted from the discretization of the BDIEs?
- What would been the convergence of the proposed numerical techniques for the problem under study?
- How to solve heat transfer equation numerically?

1.3 Objective of the study

1.3.1 General objective

The general objective of this study is to numerically solve heat transfer BVP in non-homogeneous isotropic material in plane.

1.3.2 Specific objectives

The Specific objectives of the study are to:

- find an equivalent BDIEs for the BVP in the appropriate soboleve spaces.
- identify the type of algebric system of equations that results from the discretization of the BDIEs.
- properly study the convergence of the proposed numerical techinques for the problem under study.

1.4 Significance of the study

The study of this problem will have the following importance.

- The process of the research will have greater contribution in adapting technologies for the future engineering research work in our country.
- At the same time it is hoped that the reader will also develop an ability to test the validity and accuracy of simulating results.

Chapter 2

Literature Review

In recent years, the theory and the application of BIE is well studied, see e.g.,(Costabel, 1988, Constanda, 2000, and Stephan 1987.) The background of the BIE by examining its mathematical foundation from potential theory, boundary value problems, Greens functions, Greens identities, to Fredholm integral equations was explored in e.g.(M.Costabel 1988). It was in the early 19th century when a German mathematician, Gauss introduced the first kind of integral equations of the single layer potential as a tool in numerical computation, see in Neumann. At that time, he produced all the numerical calculations by hand since computers were yet to be invented in the 1940s. Later, another German mathematician, Carl Neumann studied the double layer potential, see in Neumann. The Neumann iteration and Neumann boundary condition for Ordinary Differential Equations (ODEs) and PDEs were named after him. In (Stephan 1987), boundary integral methods has been implemented to two-dimensional screen and crack problems, a Galerkin scheme on the boundary has been implemented for the numerical solution of the BIE. The most important advantage of this method is that, it reduces a boundary value problem for linear partial differential equation in a domain to an equivalent integral equation on the boundary of the domain. In (S.Mikhailov, 2002, 2005, 2005) localized parametrixes were implemented two reduce a BVP and with variable coefficient to a localized BDIDE in (O. Chkadua, S. E. Mikhailov, and D. Natroshvili.2013), the parametrix-based direct boundary-domain integral equations for mixed BVPs variable coeffients has been analyzed. In the work, the equivalence of the BDIEs to the original mixed boundary value problem has been proved and the invertiblity of the corresponding operators has been shown for appropriate sobolev spaces. The same problem has been investigated for exterior domain in (O. Chakuda, S.E. Mikhailov and D. Natroshvili, 2009). In (J. Giroire and J.C. Nedelec, 1978), numerical solution of an exterior Neumann problem using a double layer potential has been used. We found the formulation of BDIE and BDIDE for the numerical solution of two-dimensional Helmholtz equation with variable coefficients in (AL-Jawary MA, Wrobel LC, 2011). The numerical implementation of boundary-domain integral equation for the PDE in 3D has been studied R. Grzhibovskis, S. E. Mikhailov and S. Rjasanow,(2013). In S. E. Mikhailov and N. A. Mohamed.(2012), we found a numerical implementation of a direct united boundary-domain integral equation related to Neumann boundary value problem for a scalar elliptic partial differential equation with a variable coefficients, the corresponding BDIE where discretized with quadrilateral domain elements which leads to a system of linear algebraic equation. The system were solved by LU decomposition. More numerical implementation of boundary integral equations can be found in (Attinson, 2012).

The most important work related to the boundary integral equation for solving boundary value problems comes from George Green. Green in (1828) presented the three Greens identities. Greens identities are a set of three vector derivative and integral identities which can be derived with the vector derivatives identities on its domain. In the late eighties, one could find numerous publications in the literature, where the boundary integral equation method was applied to a wide variety of engineering problems,among them are static and dynamic linear or non-linear problems of electricity/ problems of elastic dynamics, fluid dynamics, electricity and electromagnetism heat conduction etc. It could be said that today the boundary integral equation method has matured and become a power full methods for the analysis of engineering problems,and alternative to the domain methods (E. Stephan, 1987 and Jawary, 2002. This study will contribute to improve our understanding how to transform boundary value problem to boundary domain integral equation/ boundary domain integro differential equation and showing equivalence of boundary value problem with boundary integral equation.

Chapter 3

Research Methodology

3.1 Study Design

The study involves solving the heat transfer BVP for non-homogeneous isotropic material in a plane using BEM. We will transform the boundary value problem to an equivalent BDIE. For the study, mesh-based discritization of BDIEs with collecation method will be used. Then we will implement the BDIE to simulate the result.

3.2 Study procedures

To attain the objective of the study, the following procedures will be undertaken:

- Reducing the BVP to its equivalent BDIEs in the appropraite sobolevs spaces.
- Discretization of the resulting BDIEs using collocation method.
- Solving the resulting system of linear equation using direct method .
- Simulating the result in matlab graphics.

The basic steps of solving a boundary value problem using the BEM are as follows:
1) Determine a partial differential equation and boundary conditions (also, any initial conditions) of the boundary value problem.
2) Transform the region (by a region we mean a domain in 2-D, or 3-D space), the partial differential equations, and boundary conditions into a set of boundary domain integral

equations.

3) We divide the boundary of the domain into finite elements and assume a zero, or linear, second, or higher order interpolation over an element as appropriate for the particular domain of interest. Add up the integrals over each element, such that the boundary domain integral equation is discretized into a system of linear equations.

4) Solving the resulting system of equations, we obtain a function value at each node or the numerical approximation to the solution of our boundary value problem.

Chapter 4

Numerical Solution of Heat Transfer Boundary Value Problems For Non-Homogeneous Isotropic Material in a Plane

4.1 Introduction

Numerical methods are mathematical techniques used for solving mathematical problems that cannot be solved or difficult to be solved analytically. An analytical solution is an exact answer in the form of a mathematical expression in terms of the variables associated with the problem that is being solved. There are various methods for solving BVPs for PDE analytically e.g. methods of separation of variables, Fourier and Laplace transforms, integral transforms and variation of parameters. However, many problems encountered in applications cannot be solved using analytical methods. Therefore, it is necessary to resort to approximate solution methods. Many methods have been developed for the numerical solution of partial differential equations and amongst the commonly used methods e.g. finite difference method, finite volume method and finite element method. In this study we use another numerical method that can give a comparable efficiency to volume-discretisation methods is called Boundary Element Method (BEM). Boundary element methods (BEM) is attractive in solving several engineering problems. The BEM has been applied to a variety of heat transfer problems in the last thirty years. The problem of solving heat transfer problems is of great interest to a wide-range of engineers and scientists.

Green's function

In transforming boundary value problems to integral equation problems, Greens formula is a very useful tool. We state the result. It will be very useful in the derivation of latter results to follow. Suppose u and v are continuous functions in a domain $\Omega \in \Re^2$, with continuous first and second derivatives in Ω as well. Then functions u and v satisfy the Greens formula,

$$\int_{\Omega}[u\nabla^2 v - v\nabla^2 u]d\Omega = \int_{\Gamma}[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}]d\Gamma,$$

where $\Gamma = \partial\Omega$ is the boundary of domain Ω and $\frac{\partial u}{\partial n}$, and $\frac{\partial v}{\partial n}$ are the outward normal derivatives of v and u, respectively on the boundary of domain Ω .

4.2 Boundary Value Problem

Boundary value problems (BVPs) arise when modeling many kinds of natural phenomena, including electromagnetic, acoustics, and optics. A BVP seeks a function that satisfies a partial differential equation (PDE) within a domain, and that satisfies boundary conditions along the boundary of the domain.

Assume $\Omega \subset \Re^2$ is a connected domain of the Dirichlet boundary value problem (first boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\Delta u = f \text{ in } \Omega;$$
$$u = g \text{ on } \Gamma;$$

where u is specified along the whole boundary and g is given and continuous on Γ. f is a prescribed (ordered) continuous function on the boundary $\overline{\Omega}$ of the domain Ω. $\overline{\Omega}$ is closure of domain Ω. We may interpret the solution u of the Dirichlet problem as the steady state temperature distribution in a body, with the temperature prescribed at all points on the boundary.

4.3 Boundary Element Method

The boundary element method (BEM) is an important computational tool used by researchers in many fields in the physical and engineering sciences (Paris F., Canas J. 1997). The BEM is one of the methods for the numerical solution of boundary integral equations. It presents many advantages, first of all, only the boundary of the domain needs to be discretised. In BEM, discretization is made on the boundary by dividing it into small portions which are called *boundary elements*, (Brebbia C . A and Dominguez, J. 1992). The points where the unknown values are considered are called *nodes*. If the node is taken in the middle of the element, the method is called *constant element method*. The method with the nodes at the extremes or ends of the element is the *linear element method* (see Figure 4.2). Higher order elements can be obtained by adding nodes to the elements (e.g. center and end points for a cubic element). Especially in two dimensions, where the boundary is just a curve, this allows very simple data input and storage methods. Also, the smaller computer time due to a lesser number of nodes and elements (but a fully populated matrix). On each element, we make out the extreme (end) points and the nodal points. The later are the points at which values of the boundary quantities are assigned.

Case I (Constant elements)

- The boundary segment is approximated by a straight line which connects its extreme points.
- The node is placed at the midpoint of the straight line.
- The boundary quantity is assumed to be constant along the element and equal to its value at the nodal point fig (4.1)

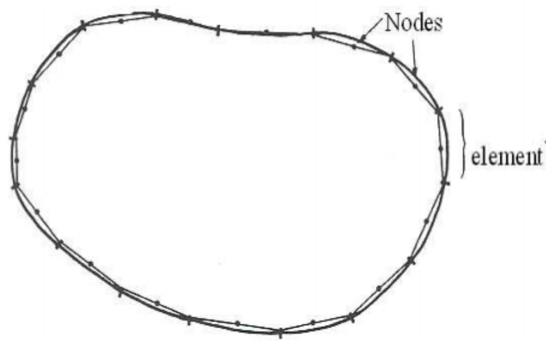

Figure 4.1: Constant element

CaseII (Linear elements)

- The boundary segment is again approximated by a straight line connecting its extreme points.
- The element has two nodal points usually placed at the end points.
- The boundary quantity is assumed to vary linearly between the nodal points fig (4.2)

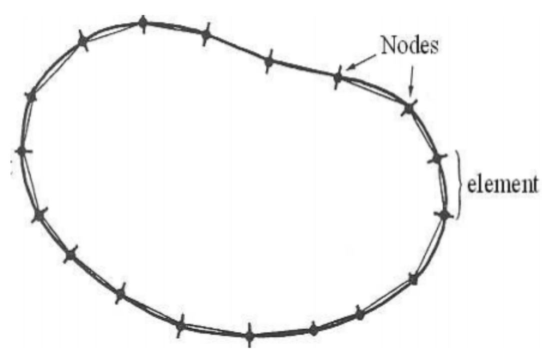

Figure 4.2: Linear element

Case III (Quadratic elements)

- The geometry is approximated by a parabolic arc.
- The element has three nodal points, two of which are placed at the end and the third somewhere in-between, usually at the midpoint fig (4.3).

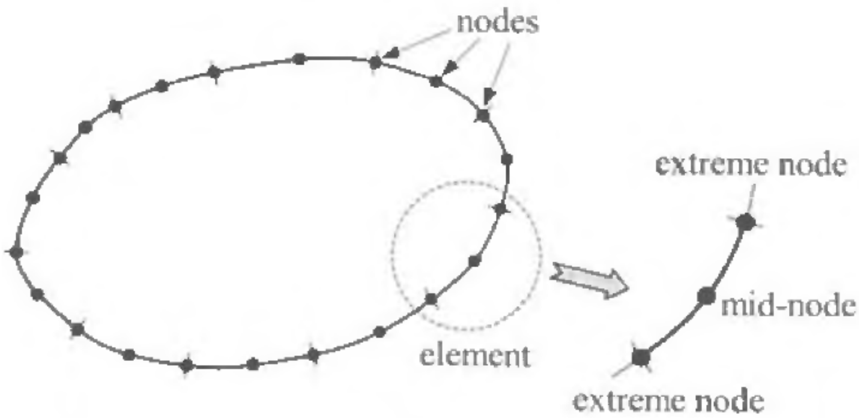

Figure 4.3: Quadratic element

4.4 Reduction of the BVP to a BDIE

Let us consider the following heat transfer BVP in an isotropic inhomogeneous medium for a two-dimensional bounded body Ω, with prescribed temperature $u(x)$ on part $\partial_D\Omega$ of the boundary $\partial\Omega$. We consider the PDE

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(k(x)\frac{\partial u}{\partial x_j}\right) + g(x) = 0. \tag{4.4.1}$$

with the boundary condition

$$u(x) = \overline{u}(x), \quad x \in \partial_D\Omega. \tag{4.4.2}$$

where Ω is a bounded domain, $u(x)$ the temperature, $k(x)$ a known variable thermoconductivity coefficient, $g(x)$ a known heat source and $\overline{u}(x)$ is known function.

The BVP (4.4.1) - (4.4.2) appears particularly when modelling stationary heat transfer, elastostatics, electrostatics, and diffusion problems for functionally graded materials, as well as in flow in porous media.

The Green formula for the differential operator L

$$\int_{\Omega}[uLv - vLu]d\Omega = \int_{\partial\Omega}[uTv - vTu]d\Gamma, \tag{4.4.3}$$

where u and v are arbitrary twice differentiable functions.

Let L be a linear operator and $F(x,y)$ be its fundamental solution, i.e.

$$L_x F(x,y) = \delta(x-y)$$

where δ is the Dirac delta function. Then one could take $v(x) = F(x,y)$, identify u(x) with a solution of Eq.(4.4.1), and thus arrive at the third Green identity

$$c(y)u(y) - \int_{\partial\Omega} [u(x)T_x(x,y) - F(x,y)Tu(x)]d\Gamma(x) = \int_{\Omega} F(x,y)g(x)d\Omega(x) \qquad (4.4.4)$$

where $c(y)$ is given by

$$\text{c(y)} = \begin{cases} 1 & \text{if } y \in \Omega \\ \frac{1}{2} & \text{if } y \in \partial\Omega \text{ and } \partial\Omega \text{ smooth at y} \\ 0 & \text{if } y \notin \Omega \cup \partial\Omega \end{cases}$$

which has only a weak singularity at $x = y$. Substituting the boundary condition in the Green identity Eq.(4.4.4) and applying it for $y \in \partial\Omega$, (AL.Jawary and LC Wrobel. 2012). For partial differential operators with variable coefficients, like L in Eq.(4.4.1), a fundamental solution is generally not available in explicit form. However, a parametrix is often available, which is a function $P(x,y)$ satisfying the equation (S. E. Mikhailov. 2002).

$$L_x p(x,y) = \delta(x-y) + R(x,y) \qquad (4.4.5)$$

The fundamental solution of the operator with frozen coefficients $a(x) = a(y)$ corresponding to the operator L defined in Eq.(4.4.1), can be used as a parametrix, in the two-dimensional case (S. E. Mikhailov. 2002)

$$p(x,y) = \frac{1}{2\pi a(y)} \ln|x-y| \qquad (4.4.6)$$

substituting eq.(4.4.6) in.eq (4.4.5), we obtain

$$\textstyle\sum_{j=1}^{2} \left[\frac{\partial}{\partial x_j} a(x) \frac{\partial}{\partial x_j} \left[\frac{1}{2\pi a(y)} \ln|x-y| \right] \right] = \delta(x-y) + R(x,y).$$

By applying product rule,we get:

$$\sum_{j=1}^{2}\left[\frac{1}{a(y)}.\frac{\partial a(x)}{\partial x_j}.\left[\frac{1}{2\pi}\ln|x-y|\right]\right]+\frac{a(x)}{a(y)}.\frac{\partial^2}{\partial x_j^2}\left[\frac{1}{2\pi a(y)}\ln|x-y|\right]=\delta(x-y)+R(x,y).$$

since $a(x)=a(y)$ and $\sum_{j=1}^{2}\frac{\partial^2}{\partial x_j^2}\left[\frac{1}{2\pi a(y)}\ln|x-y|\right]=\delta(x-y)$, we have

$$\delta(x-y)=\left[\frac{a(x)}{a(y)}\right].\sum_{j=1}^{2}\frac{\partial^2}{\partial x_j^2}\left[\frac{1}{2\pi}\ln|x-y|\right]$$

and

$$R(x,y)=\sum_{j=1}^{2}\frac{1}{a(y)}.\frac{\partial a(x)}{\partial x_j}.\frac{\partial}{\partial x_j}\left[\frac{1}{2\pi}\ln|x-y|\right]$$

Since

$$\frac{\partial}{\partial x_j}\left[\frac{1}{2\pi}\ln|x-y|\right]=\frac{1}{2\pi r}\frac{\partial r}{\partial x_j}=\frac{x_j-y_j}{2\pi r^2},\quad r=|x-y|=\sqrt{(x_i-y_i)(x_i-y_i)}$$

The remainder will be,

$$R(x,y)=\sum_{j=1}^{2}\frac{x_j-y_j}{2\pi a(y)|x-y|^2}\frac{\partial a(x)}{\partial x_j}\quad ,x,y\in\Re^2 \tag{4.4.7}$$

substituting $p(x,y)$ for $v(x)$ in Eq.(4.4.3) and taking $u(x)$ as a solution to Eq.(4.4.1), we obtain the integral equation.

$$\begin{aligned}c(y)u(y)-\int_{\partial\Omega}[u(x)T_x(x,y)P(x,y)-P(x,y)Tu(x)]d\Gamma(x)+\int_{\Omega}R(x,y)u(x)d\Omega(x)\\=\int_{\Omega}P(x,y)g(x)d\Omega(x)\end{aligned} \tag{4.4.8}$$

The use of Greens identity in the form of Eq.(4.4.8) for heat conduction problems with variable coefficients has two main drawbacks. The first is that it contains the variable coefficient $a(x)$ in each integrand, which makes the implementation of the formulation and the resulting computer code less general and difficult to develop as a unified code, as the function $a(x)$ will change for different problems. The second main drawback is that there are two domain integrals with known integrand on the right-hand side (coming from heat source effects), and the second domain integral with unknown integrand on the left hand side coming from the remainder $R(x,y)$.

Eq.(4.4.8) can be used for formulating either a BDIE or a BDIDE, with respect to u and its derivatives. Let us consider the BDIE forms.

4.5 Boundary-domain integral equations (BDIEs)

Substituting the boundary condition (4.4.2) into (4.4.8) introducing a new variable $t(x) = Tu(x)$ for the unknown $Tu(x)$ on $\partial\Omega$, we can reduce the BVP (4.4.1)-(4.4.2) to the following system of BDIEs for $u(x)$ at $x \in \Omega$, and $t(x)$ at $x \in \partial\Omega$,

$$c(y)\bar{u}(y) - \int_{\partial\Omega} [\bar{u}(y)T_x(x,y)P(x,y) - P(x,y)Tt(x)]d\Gamma(x) + \int_{\Omega} R(x,y)u(x)d\Omega(x) \\ = \int_{\Omega} P(x,y)g(x)d\Omega(x) \quad (4.5.1)$$

$$\int_{\partial\Omega} P(x,y)Tt(x)d\Gamma(x) + \int_{\Omega} R(x,y)u(x)d\Omega(x) = -c(y)\bar{u}(y) + \int_{\partial\Omega} \bar{u}(y)T_x(x,y)P(x,y)d\Gamma(x) \\ + \int_{\Omega} P(x,y)g(x)d\Omega(x) \quad y \in \partial\Omega. \quad (4.5.2)$$

$$u(y) + \int_{\partial\Omega} p(x,y)t(x)d\Gamma(x) + \int_{\Omega} R(x,y)u(x)d\Omega(x) = \int_{\partial\Omega} \bar{u}(x)T_x p(x,y)d\Gamma(x) \\ + \int_{\Omega} p(x,y)g(x)d\Omega(x), \quad y \in \Omega. \quad (4.5.3)$$

In the above BDIE does not lead to a BIE as in the case when the parametrix is a fundamental solution. The systems of equations (4.5.2)-(4.5.3) will lead, after discretisation, to fully populated systems of linear algebraic equations[**S.Mikhailov**].

4.6 Discretisation of the BDIE with constant element

Let us discretise the domain Ω into a mesh of triangular elements $T_k, k = 1, 2, ..., N$, $T_h \cap T_m = \varnothing$, $h \neq m$. Let J be the total number of nodes $x^i, i = 1, ..., J$, at the vertices of triangles, from which there are J_D nodes on $\partial\Omega$. To obtain a system of linear algebraic equations by the collocation method, we collocate at the nodes $x^i, i = 1, ..., J$, and substitute an interpolation of $u(x)$ of the form

$$u(x) \approx \sum_{\omega_j \ni x} u(x^j)\Phi_j(x), \quad \Phi_j(x) = \begin{cases} \Phi_{kj}(x) & \text{if } x, x^j \in \bar{T}_k, \\ 0 & otherwise \end{cases} \quad (4.6.1)$$

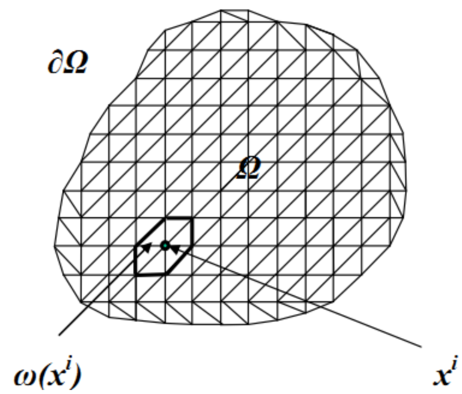

Figure 4.4: Discrtization of the domain Ω

where $\bar{\omega_j}$ is the support of $\Phi_j(x)$, which consists of all triangular elements that have x^j as a vertex; $\Phi_{kj}(x)$ are the shape functions localized on an element T_k, and associated with the node x^j. For the triangular elements, $\Phi_{kj}(x)$ can be chosen as linear functions. We can also use an interpolation of $t(x) = (Tu)(x^j)$ along only boundary nodes

$$t(x) = \sum_{x^j \in \partial\Omega} t(x^j) v_j(x), \quad x \in \partial\Omega \tag{4.6.2}$$

Here, $v_j(x)$ are boundary shape functions, taken now as constant.

$$v_j(x) = \begin{cases} 1 & \text{if } x^j \in \Gamma_j, \quad \Gamma_j \textit{are boundary elements.} \\ 0 & \textit{otherwise} \end{cases}$$

Substituting the interpolations (4.6.1) and (4.6.2) in BDIE (4.5.1) and applying the collocation method, we arrive at the following system of J_D linear algebraic equations,

$$\sum_{x^j \in \Omega} \int_{\omega_j} \Phi_j(x) R(x, x^i) u(x^j) d\Omega(x) + \sum_{x^j \in \partial\Omega} \int_{\partial\Omega \cap \bar{\omega}_j} p(x, x^i) v_j(x) t(x^j) d\Gamma(x)$$
$$= c(x^i)\bar{u}(x^i) - \sum_{x^j \in \partial\Omega} \int_{\omega_j} \Phi_j(x) R(x, x^i) \bar{u}(x^j) d\Omega(x) + \int_{\partial\Omega} \bar{u}(x) T_x p(x, x^i) d\Gamma(x)$$
$$+ \int_{\Omega} p(x, x^i) g(x) d\Omega(x), \quad x^i \in \partial\Omega. \tag{4.6.3}$$

$$\sum_{x^j \in \Omega} K'_{ij} u(x^j) + \sum_{x^j \in \partial\Omega} Q_{ij} t(x^j) = c(x^i)\bar{u}(x^i) - \sum_{x^j \in \partial\Omega} K'_{ij} \bar{u}(x^j) + \Psi D(x^i), \quad x^i \in \partial\Omega. \tag{4.6.4}$$

Discretising the BDIE (4.5.2), by substitution of interpolations (4.6.1) and (4.6.2), we arrive at the following system of $J - J_D$ linear algebraic equations,

$$u(x^i) + \sum_{x^j \in \Omega} K'_{ij} u(x^j) + \sum_{x^j \in \partial\Omega} Q_{ij} t(x^j) = - \sum_{x^j \in \partial\Omega} K'_{ij} \bar{u}(x^j) + \Psi D(x^i), \quad x^i \in \Omega. \tag{4.6.5}$$

Here,

$$K'_{ij} = \int_{\omega_j} \Phi_j(x) R(x, x^i) d\Omega(x) \tag{4.6.6}$$

$$\Psi D(x^i) = \int_{\partial\Omega} \bar{u}(x) T_x p(x, x^i) d\Gamma(x) + \int_{\Omega} p(x, x^i) g(x) d\Omega(x) \tag{4.6.7}$$

$$Q_{ij}(x^i) = \int_{\partial\Omega \cap \bar{\omega}_j} p(x, x^i) v_j(x) d\Gamma(x) \tag{4.6.8}$$

The calculations of boundary domain integrals and the treatment of weak singularity using numerical integration techniques for domain integrals is listed below.

4.7 Computation of matrix A

In this section, we are going to explain how the elements of matrix A can be computed for some boundary and domain integrals. Let us begin by introducing the reference triangular element $T_r = (t_1, t_2) \mid 0 \leqslant t_1, t_2 \leqslant 1, t_1 + t_2 \leqslant 1$ and defining a transformation

$$F : \begin{cases} T_r \Rightarrow T \\ t \Rightarrow x \end{cases}$$

such that $T \subset \Omega \subset R^2$ and T is any triangle in our domain Ω. Also, if u is a function defined on $\Omega \subset R^2$, we have $u(x) = \overline{u}(F^{-1}(x)) = \overline{u} \circ F^{-1}(x)$, where $\overline{u}$ is a function defined on T_r. So, we have with $x = F(t) \Rightarrow dx = |\frac{\partial F}{\partial t}| dt$. Since an affine transformation makes it possible to transform a reference triangle T_r to any triangle T, we have just to consider numerical integration on T_r.

4.7.1 Computation of sub-matrix Q'

$$Q'_{ij} = \int_{\partial\Omega \cap \bar{\omega}_j} p(x, x^i) T u(x) d\Gamma(x) \tag{4.7.1}$$

which can be written as

$$Q'_{ij} = \int_{\partial\Omega \cap \bar{\omega}_j} p(x, x^i) T \bar{u}(x) d\Gamma(x) \tag{4.7.2}$$

since $u(x) := \bar{u}(x)$

Now our goal is to calculate the $u(x)$, by using slightly the same method in [**Maischak**], which has been implemented for the finite element method.

4.8 Assembling the system matrix and right-hand side for BDIE

In this section we shall discuss the assembling of matrix A and right-hand side b for the Dirichlet problems.
Let us start with a Laplace equation with a mesh of eight boundary elements, nine nodes and eight triangular cells, as shown below. For the BDIE method, the system of algebraic

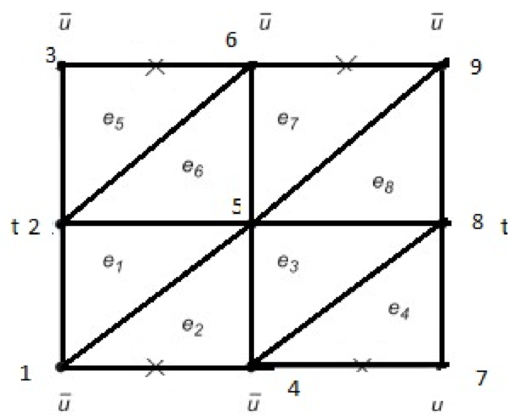

Figure 4.5: Simple mesh

equations resulting from Eq.(4.6.5) has two unknown variables t and u. The collocation nodes for calculating t on Dirichlet boundaries are taken at the mid point of the boundary elements, while in the second case the collocation points are at the end nodes. Therefore, the system $Ax = b$ is given by

$$A = \begin{bmatrix} A11 & A12 & A13 \\ A21 & D + A22 & A23 \\ A31 & A32 & I + A33 \end{bmatrix}_{7\times 7} \qquad x = \begin{bmatrix} t \in \partial_D\Omega \\ u \in \partial_D\Omega \\ u \in \Omega \end{bmatrix}_{7\times 1}$$

$$b = \begin{bmatrix} cvec + b11 + b12 + b13 \\ b21 + b22 + b23 \\ b31 + b32 + b33 \end{bmatrix}_{7\times 1}$$

where $[A11]_{4\times4}$, $[A21]_{2\times4}$, $[A31]_{1\times4}$ are the integrals in Eq.(4.6.8) with the collocation nodes belonging to $\partial_D\Omega$ and interior nodes in Ω and integration nodes x^j belonging to $\partial_D\Omega$, respectively. Also, $[A12]_{4\times2}$, $[A22]_{2\times2}$, $[A32]_{1\times2}$ are the second integral in Eq.(4.6.6) with the collocation nodes x^i belonging to $\partial_D\Omega$ and interior nodes in Ω and integration nodes x^j belonging to respectively. Moreover, $[A13]_{4\times1}$, $[A23]_{2\times1}$, $[A33]_{1\times1}$ are the second integral in Eq.(4.6.6) with the collocation nodes x^i belonging to $\partial_D\Omega$ and interior nodes in Ω and integration nodes x^j belonging to Ω, respectively. In a general mesh the dimensions of the matrix are $[A]_{J-2\times J-2}$, and right-hand side $[b]_{J-2\times1}$, where J is the total number of nodes.

The matrix D in this simple case is given by

$D = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, and the matrix I is the identity matrix, in this case just equal 1. The right-hand side can be assembled in the same way as matrix A, where $[b11]_{4\times1}$, $[b21]_{2\times1}$, $[b31]_{1\times1}$ are the first integral in Eq.(4.6.7) with the collocation nodes belonging to $\partial_D\Omega$ and interior nodes in Ω, respectively. Also, $[b12]_{4\times1}$, $[b22]_{2\times1}$, $[b32]_{1\times1}$ are the second integral in Eq.(4.6.8) with the collocation nodes belonging to $\partial_D\Omega$ and interior nodes in Ω respectively. Moreover, $[b13]_{4\times1}$, $[b23]_{2\times1}$ and $[b33]_{1\times1}$ are equal to -$\sum_{x^j\in\partial_D\Omega} k_{ij}\bar{u}(x^j)$ (with only the second integral in Eq.(4.6.6), since $R = 0$ for the Laplace equation, therefore, the first integral disappears), with the collocation nodes belonging to $\partial_D\Omega$ and interior nodes in Ω, respectively. In addition, $[cvec]_{4\times1}$ is a vector equal to $-c(x^i)\bar{u}(x^i)$, with the collocation nodes belonging to and the values of c given in Eq.(4.4.4)

Chapter 5

Result and Discussion

In this section various particular boundary value problems are solved numerically by employing the integral equations obtained in the previous section.

5.1 Numerical Solution using matlab

It is clear from the above discussion that a method such as the BEM would best be implemented with a tool that can perform matrix computations and numerical integration efficiently and produce graphical output. Matlab is one such tool that can perform all these tasks with a simple code. we shall examine some test examples to assess the performance of the BDIE formulations with Dirichlet boundary conditions. To verify the convergence of the methods, we applied them to some test problems on square domains, for which an exact analytical solution, u_{exact}, is available.

We calculate the exact solution and the difference between the exact and approximate solutions. We can draw and generate a surface plot for the exact and approximate solutions and the difference between them, also the relative error is calculated as

$$r(J) = \frac{|u_{approx}(x^j) - u_{exact}(x^j)|}{|u_{exact}(x^j)|} \tag{5.1.1}$$

where u_{approx}, u_{exact} are the numerical and exact solutions, respectively. In order to verify the convergence of the method, we applied the BEM to some test problems with different domains as discussed below. The surface plots of the numerical solutions were obtained with the most refined mesh in each example. The following two Dirichlet boundary value problems are taken into account in our numerical experiments:

Examples 1

The starting point for testing the BDIE formulations is to consider our original BVP (4.4.1-4.4.2), Square domain $\bar{\boldsymbol{\Omega}} = \{(\boldsymbol{x_1}, \boldsymbol{x_2}) : \boldsymbol{2} \leq \boldsymbol{x_1}, \boldsymbol{x_2} \leq \boldsymbol{3}\}$, $\boldsymbol{k(x)} = \boldsymbol{2(x_1 + x_2)}$, $\boldsymbol{g(x)} = 4$ for $\boldsymbol{x} \in \bar{\boldsymbol{\Omega}}$ with boundary conditions and $\bar{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{x_1 + x_2}$.

The exact solution for this problem is $\boldsymbol{u_{exact}(x)} = [\frac{\mathbf{16}}{\mathbf{85}}([\boldsymbol{x_1^2} + \boldsymbol{x_2^2}]^{\mathbf{2}} - \frac{\mathbf{1}}{[\boldsymbol{x_1^2} + \boldsymbol{x_2^2}]^{\mathbf{2}}} - \frac{\mathbf{16}}{\mathbf{255}}(\frac{[\boldsymbol{x_1^2} + \boldsymbol{x_2^2}]^{\mathbf{2}}}{\mathbf{16}}) - \frac{\mathbf{16}}{[\boldsymbol{x_1^2} + \boldsymbol{x_2^2}]^{\mathbf{2}}})] \cos(\mathbf{4} \arctan(\frac{\boldsymbol{x_2}}{\boldsymbol{x_1}}))$.

Table 5.1: Computed Temperature along line of $\boldsymbol{x_2} = 2.5$

$\boldsymbol{x_1}$	8	16	32	Exact
2	-0.022383	-0.022375	-0.022373	-0.022371
2.125	-0.047005	-0.046981	-0.046973	-0.046969
2.25	-0.076310	-0.076263	-0.076248	-0.076241
2.375	-0.113207	-0.113134	-0.113110	-0.113099
2.50	-0.161361	-0.161261	-0.161228	-0.161212
2.625	-0.225558	-0.225431	-0.225388	-0.225367
2.75	-0.312171	-0.312024	-0.311973	-0.311948
2.875	-0.429788	-0.429648	-0.429597	-0.429571
3	-0.590047	-0.589986	-0.589958	-0.589941

Table 5.1 lists the computed values of $u(x)$ along the line $x_2 = 2.5$. The values at interior points are obtained by using 8 to 32 boundary elements, compared with the exact solution. The improvement in accuracy of the numerical results can be seen clearly when the number of boundary elements is increased from 8 to 32.

By applying the Matlab program (Appendix), a surface plot of the exact and approximate solutions, see figure 5.1.

Table 5.2 shows the values of the relative error with different BEM discretisations.

Number of elements	Relative error
8	0.0437
16	0.0145
32	0.0030

Table 5.2: Convergence of the approximate solution of Example 1

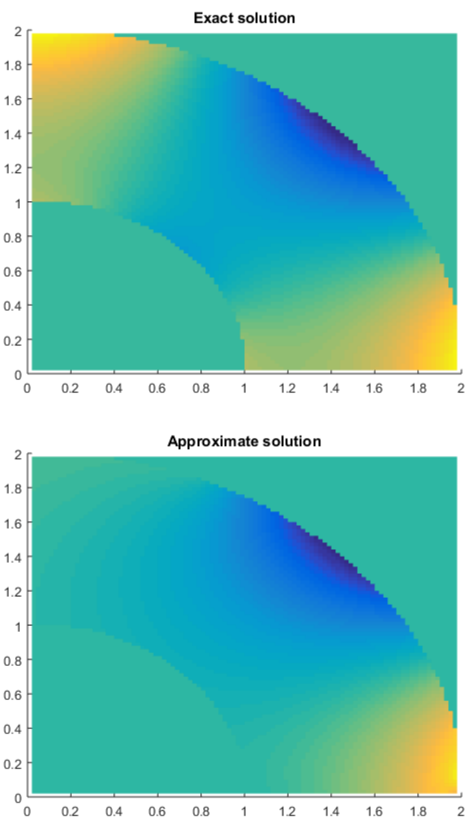

Figure 5.1: Surface plot of solution for Example 1

Examples 2

Square domain $\bar{\Omega} = \{(x_1, x_2) : 1 \leq x_1, x_2 \leq 2\},\ k(x) = x_1^2 + x_2^2,\ g(x) = 2(x_1^2 + x_2^2)$ for $x \in \bar{\Omega}$ with boundary conditions and $\bar{u}(x) = 1 + x_1$.

The exact solution for this problem is $u_{exact}(x) = x_1 + x_2$

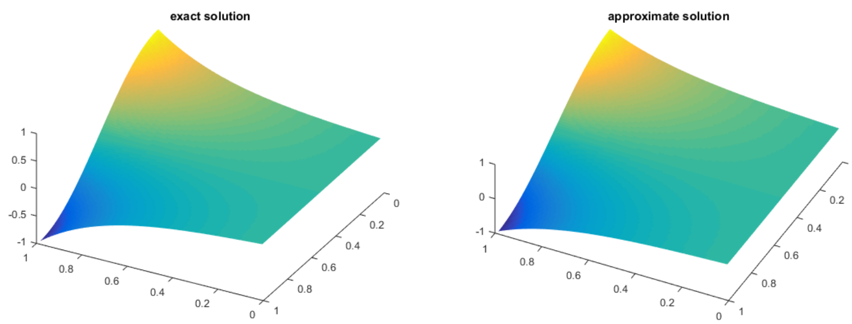

Figure 5.2: Surface plot of solution for Example 2

Chapter 6

Conclusions

In this thesis, we have given brief discussion on heat transfer boundary value problem for PDE with variable coefficients and stated various methods in solving BVPs for PDE numerically. We focused on one of the numerical method in solving BVPs, i.e., Boundary Element Method (BEM). We discussed the formulations of Boundary Domain Integral Equation (BDIE) related with Dirichlet problems and then implement the boundary element method to provide the numerical solution for PDEs with variable coefficients. Greens second identity and parametrix(levi functions) are the concepts used to transform boundary value problems to boundary integral equation. This aim was achieved by studying steady state heat conduction in an isotropic inhomogeneous medium. The behaviour of many modern industrial materials, for instance functionally graded materials, can be mathematically modeled by PDEs with variable coefficients. The solution of PDEs with variable coefficients is therefore important in many practical engineering problems. The application of the BEM to PDEs with variable coefficients is hampered by the need to find appropriate fundamental solutions. It is difficult or impossible to derive an analytic expression for the fundamental solution for general PDEs with variable coefficients, except for some special cases. The ability of finding the fundamental solution for this type of problem has been restricted to only very specific cases of variable coefficients. Even for such simple cases the mathematical procedures are very complicated. As a result, if the fundamental solution cannot be found for PDEs with general types of variable coefficients, domain integrals will remain in the BEM formulation. The method is generally easy to implement to obtain numerical values for particular problems. It can be applied to a wide class of practical problems for inhomogeneous isotropic as a special case. The numerical results obtained using the method indicate that it can provide accurate numerical solutions to particular boundary value problems. Numerical test examples show that accurate computational results can be achieved using BDIE methods. As observed from tables the number of node increase in the boundary element method, the approximate solution approaches to the exact solution.

6.1 Recommendation

We will now suggest some ideas on how this thesis might be modified and extended, including some ideas for future work, Theoretically, it is straightforward to extend the BDIE and BDIDE methods to three-dimensional problems of heat conduction, non-homogeneous Helmholtz and diffusion equations in an isotropic non-homogeneous medium using the fundamental solution for the three-dimensional problems and compare with other numerical methods.

Bibliography

[1] Divo E.A. and Kassab A.J.(2003) Boundary element methods for heat conduction: With applications in non-homogeneous media, WIT Press: Southampton.

[2] Paris F., Canas J.(1997) Boundary element method fundamentals and applications, New York: Oxford University Press.

[3] Attinson, K. E. . The Numerical solution of the two-Dimensional helmholtz equation with variable coefficients by the radial integration boundary integral and integro differential equation methods. International of the second of computer Mathematics, 89(11): 1463-1487.

[4] Carslaw, H . S and Jaeger, J . C. (1959). Conduction of heat in solids. *Oxford: Clarendon Press, 2nd ed.*

[5] Costabel.M, (1988). Boundary integral operators on Lipschitz domains: Elementary results,*SIAM J. Math. Anal.* 19:613-626.

[6] Brebbia C . A and Dominguez, J. (1992). Boundary Elements: An Introduction Course. *WIT press Computational Mechanics Publlications, 2nd editon.*

[7] AL.Jawary and LC Wrobel. (2012). Radial integration boundary integral and integro-differential equation methods for two-dimensional heat conduction problems with variable coefficients.

[8] O. Chkadua, S. E. Mikhailov, and D. Natroshvili.(2013). Analysis of direct segregated boundary-domain integral equations for variable-coefficient mixed BVPs in exterior domains. *Anal. Appl.*, 11(4).

[9] O. Chakuda, S.E. Mikhailov and D. Natroshvili, (2009) Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficent, I: Equivalence and Invertibility. *J. Int. Equat. Appl.*, 21.

[10] C. Constanda,(2000) Direct and Indirect Boundary Integral Equations Methods. Chapman & Hall/CRC,.

[11] E. Stephan,(1987). Boundary integral equations for screen problems in R^3, *Integral Equations Operator Theory* 10, 236-257.

[12] R. Grzhibovskis, S. E. Mikhailov and S. Rjasanow,(2013). Numerics of boundary-domain integral and integro-differential equations for BVP with variable coefficient in 3D,*Computational. Mechanics.* 51.

[13] E. Stephan, (1984). Galerkin Collocation for an Improved Boundary Element Method for a Plane Mixed Boundary Value Problem.

[14] S. E. Mikhailov.(2002). Localized boundary-domain integral formulations for problems with variable coefficients.Eng. Anal. Bound. Elem., 26:681-690.

[15] S.E. Mikhailov,(2005). Localized direct boundary-domain integro-differential formulations for scalar nonlinear BVPs with variable coeffecients. *Jornal of Engineering Mathematics*, 51, 283-302

[16] S.E. Mikhailov,(2006).Analysis of united boundary-domain integral and integro-differential equations for a mixed BVP with variable coeffecients.Meth. Appl. Sci. 29, 715-739.

[17] J. Giroire and J.C. Nedelec,(1978). Numerical solution of an exterior Neumann problem using a double layer potential, *Math. of Comp.* 32 , 973-990.

[18] U. Lamp; T. Schleicher; E. Stephan; W. L. & Wendland, (1984). Galerkin collocation for an improved boundary element method for a plane mixed boundary value problem.

[19] S. E. Mikhailov and N. A. Mohamed.(2012). Numerical solution and spectrum of boundary-domain integral equation for the Neumann BVP with variable coefficient. *Internat. J. Comput. Math.* 89(11).

[20] AL-Jawary MA, Wrobel LC, (2011). Numerical solution of a Dirichlet problem with variable coefficients by the boundary-domain integro-differential equation method. *Eighth UK Conference on Boundary Integral Methods, University of Leeds Press; p. 25-32.*

[21] Paris F., Canas J.(1997) Boundary element method fundamentals and applications, NewYork: Oxford University Press.

[22] Wrobel L.C.(2002) The boundary element method, vol.1, Chichester.

[23] AL-Jawary MA, Ravnik LB, Wrobel LC and Skerget L Boundary element formulations for numerical solution of two-dimensional diusion problems with variable coecients.

[24] M. Costabel.(2007) Some historical remarks on the positivity of boundary integral operators. In Boundary element analysis, Lect. Notes Appl. Comput. Mech., volume 29.

[25] C. Neumann. Untersuchungen uber das logarithmis che und Newtonsche Potential.

[26] Gao XW. A meshless BEM for isotropic heat conduction problems with heat generation and spatially varying conductivity. *International Journal for Numerical Methods in Engineering; 1411-1431.*

[27] Maischak M(2008/2009). Lecture notes in computational mathematics. *Brunel University*;

APPENDIX A

A.1 Mapping the reference triangle to an arbitrary triangle

Given the reference triangle T_r with vertices (0,0), (1,0), (0,1) and a triangle T of the mesh in Ω of vertices p_1, p_2, p_3, the transformation $F\colon T_r \to T$ is given by [**Maischak**]
$F(t) = a + a_1 t_1 + a_2 t_2$
and its coefficients depend on T. Using F, we can write that $p_1 = a, p_2 = a+a_1, p_3 = a+a_2$, and this gives $a = p_1, a_1 = p_2 p_1, a_2 = p_3 p_1$.
Therefore, we get the Jacobian matrix

$$\frac{\partial F}{\partial t} = \begin{bmatrix} \frac{\partial F_1}{\partial t_1} & \frac{\partial F_1}{\partial t_2} \\ \frac{\partial F_2}{\partial t_1} & \frac{\partial F_2}{\partial t_2} \end{bmatrix} = \begin{bmatrix} a_1(1) & a_2(1) \\ a_1(2) & a_2(2) \end{bmatrix}$$

For the determinant we get,

$$\frac{\partial F}{\partial t} = a_1(1)a_2(2) - a_1(2)a_2(1).$$

A.2 Numerical integration of boundary integrals

The standard BEM for two-dimensional problems requires the numerical integration of shape functions, and the product of shape functions with the fundamental solution or its derivative. The Gauss quadrature formula is widely used see for example[**Maischak**]

A.2.1 Mapping reference interval to arbitrary interval

Suppose the reference interval I_r = [-1, 1], and arbitrary interval $I = [p_1,\, p_2]$, the transformation $\tilde{D}\colon I_r \to I$ is given by

$$\tilde{D}\colon = a + a_1 t$$

where $a = \frac{p_1 + p_2}{2}$, and $a_1 = \frac{p_2 - p_1}{2}$. Therefore we get the Jacobian, $|\frac{d\tilde{D}}{dt}| = |a_1|$
Therefore, to find the value of one boundary integral given in chapter 4 by the Gauss quadrature formula:

$$\int_{\partial\Omega} \bar{u}(x)T_x p(x,x^i)d\Gamma(x) = \sum_{s=1}^{2} \int_{\Gamma_s} \sum_{m=k}^{2} \bar{u}(x)\tilde{D}(t)T_x p(\tilde{D}(t),x^i)|a_1|w(m)dt$$

where N is the number of boundary elements, K is the number of Gauss points and w are the Gauss weights. In our work, we used an eight points Gaussian quadrature rule to get sufficiently accurate results.

A.3 Numerical integration of domain integrals

Applying the BEM for problems with internal sources requires a domain discretization into a number of triangular elements, and then numerical integration over such triangles. Since an affine transformation makes it possible to transform the reference triangle T_r to any triangle T, we have just to consider numerical integration on T_r as discussed in subsection B.2. The integral of an arbitrary function f, over the reference triangle T_r is given by

$$\iint_{T_r} f(t_1,t_2)dt_1dt_2 = \int_0^1 dt_1 \int_0^{0-t_1} f(t_1,t_2)dt_1dt_2 = \int_0^1 dt_2 \int_0^{0-t_2} f(t_1,t_2)dt_1dt_1$$

Therefore, in order to find the value of the integral we use Gauss quadrature formula The quadrature formula for numerical integration of the integral in above Eqns, can be written in the standard form

$$I = \sum_{m=1}^{k} w(m)f(t_1(m),t_2(m)),$$

where $w(m)$ are the weights associated with specific points$(t1(m),t2(m))$ and K is the number of points.

Matlab code

Matlab program for example 1

```
clear;
clc;
for i=1:4
r=2^i;
N =2*r;
end
for j = 1:99
y(j) = 0.01*j;
```

```
for i = 1:99
x(i) = 0.01*i;
end
end
for i=1:r
xb(i)=1+((i-1)/r);
yb(i)=0;
bt(i)=1;
bv(i)= 0;
end
for i=1:4*r
x1=(pi)/(8*r);
xb(i+r)=2*cos((i-1)*x1);
yb(i+r)=2*sin((i-1)*x1);
bt(i+r)=0;
mx=2*cos(((i+0.5)-1)*x1);
my=2*sin(((i+0.5)-1)*x1);
bv(i+r)=3*cos(4*atan(my/mx));
end
for i=1:r
xb(i+5*r)= 0;
yb(i+5*r)=2-((i-1)/(r));
bt(i+5*r)=1;
bv(i+5*r)=0;
end
for i=1:2*r
x2=(pi)/(4*r);
xb(i+6*r)=sin((i-1)*x2);
yb(i+6*r)=cos((i-1)*x2);
bt(i+6*r)=0;
mx=sin(((i+0.5)-1)*x2);
my=cos(((i+0.5)-1)*x2);
bv((i+6*r))=cos(4*atan((my)/(mx)));
xb(8*r+1)=xb(1);
yb(8*r+1)=yb(1);
bt(8*r+1)=bt(1);
```

```
bv(8*r+1)=bv(1);
end
n = 64;
for i = 1:n
xm (i) = 0.5*(xb (i) + xb(i + 1));
ym (i) = 0.5*(yb(i) + yb(i + 1));
lm(i) = sqrt((xb(i + 1) - xb(i))^2 + (yb(i + 1)- yb(i))^2);
nx(i) = (yb(i + 1) - yb(i))/lm (i);
ny(i) = (xb(i) - xb (i + 1))/lm(i);
end
for m = 1:n
b(m) = 0;
for k = 1:n
if(k == m)
G = 0.0;
F = lm(k)/(2.0*pi)*(log(lm(k)/2.0) - 1.0);
del = 1.0;
else
[F, G] =findfg (xm(m), ym(m), xb(k), yb(k), nx(k), ny(k),lm(k));
del = 0.0;
end
if (bt (k) == 0)
A (m, k) = -F;
b(m) = b(m) + bv(k)*(- G + 0.5*del);
else
A(m, k) = G -0.5*del;
b(m) = b(m) + bv(k)*F;
end
end
z=A\b';
end
for m = 1:n
u (m) = (1 - bt (m))*bv (m) + bt (m)*z(m);
q(m) = (1 - bt (m))*z(m) + bt(m)*bv (m);
end
for j = 1:99
```

```
y(j) = 0.02*j;
for i = 1:99
x(i) = 0.02*i;
y1=x(i);
y2=y(j);
s1(j,i) = 0;
for k = 1:n
if ((y1.^2+y2.^2)>1)&&((y1.^2+y2.^2)<4)
[F, G] =findfg (x (i), y (j), xb (k), yb (k), nx (k), ny (k),lm(k));
s1(j,i) = s1(j,i) + u(k)*G - q(k)*F;
else s1(j,i)=0;
end
z1(j,i)=0;
if ((y1.^2+y2.^2)>1)&&((y1.^2+y2.^2)<4)
e1=(16/85*((y1.^2+y2.^2)^2-(1/(y1.^2+y2.^2)^2))-(16/255)*(((y1.^2+y2.^2
-(16)/(y1.^2+y2.^2)^2)).*cos(4*atan(y2/y1));
z1(j,i)=z1(j,i)+(e1);
else z1(j,i)=0;
end
end
end
end
d1=z1-s1;
figure(1)
surface (x, y, z1,'EdgeColor','none')
title('Exact solution')
figure(2)
surface (x, y, s1,'EdgeColor','none')
title('Approximate solution')
```

When $k \neq m$, the integrals can be evaluated by using numerical methods such as Gauss quadrature which can be achieved by calling a function quadl and the code is:

```
function [F,G] =findfg(xi, eta, xk, yk, nkx, nky, lk)
F = (lk/(4.0*pi))*quadl(@(t) intf (t, xi, x, xk, yk, nkx, nky, lk),-1,
```

```
G = (lk/(2.0*pi))*quadl(@(t) intg (t, xi, x, xk, yk, nkx, nky,lk),-1, 1
```

where intf and intg are defined respectively as:

```
function y = intf (t, xi, x, xk, yk, nkx, nky, lk)
y = log ((xk - t*lk*nky - xi).^2 + (yk + t*lk*nkx - x).^2);
end
function y = intg(t, xi, eta, xk, yk, nkx, nky, lk)
y = (nkx*(xk -t*lk*nky - xi) + nky*(yk + t*lk*nkx - eta))./((xk ...
- t*lk*nky -xi).^2 + (yk + t*lk*nkx-eta).^2);
end
```

Matlab program for example 2

```
clear;
clc;
for i=1:4
r=2^i;
N(i)=4*r;
end
for j = 1:99
y(j) = 0.01*j;
for i = 1:99
x(i) = 0.01*i;
end
end
for i=1:r
xb(i)=1+((i-1)/r);
yb(i)=0;
bt(i)=1;
bv(i)= 0;
end
n = 4*r;
for i = 1:n
xm (i) = 0.5*(xb (i));
ym (i) = 0.5*(yb(i));
lm(i) = sqrt((xb(i + 1) - xb(i))^2 );
nx(i) = (yb(i + 1) - yb(i))/lm (i);
```

```
ny(i) = (xb(i) - xb (i + 1))/lm(i);
end
for m = 1:n-1
b(m) = 0;
for k = 1:n
if(k == m)
G = 0.0;
F = lm(k)/(2.0*pi)*(log(lm(k)/2.0) - 1.0);
del = 1.0;
else
[F, G] =findfg (xm(m), ym(m), xb(k), yb(k), nx(k), ny(k),lm(k));
del = 0.0;
end
if (bt (k) == 0)
A (m, k) = -F;
b(m) = b(m) + bv(k)*(-G + 0.5*del);
else
A(m, k) = G -0.5*del;
b(m) = b(m) + bv(k)*F;
end
end
z=A\b';
end
for m = 1:n
u(m) = (1 - bt (m))*bv (m) + bt (m)*z(m);
q(m) = (1 - bt (m))*z(m) + bt(m)*bv (m);
end
for j = 1:99
y(j) = 0.01*j;
for i = 1:99
x(i) = 0.01*i;
y1=x(i);
y2=y(j);
s1(j,i) = 0;
for k = 1:n
if (y1^2+y2^2<1)
```

```
[F, G] = findfg (x (i), y (j), xb (k), yb (k), nx (k), ny ...
(k),lm(k));
s1(j,i) = s1(j,i) + u(k)*G - q(k)*F;
else s1(j,i)=0;
end
z1(j,i)=0;
if (y1^2+y2^2<1)
z1(j,i)=z1(j,i)+(y1+y2);
else z1(j,i)= 0;
end
end
end
end
d1=z1-s1;
subplot(2,2,1); surface (x, y, z1,'EdgeColor','none')
title('Exact solution')
subplot(2,2,2); surface (x, y, s1,'EdgeColor','none')
title('Approximate solution')
```