MODELING THE EFFECTS OF GLOBAL CLIMATE CHANGE AND DECREASING FOREST COVER ON SEASONAL RAINFALL

By

Zenebe Shifaraw

A Thesis Submitted to

The Program of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

Adama, Ethiopia
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Abstract

In this research work we develop a mathematical model that describe the effects of global climate change and decreasing of forest cover on seasonal rainfall. Here, we assume that global temperature, the amount of forest cover and the amount of rainfall are related. A non linear differential equation model was developed and analyzed using the stability theory of ordinary differential equations. Both local and global stability was studied using linearizing and lyapunov function approach respectively. The rise in global temperature was represented by energy balance equation and the decline in forest area was represented by logistic equations. Rainfall is, however, represented as a periodic function; hence, second order differential equation, of which the solution is periodic, is used to represent the rate of change in the amount of rainfall. From the obtained results, it is inferred that the decreasing in the amount of forest cover and the rise in global temperature affects the fluctuation of rainfall. Furthermore, the amount of rainfall is directly depend on the amount of forest cover. Numerical simulations are carried out to verify the analytical results and confirm this results. We also concluded that in order to avoid the fluctuation of rainfall, efforts should be made to limit the deforestation and releasing green house gases into atmosphere.

Key Words: *Modeling, Climate Change, Global mean temperature, Forest, Rainfall*

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Adama Science and Technology University Zenebe Shifaraw
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Notations and Abbreviations

EBM	Energy Balance Model
GHGs	Green House Gases
IA	Integrated Assessment
IVP	Initial Value Problem
IPCC	Intergovernmental Panel on Climate Change
ODE	Ordinary Differential Equation
OLR	Outgoing Longwave Radiation
UN	United Nations
T	Global mean temperature
F	Amount of forest cover
R	Amount of rainfall
S	The rate of change of rainfall

Chapter 1

Introduction

Human beings have become a component in the earth's system, driving and accelerating global warming through the rapid release of green house gases (GHGs) into the atmosphere. Climate change is a major concern affecting all organisms. Over the past 30 years, global surface temperature has been rising approximately 0.2^0C per decade [1]. Severe weather conditions such as drought and flood have been known to result from high global temperature. Global climate change affects the fluctuations in rainfall patterns. The amount of greenhouse gases (especially carbon dioxide) in the atmosphere depends inversely on the amount of forest area. Today, forest cover has decreased from the past due to timber demands for building and manufacturing [2] and to land usage for agriculture.

Forests may be divided into three main groups; boreal forests, temperate forests and tropical rain forests [3]. Boreal forests are dominated by Larch which is deciduous, and drops its needles in winter. Temperate forests have a number of sub-classes and include both deciduous trees (which lose their leaves during winter) and evergreens. Tropical rain forests are characterized by very high annual rainfall and warm temperatures. The climate within tropical forests is stable, and humidity levels are usually very high.

Analysis of climate change and its extreme is becoming more important as it clearly affect the human society and are essential for exploration of ecological and societal changes. Climate change is an irreversible consequence of the global warming phenomenon. Global warming, or the greenhouse effect, has been brought about by an increase in greenhouse gases (GHG) in the atmosphere [4]. Climate change will impact on bio-physical systems and ultimately will have consequences to human well being. Understanding climate change is fundamental to prepare and to cope with future risk. Climate change is not uniform over space and time and its impact on bio-physical system varies from place to place. Therefore, it is necessary to understand climate change at the local scale and aim to get site-specific information. Climate change, which is induced by global warming, has become a global concern because it has the potential to impact many systems and sectors which would threaten human well being.

The El Nino phenomenon has played an important role on the behavior of rainfall in different areas of the world; it is also a major cause of extreme weathers in many regions of the world [5]. Changes in global temperature, therefore, lead to variations in the amount of rainfall in different areas of the world, an amount which depends on the level of forest transpiration. Deforestation is the single largest cause of forest area diminution, even compared to natural disasters such as wildfire. Evapotranspiration from the forest acts to reduce surface temperatures. After deforestation, evaporation will be reduced, meaning more energy is emitted from the ground as sensible heat instead of latent heat, which acts to warm the surface. The amount of rainfall is an important indicator for severe weather conditions such as drought and flood.

As presented in [3], in all forest types, a warming climate could promote the frequency and severity of pest outbreaks, which will damage the health of the forest and may reduce the numbers of certain species, allowing others to increase in numbers. Forest fires may become more frequent and larger in extent, particularly if rising temperatures are accompanied by an increase in droughts. The species composition of the forests in boreal and temperate regions could be changed by warmer temperatures, where cold-adapted species could be replaced by those trees which occur in a warmer climate, or drought-resistant species could increase in number. The impact of climate change on forests in large scale circulation could reduce rainfall and cause dieback of the forest. Dieback of the tropical forests would release large amounts of carbon (as CO_2) into the atmosphere, amplifying the warming [6]. The evaporation from the surface would also be smaller, resulting in reduced cloud amounts and rainfall and increased surface warming.

Forests play a vital role in the functioning of the earth's natural systems. The forests regulate local and global weather through their absorption and creation of rainfall and their exchange of atmospheric gases. Cutting the rainforests changes the reflexivity of the earth's surface, which affects global weather by altering wind and ocean current patterns, and changes rainfall distribution [7]. If the forests continue to be destroyed, global weather patterns may become more unstable and extreme. Rainfall is also affected when forest clearing fires create air pollution and release tiny particles, known as aerosols, into the atmosphere. The rainfall of a place varies from year to year, both in total amount and in distribution over a year. In addition, seasonal rainfall affects not only agriculture but also other industries such as river transportation and tourism.

1.1 Statement of the Problem

Changes in global climate would significantly affect human health, natural aquatic and terrestrial ecosystems, and agricultural ecosystems. World-wide attention recently has turned to these issues and scientists from many disciplines and many countries are working to assess the potential magnitude and direction of the changes and the risks to the biota. This study is intended to answer the following basic questions:

1. How can we formulate a mathematical model that describe the relationship between the dynamics of global temperature, forest area and amounts of rainfall?

2. What are the impacts of the rise in global temperature and decline in forest area on fluctuation of seasonal rainfall?

3. What are the ecological implication of the interactions between global temperature, forest area and rainfall?

1.2 Objective of the Study

1.2.1 General Objective:

The general objective of this study is to develop a mathematical model that describe the effects of global climate change and decreasing in forest cover on seasonal rainfall.

1.2.2 Specific Objectives:

The specific objectives of this study are to:

- develop a mathematical model that describe the relationship between global climate change and decreasing forest area on seasonal rainfall,
- study the effects of the rise in global temperature and decline of forest area on the fluctuation of seasonal rainfall,
- investigate the ecological implication of interactions between global temperature, forest area and rainfall.

1.3 Methodology

In this study first we develop a mathematical model which describes the effects of global climate change and decreasing forest cover on seasonal rainfall. We developed a system of non-linear ordinary differential equation. We study the formulated governing mathematical model analytically to understand the physical meaning of the governing equation. Both local and global stability analysis of the equilibrium points of the model equations was done using Jacobian matrix and Lyapunov functions respectively and also the boundedness was studied. The analytical solutions of the model equations was supplemented by numerical simulations by appropriately choosing the system parameter values using MATLAB.

1.4 Significance and Beneficiaries

This study provides a reliable information on how we can use the mathematical model to know the effects of global temperature and decreasing forest cover on seasonal rainfall. It also improve our understanding of the effects of climate change. The outcome of the study benefits all the policy makers environmentalists to develop awareness about climate change. Also, initiates other researchers to undertake further extension and rigorous mathematical analysis on climate change.

1.5 Expected Outcomes

The expected outcomes of the proposed study are:

- A mathematical model that describe the effects of global climate change and its consequences,
- Propose cost effective climate change prevention strategies,
- Increase peoples degree of participation to prevent and reduce the expansion of global climate change.

1.6 Limitation and Delimitation of the Study

Models of climate changes are still evolving because we do not yet completely understand or model everything that can or will affect climate. Scientific uncertainty will always be a component of modeling climate change. Our challenge is to reduce this uncertainty. Due to the lack of registered data for global mean temperature in our country, the proposed model is not validated. Further, the shortage of time is another factor to validate the developed models as expected. This study doesn't not focus directly on meteorology, socio-economic impacts of climatic changes, although all of these disciplines and many others will be drawn upon in our descriptions.

1.7 Mathematical Preliminaries

In this chapter, we state some important theorems and give the definitions of terminologies which are most crucial for the thesis as an input from existing literature.

1.7.1 Existence and Uniqueness Theorem

Before one spends much time attempting to solve a given differential equation, it is wise to know that solutions actually exist. We may also want to know whether the solution is uniquely satisfy the equation with the given initial condition. We consider here the autonomous system in R^n i.e, a collection of equations that do not explicitly contain the independent variable. More generally, autonomous systems have the form

$$x' = f(x) \tag{1.1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$f = \begin{bmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_n(x_1, x_2, ..., x_n) \end{bmatrix}$$

The following theorem and its prove can be found in [22].

Theorem 1.7.1. *(Existence and Uniqueness Theorem): Assume D is an open subset of $R \times R^n$, $f : D \to R^n$ is a continuous function. Then for each $t_0 \in R$ and $x_0 \in R^n$, the IVP*

$$x' = f(t, x), \ \ x(t_0) = x_0 \tag{1.2}$$

has a solution x. If in addition, f has continuous first order partial derivatives with respect to $x_1, x_2, ..., x_n$ on R^n, then the above IVP has a unique solution.

Definition 1.7.1. Let f be a real valued function defined on a domain D. The function f is said to be **bounded** on D if and only if there is a positive number M such that $|f(x, y)| \leq M$ for all $(x, y) \in D$.

Definition 1.7.2. A solution $\varphi : t \to \varphi(t)$ of $x' = f(x)$ is called **periodic** with period T or T-periodic if $\varphi(t + T) = \varphi(t)$ for all t. If, moreover, $\varphi(t + \tau) \neq \varphi(t)$ for any $\tau \in (0, T)$, then T is called the minimal period.

Definition 1.7.3. Suppose that the function $f(x)$ is twice continuously differentiable, and that x_0 is an equilibrium point of $\frac{dx}{dt} = f(x)$, so that $f(x_0) = 0$. The linear system $\frac{dx}{dt} = J(x_0)(x - x_0)$ is the **linearization** of f at x_0, where $J(x_0)$ is the $n \times n$ matrix of partial derivatives of f, $J = (\partial f_i / \partial x_j)^n$, evaluated at x_0.

Definition 1.7.4. A domain $\Omega \subseteq \Re^n$ is said to be **positively invariant** for the system $\frac{dx}{dt} = f(x)$ if for all $x(t_0) \in \Omega$, then $x(t) \in \Omega$, for all $t \geq t_0$.

1.7.2 Equilibrium Solutions and Stability

For the following basic points we may rely on [22, 23, 24, 25] and references there in. Consider the following autonomous first order ordinary differential equation one in which the independent variable t does not appear explicitly.

$$\frac{dx}{dt} = f(x) \tag{1.3}$$

The solutions of the equation $f(x) = 0$ play an important role and are called **critical points** of the autonomous differential equations (1.3). If $x = c$ is a critical point of (1.3), then the differential equation has the constant solution $x(t) \equiv c$. A constant solution of a differential equation is sometimes called an **equilibrium solution** (one may think of a population that remains constant because it is in "equilibrium" with its environment). Thus the critical point $x = c$, a number, corresponds to the equilibrium solution $x(t) \equiv c$, a constant valued function.

Definition 1.7.5. Stability of critical points
A critical point $x = c$ of an autonomous first order ordinary differential equation is said to be *stable* provided that, if the initial value x_0 is sufficiently close to c, then $x(t)$ remains close to c for all $t > 0$. More precisely, the critical point c is **stable** if, for each $\epsilon > 0$,there exists $\delta > 0$ such that

$$|x_0 - c| < \delta$$

implies that $|x(t) - c| < \epsilon$ for all $t > 0$. The critical point $x = c$ is **unstable** if it is not stable.

Theorem 1.7.2. *(Routh-Hurwitz Criteria)*
Given the polynomial,

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n,$$

where the coefficients a_i are real constants, $i = 1, ..., n$, define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial:

$$H_1 = (a_1), \; H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \; \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix},$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & ... & 0 \\ a_3 & a_2 & a_1 & 1 & ... & 0 \\ a_5 & a_4 & a_3 & a_2 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & 0 & ... & a_n \end{pmatrix},$$

where $a_j = 0$ if $j > n$. All of the roots of polynomial $p(\lambda)$ are negative or have negative real part iff the determinants of all Hurwitz matrices are positive:

$$det\ H_j > 0,\ j = 1, 2, ..., n.$$

When $n = 2$, the Routh-Hurwitz criteria simplify to det $H_1 = a_1 > 0$ and

$$det\ H_2 = det \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} = a_1 a_2 > 0$$

or $a_1 > 0$ and $a_2 > 0$. For polynomials of degree $n = 2, 3, 4$ and 5, the Routh-Hurwitz citeria are summarized as follows.
$n = 2 : a_1 > 0$ and $a_2 > 0$.
$n = 3 : a_1 > 0,\ a_3 > 0,$ and $a_1 a_2 > a_3$.
$n = 4 : a_1 > 0,\ a_3 > 0,\ a_4 > 0,$ and $a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$.
$n = 5 : a_i > 0; i = 1, 2, 3, 4, 5,\ a_1 a_2 a_3 > a_3^2 + a_1^2 a_4,$ and
$(a_1 a_4 - a_5)\ (a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) > a_5 (a_1 a_2 a_3)^2 + a_1 a_5^2$.

The proof of the above theorem can be found in [23] and references there in.

Definition 1.7.6. Let x_0 be an equilibrium point for the autonomous systems $x' = f(x)$. A continuously differentiable function V defined on an open set $U \subset R^n$ with $x_0 \in U$ is called a Lyapunov function for $x' = f(x)$ on U provided $V(x_0) = 0$, $V(x) > 0$ for $x \neq x_0$, $x \in U$, and

$$grad\ V(x).f(x) \leq 0, \tag{1.4}$$

for $x \in U$. If the inequality (1.4) is strict for $x \in U, x \neq x_0$, then V is called a strict Lyapunov function for $x' = f(x)$ on U.

Note that (1.4) implies that if $x \in U$, then

$$\frac{d}{dt} V(\phi(t, x)) = grad\ V(\phi(t, x)).f(\phi(t, x)) \leq 0,$$

as long as $\phi(t, x)$ remains in U, so V is decreasing along orbits as long as they stay in U.

The proves of the following two theorems can be founded in [25] and references there in.

Theorem 1.7.3. *(Lyapunov's Stability Theorem) If V is a Lyapunov function for the autonomous systems $x' = f(x)$ on an open set U containing an equilibrium point x_0, then x_0 is stable. If V is a strict Lyapunov function, then x_0 is asymptotically stable.*

Theorem 1.7.4. *Let $x_0 = 0$, ($x_0 \in R^n$) be an equilibrium point of the system $\frac{dx}{dt} = f(x)$, ($x \in R^n$). Let $V : R^n \to R$ be continuously differentiable function such that*

(1) $V(x_0) = 0$

(2) $V(x) > 0$, *for all* $x \neq x_0$

(3) $\dot{V}(x) < 0$, *for all* $x \neq x_0$

(4) $V(x) \to \infty$ *as* $||x|| \to \infty$,

then the equilibrium point x_0 is a globally asymptotically stable.

1.7.3 Logistic Equation

The following description of logistic equation is obtained from [24, 29] and references there in. In their beginnings, before environmental limitations become significant, populations will grow in an almost exponential fashion. As time goes on, the population growth rate will slow, in a manner similar to limited growth functions, until the size of the population reaches an equilibrium, a maximum population size that is sustainable in a given environment. A logistic growth function incorporates both of these aspects. Assuming a maximum population size K, rate of change of population $P(t)$ is proportional to $P(t)$ and difference between equilibrium amount and $P(t)$.

$$\frac{dP}{dt} = \frac{a}{K}P(K - P) \tag{1.5}$$

In a population scenario, rate of growth is proportional both to the number in the current population and to the limitations imposed by environmental factors such as availability of food and the presence of natural predators. In this model, a is the *intrinsic rate of growth* (rate of growth if not limited by outside factors) and K is called the *carrying capacity* (maximum sustainable population).

Distributing $\frac{1}{K}$ gives the logistic growth model,

$$\frac{dP}{dt} = aP(1 - \frac{P}{K}) \tag{1.6}$$

At the beginning, when P is near 0, $\frac{dP}{dt} \approx a(1)P = aP$ (exponential growth) and as $P \longrightarrow K$, $\frac{dP}{dt} \approx a(1-1)P = 0$ (like the equilibrium stage of the limited growth model).

We can rewrite equation (1.6) as

$$\frac{dP}{dt} = aP - bP^2 \tag{1.7}$$

where $a > 0, b = \frac{a}{K} > 0$.

The solution of the logistic initial value problem

$$\frac{dP}{dt} = aP - bP^2, P(t_0) = P_0 \tag{1.8}$$

is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-a(t-t_0)}} \tag{1.9}$$

One method of solving (1.8) is separation of variables. This solution represents the logistic function and it has an upper limit of

$$\lim_{t\to\infty} P(t) = \frac{aP_0}{bP_0 + 0} = \frac{a}{b}.$$

Thus a population that satisfies the logistic equation does not grow without bound like a naturally growing population modeled by the exponential equation $P' = kP$. Instead, it approaches the finite limiting population $\dfrac{a}{b}$ as $t \to \infty$.

Chapter 2

Literature Review

The first global models (models of world dynamics) were developed in the 1970's. These models produced mostly qualitative results and represented first systematic attempts to analyze global trends. These models evolved into a more quantitative integrated assessment(IA) approach in global modeling [10].

As presented in [12], the Intergovernmental Panel on Climate Change (IPCC) was jointly established by the world meteorological organization and the united nations environment programme in 1988 in order to:

(i) assess available scientific information on climate change,

(ii) assess the environmental and socio-economic impacts of climate change, and

(iii) formulate response strategies.

The IPCC first assessment report was completed in 1990 and served as the basis for negotiating the UN framework convention on climate change. It also completed its 1992 supplement and "climate change 1994: radiative forcing of climate change and an evaluation of the emission scenarios"to assist the convention process further. Its first assessment report of 1990 concluded that continued accumulation of anthropogenic greenhouse gases in the atmosphere would lead to climate change whose rate and magnitude were likely to have important impacts on natural and human systems [12].

The IPCC supplementary report of 1992, timed to coincide with the final negotiations of the united nations framework convention on climate change, added new quantitative information on the climatic effects of aerosols but confirmed the essential conclusions of the 1990 assessment concerning our understanding of climate and the factors affecting it. The 1994 report examined in depth the mechanisms that govern the relative importance of human and natural factors in giving rise to radiative forcing, the "driver"of climate change. This report incorporated further advances in the quantification of the climatic effects of aerosols, but it also found no reasons to alter in any fundamental way those conclusions of the 1990 report which it addressed [12].

The dynamical description of climate model is given for example in [14, 13]. The mathematical model proposed in [13] is extend to a dynamical climate change model in [14] as follows. Global climate is determined by the radiation balance of the planet. The Earth warms through the absorption of incoming solar radiation (or insolation). For the global climate model, the variable of interest is the annual global mean surface temperature $T = T(t)(^oC)$. The annual global mean insolation is known as the solar constant Q, with current value $Q = 343\ W/m^2$. The incoming energy absorbed by the Earth is then modeled by the term $Q(1-\alpha)$, where α is the average global insolation reflected back into space. The loss of energy is modeled by a linear approximation $A + BT$, with parameters A and B estimated from satellite measurements to be $A = 202\ W/m^2$ and $B = 1.9\ W/(m^2\ ^oC)$. Finally the model equation for the annual global mean temperature is then given in [14] by

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

where the left-hand side of this equation represents the change in energy stored in the Earth's surface. The parameter R is the heat capacity of the Earth's surface, with units $J/(m^2\ ^oC)$. The units on each side of the equation are those of energy, namely W/m^2 (equivalently, $J/(sm^2)$).

The mathematical model that describe the competition among the various diversity of rain forest was introduced in [15] using the Lotka-Volterra predator-prey model. In [16] a mathematical model that describe the dynamic of the relationship between biomass, industrialization, population, pollution, and pollution released by the biomass resources was written in the form of a system of differential equations and found that the resource biomass decreases as the density of industrialization, population and pollution increases. In [17] found that the forestry resources lead to extinction on the account of population and industrialization driven by population pressure. In these study the control measures for maintaining the ecological stability was discussed.

In [18] a mathematical model that compute the crucial roles of water vapor in global warming by expressing the temperature T as a function of carbon dioxide CO_2, that is $T(t) = f(C)$, where C denotes CO_2, was developed. The developed differential equation was,

$$\frac{dT}{dt} = \frac{dT}{dC}\frac{dC}{dt} = k\frac{dC}{dt}$$

The term $\frac{dT}{dC}(= k)$ denotes the change of temperature per unit change in CO_2 concentration in the atmosphere. This model of global warming suggests that the water vapor has greatest contribution in increasing earth temperature.

In [19] a mathematical model of forestry resources conservation considering wood based industries and synthetic industries was studied. In this study it is shown that forestry resources are being depleting through wood based industries and also discussed the conservation of forestry resources through the replacement of wood industries by synthetic industries. But, the negative effects of synthetic industries on the forestry resources are not considered. On the other hand in [20] a mathematical model that study the simultaneous effect of pollutants emitted from wood and non-wood based industries on forest resources was proposed and it concluded that wood-based industries deplete the forest resources both directly (through harvesting) and indirectly (through pollutants), but that non-wood industries only deplete the forest resources indirectly (through pollution).

A mathematical model that simultaneously represents the relationship between global temperature, amount of forest area and amount of rainfall was proposed in [21] using a system of non linear ODE. The governing equation in the model was

$$\frac{dT}{dt} = a_T(1 - \frac{T - m_T}{b_T})(T - m_T) \tag{2.1}$$

$$\frac{dF}{dt} = -a_F(1 - \frac{F - m_F}{b_F})(F - m_F) \tag{2.2}$$

$$\frac{dR}{dt} = S \tag{2.3}$$

$$\frac{dS}{dt} = -e_S^2(R - c) \tag{2.4}$$

where the model variables and parameters are defined as $T(t)$ global temperature at time t, $F(t)$ the amount of forest area at time t, $R(t)$ the amount of rainfall at time t, S the rate of change of rainfall, a_T rate of temperature, m_T minimum global temperature, a_F rate of forest area, m_F minimum forest area, b_T and b_F are the difference between equilibrium points, e_s the period of the predicted solution , c parameter for shifting the predicted solution. The stability analysis of all the models were conducted. The model has the problem of the cross products or predator-prey terms between the rate of change in forest area and global temperature, the rate of change in rainfall and global temperature. Finally this model was modified based on statistical data with high correlation coefficients as follows in [21].

$$\begin{aligned}
\frac{dT}{dt} &= a_T(1 - \frac{T - m_T}{b_T})(T - m_T) \\
\frac{dF}{dt} &= -a_F(1 - \frac{F - m_F}{b_F})(F - m_F) + e_F R + \alpha_F FR + \beta_F TF - \pi_F TR \\
\frac{dR}{dt} &= f_R S - \alpha_R FR - \pi_R TR + \gamma_R FS + \mu_R TS + \delta_R RS \\
\frac{dS}{dt} &= -e_S^2(R - c) + \beta_S TF + \pi_S TR - \gamma_S FS - \mu_S TS - \delta_S RS
\end{aligned} \tag{2.5}$$

where the new parameters e_F the rate of change in forest area caused by the amount of water (rainfall) absorption, α_F growth rate of the forest affected by the proportion of rainfall in the forest, α_R is the rate of decrease that indicate some amount of rainfall remaining in the forest, γ_R and γ_S are both the difference rates representing the changing behavior of the rain falling in the forest, π_R and π_S are the decreasing and increasing rainfall difference rates, respectively, affecting by the proportion of rainfall at each temperature level and FR the proportion of the amount of rainfall in the forest and FS is the proportion of the rainfall difference rate (or the rate of change in rainfall) in the forest, TF the proportion of the forest cover at each temperature level. They concluded that the rainfall difference rate is affected by the proportion of rainfall at each temperature level

However, due to large number of variables and parameters, the mathematical analysis of the models was not studied in any of the reviewed literature. Thus from mathematical point of view, the proposed model is not tractable. That is it is difficult to study basic properties like stability analysis of this mathematical model. Here, we propose and analyze a non-linear mathematical model to study the effects of global climate change and decreasing forest cover on seasonal rainfall.

Chapter 3

Model Formulation and Analysis

In this chapter, we present model description and formulation to complement and extend the work in [21]. The mathematical theory of climate is a branch of the theory of climate, which investigates the behavior of the climate models solutions on the arbitrarily large time scales by the use of a collection of mathematical methods. This research work relies on the hypothesis that changes in global mean temperature, forest area and amount of rainfall are related. Using the stability theory of differential equations, the developed model has been analyzed qualitatively.

3.1 Model Assumptions and Definition of Variables

3.1.1 The model assumptions

The following assumptions are made in order to construct the model:

(i) The temperature of the earth receives incoming insolation (solar energy) but radiates some of it back into space.

(ii) The forest area cannot decrease below zero and cannot be larger than the fixed area, which is an example of a logistic decay.

(iii) The amount of rainfall is apparently seasonal and therefore periodic. The second order differential equation with periodic solution should be considered to capture the seasonal rainfall.

3.1.2 The model variables

The state variables of the model are:

(i) $T(t)$ - the global average temperature in the Earth's photosphere at time t.

(ii) $F(t)$ - the amount of forest at time t.

(iii) $R(t)$ - the amount of rainfall at the time t.

3.2 Model Formulation

From the model description, assumptions and definition of variables in section (4.1), and following the model given in [30], the global mean temperature T can be modeled by the energy balance equation (EBM).

$$C\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4. \tag{3.1}$$

In this model, the first term on the right is incoming heat absorbed by the Earth and its atmosphere system. The second term is heat radiating out as if the Earth were a blackbody with all of the outgoing longwave radiation (OLR) escaping to space.

- $T(K, kelvins)$ is the average temperature in the Earth's photosphere (upper atmosphere, where the energy balance occurs in this mode) $(1kelvin = 1^oC)$;
- t (years) is time;
- $C(W - yr/m^2K)$ is the averaged heat capacity of the Earth/ atmosphere system (heat capacity is the amount of heat required to raise the temperature of an object or substance 1 kelvin $(= 1^oC)$), where $Wyrm^{-2}$ is watt-years per square meter;
- $Q(W/m^2)$ is the annual global mean incoming solar radiation (or insolation) per square meter of the Earth's surface;
- α (dimensionless) is planetary albedo (reflectivity), and
- $\sigma(W/m^2K^4)$ is a constant of proportionality, the Stefan-Boltzmann constant.

Note that (3.1) is an autonomous ordinary differential equation (ODE), meaning that the expression for the derivative does not explicitly involve the independent variable t. Values for the parameters are:

$$C = 2.912\ W - yr/m^2K,\ Q = 242\ W/m^2,\ \alpha = 0.30\ \ [28,\ 30]$$

and $\sigma = 5.67 \times 10^{-8}\ W/m^2K^4$.

The dynamics of forest area are assumed by a logistic equation instead of an exponential equation [20]

$$\frac{dF}{dt} = -r(1 - \frac{F}{L})F \tag{3.2}$$

where r and L are the decreasing rate and carrying capacity of the forest resources respectively. The amount of periodic rainfall will be represented by the following second order differential equation [21]

$$\frac{d^2R}{dt^2} = -e_S^2(R - c) \tag{3.3}$$

Since the other two variables can be represented by first order differential equations, this second order equation representing the behavior of rainfall will be reduced to two first order differential equations. Introducing an intermediate variable S as the rate of change of rainfall, the amount of periodic rainfall will be represented by the following system of first order differential equations.

$$\begin{aligned} \frac{dR}{dt} &= S \\ \frac{dS}{dt} &= -e_S^2(R - c) \end{aligned} \tag{3.4}$$

where e_s is the period of the predicted solution and c is the parameter for shifting the predicted solution.

Today a global climate change with a continuously rise in global average temperature with a continuous decline in forest cover. Two factors have a direct influence on rainfall fluctuations. As the forest area have been decreased exponentially from the past and since the forest area cannot decrease below zero and cannot be larger than the maximum area of the studied region, we use the logistic equation in which the solution can represent the behavior of the forest. A mathematical model representing the relationship between global temperature, forest area and seasonal rainfall is studied in the form of a system of first order differential equations [21] with the correlation coefficient (r) equal to 0.9106, 0.9743 and 0.9669, respectively without any analysis using the stability theory of DEs.

In this thesis we propose a first order differential equation whose solutions for annual amount of global average temperature and forest area can capture the actual global average temperature and forest area patterns as follows:

$$\begin{aligned}
\frac{dT}{dt} &= \frac{1}{C}\left(Q(1-\alpha)-\sigma T^4\right)\\
\frac{dF}{dt} &= -r(1-\frac{F}{L})F + e_F R - \pi_F T R\\
\frac{dR}{dt} &= f_R S - \alpha_R F R - \pi_R T R\\
\frac{dS}{dt} &= -e_S^2(R-c) + \beta_S T F
\end{aligned} \tag{3.5}$$

where the decreasing/increasing rates of each variable are r and f_R . e_S is the annual rainfall period and c is the parameter for adjusting the minimum of rainfall solution. L is the carrying capacity of the forest cover. e_F is the rate of change in forest area caused by the amount of water (rainfall) absorption, π_F is the decay rate of forest cover caused by water absorption at each temperature level, α_R is the rate of decrease that indicate some amount of rainfall remaining in the forest, π_R is the decreasing rainfall difference rates affecting by the proportion of rainfall at each temperature level and β_S is the difference rate of the rainfall difference rate. TF is the proportion of the forest cover at each temperature level, TR represent the proportion of rainfall at each temperature level and FR is the proportion of the amount of rainfall in the forest.

3.3 Qualitative analysis

Here, we analyze the model (3.5) by using stability theory of differential equations. For this in the next sections, first we show that all solutions of model (3.5) exists, unique, nonnegative and bounded.

3.3.1 Existence and uniqueness of the solutions of the model

Theorem 3.3.1. *The system of equation in the model (3.5) with the non-negative initial value* $X(t_o) = (T_o, F_o, R_o, S_o)$ *has a unique solution.*

Proof. (i) Existence
Assume (t, T, F, R, S) is an open subset of $R \times R^4$ and let

$$f_1(T,F,R,S) = \frac{1}{C}\left(Q(1-\alpha) - \sigma T^4\right)$$
$$f_2(T,F,R,S) = -r(1-\frac{F}{L})F + e_F R - \pi_F TR$$
$$f_3(T,F,R,S) = f_R S - \alpha_R FR - \pi_R TR$$
$$f_4(T,F,R,S) = -e_S^2(R-c) + \beta_S TF$$

Since all $f_i(i = 1, 2, 3, 4)$ are continuous in $(t, T, F, R, S) \subset R \times R^4$, then by Theorem (1.7.1) for each $t_o \in R$ and $(T_o, F_o, R_o, S_o) \in R^4$, there exist the solution of the system of equation in the model (3.5) with the given initial value.

(ii) Uniqueness
In addition to the existence of the solution of the system of equation in the model (3.5), each $f_i(i = 1, 2, 3, 4)$ has a continuous first order partial derivatives with respect to T, F, R and S on R^4. Therefore by Theorem (1.7.1) the system of equation in the model with the given initial value has a unique solution.
Hence the theorem follows. □

3.3.2 Positivity of the solutions of the model

Theorem 3.3.2. *Given the initial conditions $T(t) = 0$, $R(t) = 0$ and $S(t) = 0$, then all solutions $(T(t),\ F(t),\ R(t),\ S(t))$ of the model (3.5) are positively invariant for all $t > 0$.*

Proof.
$$\frac{dT}{dt} = \frac{1}{C}\left(Q(1-\alpha) - \sigma T^4\right)$$

$$= a - bT^4$$

where $a = \frac{Q}{C}(1-\alpha) > 0$ since $\alpha = 0.3$ and $b = \frac{\sigma}{C} > 0$.
By using separation of variables we get

$$\frac{dT}{a - bT^4} = dt$$

which is the same with
$$\frac{dT}{(\sqrt{a})^2 - (\sqrt{b}T^2)^2} = dt$$

Integrating both sides with $T(0) = 0$ we get

$$\frac{1}{2\sqrt{a}} ln\left(\frac{\sqrt{a} + \sqrt{b}T^2}{\sqrt{a} - \sqrt{b}T^2}\right) = t$$

Solving this equation we have
$$T^2(t) = \frac{\sqrt{a}(e^{rt} - 1)}{\sqrt{b}(1 + e^{rt}}$$

where $r = 2\sqrt{a}$. Solving for T we get

$$T(t) = \left(\frac{\sqrt{a}(e^{rt} - 1)}{\sqrt{b}(1 + e^{rt})} \right)^{\frac{1}{2}}$$

which is positive for all $t \geq 0$ since $r > 0$ gives $e^{rt} \geq 1$.
Therefore $T(t) \geq 0$ for all $t \geq 0$.

Since from our assumptions the amount of forest cover cannot be decrease below zero, we have

$$F(t) \geq 0$$

for all $t \geq 0$. Consider the last equation of the model

$$\frac{dS}{dt} = -e_s^2(R - c) + \beta_S TF$$

which implies

$$dS = \left(-e_s^2(R - c) + \beta_S TF\right) dt$$

By integrating both sides we get

$$S(t) = -e_s^2 \int (R - c)dt + \beta_S \int TFdt$$

Since both $T(t) \geq 0$ and $F(t) \geq 0$ for all $t \geq 0$, then $S(t) \geq 0$ for all $t \geq 0$ if $c \geq R$.

Now consider the third equation of the model

$$\frac{dR}{dt} = f_R S - \alpha_R FR - \pi_R TR.$$

This implies that

$$\frac{dR}{dt} + (\alpha_R F + \pi_R T)R = f_R S$$

which is the first order ODE. Integrating factor becomes

$$\mu(t) = e^{\int (\alpha_R F + \pi_R T)dt}$$

Multiplying by $\mu(t)$ and integrating both sides we get

$$R(t)e^{\int (\alpha_R F + \pi_R T)dt} = f_R \int Se^{\int (\alpha_R F + \pi_R T)dt}dt + c_1$$

where c_1 is an arbitrary constant. Solving for $R(t)$ we get

$$R(t) = f_R e^{-\int (\alpha_R F + \pi_R T)dt} \int Se^{\int (\alpha_R F + \pi_R T)dt}dt + c_1 e^{-\int (\alpha_R F + \pi_R T)dt}.$$

Since all solutions $T(t) \geq 0$, $F(t) \geq 0$ and $S(t) \geq 0$ for all $t \geq 0$, then $R(t) \geq 0$ for all $t \geq 0$ if $c \geq R$.
Hence the theorem follows.

$\square$

3.3.3 Boundedness of the solutions of the model

Theorem 3.3.3. *The set*
$\Omega = \left\{(T, F, R, S) : 0 \leq T \leq \left(\frac{Q}{\sigma}\right)^{\frac{1}{4}}, 0 \leq F \leq L, 0 \leq R \leq R_M, 0 \leq S \leq S_M, \right\}$, *where* $R_M = f_R S_M T_{max}$ *and* $S_M = (e_s^2 c + \beta_S \left(\frac{Q}{\sigma}\right)^{\frac{1}{4}} L)T_{max}$ *attracts all solutions initiating in the interior of the positive octant.*

Proof. Here we have to show the boundedness of each model over a certain period of time T_{max}, since the models of each variables are over a certain period of time. From the first equation of the model (3.5), we have

$$\frac{dT}{dt} = \frac{1}{C}\left(Q(1-\alpha) - \sigma T^4\right)$$

$$\leq \frac{Q}{C} - \frac{\sigma}{C}T^4 = \frac{1}{C}\left(Q - \sigma T^4\right)$$

which implies that

$$\frac{dT}{Q - \sigma T^4} \leq \frac{dt}{C}$$

This gives

$$\frac{dT}{(\sqrt{Q})^2 - (\sqrt{\sigma T^2})^2} \leq \frac{dt}{C}$$

Integrating both sides we get

$$\frac{1}{2\sqrt{Q}} ln\left(\frac{\sqrt{Q} + \sqrt{\sigma}T^2}{\sqrt{Q} - \sqrt{\sigma}T^2}\right) \leq \frac{t}{C}$$

Solving this equation we have

$$T^2(t) \leq \frac{\sqrt{Q}(1 - e^{-at})}{\sqrt{\sigma}e^{-at} + \sqrt{\sigma}}$$

where $a = \frac{2\sqrt{Q}}{C}$. Solving for T we get

$$T(t) \leq \left(\frac{\sqrt{Q}(1 - e^{-at})}{\sqrt{\sigma}e^{-at} + \sqrt{\sigma}}\right)^{\frac{1}{2}}$$

Now

$$\sup_{t \in [0, T_{max}]} T(t) \leq \left(\frac{\sqrt{Q}}{\sqrt{\sigma}}\right)^{\frac{1}{2}} = \left(\frac{Q}{\sigma}\right)^{\frac{1}{4}}$$

Therefore

$$0 \leq T \leq \left(\frac{Q}{\sigma}\right)^{\frac{1}{4}}.$$

Since from our assumptions the amount of forest cover cannot decrease below zero and cannot be larger than the fixed area, we have

$$0 \leq F(t) \leq L,$$

where L is the larger fixed area of the amount of forest.

From the last equation of the model we have

$$\frac{dS}{dt} = -e_S^2(R-c) + \beta_S TF$$

$$\leq e_s^2 c + \beta_S TF$$

This implies

$$\frac{dS}{dt} \leq e_s^2 c + \beta_S T_M L,$$

where $T_M = \left(\frac{Q}{\sigma}\right)^{\frac{1}{4}}$. Now integrating both sides we get

$$S(t) \leq (e_s^2 c + \beta_S T_M L)t + c_1$$

for arbitrary positive constant c_1.

$$\sup_{t\in[0,T_{max}]} S(t) \leq (e_s^2 c + \beta_S T_M L)T_{max} + c_1 = S_M$$

Therefore

$$0 \leq S \leq S_M.$$

From the third equation of the model we have

$$\frac{dR}{dt} = f_R S - \alpha_R FR - \pi_R TR$$

$$\leq f_R S$$

This implies

$$\frac{dR}{dt} \leq f_R S_M$$

Integrating both sides we get

$$R(t) \leq f_R S_M t + c_2$$

for arbitrary positive constant c_2.

$$\sup_{t\in[0,T_{max}]} R(t) \leq f_R S_M T_{max} + c_2 = R_M$$

Hence

$$0 \leq R \leq R_M.$$

Therefore the above system of equation is bounded in the set Ω.

□

3.3.4 Equilibrium points and their existence

There are two equilibrium points.

Theorem 3.3.4. *1. The equilibrium points $E_1(T^*, F_1^*, R_1^*, S_1^*)$ and $E_2(T^*, F_2^*, R_2^*, S_2^*)$ are exists provided the following conditions are satisfied.*

$$\frac{e_F}{\pi_F} > T^*,$$

$$r + \frac{\beta_S T^*}{e_s^2}(\pi_F T^* - e_F) > 0$$

and

$$\left(r + \frac{\beta_S T^*}{e_s^2}(\pi_F T^* - e_F)\right)^2 \geq 4\frac{r}{L}c(e_F - \pi_F T^*)$$

Proof. The equilibrium points of the model (3.5) may be obtained by solving the following algebraic equations:

$$\frac{1}{C}\left(Q(1-\alpha) - \sigma T^4\right) = 0 \tag{3.6}$$

$$-r(1 - \frac{F}{L})F + e_F R - \pi_F T R = 0 \tag{3.7}$$

$$f_R S - \alpha_R F R - \pi_R T R = 0 \tag{3.8}$$

$$-e_S^2(R - c) + \beta_S T F = 0 \tag{3.9}$$

From equation (4.6), solving for T we get $T = \left(\frac{Q(1-\alpha)}{\sigma}\right)^{\frac{1}{4}}$. Thus by substituting the values of the parameters in section (4.2) the basic model gives the equilibrium temperature $T^* = 234.8K$. The actual value of the surface temperature is $287.7K$. A good part of the difference can be attributed to the greenhouse effect of the Earth's atmosphere.

From equation (4.9), solving for R we get

$$R = c + c_o F \tag{3.10}$$

where $c_o = \frac{\beta_S T^*}{e_s^2}$.

From equation (4.8), solving for S we get

$$f_R S = R(\alpha_R F + \pi_R T^*)$$

Substituting the values of R in equation (3.10) we get

$$S = \frac{(c + c_o F)(\alpha_R F + \pi_R T^*)}{f_R}$$

which gives

$$S = \frac{c_o \alpha_R}{f_R}F^2 + \frac{c\alpha_R + c_o \pi_R T^*}{f_R}F + \frac{c\pi_R T^*}{f_R} \tag{3.11}$$

From equation (4.7), we have

$$-r(1-\frac{F}{L})F+e_F R-\pi_F T R=0$$

Substituting the values of R in equation (3.10) we get

$$-r(1-\frac{F}{L})F-(c+c_o F)(\pi_F T-e_F)=0$$

which gives the following quadratic equation

$$\frac{r}{L}F^2-(r+c_o(\pi_F T^*-e_F))F+c(e_F-\pi_F T^*)=0$$

Solving this for F we get

$$F=\frac{(r+c_o(\pi_F T^*-e_F))\pm\sqrt{(r+c_o(\pi_F T^*-e_F))^2-4\frac{r}{L}c(e_F-\pi_F T^*)}}{2\frac{r}{L}}$$

Now we get the following two solutions

$$F_1=\frac{(r+c_o(\pi_F T^*-e_F))+\sqrt{(r+c_o(\pi_F T^*-e_F))^2-4\frac{r}{L}c(e_F-\pi_F T^*)}}{2\frac{r}{L}}=F_1^*$$

and

$$F_2=\frac{(r+c_o(\pi_F T^*-e_F))-\sqrt{(r+c_o(\pi_F T^*-e_F))^2-4\frac{r}{L}c(e_F-\pi_F T^*)}}{2\frac{r}{L}}=F_2^*.$$

The following condition satisfied

F_1^* and F_2^* are nonnegative iff

$$(e_F-\pi_F T^*)>0,$$

$$r+\frac{\beta_S T^*}{e_s^2}(\pi_F T^*-e_F)>0$$

and

$$\left(r+\frac{\beta_S T^*}{e_s^2}(\pi_F T^*-e_F)\right)^2\geq 4\frac{r}{L}c(e_F-\pi_F T^*)$$

Substituting this values of F_1^* and F_2^* into equations (3.10) and (3.11) above we get

$$R_1=c+c_o F_1^*=R_1^*$$

and

$$R_2=c+c_o F_2^*=R_2^*.$$

$$S_1=\frac{c_o\alpha_R}{f_R}F_1*^2+\frac{c\alpha_R+c_o\pi_R T^*}{f_R}F_1^*+\frac{c\pi_R T^*}{f_R}=S_1^*$$

and

$$S_2=\frac{c_o\alpha_R}{f_R}F_2*^2+\frac{c\alpha_R+c_o\pi_R T^*}{f_R}F_1^*+\frac{c\pi_R T^*}{f_R}=S_2^*.$$

Therefore, the equilibrium points $E_1(T^*, F_1^*, R_1^*, S_1^*)$ and $E_2(T^*, F_2^*, R_2^*, S_2^*)$ are positive if the following conditions are satisfied.

$$\frac{e_F}{\pi_F} > T^*,$$

$$r + \frac{\beta_S T^*}{e_s^2}(\pi_F T^* - e_F) > 0$$

and

$$\left(r + \frac{\beta_S T^*}{e_s^2}(\pi_F T^* - e_F)\right)^2 \geq 4\frac{r}{L}c(e_F - \pi_F T^*)$$

□

Remark 1. From equation (3.10) we get the following result.

$\frac{dR}{dF} = c_o > 0$, this implies that the amount of rainfall increases as the amount of forest cover increases, because the increasing of amount of rainfall is dependent on the amount of forest cover.

3.3.5 Stability analysis

In this part of the research we present the stability of equilibrium points E_1 and E_2.

Local stability analysis

The local stability behavior of equilibrium points may be determined by eigenvalues of the corresponding Jacobian matrix J . The general Jacobian matrix J for the model (3.5) is given by

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial T} & \frac{\partial f_1}{\partial F} & \frac{\partial f_1}{\partial R} & \frac{\partial f_1}{\partial S} \\ \frac{\partial f_2}{\partial T} & \frac{\partial f_2}{\partial F} & \frac{\partial f_2}{\partial R} & \frac{\partial f_2}{\partial S} \\ \frac{\partial f_3}{\partial T} & \frac{\partial f_3}{\partial F} & \frac{\partial f_3}{\partial R} & \frac{\partial f_3}{\partial S} \\ \frac{\partial f_4}{\partial T} & \frac{\partial f_4}{\partial F} & \frac{\partial f_4}{\partial R} & \frac{\partial f_4}{\partial S} \end{pmatrix}$$

which gives

$$J = \begin{pmatrix} \frac{-4\sigma}{C}T^3 & 0 & 0 & 0 \\ -\pi_F R & 2\frac{r}{L}F - r & e_F - \pi_F T & 0 \\ -\pi_R R & -\alpha_R R & -\alpha_R F - \pi_R T & f_R \\ \beta_S F & \beta_S T & -e_s^2 & 0 \end{pmatrix}$$

Theorem 3.3.5. *The equilibrium E_1 is locally asymptotically stable if the following condition is satisfied:*
$2\frac{r}{L}F_1^* - r < 0$.

Proof. Let $J(E_i)$ be the matrix J evaluated at the equilibrium $E_i (i = 1, 2)$.

$$J(E_1) = \begin{pmatrix} \frac{-4\sigma}{C}T*^3 & 0 & 0 & 0 \\ -\pi_F R_1^* & 2\frac{r}{L}F_1^* - r & e_F - \pi_F T^* & 0 \\ -\pi_R R_1^* & -\alpha_R R_1^* & -\alpha_R F_1^* - \pi_R T^* & f_R \\ \beta_S F_1^* & \beta_S T^* & -e_s^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -a_{11} & 0 & 0 & 0 \\ -a_{21} & a_{22} & -a_{23} & 0 \\ -a_{31} & -a_{32} & -a_{33} & f_R \\ a_{41} & a_{42} & -e_s^2 & 0 \end{pmatrix}$$

where $a_{11} = \frac{4\sigma}{C}T*^3$, $a_{21} = \pi_F R_1^*$, $a_{22} = 2\frac{r}{L}F_1^* - r$, $a_{23} = e_F - \pi_F T^*$, $a_{31} = \pi_R R_1^*$, $a_{32} = \alpha_R R_1^*$, $a_{33} = \alpha_R F_1^* + \pi_R T^*$, $a_{41} = \beta_S F_1^*$, and $a_{42} = \beta_S T^*$.

Now the corresponding characteristic equation of the Jacobian matrix $J(E_1)$ is given as,

$$\lambda^4 + r_1\lambda^3 + r_2\lambda^2 + r_3\lambda + r_4 = 0$$

where

$$r_1 = a_{11} + a_{33} - a_{22}$$

$$r_2 = a_{11}(a_{33} - a_{22}) + f_R e_s^2 - a_{23}a_{32} - a_{22}a_{33}$$

$$r_3 = f_R(a_{23}a_{42} - a_{22}e_s^2) - a_{11}(a_{22}a_{33} + a_{23}a_{32} - f_R e_s^2)$$

$$r_4 = a_{11}f_R(a_{23}a_{42} - a_{22}e_s^2)$$

By using Routh-Hurwitz criterion the equilibrium point E_1 is locally asymptotically stable if the following conditions are satisfied:
$r_i > 0 (i = 1, 2, 3, 4)$, $r_1r_2 - r_3 > 0$, $r_3(r_1r_2 - r_3) - r_1^2 r_4 > 0$, where r_1, r_2, r_3 and r_4 are defined as above. Here it is easy to see that $r_1 > 0$ if and only if $a_{22} = 2\frac{r}{L}F_1^* - r < 0$.

□

Theorem 3.3.6. *The equilibrium E_2 is locally asymptotically stable if the following condition is satisfied:*
$2\frac{r}{L}F_2^* - r < 0$.

Proof. The matrix at the equilibrium point E_2 is given as,

$$J(E_2) = \begin{pmatrix} \frac{-4\sigma}{C}T*^3 & 0 & 0 & 0 \\ -\pi_F R_2^* & 2\frac{r}{L}F_2^* - r & e_F - \pi_F T^* & 0 \\ -\pi_R R_2^* & -\alpha_R R_2^* & -\alpha_R F_2^* - \pi_R T^* & f_R \\ \beta_S F_2^* & \beta_S T^* & -e_s^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -a_{11} & 0 & 0 & 0 \\ -a_{21}^* & a_{22}^* & -a_{23} & 0 \\ -a_{31}^* & -a_{32}^* & -a_{33}^* & f_R \\ a_{41}^* & a_{42} & -e_s^2 & 0 \end{pmatrix}$$

where $a_{11} = \frac{4\sigma}{C}T*^3$, $a_{21}^* = \pi_F R_2^*$, $a_{22}^* = 2\frac{r}{L}F_2^* - r$, $a_{23}^* = e_F - \pi_F T^*$, $a_{31}^* = \pi_R R_2^*$, $a_{32}^* = \alpha_R R_2^*$, $a_{33}^* = \alpha_R F_2^* + \pi_R T^*$, $a_{41}^* = \beta_S F_2^*$, and $a_{42} = \beta_S T^*$.

Now the corresponding characteristic equation of the Jacobian matrix $J(E_2)$ is as follows,

$$m^4 + q_1 m^3 + q_2 m^2 + q_3 m + q_4 = 0$$

where

$$q_1 = a_{11} + a_{33}^* - a_{22}^*$$

$$q_2 = a_{11}(a_{33}^* - a_{22}^*) + f_R e_s^2 - a_{22}^* a_{33}^* - a_{23} a_{32}^*$$

$$q_3 = f_R(a_{23}a_{42} - a_{22}^* e_s^2) - a_{11}(a_{22}^* a_{33}^* + a_{23} a_{32}^* - f_R e_s^2)$$

$$q_4 = a_{11} f_R (a_{23} a_{42} - a_{22}^* e_s^2)$$

By using Routh-Hurwitz criterion the equilibrium point E_2 is locally asymptotically stable if the following conditions are satisfied:
$q_i > 0 (i = 1, 2, 3, 4)$, $q_1 q_2 - q_3 > 0$, $q_3(q_1 q_2 - q_3) - q_1^2 q_4 > 0$, where q_1, q_2, q_3 and q_4 are defined as above. Here it is easy to see that $q_1 > 0$ if and only if $a_{22}^* = 2\frac{r}{L}F_2^* - r < 0$.

□

Global stability analysis

Theorem 3.3.7. *Let $F \geq F_2^*$ $R \geq R_2^*$ and $S \geq S_2^*$. Then the equilibrium point $E_2(T^*, F_2^*, R_2^*, S_2^*)$ is globally asymptotically stable in the region $\Omega \subseteq \Re_+^4$ if the following inequalities hold in Ω.*
i) $\alpha_R F_2^* < \pi_R T^*$
ii) $e_F > \pi_F T^*$
iii) $c < R_2^*$
iv) $F_2^* < L$

Proof. To prove this theorem we consider a positive definite function as follows and under some conditions, we show that its derivative is negative.

$$V(T, F, R, S) = \frac{C}{2}(T - T^*)^2 + \frac{1}{2}(F - F_2^*)^2 + \frac{1}{2}(R - R_2^*)^2 + \frac{1}{2e_s^2}(S - S_2^*)^2 \tag{3.12}$$

V is a positive definite function, since $V > 0$ for all $(T, F, R, S) \in \Omega \diagup (T^*, F_2^*, R_2^*, S_2^*)$ and $V(T^*, F_2^*, R_2^*, S_2^*) = 0$. On differentiating equation (3.12) with respect to t we get,

$$\frac{dV}{dt} = C(T - T^*)\frac{dT}{dt} + (F - F_2^*)\frac{dF}{dt} + (R - R_2^*)\frac{dR}{dt} + \frac{1}{e_s^2}(S - S_2^*)\frac{dS}{dt}$$

Substituting the values of the derivative from the model (3.5), we have

$$\frac{dV}{dt} = (T - T^*)(Q(1-\alpha) - \sigma T^4) + (F - F_2^*)(-r(1 - \frac{F}{L})F + e_F R - \pi_F TR)$$

$$+(R - R_2^*)(f_R S - \alpha_R FR - \pi_R TR) + \frac{1}{e_s^2}(S - S_2^*)(-e_s^2 R + e_s^2 c + \beta_S TF).$$

From this, we get,

$$\frac{dV}{dt} = (T - T^*)(\sigma T^{*4} - \sigma T^4) + (F - F_2^*)\left(-\frac{R_2^*(e_F - \pi_F T^*)}{F_2^*(1 - \frac{F_2^*}{L})}(1 - \frac{F}{L})F + R(e_F - \pi_F T)\right)$$

$$+(R - R^*)\left(\frac{R^*}{S^*}(\alpha_R F_2^* - \pi_R T^*) - \alpha_R FR - \pi_R TR\right) + (S - S_2^*)\left((R - c) + \frac{(R_2^* - c)}{T^* F_2^*}TF\right)$$

Now $\frac{dV}{dt}$ can be simplified to,

$$\frac{dV}{dt} = -\sigma(T - T^*)^2(T^* + T)(T^{*2} + T^2) - (F - F_2^*)\left(-\frac{R_2^*(e_F - \pi_F T^*)}{F_2^*(1 - \frac{F_2^*}{L})}(1 - \frac{F}{L})F\right)$$

$$-(F - F_2^*)(R(e_F + \pi_F T)) + (R - R_2^*)\left(\frac{R_2^*}{S_2^*}(\alpha_R F_2^* - \pi_R T^*) - (\alpha_R FR + \pi_R TR)\right)$$

$$+(S - S_2^*)\left((R - c) + \frac{(c - R_2^*)}{T^* F_2^*}TF\right)$$

We note that $\dfrac{dV}{dt} < 0$ if the following condition holds:
$\alpha_R F_2^* < \pi_R T^*$, $e_F > \pi_F T^*$, $F_2^* < L$, $c < R_2^*$.
Hence all conditions of the theorem are satisfied for $\frac{dV}{dt}$ to be negative in the region Ω. Therefore the equilibrium E_2 is globally asymptotically stable in the region Ω. □

Interactions between variables

Consider the dynamics of amount of forest cover

$$\frac{dF}{dt} = -r(1 - \frac{F}{L})F - \pi_R TR + e_F R$$

When $R = 0$, this equation becomes

$$\frac{dF}{dt} = -r(1 - \frac{F}{L})F$$

which is the logistic equation. The equilibrium point $F = 0$ of this equation implies that there is no amount of forest cover.

Solving this equation for F we get

$$F(t) = \frac{1}{e^{rt} + \frac{1}{L}}$$

$\lim_{t\to\infty} supF(t) = 0$ which implies that as time increases with no rainfall, the amount of forest cover becomes extinct.

Theorem 3.3.8. *If there is no amount of rainfall, the rate of change of forest cover has stable equilibrium point.*

Proof. Let there is no amount of rainfall *i.e* $F = 0$, then from the second equation of the model (3.5) we get

$$\frac{dF}{dt} = -r(1 - \frac{F}{L})F$$

which is the logistic equation. This equation has the equilibrium points at $F = 0$ and $F = L$. Solving the equation for F we get

$$F(t) = \frac{1}{e^{rt} + \frac{1}{L}}$$

.
We obtain $\lim_{t\to\infty} F(t) = 0$. Hence, $F = 0$ is the stable equilibrium point and also there is no forest area which may be due to no rainfall.

□

Consider the model for the amount of periodic rainfall

$$\begin{aligned} \frac{dR}{dt} &= f_R S - \alpha_R F R - \pi_R T R \\ \frac{dS}{dt} &= -e_S^2(R - c) + \beta_S T F \end{aligned} \tag{3.13}$$

Assume there is no forest cover and no change in global temperature (i.e $F = 0$ and $T = 0$), then we get the model

$$\begin{aligned} \frac{dR}{dt} &= f_R S \\ \frac{dS}{dt} &= -e_S^2(R - c) \end{aligned} \tag{3.14}$$

Now we have the following theorem.

Theorem 3.3.9. *The system of equation (3.14) has a periodic solution and its steady state is center.*

Proof. This equation can be written in the form

$$\begin{pmatrix} R' \\ S' \end{pmatrix} = \begin{pmatrix} 0 & f_R \\ -e_s^2 & 0 \end{pmatrix} \begin{pmatrix} R \\ S \end{pmatrix} + \begin{pmatrix} 0 \\ ce_s^2 \end{pmatrix}$$

The exact solution for equations (3.14) is

$$R(t) = c + C_1 cos(e_s\sqrt{f_R}t) + C_2 sin(e_s\sqrt{f_R}t)$$

where $C_1 = R_0 - c$, $C_2 = S_0/e_S$, $R(0) = R_0$, $S(0) = S_0$
The critical point is the solution to the system

$$\begin{cases} f_R S = 0 \\ -e_s^2 R + ce_s^2 = 0 \end{cases}$$

Solving the system we obtain the critical point $(c, 0)$
Now use the change of variable

$$R = u + c, S = v$$

to translate the critical point to the origin $(0, 0)$

$$\begin{aligned} \frac{dR}{dt} &= \frac{du}{dt} = f_R v \\ \frac{dS}{dt} &= \frac{dv}{dt} = -e_S^2(u + c) + ce_S^2 = -e_S^2 u \end{aligned} \tag{3.15}$$

Then we get

$$\begin{aligned} \frac{du}{dt} &= f_R v \\ \frac{dv}{dt} &= -e_S^2 u \end{aligned} \tag{3.16}$$

The characteristic equation of the coefficient matrix, say

$$A = \begin{pmatrix} 0 & f_R \\ -e_s^2 & 0 \end{pmatrix}$$

is $det(A - \lambda I) = \lambda^2 + e_s^2 f_R = 0$
Solving we obtain $\lambda_{1,2} = \pm e_s \sqrt{f_R} i$. Since all the eigen values of A has zero real parts, then the solution of the above system is periodic with the steady state center. □

From the last equation of the model (3.5) we have

$$\frac{dS}{dt} = -e_s^2(R - c) + \beta_S TF$$

When $R = 0$, this equation becomes

$$\frac{dS}{dt} = e_s^2 c + \beta_S TF$$

Now as T increases, the rate of change of rainfall S increases. From this we can see that the rainfall difference rate β_S is affected by the rise in global temperature.

Chapter 4

Numerical Simulation

In order to check the feasibility of the results obtained in the previous chapter for the existence of E_2 we conduct some numerical computations of the model (3.5) using Runge-Kutta 4-th order and ode45 MATLAB codes. The following parameter values have been chosen [21, 28, 30]:

No.	Parameters	values of parameters	No.	Parameters	values of parameters
1	C	2.912	8	π_F	0.001199
2	Q	242	9	f_R	1.0036
3	α	0.3	10	α_R	0.00106
4	σ	5.67×10^{-8}	11	π_R	0.000097
5	r	4	12	e_s	6.2832
6	L	200	13	c	0.3117
7	e_F	0.3201	14	β_S	0.01

Table 4.1: Parameters value

It is found that under the above mentioned numerical parameters, the equilibrium values of $E_2(T^*, F_2^*, R_2^*, S_2^*)$, are obtained as:

$$T^* = 234.8, F_2^* = 0.0030077, R_2^* = 0.3118789, S_2^* = 0.0070787.$$

The eigenvalues of the Jacobian matrix $J(E_2)$ corresponding to the equilibrium $E_2(T^*, F_2^*, R_2^*, S_2^*)$ for the model system (3.5) are:

$$-0.0122 - 6.294i, -0.0122 + 6.294i, -3.9982, -1.0082.$$

Here, it may be noted that two eigenvalues of matrix are negative and the other two eigenvalues are with negative real parts. Therefore, it implies that the interior equilibrium $E_2(T^*, F_2^*, R_2^*, S_2^*)$

is locally asymptotically stable.

For the above set of parameters, the computer generated graphs are plotted in Figures (4.1 - 4.7) to predict the sensitivity of the model system under the change of different parameter values. In Figures (4.1-4.3) the graphs of the variables of model are plotted with time t. Figure 4.1 shows the change in global mean temperature as time increases. Here, it may be noted that T increases as time t increases and finally attains its equilibrium.

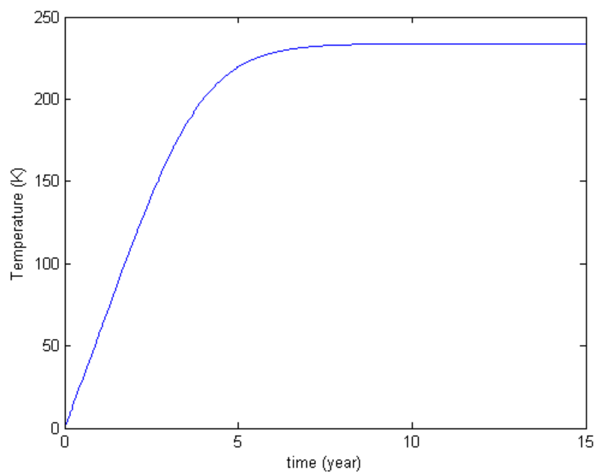

Figure 4.1: Global mean temperature with time

From Figure 4.2 we observe that as time increases the amount of forest cover decreases to zero. Figure 4.3 shows that the amount of rainfall oscillates with high amplitude as time increases, since its behavior is seasonal and its solution is periodic.

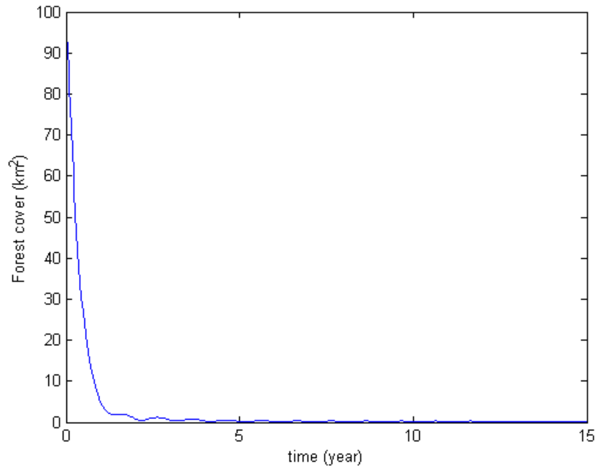

Figure 4.2: The amount of forest cover with time

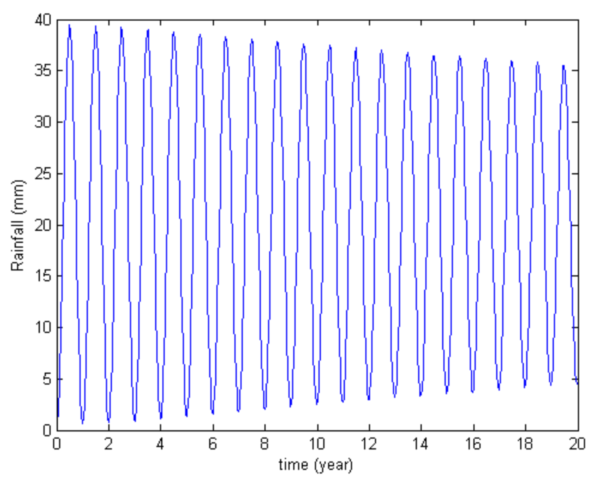

Figure 4.3: The amount of rainfall with time

Figure 4.4 shows that the change in the amount of forest cover F with respect to time t for different values of π_F. Here, it may be noted that F decreases as π_F increases and finally approaches to the minimum forest cover which is zero. The change in the decreasing rate of forest cover can be due to decay of forest cover caused by water absorption at each temperature level. Thus, π_F is an important parameter and it should remain under control in order to have the stable system i.e. the decreasing of forest cover should be held in check.

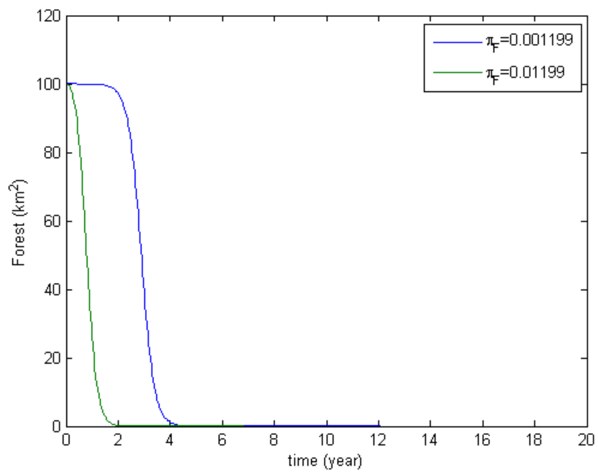

Figure 4.4: Effect of π_F on forest

The variation of the amount of rainfall R with time for different values of α_R are plotted in Figure 4.5. From Figure 4.5 one can observe that as the value of the parameter α_R increases the amplitude of the oscillations for the amount of rainfall increases. This implies that the amount of rainfall R is directly dependent on the forest cover F and decreasing the amount of forest cover affects the rainfall fluctuation.

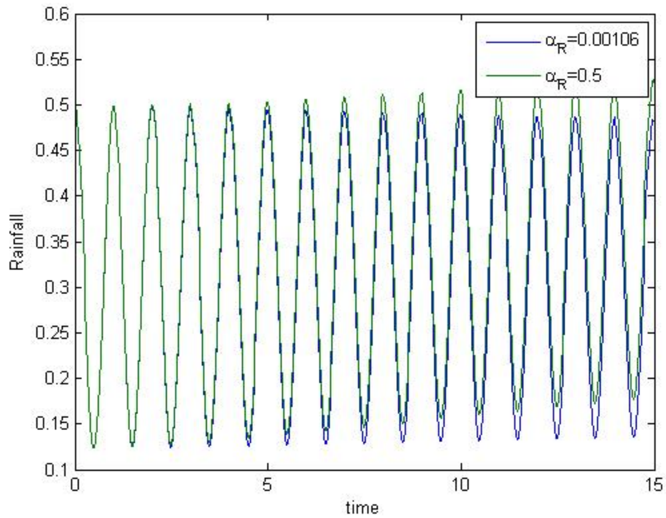

Figure 4.5: Effect of α_R on rainfall

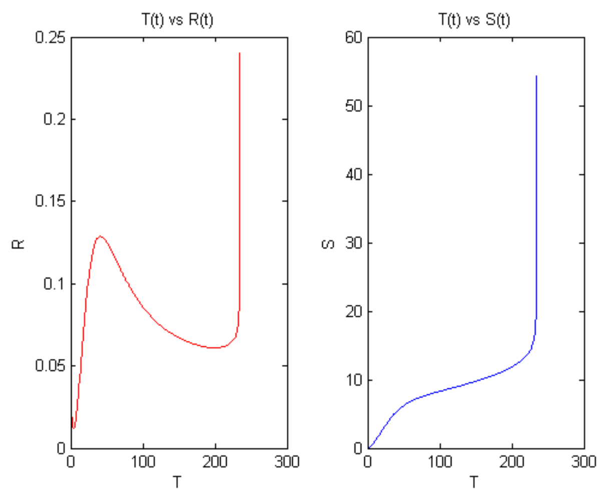

Figure 4.6: Interactions between Temperature, Rainfall and Rate of change of Rainfall

From the first graph of Figure 4.6 we observe that as the global temperature increases the amount of rainfall oscillates with high amplitude until the equilibrium point of global mean temperature is reached. Here, we can see that the global temperature affects the fluctuation of rainfall. From the second graph of Figure 4.6 we see that the rate of change of rainfall S increases as the global mean temperature increases.

From the first graph of Figure 4.7 we see that as the amount of forest cover increases, the global warming decreases to zero. Inversely as the amount of forest cover decreases the global warming increases, i.e., decreasing of forest cover causes the rise in global mean temperature. From this discussion, it can be said that F has an important role in the above model system and it should be controlled, otherwise the global temperature become high.

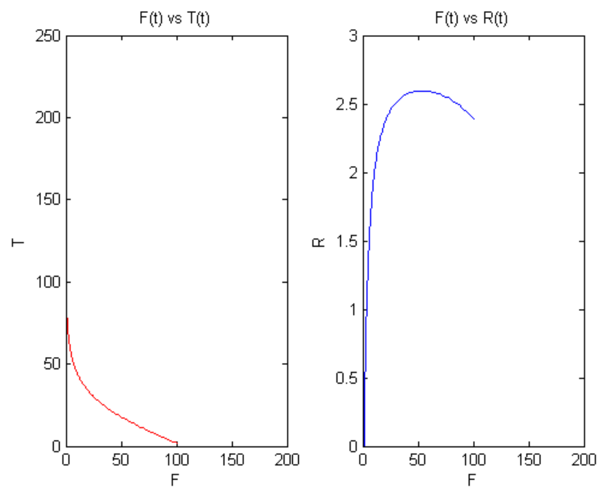

Figure 4.7: Interactions of forest cover with mean temperature and amount of rainfall

From the second graph of Figure 4.7 one can easily observe that as the amount of forest cover increases the amount of rainfall also increases and finally attains its equilibrium. This graphs describes that there is the dependence between forest and rainfall, so as the amount of forest cover decreases, the amount of rainfall decreases. From this discussion, it can be said that F has an important role in the above model system and it should be controlled, otherwise the fluctuation of rainfall can be affected.

The second graph of Figure 4.7 also reveals that as R decreases beyond certain level, the amount of forest cover F leads to extinction. One major observation from results of numerical simulation (as showed by Figure 4.6 and 4.7) is that the rate of change of rainfall is affected by the proportion of forest cover at each temperature level.

Chapter 5

Conclusions

In this research, we have developed a mathematical model that helps to study the effects of global climate change and decreasing forest cover on seasonal rainfall. The model is developed using physically-based formulations of the processes that make up the climate system. In the modeling process, logistic decay of forest cover, energy balance equation of the rise in global mean temperature and second order equation with periodic solution of rainfall have been considered. Using the stability theory of differential equations, the proposed model has been analyzed qualitatively to establish the following:

1) existence, boundedness and non negativeness of solutions

2) identification of equilibrium points along with their existence

3) the stability of each equilibrium point.

The qualitative analysis shows that the equilibrium points E_1 and E_2 are stable under certain conditions. Numerical simulation has also been performed to illustrate the feasibility of the obtained results.

The developed model was refined to cover ecological implication of interactions between variables. As the amount of forest cover decreases the global warming increases, i.e., decreasing of forest cover causes the rise in global mean temperature. Furthermore as the amount of forest cover decreases, the amount of rainfall decreases and so it reveals that as R decreases beyond certain level, the amount of forest cover F leads to extinction. This describes that there is the dependence between forest and rainfall. From this discussion, it can be said that F has an important role in the above model system and it should be controlled, otherwise the global temperature become high and affects the fluctuation of rainfall.

From numerical simulation results, the parameter α_R indicates that the amount of rainfall is affected by the proportion of the amount of rainfall in the forest and the parameter β_S indicates that the rainfall difference rate is affected by the proportion of forest cover at each temperature level. Therefore, it can be concluded that decreasing of the amount of forest cover and the rise in global temperature influences the fluctuation of rainfall. So in order to avoid this, efforts should be made to limit the deforestation and releasing green house gases into atmosphere, since when the amount of forest cover decrease the releasing of green house gases into atmosphere will increase the rate of climate change, and thus increase unexpected societal and ecological damage by affecting seasonal rainfall. The model requires bifurcation analysis and it would also be interesting to validate the model by collecting the data which are left for the future work.

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Declaration

I here under signed declare that this thesis is my original work and has not been presented for a degree in any other university, and that all sources of material used for the thesis have been fully acknowledged.

(Name)	Signiture

This thesis has been submitted for the examination with my Approves as Adama Science and Technology university advisor.

(Name)	Signiture