A HYBRID NUMERICAL SCHEME FOR SOLVING SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS WITH DISCONTINUOUS TERMS

Yisak Amana Kuma

A Thesis Submitted to
The Department of Applied Mathematics
School of Applied Natural Science
Presented in Partial Fulfillment of the Requirement for the
Degree of Master's in Mathematics (Numerical Analysis)

Office of Graduate Studies
Adama Science and Technology University

June, 2024
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Co-Advisor: Dr. Tekle Gemechu Dinka

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Declaration

I hereby declare that this master thesis entitled ***"A hybrid numerical scheme for solving singularly perturbed convection-diffusion problems with discontinuous terms"*** is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

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We, the advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"A hybrid numerical scheme for solving singularly perturbed convection-diffusion problems with discontinuous terms"*** prepared under our guidance by Yisak Amana Kuma submitted in partial fulfillment of the requirements for the degree of Mater's of Science in Applied Mathematics.

Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures..

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled **"A hybrid numerical scheme for solving singularly perturbed convection-diffusion problems with discontinuous terms"** and developed by Yisak Amana Kuma , hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

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Major Advisor	Signature	Date
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Co-advisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the thesis by Yisak Amana Kuma have read and evaluated the thesis entitled ***"A hybrid numerical scheme for solving singularly perturbed convection-diffusion problems with discontinuous terms"*** and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics (Numerical Analysis).

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Acronyms

BVP Boundary value problem

ODE Ordinary differential equation

FDM Finite difference method

FDS Finite difference scheme

NSFDM Non-standard finite difference method

PDE Partial differential equation

SPBVP Singularly perturbed boundary value problem

SPBVPs Singularly perturbed boundary value problems

SPCDBVPs Singularly perturbed convection diffusion boundary value problems

SPCDPs Singularly perturbed convection-diffusion problems

SPP Singularly perturbed problem

Abstract

This thesis presents a hybrid numerical scheme for solving singularly perturbed convection-diffusion boundary value problems with discontinuous convection coefficient and source terms. A hybrid numerical scheme was constructed using the non-standard finite-difference method on the Shishkin mesh and midpoint upwind finite difference schemes. The solutions of the problem considered exhibits an interior layer. Stability and uniform convergence analysis are investigated for the proposed hybrid scheme. Maximum absolute errors and convergence rates are computed for various perturbation parameter values and mesh sizes, using two numerical test examples. The proposed scheme is shown to be almost first-order uniformly convergent.

Keywords: Discontinuous terms; Hybrid numerical scheme; Interior layer; Shishkin mesh; Singularly perturbed problem; Uniformly convergent

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

In the real world, we often encounter a variety of problems that can be described as parameter-dependent differential equations. Singularly perturbed problems (SPPs) involve phenomena where a parameter, typically representing a physical property, becomes relatively small compared to other terms in the governing equations. Singularly perturbed problems were defined by a large number of authors. It was defined by Roos (2008) as ordinary or partial differential equations with a small positive parameter and solutions (or derivatives) that approach a discontinuous limit as the parameter approaches zero.

Shishkin and Shishkina (2008); Chandru, Prabha, and Shanthi (2017); Woldaregay and Duressa (2021); R. Ranjan and Prasad (2022) were studied singularly perturbed differential equation (SPDE) as differential equations in which the highest-order derivative is multiplied by a small positive perturbation parameter $\varepsilon(0 < \varepsilon << 1)$. Problems of this nature are referred to as singularly perturbed. The perturbation parameter, ε, can have any value in the open-closed interval$(0, 1]$ (Shishkin & Shishkina, 2008).

Singularly perturbed differential equations (SPDEs) are frequently used to model a wide range of physical processes. These equations can be found in fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory (Priyadharshini, Ramanujam, & Valanarasu, 2010), mechanical and electrical systems, celestial mechanics, particle and light physics, as well as in semi and superconductors, extreme thermal processes, and chemical and biochemical reactions (D. Kumar & Kadalbajoo, 2011). Other areas where these equations are utilized include convective heat transport with a large Peclet number, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory, meteorology, oceanography, rarefied gas dynamics, diffraction theory, reaction-diffusion process, non-equilibrium and radiating flows, and Navier-Stokes equations of fluid flow at high Reynolds number (Chandru et al., 2017).

Based on the statements above, we can conclude that singularly perturbed problems (SPPs) are ordinary or partial differential equations that involve a small positive parameter ε ($0 < \varepsilon \ll 1$), which cannot be approximated by setting the parameter value to zero. This small parameter is known as the singular perturbation parameter. Naively setting the parameter to zero changes the nature of the problem. In the case of differential equations, boundary conditions cannot be satisfied whereas in the case of algebraic equations, the number of viable solutions is reduced (M. Kumar et al., 2011). The behavior of the solutions of these types of differential equations is dependent on the magnitude of the parameters, and there are thin transition layers where the solution fluctuates extremely quickly, while the solution behaves regularly and slowly away from these layers.

The solutions for these problems usually involve boundary layers or interior layers. These are essentially thin regions inside the domain or near its boundary, particularly when ε approaches zero (D. Kumar & Kadalbajoo, 2011). Sharp changes often occur inside the domain of interest in many physical situations. In fluid and solid mechanics, these narrow regions are known as shock layers,in electrical applications, they are described as skin layers ,in quantum mechanics, they are called transition points, and in mathematics, they are referred to as stroke lines and surfaces (Kadalbajoo & Kumar, 2008). Boundary and/or interior layers in the solution of these problems generally lead to sharp gradients and challenging numerical computations.

The desire to comprehend such challenging numerical computations, develop novel analytical and numerical methods, and tackle real-world problems in numerous areas of science and engineering are some of the main motivations for studying singularly perturbed differential equations (SPDEs).

1.2 Convection-Diffusion Problems

Boffi (2020) defined the Convection-Diffusion problem as a differential equation whose second-order derivatives are multiplied by some parameter ε that is allowed to be close to zero.

Imagine a river that flows powerfully and smoothly. At a certain point, liquid pollution seeps into the water. Here, two physical processes are at work: the pollution diffuses slowly through the water, while the river's rapid flow is the main driver, convecting the pollutants downstream quickly. Diffusion gradually widens that curve, creating a long, thin, curved wedge form, whereas convection alone would have carried the pollutants along a one-dimensional curve on the surface. We have a convection-diffusion problem when convection and diffusion are both present in a linear differential equation and convection dominates.

Convection is the upward movement of hotter and less denser substances caused by heat, whereas cooler and dense substances sink. Meanwhile, diffusion occurs concurrently with convection and involves the transfer of particles from high-concentration regions to low-concentration regions (Gregory & Wickramasinghe, 2023). Convection-diffusion problems (CDPs) are widely encountered in various fields of science, engineering, and other fields where mathematical modeling is important. It has applications in dispersing pollutants in river estuaries, financial modeling, atmospheric pollution, Semi-conductor equations, Fokker-Planck equation,transport of groundwater, Stefan problems on a variable mesh, transport of turbulence (Morton & Kellogg, 1997). These problems involve the combined effects of convection, diffusion, and source/sink terms and are described by ordinary differential equations (ODEs) or partial differential equations (PDEs). In the context of mathematical modeling, a singularly perturbed convection-diffusion problem refers to a class of differential equations that involve convection and diffusion processes, where the existence of a small parameter generates unusual behavior in the solution.

Understanding the behavior of solutions to such problems is crucial for accurate modeling and prediction in practical applications. Let us denote the independent variable as x and the dependent variable as $y(x)$. The general form of one-dimensional stationary singularly perturbed convection-diffusion problem (SPCDP) can be expressed as:

$$\varepsilon y''(x) + p(x)y'(x) = f(x) \tag{1.1}$$

where, $0 < \varepsilon << 1$, is singular perturbation parameter. $p(x)$ is convection coefficient, and $f(x)$ represents any source or forcing term present in the problem. If the order of the differential equation (1.1) is reduced by one when the perturbation parameter ε is set to zero, the singularly perturbed problem (SPP) is said to be of the convection-diffusion type. If the order is reduced by two, it is known as a reaction-diffusion-type problem (Siraj, Duressa, & Bullo, 2019). In this study, the first type is considered where the convection coefficient and source terms have a discontinuity in the domain.

In many real-world scenarios, the convection coefficient, which represents the strength of the convective flow, exhibits discontinuities or sharp changes across the domain of interest. The presence of such discontinuities can significantly affect the behavior and accuracy of the solution to the convection-diffusion problem (P. A. Farrell, Hegarty, Miller, O'Riordan, & Shishkin, 2004). It is therefore essential to investigate singularly perturbed convection-diffusion boundary value problems where the convection coefficient exhibits discontinuity. Analytically solving the singularly perturbed convection-diffusion boundary value problems (SPCDBVPs) becomes complex due to boundary and interior layers. Accordingly, the attention is often shifted towards numerical approaches. To overcome these associated difficulties through the use of numerical approaches, the fitted mesh methods and the fitted operator methods are two common strategies to develop the parameter uniform robust numerical techniques for singularly perturbed problems (Woldaregay, Duressa, et al., 2022). Therefore, it is important to develop a robust numerical approach that can handle the considered problems to provide accurate numerical solutions by using one of those strategies. For small ε the problem, (1.1) exhibits a boundary layer depending on the value of the coefficient of the convective term, $p(x)$ (Woldaregay & Duressa, 2022). We have three cases, if $p(x) < 0$, (1.1) has right-end boundary layer, and if p(x) > 0, it has left-end boundary layer. If $p(x) = 0$, (1.1) becomes oscillatory.

1.3 Statement of the Problem

In singularly perturbed problems, using numerical methods developed for regular problems leads to solution errors that depend on the parameter ε. The errors of the numerical solution depend on the distribution of the mesh points and become small only when the effective mesh size in the layer is much less than the value of the parameter ε (Gupta, Sahoo, & Dubey, 2021). Such numerical methods turn out to be inapplicable for singularly perturbed problems (Gupta et al., 2021). For singularly perturbed convection-diffusion problems (SPCDPs) with discontinuous convection coefficient and source terms, the jump discontinuity at one or more points in the interior of the domain gives rise to an interior layer in the exact solution of the problem (P. A. Farrell et al., 2004). Hence, the numerical treatment of singularly perturbed convection-diffusion boundary value problems (SPCDBVPs) having discontinuity in the convection coefficient and source terms give rise to computational difficulties.

In general, since the convection coefficient and source terms have a discontinuity inside the domain, the numerical treatment of singularly perturbed convection-diffusion problems (SPCDPs) with discontinuous terms becomes more difficult than singularly perturbed convection-diffusion problems (SPCDPs) with continuous terms.

Thus, in order to get accurate and uniformly convergent solutions, specially designed numerical methods are required.

To the best of our knowledge, the problem considered is not solved using the nonstandard finite difference method on the Shishkin mesh and the midpoint upwind finite difference scheme. The goal of this scheme in our study is to obtain an accurate and uniform convergent numerical solution throughout the domain.

1.4 Objectives of the Study

1.4.1 General objective

The general objective of this study is to develop a hybrid numerical scheme for solving a class of singularly perturbed convection-diffusion problems with a discontinuous terms.

1.4.2 Specific objectives

The specific objectives of this study are to:

- construct an ε-uniform numerical method for solving the singularly perturbed convection-diffusion problem with discontinuous terms,
- analyze the stability of the proposed numerical scheme,
- establish the convergence of the proposed numerical scheme.

1.5 Significance of the Study

The results of this study will contribute to research activities in the following areas:

- Provide a numerical method for solving singularly perturbed convection-diffusion boundary value problems (SPCDBVPs) with discontinuous term.
- Help graduate students with developing scientific approaches and research methodologies.
- Provide some background information for other researchers who work in this area.

CHAPTER 2

REVIEW OF RELATED LITERATURE

Singularly perturbed convection-diffusion boundary value problems (SPCDVBPs) arise in various fields of science and engineering (Prabha, Chandru, & Shanthi, 2017). These problems involve a convection term, a diffusion term, and a source term, and sometimes exhibit discontinuities in the convection coefficient and source terms. The behavior of the SPCDP solution highly depends on the nature of the convection coefficient, whether it is vanishing at some points of the problem domain or not (Gupta et al., 2021). Solving such problems accurately and efficiently is crucial for obtaining reliable results. The existing literature primarily focuses on the development and analysis of numerical methods for singularly perturbed convection-diffusion problems (SPCDPs) with continuous terms (problems with only boundary layers in solutions). Although these methods have proven to be effective in handling such problems, they may encounter difficulties when dealing with discontinuous terms, which are encountered in practical applications. Numerous numerical methods have been developed for solving singularly perturbed convection-diffusion problems with smooth data. For example; Kadalbajoo and Gupta (2009) developed a numerical solution for the singularly perturbed convection-diffusion problem using the parameter uniform B-spline collocation approach. Furthermore, the approach was shown to be boundary layer resolving as well as second-order uniformly convergent. Kaushik, Kumar, and Vashishth (2012) presented an efficient mixed asymptotic-numerical technique for solving singularly perturbed convection-diffusion problems. Using the non-polynomial spline approach, Khandelwal and Khan (2017) developed a solution for singularly perturbed convection-diffusion boundary value problems with two small parameters. Melesse, Tiruneh, and Derese (2020) presented solutions for systems of singularly perturbed convection-diffusion problems using the initial value method.

For a class of second-order singularly perturbed boundary value problems with a simple turning point displaying twin boundary layers, R. Ranjan and Prasad (2022) developed a novel exponentially fitted finite difference approach. Through numerical validation, they showed that the approach provides highly accurate and consistently convergent results with a linear rate for all values of the mesh size $h >> \varepsilon$. They also noted that the method is appropriate for $\varepsilon \leq 10^{-5}$. Gregory and Wickramasinghe (2023) proposed an upwind finite difference method and used the Shishkin mesh scheme to capture the solution near boundary layers for singularly perturbed convection diffusion problems (SPCDPs).

However, the majority of existing standard finite difference approaches for dealing with singular perturbation problems (SPPs) produce satisfactory results only when the mesh size is significantly smaller than the perturbation parameter, making it both costly and time-intensive.

Convection-diffusion problems (CDPs) with discontinuous terms have been considered difficult to handle analytically and numerically in singular perturbation theory, consequently, few researchers have presented numerical methods for dealing with it. P. A. Farrell et al. (2004) presented ε-uniformly convergent numerical method. The author utilized a piecewise uniform mesh fitted to the inner layers and the standard upwind finite-difference operator on this mesh.

P. Farrell, Hegarty, Miller, O'Riordan, and Shishkin (2004) studied Singularly perturbed convection-diffusion problems with discontinuous source terms. The authors used a piecewise uniform mesh, which is fitted to the interior and boundary layers, and the standard upwind finite-difference method on this mesh.

Cen (2005) Considered singularly perturbed convection-diffusion problem with discontinuous convection coefficient and solved on a Shishkin mesh using a hybrid difference approach. Furthermore, the author showed that the approach is almost second-order convergent in the maximum norm (discrete L_∞ norm). To obtain the numerical solution and the numerical derivative, Priyadharshini and Ramanujam (2009) used a hybrid finite difference scheme on a Shishkin mesh. The authors applied the hybrid finite difference operator on a piecewise uniform mesh fitted to the inner layers. Priyadharshini et al. (2010) presented a hybrid difference schemes on the Shishkin mesh for convection-diffusion and reaction-diffusion type equations with discontinuous source terms. To solve singularly perturbed non-stationary parabolic problems with discontinuous source terms Mukherjee and Natesan (2011), presented ε-uniform error estimate of the hybrid numerical scheme. The authors discretized the time derivative by the backward-Euler method and the spatial derivatives by a proper combination of the midpoint upwind scheme in the outer regions and the classical central difference scheme in the interior layer regions on a layer resolving piecewise-uniform Shishkin mesh.

Rao, Chawla, and Chaturvedi (2022) presented a parameters-uniform numerical method for a class of coupled system of singularly perturbed time-dependent convection-diffusion equations with a discontinuous source term. The authors used an appropriate Shishkin mesh, and discretized the considered problem using an upwind central difference scheme away from the line of discontinuity, and a particular upwind central difference scheme along the line of discontinuity. Using a first-order accurate, simple upwind approach on a specially designed piecewise uniform Shishkin mesh, Sahoo and Gupta (2022) constructed almost second-order ε-uniform convergent methods for a singularly perturbed convection-diffusion problem (SPCDP) having discontinuous convective and source terms. In addition, the author constructed a Richardson extrapolation method to improve the order of convergence and accuracy.

For the singularly perturbed convection-diffusion problem with discontinuous convection coefficient, K. R. Ranjan and Gowrisankar (2022) developed the discontinuous Galerkin finite element method (DGFEM). Authors used a typical Shishkin mesh, to discritize the domain, and obtained the convergence of order $\mathcal{O}(N^{-1}lnN)$.
Mai, Zhu, and Liu (2023) developed an adaptive grid scheme based on an upwind finite difference method on an arbitrary nonuniform grid.
The present study entails a novel approach to solving the singularly perturbed convection-diffusion problem (SPCDP) with a discontinuous convection and source terms. Numerical methods often encounter significant challenges in dealing with discontinuities in the convection coefficient and source terms, necessitating specialized treatment. Our hybrid numerical scheme aims to address these challenges effectively. The proposed scheme involves a combination of numerical techniques that complement each other to achieve optimal results. The scheme leverages the strengths of both finite difference and finite element methods to solve the SPCDP with a high degree of accuracy and efficiency.

The success of our approach lies in its ability to accommodate the unique demands of the problem at hand, such as the presence of discontinuity in the convection coefficient and source terms. By doing so, we overcome the limitations of traditional numerical methods that fail to handle such complexities. Overall, our study demonstrates the efficacy of the proposed hybrid numerical scheme in addressing the challenges posed by the SPCDP with a discontinuous convective coefficient and source terms. Our findings hold significant implications for researchers and practitioners in the field and pave the way for further advancements in the area of numerical methods.

CHAPTER 3

METHODOLOGY OF THE STUDY

This chapter presents the study design, data sources, and mathematical procedures used to achieve the objectives of the study.

3.1 Study Design

In this thesis, we applied both the documentation review and numerical experimentation on singularly perturbed boundary value problems of convection-diffusion equation with discontinuous terms.

3.2 Source of Information

The relevant sources of information from both published and unpublished documents such as books, journals, and related studies found on the internet were used to obtain the desired knowledge.

3.3 Mathematical Procedures

The study followed the following procedures in order to achieve the specified particular objectives.

- Defining the problems,
- Discretizing the domain with the use of a typical Shishkin mesh,
- Constructing hybrid numerical scheme on Shishkin mesh,
- Establishing the stability analysis of the obtained numerical scheme,
- Investigating the convergence analysis of the developed numerical scheme,
- Writing an algorithm for the formulated scheme by using Matlab software,
- Provide numerical illustrations to validate the applicability of the developed numerical scheme,
- Presenting the results using graphical representations and tables.

CHAPTER 4

DESCRIPTION AND ANALYSIS OF THE METHOD

4.1 Qualitative Properties of Continuous Problem

In this section we need to discuss some properties of the considered problem. We considered the following singularly perturbed convection-diffusion problem with a discontinuity on the coefficient of convection term and source term

$$\mathcal{L}y(x) \equiv \varepsilon y''(x) + p(x)y'(x) = f(x) \tag{4.1}$$

in the domain $\Omega = (0,1)$, with the boundary conditions

$$y(0) = y_0,\ y(1) = y_1 \tag{4.2}$$

$$p(x) < -\alpha_1 < 0,\ x < r,\ \text{and}\ p(x) > \alpha_2 > 0,\ x > r \tag{4.3}$$

$$|[p](r)| \leq C,\ \ |[f](r)| \leq C \tag{4.4}$$

where: $\Omega = (0,1), \bar{\Omega} = [0,1]$, $r \in \Omega$. C is a positive arbitrary constant independent of perturbation parameter ε.
For the convenience we introduced the notations $\Omega^- = (0,r)$, $\Omega^+ = (r,1)$.
The functions $p(x)$, $f(x)$ are assumed to be twice continuously differentiable in $\Omega^- \cup \Omega^+$.
Furthermore, we assumed the discontinuity at r, and the jump discontinuity of any function η, denoted by $[\eta]$ is defined by $[\eta](r) = \eta(r^+) - \eta(r^-)$, that is $[\eta](r) = \lim_{x\to r^+}\eta(x) - \lim_{x\to r^-}\eta(x)$. The continuous problem (4.1) with the given hypotheses guarantee the existence and uniqueness of the exact solution $y \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ (P. A. Farrell et al., 2004).

Using the decomposition technique, defined in equation (4.50), Cen (2005) had proved the following Lemma, which shows the solution, and derivative bounds for the problem (4.1) - (4.4).

Lemma 1 *For $0 \leq k \leq 4$, the regular component solution v_L and v_R, and its derivatives satisfy the bound*

$$|v_L^{(k)}(x)| \leq C(1+\varepsilon^{2-k}),\ x \in \Omega^-,\ |v_R^{(k)}(x)| \leq C(1+\varepsilon^{2-k}),\ x \in \Omega^+$$
$$|[v](r)| \leq C,\ |[v'](r)| \leq C,\ |[v''](r)| \leq C$$

While, the singular component solution w_L and w_R, and its derivatives satisfy the bound

$$|w_L^{(k)}(x)| \leq C\varepsilon^{-k}exp(-(r-x)\frac{\alpha_1}{\varepsilon}),\ x \in \Omega^-,\ |w_R^{(k)}(x)| \leq C\varepsilon^{-k}exp(-(x-r)\frac{\alpha_2}{\varepsilon}),\ x \in \Omega^+$$

where, the smooth components v_L and v_R approximate v respectively to the left and to the right of the point of discontinuity $x = r$, and the non-smooth components w_L and w_R approximate w respectively to the left and to the right of the point of discontinuity $x = r$.

Proof. The proof of this *Lemma* follows from (Cen (2005), *Lemma 1*).

4.2 Piece-wise-uniform Shishkin Mesh Generation

In this section specially designed layer adapted Piece-wise uniform Shishkin mesh is introduced to construct robust and efficient numerical methods to the considered problem. Piece-wise uniform meshes introduced by Shishkin and Shishkina (2008), are a very useful tool to construct robust and efficient numerical methods to approximate the solution of singularly perturbed problems. What distinguishes a Shishkin mesh from any other mesh is the choice of a transition parameter(s), which are the point(s) at which the mesh size changes abruptly. In our study, we denote this transition parameter by τ_1 and τ_2.

The domain Ω has N mesh intervals, which are divided into four uniform mesh sub-intervals as illustrated in the figure below.

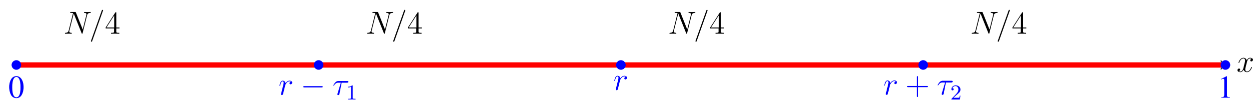

Figure 4.1: Domain decomposition of Shishkin mesh

Shishkin meshes capture the interior layer by condensing a large number of points in the interior region as ε approaches 0.

We divide the domain Ω into four uniform mesh sub-intervals as follows.

$$\bar{\Omega} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$$

$$\text{where },\Omega_1 = [0, r - \tau_1],\ \Omega_2 = [r - \tau_1, r],\ \Omega_3 = [r, r + \tau_2],\ \Omega_4 = [r + \tau_2, 1]$$

with $0 < \tau_1 \leq \frac{r}{2},\ 0 < \tau_2 \leq \frac{(1-r)}{2}$.

The transition parameters τ_1 and τ_2 are constructed as

$$\tau_1 = min\{\frac{r}{2}, z\varepsilon lnN\}, \qquad \tau_2 = min\{\frac{1-r}{2}, z\varepsilon lnN\},$$

$$\text{where, } z = 2\alpha^{-1},\ \alpha = min\{\alpha_1, \alpha_2\}.$$

Then, for each subdivision, it has a mesh width of h_i. Where,

$$h_i = \begin{cases} H_1 = \frac{4(r-\tau_1)}{N},\ \text{for } i = 1, 2, \cdots, \frac{N}{4} \\ h_1 = \frac{4\tau_1}{N},\ \text{for } i = \frac{N}{4} + 1, \cdots, \frac{N}{2} \\ h_2 = \frac{4\tau_2}{N},\ \text{for } i = \frac{N}{2} + 1, \cdots, \frac{3N}{4} \\ H_2 = \frac{4(1-r-\tau_2)}{N},\ \text{for } i = \frac{3N}{4} + 1, \cdots, N. \end{cases} \tag{4.5}$$

Now, let, $x_1 = 0 : H_1 : r - \tau_1,\ x_2 = r - \tau_1 : h_1 : r,$

$x_3 = r : h_2 : r + \tau_2,\ x_4 = r + \tau_2 : H_2 : 1$

Then, the mesh points given by

$$x_i = [x_1, x_2, x_3, x_4],\ \text{for } i = 1, 2, 3, \cdots, N.$$

That is the nodal points (the set of interior points of the mesh) are expressed as

$$\Omega^N = \{x_i : 1 \le i \le \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \le i \le N - 1\}$$

Then clearly, $x_{\frac{N}{2}} = r$, and $\bar{\Omega}^N = \{x_i\}_0^N$
If we assume $\tau = \tau_1 = \tau_2 = z\varepsilon lnN$, then, the mesh has space

$$h_i = \begin{cases} H_1 = \frac{4(r-\tau)}{N},\ \text{for } i = 1, 2, \cdots, \frac{N}{4}, \\ h = \frac{4\tau}{N},\ \text{for } i = \frac{N}{4} + 1, \cdots, \frac{3N}{4}, \\ H_2 = \frac{4(1-r-\tau)}{N},\ \text{for } i = \frac{3N}{4} + 1, \cdots, N. \end{cases} \quad (4.6)$$

4.3 Construction of the Numerical Scheme

In these subsection, we develop the proposed hybrid numerical scheme. So that we construct each method separately before combining them to construct the hybrid scheme.

4.3.1 Non-standard finite difference method

We consider singularly perturbed convection diffusion boundary value problem of the form:

$$\mathcal{L}y(x) \equiv \varepsilon y''(x) + p(x)y'(x) = f(x) \quad (4.7)$$

in the interval $\Omega = (0, 1)$ and the boundary conditions:

$$y(0) = y_0, y(1) = y_1 \quad (4.8)$$

where, $p(x)$ and $f(x)$ are known functions, and $0 < \varepsilon \ll 1$ is a small positive perturbation parameter. It is assumed that the functions $p(x)$, $f(x)$ are sufficiently smooth. To derive the non-standard finite difference method for the singularly perturbed convection-diffusion problem (4.7), we start by discretizing the domain $\Omega = (0, 1)$ into a uniform grid with step size h.
Now, we divide the interval Ω in to N equal parts with constant mesh length $h = \frac{1}{N}$. Let $0 = x_0 < x_1 < \cdots < x_N = 1$, be the mesh points. Then we have:

$$x_i = x_0 + ih, \quad i = 1, \cdots, N - 1$$

To construct exact finite difference methods, we follow (Mickens, 2005). The denominator function for the discrete derivatives in Mickens' techniques and rules to construct exact finite-difference methods for various problems must be presented in terms of more complex functions of step sizes rather than those used in the standard methods. We consider the homogeneous constant coefficient sub-equations of (4.7) given by:

$$\varepsilon y''(x) + py'(x) = 0 \tag{4.9}$$

For $p > 0$, (4.9) has two linearly independent solutions namely $e^{\lambda 1x}$ and $e^{\lambda 2x}$ with :

$$\lambda_1 = \frac{-p + \sqrt{p^2}}{2\varepsilon}, \lambda_2 = \frac{-p - \sqrt{p^2}}{2\varepsilon}$$

We denote the approximate solution of $y(x)$ at x_i by Y_i. The solution of a difference equation at the point of the grid x_i given by

$$Y_i = C_1 e^{\lambda 1x_i} + C_2 e^{\lambda 2x_i}.$$

Using the theory of difference equations for second-order linear difference equations, we have

$$\begin{vmatrix} Y_{i-1} & e^{\lambda 1x_i-1} & e^{\lambda 2x_{i-1}} \\ Y_i & e^{\lambda 1x_i} & e^{\lambda 2x_i} \\ Y_{i+1} & e^{\lambda 1x_{i+1}} & e^{\lambda 2x_{i+1}} \end{vmatrix} = 0$$

Now, we solve the determinant as follows:

$$\begin{aligned} &Y_{i-1}(e^{\lambda_1 x_i} \cdot e^{\lambda_2 x_{i+1}} - e^{\lambda_2 x_i} \cdot e^{\lambda_1 x_{i+1}}) - Y_i(e^{\lambda_1 x_{i-1}} \cdot e^{\lambda_2 x_{i+1}} - e^{\lambda_2 x_{i-1}} \cdot e^{\lambda_1 x_{i+1}}) \\ &+ Y_{i+1}(e^{\lambda_1 x_{i-1}} \cdot e^{\lambda_2 x_i} - e^{\lambda_1 x_i} \cdot e^{\lambda_2 x_{i-1}}) = 0. \end{aligned}$$

Substituting the values of $\lambda_{1,2}$ we get:

$$\begin{aligned} &Y_{i-1}[e^{\frac{-p}{2\varepsilon}(x_i+x_{i+1})}(e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)})] - Y_i[e^{\frac{-p}{2\varepsilon}(x_{i-1}+x_{i+1})}(e^{\frac{\sqrt{p^2}}{2\varepsilon}(-2h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(2h)})] \\ &+ Y_{i+1}[e^{\frac{-p}{2\varepsilon}(x_{i-1}+x_i)}(e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)})] = 0. \end{aligned}$$

Dividing both sides by $e^{\frac{-p}{2\varepsilon}(x_{i-1}+x_{i+1})}$, we get:

$$\begin{aligned} &Y_{i-1}[e^{\frac{p}{2\varepsilon}(-h)}(e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)})] - Y_i[(e^{\frac{\sqrt{p^2}}{2\varepsilon}(-2h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(2h)})] \\ &+ Y_{i+1}[e^{\frac{p}{2\varepsilon}(h)}(e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)})] = 0. \end{aligned}$$

Dividing both sides by $e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)}$,

$$e^{\frac{p}{2\varepsilon}(-h)}Y_{i-1} - [\frac{e^{\frac{\sqrt{p^2}}{2\varepsilon}(-2h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(2h)}}{e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)}}]Y_i + e^{\frac{p}{2\varepsilon}(h)}Y_{i+1} = 0 \tag{4.10}$$

Applying the identity of hyperbolic function ,

$$sinh(-2x) = -sinh2x = \frac{e^{-2x} - e^{2x}}{2},$$

$$-2sinh2x = e^{-2x} - e^{2x} \Rightarrow e^{\frac{\sqrt{p^2}}{2\varepsilon}(-2h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(2h)} = -2sinh2(\frac{\sqrt{p^2}}{2\varepsilon}h).$$

For the denominator $e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)}$, of (4.10) we use the hyperbolic identity, $-2sinhx = e^{-x} - e^{x}$, then we have

$$e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)} = -2sinh(\frac{\sqrt{p^2}h}{2\varepsilon}).$$

Now, $$\frac{e^{\frac{\sqrt{p^2}}{2\varepsilon}(-2h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(2h)}}{e^{\frac{\sqrt{p^2}}{2\varepsilon}(-h)} - e^{\frac{\sqrt{p^2}}{2\varepsilon}(h)}} = \frac{-2sinh2(\frac{\sqrt{p^2}}{2\varepsilon}h)}{-2sinh(\frac{\sqrt{p^2}}{2\varepsilon}h)} = \frac{sinh2(\frac{\sqrt{p^2}}{2\varepsilon}h)}{sinh(\frac{\sqrt{p^2}}{2\epsilon}h)}.$$

Recall the identity of hyperbolic function : $sinh2x = 2sinhxcoshx$, then

$$\frac{sinh2(\frac{\sqrt{p^2}}{2\varepsilon}h)}{sinh(\frac{\sqrt{p^2}}{2\varepsilon}h)} = \frac{2sinh(\frac{\sqrt{p^2}}{2\varepsilon}h)cosh\frac{\sqrt{p^2}}{2\varepsilon}h}{sinh(\frac{\sqrt{p^2}}{2\varepsilon}h)} = 2cosh\frac{\sqrt{p^2}}{2\varepsilon}h$$

Then, (4.10) becomes :

$$e^{\frac{p}{2\varepsilon}(-h)}Y_{i-1} - 2cosh(\frac{\sqrt{p^2}}{2\varepsilon}h)Y_i + e^{\frac{p}{2\varepsilon}h}Y_{i+1} = 0 \quad (4.11)$$

Equation (4.11) is an exact difference scheme of equation (4.9). Multiplying both sides of equation (4.11) by $e^{\frac{p}{2\varepsilon}h}$, we obtain

$$Y_{i-1} - 2e^{\frac{p}{2\varepsilon}h} \cdot cosh(\frac{\sqrt{p^2}}{2\varepsilon}h)Y_i + e^{\frac{p}{2\varepsilon}h} \cdot e^{\frac{p}{2\varepsilon}h}Y_{i+1} = 0.$$

By using the identity, $cosh(\frac{\sqrt{p^2}}{2\varepsilon}h) = \frac{e^{\frac{\sqrt{p^2}}{2\varepsilon}h} + e^{\frac{-\sqrt{p^2}}{2\varepsilon}h}}{2}$ and approximating $\sqrt{p^2} = p$, we have

$$Y_{i-1} - e^{\frac{p}{2\epsilon}h}(e^{\frac{p}{2\epsilon}h} + e^{-\frac{p}{2\epsilon}h})Y_i + e^{\frac{p}{\epsilon}h}Y_{i+1} = 0,$$
$$Y_{i-1} - [e^{\frac{p}{2\varepsilon}h + \frac{p}{2\varepsilon}h} + e^{-\frac{p}{2\varepsilon}h + \frac{p}{2\varepsilon}h}]Y_i + e^{\frac{p}{\varepsilon}h}Y_{i+1} = 0,$$

$$Y_{i-1} - [e^{\frac{p}{\varepsilon}h} + 1]Y_i + e^{\frac{p}{\varepsilon}h}Y_{i+1} = 0. \quad (4.12)$$

We can write the exponential term as $e^{\frac{p}{\varepsilon}h} = e^{\frac{p}{\varepsilon}h} - 1 + 1$, substituting in (4.12), it becomes

$$Y_{i-1} - [e^{\frac{p}{\varepsilon}h} - 1]Y_i - 2Y_i + [e^{\frac{p}{\varepsilon}h} - 1]Y_{i+1} + Y_{i+1} = 0.$$

Which is equivalent with:

$$Y_{i-1} - 2Y_i + Y_{i+1} = [e^{\frac{p}{\varepsilon}h} - 1](Y_i - Y_{i+1}),$$
$$\frac{Y_{i-1} - 2Y_i + Y_{i+1}}{[e^{\frac{p}{\varepsilon}h} - 1]} - (Y_i - Y_{i+1}) = 0,$$
$$\frac{Y_{i-1} - 2Y_i + Y_{i+1}}{[e^{\frac{p}{\varepsilon}h} - 1]} + (Y_{i+1} - Y_i) = 0.$$

Multiplying both sides by $\varepsilon\gamma^2$, where $\gamma = \sqrt{\frac{p}{\varepsilon h}}$ we get

$$\begin{aligned}
&\varepsilon\gamma^2 \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{[e^{\frac{p}{\varepsilon}h} - 1]} + \varepsilon\gamma^2(Y_{i+1} - Y_i) = 0,\\
&\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\frac{1}{\gamma^2}(e^{\frac{p}{\varepsilon}h} - 1)} + \varepsilon\gamma^2(Y_{i+1} - Y_i) = 0,\\
&\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\frac{\varepsilon h}{p}(e^{\frac{p}{\varepsilon}h} - 1)} + p\frac{Y_{i+1} - Y_i}{h} = 0.
\end{aligned}$$

For the second derivative approximation, let's denote the denominator function $\frac{\epsilon h}{p}(e^{\frac{p}{\varepsilon}h} - 1) = \psi^2$. Adopting it for the variable coefficient problem (4.7) expressed as:

$$\psi_i^2 = \frac{\varepsilon h}{p(x_i)}(e^{\frac{p(x_i)}{\varepsilon}h} - 1). \tag{4.13}$$

Then, the nonstandard finite difference scheme of the non homogeneous SPCDBVP, (4.7) with the given boundary conditions becomes

$$\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\psi_i^2} + p(x_i)\frac{Y_{i+1} - Y_i}{h} = f(x_i), \tag{4.14}$$

$$Y(0) = Y_0,\ Y(1) = Y_1. \tag{4.15}$$

4.3.2 Mid-point upwind scheme

To derive the midpoint upwind finite difference scheme for the given singularly perturbed convection-diffusion problem, we start by discretizing the domain $\Omega = (0, 1)$ into a uniform grid with N equally spaced points. Let $\Delta x = h = \frac{1}{N}$ be the grid spacing. Let our (arbitrary) mesh $0 = x_0 < x_1 < \ldots < x_N = 1$, be the mesh points. Then we have:

$$x_i = x_0 + ih, \quad i = 0, 1, \ldots, N,$$

where N is positive integer. For each $i \geq 1$, we set $h_i = x_i - x_{i-1}$, and $\hbar_i = h_{i+1} + h_i$.
To approximate first order derivative $y'(x)$ and second order derivative $y''(x)$ in (4.7), using Taylor series approximation

$$y_{x_{i+1}} = y(x_{i+h_{i+1}}) = y_{i+1} = y_i + h_{i+1}y_i' + \frac{h_{i+1}^2}{2}y_i'' + \frac{h_{i+1}^3}{6}y_i''' + \frac{h_{i+1}^4}{24}y_i^{(4)} + \frac{h_{i+1}^5}{120}y_i^{(5)} + \cdots \tag{4.16}$$

$$y_{x_{i-1}} = y(x_{i-h_i}) = y_{i-1} = y_i - h_iy_i' + \frac{h_i^2}{2}y_i'' - \frac{h_i^3}{6}y_i''' + \frac{h_i^4}{24}y_i^{(4)} - \frac{h_i^5}{120}y_i^{(5)} + \cdots \tag{4.17}$$

From Taylor series expansions, (4.16) and (4.17), we get:

$$D_x^- y(x_i) = \frac{Y_i - Y_{i-1}}{h_i}, \qquad D_x^+ y(x_i) = \frac{Y_{i+1} - Y_i}{h_{i+1}}, \qquad D_x^0 y(x_i) = \frac{Y_{i+1} - Y_{i-1}}{\hbar_i}$$

where, $D_x^- y(x_i)$, $D_x^+ y(x_i)$ and $D_x^0 y(x_i)$ are back ward, forward and central difference operators respectively.

Multiplying (4.17) by $\frac{h_{i+1}}{h_i}$, and adding the product to equation (4.16), we shall approximate the second-order derivative at x_i by

$$\begin{aligned} D^+D^-y(x_i) &= \frac{2}{\hbar_i}(D^+y(x_i) - D_x^-y(x_i)) \\ &= \frac{2}{\hbar_i}(\frac{Y_{i+1}-Y_i}{h_{i+1}} - \frac{Y_i - Y_{i-1}}{h_i}) \end{aligned} \tag{4.18}$$

The upwind scheme for (4.7) becomes

$$\begin{aligned} \mathcal{L}_{up}Y_i \equiv \varepsilon D^+D^-y(x_i) + p_iD^M = f_i, \\ Y(0) = Y_0, Y(1) = Y_1. \end{aligned} \tag{4.19}$$

where,

$$D^M = \begin{cases} D_x^+y(x_i), \text{ if } p_i > 0, \\ D_x^-y(x_i), \text{ if } p_i < 0 \end{cases}$$

Now, we focus on the midpoint upwind scheme for equation (4.7).
$0 = x_0 < x_1 < ... < x_N = 1$, be the mesh points, then

$$x_i = x_0 + ih, \quad i = 0, 1, \ldots, N.$$

For each $i \geq 1$, we set

$$x_{i-\frac{1}{2}} = \frac{x_{i-1}+x_i}{2}, \qquad x_{i+\frac{1}{2}} = \frac{x_{i+1}+x_i}{2}, \qquad p_{i-\frac{1}{2}} = \frac{p_{i-1}+p_i}{2},$$

$$p_{i+\frac{1}{2}} = \frac{p_{i+1}+p_i}{2}, \qquad f_{i-\frac{1}{2}} = \frac{f_{i-1}+f_i}{2}, \qquad f_{i+\frac{1}{2}} = \frac{f_{i+1}+f_i}{2}.$$

Then, the midpoint upwind scheme is given by

$$\begin{aligned} \mathcal{L}_{md}Y_i \equiv \varepsilon D^+D^-y(x_i) + p_{(x_i)}D^M = f_i \\ Y(0) = Y_0, Y(1) = Y_1 \end{aligned} \tag{4.20}$$

where,

$$D^M = \begin{cases} D_x^+y(x_i), \text{ if } p_i > 0 \\ D_x^-y(x_i), \text{ if } p_i < 0 \end{cases}, \qquad p_{(x_i)} = \begin{cases} p_{i+\frac{1}{2}}, \text{ if } p_i > 0 \\ p_{i-\frac{1}{2}}, \text{ if } p_i < 0 \end{cases}, \qquad f_i = \begin{cases} f_{i+\frac{1}{2}}, \text{ if } p_i > 0 \\ f_{i-\frac{1}{2}}, \text{ if } p_i < 0. \end{cases}$$

4.4 Hybrid Numerical Scheme

In this subsection, we present a hybrid numerical scheme on both uniform and Shishkin meshes. Our hybrid numerical scheme combines non-standard finite difference and midpoint upwind schemes. Since, the hybrid numerical scheme on Shishkin mesh is our main focus, this thesis delves greater detail about it.

4.4.1 Hybrid numerical scheme on uniform mesh

To develop hybrid numerical scheme on uniform mesh, we use the approximations for second order derivatives from equation (4.14) with the constant mesh length h.
Then, for $p_i > 0$ and $i = 3N/4, \ldots, N-1$, we use

$$\mathcal{L}^{+}_{1_{hyb}} Y_i \equiv \varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\psi_i^2} + p_{i+\frac{1}{2}} \frac{Y_{i+1} - Y_i}{h} = f_{i+\frac{1}{2}}. \tag{4.21}$$

For $p_i < 0$ and $i = 1, \ldots, N/4$, we use

$$\mathcal{L}^{-}_{1_{hyb}} Y_i \equiv \varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\psi_i^2} + p_{i-\frac{1}{2}} \frac{Y_i - Y_{i-1}}{h} = f_{i-\frac{1}{2}} \tag{4.22}$$

For second and third interval, (for $i = \frac{N}{4} + 1, \cdots, \frac{N}{2} - 1, \frac{N}{2} + 1, \cdots, \frac{3N}{4} - 1$) we use

$$\mathcal{L}^{2}_{1_{hyb}} Y_i \equiv \varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\psi_i^2} + p_i \frac{Y_i - Y_{i-1}}{h} = f_i, \text{ for } i = \frac{N}{4} + 1, \cdots, \frac{N}{2} - 1 \tag{4.23}$$

$$\mathcal{L}^{3}_{1_{hyb}} Y_i \equiv \varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\psi_i^2} + p_i \frac{Y_{i+1} - Y_i}{h} = f_i, \text{ for } i = \frac{N}{2} + 1, \cdots, \frac{3N}{4} - 1 \tag{4.24}$$

Then, the proposed hybrid numerical scheme reads, for $i = 1 : N-1$

$$\begin{cases} \mathcal{L}_{1_{hyb}} Y_i = \mathcal{F}_{1_{hyb}} \\ Y(0) = Y_0, \qquad Y(1) = Y_1 \end{cases} \tag{4.25}$$

where,

$$\mathcal{L}_{1_{hyb}} Y_i = \begin{cases} \mathcal{L}^{-}_{1_{hyb}} Y_i, \text{ for } i = 1, \cdots, \frac{N}{4}, \text{ and } p_i < 0, \\ \mathcal{L}^{2}_{1_{hyb}} Y_i, \text{ for } i = \frac{N}{4} + 1, \cdots, \frac{N}{2} - 1, \\ \mathcal{L}^{3}_{1_{hyb}} Y_i, \text{ for } i = \frac{N}{2} + 1, \cdots, \frac{3N}{4} - 1, \\ \mathcal{L}^{+}_{1_{hyb}} Y_i, \text{ for } i = \frac{3N}{4}, \cdots, N-1, \text{ and } p_i > 0, \\ D_x^{+} y(x_i) - D_x^{-} y(x_i), \text{ for } i = \frac{N}{2} \end{cases},$$

$$\mathcal{F}_{1_{hyb}} = \begin{cases} f_{i-\frac{1}{2}}, \text{ for } i = 1, \cdots, \frac{N}{4}, \text{ and } p_i < 0, \\ f_i, \text{ for } i = \frac{N}{4} + 1, \cdots, \frac{N}{2} - 1, \frac{N}{2} + 1, \cdots, \frac{3N}{4} - 1, \\ f_{i+\frac{1}{2}}, \text{ for } i = \frac{3N}{4}, \cdots, N-1, \text{ and } p_i > 0, \\ 0, \text{ for } i = \frac{N}{2} \end{cases}$$

We apply the one-sided difference formula for the treatment of the point of discontinuity r in the domain at $i = \frac{N}{2}$. The Taylor expansion of the function about the point of interest, x_i, is:

$$y(x_i + h) = y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \cdots \tag{4.26}$$

$$y(x_i+2h)=y_{i+2}=y_i+2hy'_i+\frac{4}{2!}h^2y''_i+\frac{8}{3!}h^3y'''_i+\cdots \tag{4.27}$$

To obtain the desired result for $i=\frac{N}{2}$, multiplying (4.26) by four, and subtracting (4.27), we have

$$y'_i=\frac{4Y_{\frac{N}{2}+1}-Y_{\frac{N}{2}+2}-3Y_{\frac{N}{2}}}{2h}=D_x^+y(x_i)$$

To approximate backward difference, $D_x^-y(x_i)$, using Taylor series expansion:

$$y(x_i-h)=y_{i-1}=y_i-hy'_i+\frac{h^2}{2}y''_i-\frac{h^3}{6}y'''_i+\cdots \tag{4.28}$$

$$y(x_i-2h)=y_{i-2}=y_i-2hy'_i+\frac{4}{2!}h^2y''_i-\frac{8}{3!}h^3y'''_i+\cdots \tag{4.29}$$

Multiplying (4.28) by four and subtracting (4.29), we obtain

$$y'_i=\frac{-4Y_{\frac{N}{2}-1}+Y_{\frac{N}{2}-2}+3Y_{\frac{N}{2}}}{2h}=D_x^-y(x_i)$$

$$\begin{aligned}&D_x^+y(x_i)-D_x^-y(x_i)=0\\&\frac{4Y_{\frac{N}{2}+1}-Y_{\frac{N}{2}+2}-3Y_{\frac{N}{2}}}{2h}-\frac{-4Y_{\frac{N}{2}-1}+Y_{\frac{N}{2}-2}+3Y_{\frac{N}{2}}}{2h}=0\end{aligned} \tag{4.30}$$

Now, let's consider the hybrid scheme obtained by using central finite difference to approximate first order derivative (convection term), given by

$$\mathcal{L}_{1_{hyb}}Y_i\equiv\varepsilon\frac{Y_{i-1}-2Y_i+Y_{i+1}}{\psi_i^2}+p_i\frac{Y_{i+1}-Y_{i-1}}{2h}=f_i \tag{4.31}$$

From (4.31), we calculate $Y_{\frac{N}{2}-2}$ and $Y_{\frac{N}{2}+2}$, then substitute the results in (4.30).

For $i=N/2-1$, (4.31) becomes

$$\varepsilon\frac{Y_{N/2-2}-2Y_{N/2-1}+Y_{N/2}}{\psi^2_{N/2-1}}+p_{N/2-1}\frac{Y_{N/2}-Y_{N/2-2}}{2h}=f_{N/2-1},$$

$$Y_{\frac{N}{2}-2}=\frac{4\varepsilon h}{2\varepsilon h-\psi^2_{N/2-1}p_{N/2-1}}Y_{\frac{N}{2}-1}-\frac{2\varepsilon h+\psi^2_{N/2-1}p_{N/2-1}}{2\varepsilon h-\psi^2_{N/2-1}p_{N/2-1}}Y_{\frac{N}{2}}+\frac{2h\psi^2_{N/2-1}}{2\varepsilon h-\psi^2_{N/2-1}p_{\frac{N}{2}-1}}f_{\frac{N}{2}-1}.$$

For $i=N/2+1$, (4.31) becomes

$$\varepsilon\frac{Y_{N/2}-2Y_{N/2+1}+Y_{N/2+2}}{\psi^2_{N/2+1}}+p_{N/2+1}\frac{Y_{N/2+2}-Y_{N/2}}{2h}=f_{N/2+1},$$

$$Y_{\frac{N}{2}+2}=\frac{-2\varepsilon h+\psi^2_{N/2+1}p_{N/2+1}}{2\varepsilon h+\psi^2_{N/2+1}p_{N/2+1}}Y_{\frac{N}{2}}+\frac{4\varepsilon h}{2\varepsilon h+\psi^2_{N/2+1}p_{N/2+1}}Y_{\frac{N}{2}+1}+\frac{2h\psi^2_{N/2+1}}{2\varepsilon h+\psi^2_{\frac{N}{2}+1}p_{\frac{N}{2}+1}}f_{\frac{N}{2}+1}.$$

Then, substituting $Y_{\frac{N}{2}-2}$ and $Y_{\frac{N}{2}+2}$, into (4.30), we obtain the following formula for $i = \frac{N}{2}$.

$$\mathcal{L}^r_{1_{hyb}} Y_i \equiv (\frac{-4\varepsilon h}{2\varepsilon h - \psi^2_{N/2-1} p_{N/2-1}} + 4) Y_{\frac{N}{2}-1} +$$
$$(\frac{2\varepsilon h + \psi^2_{N/2-1} p_{N/2-1}}{2\varepsilon h - \psi^2_{N/2-1} p_{N/2-1}} + \frac{2\varepsilon h - \psi^2_{N/2+1} p_{N/2+1}}{2\varepsilon h + \psi^2_{N/2+1} p_{N/2+1}} - 6) Y_{\frac{N}{2}} + (4 - \frac{4\varepsilon h}{2\varepsilon h + \psi^2_{\frac{N}{2}+1} p_{\frac{N}{2}+1}}) Y_{\frac{N}{2}+1}$$
$$= \frac{2h\psi^2_{N/2-1}}{2\varepsilon h - \psi^2_{N/2-1} p_{N/2-1}} f_{\frac{N}{2}-1} + \frac{2h\psi^2_{N/2+1}}{2\varepsilon h + \psi^2_{N/2+1} p_{N/2+1}} f_{\frac{N}{2}+1} \tag{4.32}$$

Substituting the expression (4.32) into (4.25), our hybrid scheme's final formula on the uniform mesh for $i = 1 : N-1$, is given as follows:

$$\begin{cases} \mathcal{L}^N_{1_{hyb}} Y_i = \bar{\mathcal{F}}_{hyb}, \\ Y(0) = Y_0, \qquad Y(1) = Y_1. \end{cases} \tag{4.33}$$

where,

$$\mathcal{L}^N_{1_{hyb}} Y_i = \begin{cases} \mathcal{L}^-_{1_{hyb}} Y_i, \text{ for } i = 1, \cdots, \frac{N}{4}, \text{ and } p_i < 0, \\ \mathcal{L}^2_{1_{hyb}} Y_i, \text{ for } i = \frac{N}{4}+1, \cdots, \frac{N}{2}-1, \\ \mathcal{L}^3_{1_{hyb}} Y_i, \text{ for } i = \frac{N}{2}+1, \cdots, \frac{3N}{4}-1, \\ \mathcal{L}^+_{1_{hyb}} Y_i, \text{ for } i = \frac{3N}{4}, \cdots, N-1, \text{ and } p_i > 0, \\ \mathcal{L}^r_{1_{hyb}} Y_i, \text{ for } i = \frac{N}{2}, \end{cases}$$

$$\bar{\mathcal{F}}_{1_{hyb}} = \begin{cases} f_{i-\frac{1}{2}}, \text{ for } i = 1, \cdots, \frac{N}{4}, \text{ and } p_i < 0, \\ f_{i+\frac{1}{2}}, \text{ for } i = \frac{3N}{4}, \cdots, N-1, \text{ and } p_i > 0, \\ f_i, \text{ for } i = \frac{N}{4}+1, \cdots, \frac{N}{2}-1, \frac{N}{2}+1, \cdots, \frac{3N}{4}-1, \\ \frac{2h\psi^2_{N/2-1}}{2\varepsilon h - \psi^2_{N/2-1} p_{N/2-1}} f_{\frac{N}{2}-1} + \frac{2h\psi^2_{N/2+1}}{2\varepsilon h + \psi^2_{N/2+1} p_{N/2+1}} f_{\frac{N}{2}+1}, \text{ for } i = \frac{N}{2}. \end{cases}$$

4.4.2 Hybrid numerical scheme on Shishkin mesh

For implementing a hybrid numerical scheme on Shishkin meshes, we apply equation (4.18), with the use of Shishkin meshes defined in (4.5).
When using a fitted mesh, the following modifications are required for denominator function (4.13):

$$\psi_i^2 = \frac{\varepsilon h}{p(x_i)} (e^{\frac{p(x_i)}{\varepsilon} h} - 1).$$

The required modification is

$$\Psi_i = \frac{\varepsilon}{p(x_i)} (e^{\frac{p(x_i)}{\varepsilon} h_i} - 1), \ \Psi_{i+1} = \frac{\varepsilon}{p(x_i)} (e^{\frac{p(x_i)}{\varepsilon} h_{i+1}} - 1).$$

Then interms of h_i and h_{i+1} in equation (4.18), we use Ψ_i and Ψ_{i+1} respectively. Then (4.18) becomes

$$D^+D^-_\Psi y(x_i) = \frac{2}{\hbar_i}(\frac{Y_{i+1}-Y_i}{\Psi_{i+1}} - \frac{Y_i - Y_{i-1}}{\Psi_i}). \tag{4.34}$$

Then, for $p_i > 0$, and $i = 3N/4, \ldots, N-1$, we use the approximation:

$$\begin{aligned} \mathcal{L}^+_{hyb}Y_i &\equiv \varepsilon D^+D^-_\Psi y(x_i) + p_{i+\frac{1}{2}}D^+_x y(x_i) = f_{i+\frac{1}{2}}. \\ &\equiv \frac{2\varepsilon}{\hbar_i}(\frac{Y_{i+1}-Y_i}{\Psi_{i+1}} - \frac{Y_i-Y_{i-1}}{\Psi_i}) + p_{i+\frac{1}{2}}\frac{Y_{i+1}-Y_i}{h_{i+1}} = f_{i+\frac{1}{2}}. \end{aligned}$$

For $p_i < 0$, and $i = 1, \ldots, N/4$, we use the approximation:

$$\begin{aligned} \mathcal{L}^-_{hyb}Y_i &\equiv \varepsilon D^+D^-_\Psi y(x_i) + p_{i-\frac{1}{2}}D^-_x y(x_i) = f_{i-\frac{1}{2}}. \\ &\equiv \frac{2\varepsilon}{\hbar_i}(\frac{Y_{i+1}-Y_i}{\Psi_{i+1}} - \frac{Y_i-Y_{i-1}}{\Psi_i}) + p_{i-\frac{1}{2}}\frac{Y_i-Y_{i-1}}{h_i} = f_{i-\frac{1}{2}}. \end{aligned}$$

For second and third interval we use equation (4.34) to approximate second order derivative, and to approximate first order derivative we use $D^-_x y(x_i)$ (from the left of the point discontinuity) and $D^+_x y(x_i)$ (from the right of the point discontinuity). That is given by

$$\begin{aligned} &\mathcal{L}^2_{hyb}Y_i \equiv D^+D^-_\Psi y(x_i) + p_iD^-_x y(x_i) = f_i, \text{ for } i = \frac{N}{4}+1, \cdots, \frac{N}{2}-1 \\ &\equiv \frac{2\varepsilon}{\hbar_i}(\frac{Y_{i+1}-Y_i}{\Psi_{i+1}} - \frac{Y_i-Y_{i-1}}{\Psi_i}) + p_i\frac{Y_i-Y_{i-1}}{h_i} = f_i. \end{aligned} \tag{4.35}$$

$$\begin{aligned} &\mathcal{L}^3_{hyb}Y_i \equiv D^+D^-_\Psi y(x_i) + p_iD^+_x y(x_i) = f_i, \text{ for } i = \frac{N}{2}+1, \cdots, \frac{3N}{4}-1 \\ &\equiv \frac{2\varepsilon}{\hbar_i}(\frac{Y_{i+1}-Y_i}{\Psi_{i+1}} - \frac{Y_i-Y_{i-1}}{\Psi_i}) + p_i\frac{Y_{i+1}-Y_i}{h_{i+1}} = f_i. \end{aligned} \tag{4.36}$$

Then, the proposed hybrid numerical scheme reads, for $i = 1 : N-1$

$$\begin{cases} \mathcal{L}_{hyb}Y_i = \mathcal{F}_{hyb}, \\ Y(0) = Y_0, \qquad Y(1) = Y_1, \end{cases} \tag{4.37}$$

where,

$$\mathcal{L}_{hyb}Y_i = \begin{cases} \mathcal{L}^-_{hyb}Y_i, \text{ for } i = 1, \cdots, \frac{N}{4}, \text{ and } p_i < 0, \\ \mathcal{L}^2_{hyb}Y_i, \text{ for } i = \frac{N}{4}+1, \cdots, \frac{N}{2}-1, \\ \mathcal{L}^3_{hyb}Y_i, \text{ for } i = \frac{N}{2}+1, \cdots, \frac{3N}{4}-1, \\ \mathcal{L}^+_{hyb}Y_i, \text{ for } i = \frac{3N}{4}, \cdots, N-1, \text{ and } p_i > 0, \\ D^+_x y(x_i) - D^-_x y(x_i), \text{ for } i = \frac{N}{2} \end{cases},$$

$$\mathcal{F}_{hyb} = \begin{cases} f_{i-\frac{1}{2}}, \text{ for } i = 1, \cdots, \frac{N}{4}, \text{ and } p_i < 0, \\ f_i, \text{ for } i = \frac{N}{4}+1, \cdots, \frac{N}{2}-1, \frac{N}{2}+1, \cdots, \frac{3N}{4}-1, \\ f_{i+\frac{1}{2}}, \text{ for } i = \frac{3N}{4}, \cdots, N-1, \text{ and } p_i > 0, \\ 0, \text{ for } i = \frac{N}{2}. \end{cases}$$

We apply the one-sided difference formula for the treatment of the point of discontinuity r in the domain at $i = \frac{N}{2}$. The Taylor expansion of the function about the point of interest, x_i, is:

$$y(x_i + h_i) = y_{i+1} = y_i + h_i y_i' + \frac{h_i^2}{2} y_i'' + \frac{h_i^3}{6} y_i''' + \cdots \tag{4.38}$$

$$y(x_i + 2h_i) = y_{i+2} = y_i + 2h_i y_i' + \frac{4}{2!} h_i^2 y_i'' + \frac{8}{3!} h_i^3 y_i''' + \cdots \tag{4.39}$$

To obtain the desired result for $i = \frac{N}{2}$, multiply (4.38) by four, and subtract (4.39), we have

$$y_i' = \frac{4Y_{\frac{N}{2}+1} - Y_{\frac{N}{2}+2} - 3Y_{\frac{N}{2}}}{2h_{N/2}} = D_x^+ y(x_i)$$

For backward difference, $D_x^- y(x_i)$, using Taylor series expansion:

$$y(x_i - h_i) = y_{i-1} = y_i - h y_i' + \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i''' + \cdots \tag{4.40}$$

$$y(x_i - 2h_i) = y_{i-2} = y_i - 2h_i y_i' + \frac{4}{2!} h^2 y_i'' - \frac{8}{3!} h^3 y_i''' + \cdots \tag{4.41}$$

Multiplying (4.40) by four and subtracting (4.41),

$$y_i' = \frac{-4Y_{\frac{N}{2}-1} + Y_{\frac{N}{2}-2} + 3Y_{\frac{N}{2}}}{2h_{N/2}} = D_x^- y(x_i)$$

$$\begin{aligned} & D_x^+ y(x_i) - D_x^- y(x_i) = 0 \\ & \frac{4Y_{\frac{N}{2}+1} - Y_{\frac{N}{2}+2} - 3Y_{\frac{N}{2}}}{2h_{N/2}} - \frac{-4Y_{\frac{N}{2}-1} + Y_{\frac{N}{2}-2} + 3Y_{\frac{N}{2}}}{2h_{N/2}} = 0 \end{aligned} \tag{4.42}$$

Now, let's consider the hybrid scheme obtained by using central finite difference, given by

$$\begin{aligned} \mathcal{L}_{hyb} Y_i &\equiv D^+ D_\Psi^- y(x_i) + p_i D_x^0 y(x_i) = f_i, \\ &\equiv \frac{2\varepsilon}{\hbar_i} \left(\frac{Y_{i+1} - Y_i}{\Psi_{i+1}} - \frac{Y_i - Y_{i-1}}{\Psi_i} \right) + p_i \frac{Y_{i+1} - Y_{i-1}}{\hbar_i} = f_i \end{aligned} \tag{4.43}$$

$$\equiv \left(\frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i}{\hbar_i} \right) Y_{i-1} - \left(\frac{2\varepsilon}{\hbar_i \Psi_i} + \frac{2\varepsilon}{\hbar_i \Psi_i} \right) Y_i + \left(\frac{2\varepsilon}{\hbar_i \Psi_i} + \frac{p_i}{\hbar_i} \right) Y_{i+1} = f_i, \tag{4.44}$$

Equation (4.44) is obtained from (4.43), where we utilized $\Psi_i = \Psi_{i+1}$. As it was stated in (4.6), in the interval $i = \frac{N}{4} + 1, \cdots, \frac{3N}{4}$, for $\tau_1 = \tau_2$, we have $\Psi_i = \Psi_{i+1}$, however in this instance, we utilized it to maintain the domain by eliminating the index $N/2$ from both p_i and f_i.

We calculate $Y_{\frac{N}{2}-2}$ and $Y_{\frac{N}{2}+2}$, then substitute the obtained results in (4.42).

For $i = N/2 - 1$, (4.44) becomes

$$\begin{aligned} & \left(\frac{2\varepsilon}{\hbar_{N/2-1} \Psi_{N/2-1}} - \frac{p_{N/2-1}}{\hbar_{N/2-1}} \right) Y_{\frac{N}{2}-2} - \frac{4\varepsilon}{\hbar_{N/2-1} \Psi_{N/2-1}} Y_{\frac{N}{2}-1} \\ & \qquad + \left(\frac{2\varepsilon}{\hbar_{N/2-1} \Psi_{N/2-1}} + \frac{p_{N/2-1}}{\hbar_{N/2-1}} \right) Y_{\frac{N}{2}} = f_{N/2-1} \end{aligned}$$

$$Y_{\frac{N}{2}-2}=\frac{4\varepsilon}{2\varepsilon-p_{N/2-1}\Psi_{N/2-1}}Y_{\frac{N}{2}-1}-\frac{2\varepsilon+p_{N/2-1}\Psi_{N/2-1}}{2\varepsilon-p_{N/2-1}\Psi_{N/2-1}}Y_{\frac{N}{2}}+\frac{\hbar_{N/2-1}\Psi_{N/2-1}}{2\varepsilon-p_{N/2-1}\Psi_{N/2-1}}f_{\frac{N}{2}-1}$$

For $i=N/2+1$, (4.44) becomes

$$\begin{aligned}&(\frac{2\varepsilon}{\hbar_{N/2+1}\Psi_{N/2+1}}-\frac{p_{N/2+1}}{\hbar_{N/2+1}})Y_{N/2}-\frac{4\varepsilon}{\hbar_{N/2+1}\Psi_{N/2+1}}Y_{N/2+1}\\&+(\frac{2\varepsilon}{\hbar_{N/2+1}\Psi_{N/2+1}}+\frac{p_{N/2+1}}{\hbar_{N/2+1}})Y_{N/2+2}=f_{N/2+1}\end{aligned}$$

$$Y_{\frac{N}{2}+2}=\frac{-2\varepsilon+p_{N/2+1}\Psi_{N/2+1}}{2\varepsilon+p_{N/2+1}\Psi_{N/2+1}}Y_{\frac{N}{2}}+\frac{4\varepsilon}{2\varepsilon+p_{N/2+1}\Psi_{N/2+1}}Y_{\frac{N}{2}+1}+\frac{\hbar_{N/2+1}\Psi_{N/2+1}}{2\varepsilon+p_{N/2+1}\Psi_{\frac{N}{2}+1}}f_{\frac{N}{2}+1}$$

Substituting $Y_{\frac{N}{2}-2}$ and $Y_{\frac{N}{2}+2}$, into (4.42) we get:

$$\begin{aligned}&\mathcal{L}^r_{hyb}Y_i\equiv(4-\frac{4\varepsilon}{2\varepsilon-p_{N/2-1}\Psi_{N/2-1}})Y_{N/2-1}+\\&(\frac{2\varepsilon-p_{N/2+1}\Psi_{N/2+1}}{2\varepsilon+p_{N/2+1}\Psi_{N/2+1}}+\frac{2\varepsilon+p_{N/2-1}\Psi_{N/2-1}}{2\varepsilon-p_{N/2-1}\Psi_{N/2-1}}-6)Y_{N/2}\\&+(4-\frac{4\varepsilon}{2\varepsilon+p_{N/2+1}\Psi_{N/2+1}})Y_{N/2+1}=\\&\frac{\hbar_{N/2+1}\Psi_{N/2+1}}{2\varepsilon+p_{N/2+1}\Psi_{N/2+1}}f_{N/2+1}+\frac{\hbar_{N/2-1}\Psi_{N/2-1}}{2\varepsilon-p_{N/2-1}\Psi_{N/2-1}}f_{N/2-1}\end{aligned}\tag{4.45}$$

Note that, for the above analysis we assumed $\tau_1=\tau_2=z\varepsilon lnN$, where, $z=2\alpha^{-1}$, $\alpha=min\{\alpha_1,\alpha_2\}$, in such case $h_1=h_2$.
Substituting expression (4.45) into (4.37) for $i=\frac{N}{2}$ results our hybrid scheme's final formula on the Shishkin mesh for $i=1:N-1$, as follows:

$$\begin{cases}\mathcal{L}^N_{hyb}Y_i=\bar{\mathcal{F}}_{hyb},\\ Y(0)=Y_0,\qquad Y(1)=Y_1,\end{cases}\tag{4.46}$$

where,

$$\mathcal{L}^N_{hyb}Y_i=\begin{cases}\mathcal{L}^-_{hyb}Y_i,\ \text{for } i=1,\cdots,\frac{N}{4},\ \text{and } p_i<0,\\ \mathcal{L}^2_{hyb}Y_i,\ \text{for } i=\frac{N}{4}+1,\cdots,\frac{N}{2}-1,\\ \mathcal{L}^3_{hyb}Y_i,\ \text{for } i=\frac{N}{2}+1,\cdots,\frac{3N}{4}-1,\\ \mathcal{L}^+_{hyb}Y_i,\ \text{for } i=\frac{3N}{4},\cdots,N-1,\ \text{and } p_i>0,\\ \mathcal{L}^r_{hyb}Y_i,\ \text{for } i=\frac{N}{2}.\end{cases}$$

$$\bar{\mathcal{F}}_{hyb}=\begin{cases}f_{i-\frac{1}{2}},\ \text{for } i=1,\cdots,\frac{N}{4},\ \text{and } p_i<0,\\ f_i,\ \text{for } i=\frac{N}{4}+1,\cdots,\frac{N}{2}-1,\frac{N}{2}+1,\cdots,\frac{3N}{4}-1,\\ f_{i+\frac{1}{2}},\ \text{for } i=\frac{3N}{4},\cdots,N-1,\ \text{and } p_i>0,\\ \frac{\hbar_{N/2+1}\Psi_{N/2+1}}{2\varepsilon+p_{N/2+1}\Psi_{N/2+1}}f_{N/2+1}+\frac{\hbar_{N/2-1}\Psi_{N/2-1}}{2\varepsilon-p_{N/2-1}\Psi_{N/2-1}}f_{N/2-1},\ \text{for } i=\frac{N}{2}.\end{cases}$$

4.5 Stability and Uniform Convergence Analysis

In this subsection we present the stability and uniform convergence analysis of hybrid scheme (4.46). To analysis the stability and convergence of the proposed method, we write developed scheme into diagonally dominant tri-diagonal matrix. For this reason, after rearranging the terms in scheme (4.46), we transform the system of equations into a tridiagonal system of equations on the mesh Ω^N

$$\begin{cases} \mathcal{L}^N_{hyb} Y_i = \bar{\mathcal{F}}_{hyb}, \text{ for } i = 1, 2, 3, \dots N-1 \\ Y(0) = Y_0,\ Y(1) = Y_1, \end{cases} \tag{4.47}$$

where, we define the finite difference operator, $\mathcal{L}^N_{hyb}$ as

$$\mathcal{L}^N_{hyb} Y_i = \begin{cases} s_i^- Y_{i-1} + s_i^0 Y_i + s_i^+ Y_{i+1}, \text{ for } i \neq \frac{N}{2}, \\ r_i^- Y_{i-1} + r_i^0 Y_i + r_i^+ Y_{i+1}, \text{ for } i = \frac{N}{2}. \end{cases} \tag{4.48}$$

$$\bar{\mathcal{F}}_{hyb} = \begin{cases} f_{i-\frac{1}{2}}, \text{ for } i = 1, \cdots, \frac{N}{4}, \text{ and } p_i < 0, \\ f_{i+\frac{1}{2}}, \text{ for } i = \frac{3N}{4}, \cdots, N-1, \text{ and } p_i > 0, \\ f_i, \text{ for } i = \frac{N}{4}+1, \cdots, \frac{N}{2}-1, \frac{N}{2}+1, \cdots, \frac{3N}{4}-1, \\ \frac{\hbar_{N/2+1}\Psi_{N/2+1}}{2\varepsilon + p_{N/2+1}\Psi_{N/2+1}} f_{N/2+1} + \frac{\hbar_{N/2-1}\Psi_{N/2-1}}{2\varepsilon - p_{N/2-1}\Psi_{N/2-1}} f_{N/2-1}, \text{ for } i = \frac{N}{2}. \end{cases}$$

$$\begin{cases} r_i^- = 4 - \frac{4\varepsilon}{2\varepsilon - p_{N/2-1}\psi_{N/2-1}}, \\ r_i^0 = \frac{2\varepsilon - p_{N/2+1}\Psi_{N/2+1}}{2\varepsilon + p_{N/2+1}\psi_{N/2+1}} + \frac{2\varepsilon + p_{N/2-1}\Psi_{N/2-1}}{2\varepsilon - p_{N/2-1}\psi_{N/2-1}} - 6, \\ r_i^+ = 4 - \frac{4\varepsilon}{2\varepsilon + p_{N/2+1}\Psi_{N/2+1}}. \end{cases}$$

Now, the coefficients in (4.48) for $i = 1, \dots N/4$ are given by

$$\begin{cases} s_i^- = \frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i + p_{i-1}}{2h_i}, \\ s_i^o = \frac{-2\varepsilon}{\hbar_i \Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \Psi_i} + \frac{p_i + p_{i-1}}{2h_i}, \\ s_i^+ = \frac{2\varepsilon}{\hbar_i \Psi_{i+1}}. \end{cases}$$

The coefficients in (4.48) for $i = N/4+1, \dots N/2-1$ are given by

$$\begin{cases} s_i^- = \frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i}{h_i}, \\ s_i^o = \frac{-2\varepsilon}{\hbar_i \Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \Psi_i} + \frac{p_i}{h_i}, \\ s_i^+ = \frac{2\varepsilon}{\hbar_i \Psi_{i+1}}. \end{cases}$$

The coefficients in (4.48) for $i = N/2+1, \dots, 3N/4-1$ are given by

$$\begin{cases} s_i^- = \frac{2\varepsilon}{\hbar_i \Psi_i}, \\ s_i^o = \frac{-2\varepsilon}{\hbar_i \Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i}{h_{i+1}}, \\ s_i^+ = \frac{2\varepsilon}{\hbar_i \Psi_{i+1}} + \frac{p_i}{h_{i+1}}. \end{cases}$$

Next, the coefficients in (4.48) for $i = 3N/4, \dots N-1$ are given by

$$\begin{cases} s_i^- = \frac{2\varepsilon}{\hbar_i \Psi_i}, \\ s_i^o = \frac{-2\varepsilon}{\hbar_i \Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i + p_{i+1}}{2h_{i+1}}, \\ s_i^+ = \frac{2\varepsilon}{\hbar_i \Psi_{i+1}} + \frac{p_i + p_{i+1}}{2h_{i+1}}. \end{cases}$$

Now, we start with the stability and convergence analysis of the scheme separately.

Stability analysis

In this subsection, we want to show that the discrete scheme in (4.47) and (4.48) satisfies the maximum principle, ε -uniform stability conditions and error analysis. For this, the following lemma shows that the operator $\mathcal{L}_{hyb}^N$ associated with the discrete problem (4.47) satisfies the discrete maximum principle.

Theorem 1 ***(Discrete maximum principle)***
The operator $\mathcal{L}_{hyb}^N$ defined by (4.47) satisfies a discrete maximum principle, i.e. if the solution function Y_i satisfies $Y_0 \leq 0$, $Y_N \leq 0$ and $\mathcal{L}_{hyb}^N Y_i \geq 0$ for $i = 1, 2, 3, \dots N-1$, then $Y_i \leq 0$ for $i = 0, 1, 2, \cdots, N$.

Proof. From equation (4.48) we have

$$\mathcal{L}_{hyb}^N Y_i \equiv A_{i,i-1} Y_{i-1} + A_{i,i} Y_i + A_{i,i+1} Y_{i+1}, \text{ for } i = 1, 2, 3, \dots N-1$$

where

$$\begin{aligned} &A_{i,i-1} = s_i^-, A_{i,i} = s_i^0, A_{i,i+1} = s_i^+ \text{ for } i \neq N/2 \\ &A_{\frac{N}{2},\frac{N}{2}-1} = r_{\frac{N}{2}-1}^-, A_{\frac{N}{2},\frac{N}{2}} = r_{\frac{N}{2}}^0, A_{\frac{N}{2},\frac{N}{2}+1} = r_{\frac{N}{2}+1}^+ \text{ for } i = N/2 \end{aligned}$$

Then, according to the hypothesis of the discrete maximum principle, we suppose that

$$\begin{aligned} \mathcal{L}_{hyb}^N Y_i &= \mu_i, \text{ for } i = 1, 2, 3, \dots N-1 \\ Y_0 = \mu_0, \ Y_N &= \mu_N, \mu_i \leq 0 \text{ for all } i \end{aligned} \tag{4.49}$$

For this, since the values of the difference solution at boundary conditions are non-negative, we want to show that the difference solution is non-negative at each entire domain. To do this, we consider the following three cases depending on the p_i values.

Case 1: $x_i < r$
Here, we have two intervals, which are $1 \leq i \leq \frac{N}{4}$ and $\frac{N}{4} + 1 \leq i \leq \frac{N}{2} - 1$.
For $1 \leq i \leq \frac{N}{4}$, we have

$$\begin{aligned} &|A_{i,i}| \geq |A_{i,i-1}| + |A_{i,i+1}| \geq 0, \\ &\equiv \frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i + p_{i-1}}{2h_i} + \frac{-2\varepsilon}{\hbar_i \Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \Psi_i} + \frac{p_i + p_{i-1}}{2h_i} + \frac{2\varepsilon}{\hbar_i \Psi_{i+1}} \geq 0. \end{aligned}$$

Similarly, for $N/4 \leq i \leq \frac{N}{2}-1$, $|A_{i,i}| \geq |A_{i,i-1}| + |A_{i,i+1}| \geq 0$. Hence, for $p_i \leq 0$, we have a non-negative difference coefficient.

Case 2: $x_i > r$

Here, we have also two intervals, which are $\frac{N}{2}+1 \leq i \leq \frac{3N}{4}-1$ and $\frac{3N}{4} \leq i \leq N-1$

For $\frac{N}{2}+1 \leq i \leq \frac{3N}{4}-1$, we have

$$
\begin{aligned}
&|A_{i,i}| \geq |A_{i,i-1}| + |A_{i,i+1}| \geq 0. \\
&\equiv \frac{2\varepsilon}{\hbar_i \Psi_i} + \frac{-2\varepsilon}{\hbar_i \Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i}{h_{i+1}} + \frac{2\varepsilon}{\hbar_i \Psi_{i+1}} + \frac{p_i}{h_{i+1}} \geq 0.
\end{aligned}
$$

similarly, for $\frac{3N}{4} \leq i \leq N-1$, we have $|A_{i,i}| \geq |A_{i,i-1}| + |A_{i,i+1}| \geq 0$. Thus, $|A_{i,i}| \geq |A_{i,i-1}| + |A_{i,i+1}| \geq 0$.

Case 3: $x_i = r$

In this case, we have $i = \frac{N}{2}$, and

$$
|A_{i,i}| \geq |A_{i,i-1}| + |A_{i,i+1}| \geq 0 \equiv 4 - \frac{4\varepsilon}{2\varepsilon - p_{N/2-1}\psi_{N/2-1}} + \frac{2\varepsilon - p_{N/2+1}\Psi_{N/2+1}}{2\varepsilon + p_{N/2+1}\psi_{N/2+1}} + \frac{2\varepsilon + p_{N/2-1}\Psi_{N/2-1}}{2\varepsilon - p_{N/2-1}\psi_{N/2-1}} - 6 + 4 - \frac{4\varepsilon}{2\varepsilon + p_{N/2+1}\Psi_{N/2+1}} \geq 0.
$$

Therefore, in all the cases, it can be shown that the $(N+1)\times(N+1)$ matrix formed by the coefficients of the mesh function Y_i in (4.49) for $i = 0, 1, 2, \ldots, N$ forms a tri-diagonal and diagonally dominant matrix. Hence, the proof is complete.

Local Error Analysis

In this subsection, we establish the ε-uniform error estimate of the hybrid scheme in (4.47) and (4.48). It is well known that the solution y of (4.1) - (4.4) is decomposed into the smooth and non-smooth components P. A. Farrell et al. (2004). Here, we decompose the discrete solution y into the smooth and the non-smooth components as

$$
y_i = \begin{cases} v_{L_i} + w_{L_i}, \text{ for } 1 \leq i \leq \frac{N}{2}-1, \\ v_{L_i} + w_{L_i} = v_{R_i} + w_{R_i}, \text{ for } i = \frac{N}{2}, \\ v_{R_i} + w_{R_i}, \text{ for } \frac{N}{2}+1 \leq i \leq N-1. \end{cases} \tag{4.50}
$$

where, the smooth components v_L and v_R approximate v respectively to the left and to the right of the point of discontinuity $x = r$, and the non-smooth components w_L and w_R approximate w respectively to the left and to the right of the point of discontinuity $x = r$.

The smooth components are defined as the solutions of the discrete problems

$$
\begin{cases} \mathcal{L}^N_{hyb} V_{L_i} = \bar{\mathcal{F}}_{hyb}, \text{ for } 1 \leq i \leq \frac{N}{2}-1, \\ \mathcal{L}^N_{hyb} V_{R_i} = \bar{\mathcal{F}}_{hyb}, \text{ for } \frac{N}{2}+1 \leq i \leq N-1, \\ V_L(0) = v(0), V_L(N/2) = v(r^-), \\ V_R(0) = v(0), V_R(N/2) = v(r^+). \end{cases}
$$

The non-smooth components satisfy the following systems of equations

$$\begin{cases} \mathcal{L}_{hyb}^{N} W_{L_i} = 0, \text{ for } 1 \leq i \leq \frac{N}{2} - 1, \\ \mathcal{L}_{hyb}^{N} W_{R_i} = 0, \text{ for } \frac{N}{2} + 1 \leq i \leq N - 1, \\ W_L(0) = 0, W_R(N) = 0, \\ V_L(\frac{N}{2}) + W_L(\frac{N}{2}) = V_R(\frac{N}{2}) + W_R(\frac{N}{2}). \end{cases}$$

Now, we analyze the errors separately in the interior layer region (i.e., for $x_i \in \bar{\Omega}^N \cap (r - \tau_1, r + \tau_2)$ and in the outer region (i.e., for $x_i \in \bar{\Omega}^N \cap \{[0, r - \tau_1] \cup [r + \tau_2, 1]\}$.

Error in the outer region

In the following lemma, we obtain the error bounds associated with the smooth components.

Lemma 2 *The local truncation errors associated with the smooth components satisfy the following estimates*

$$\begin{cases} |V_{L_i} - v(x_i)| \leq CN^{-1}, \textit{ for } 1 \leq i \leq \frac{N}{2} - 1, \\ |V_{R_i} - v(x_i)| \leq CN^{-1}, \textit{ for } \frac{N}{2} + 1 \leq i \leq N - 1. \end{cases}$$

where, $v(x_i)$ is assumed as an exact solution.

Proof. In this proof, we consider the following two cases depending on the value of p_i.

Case 1: For $p_i > 0$, we have $\frac{N}{2} + 1 \leq i \leq N - 1$, and it is on the right of the point of discontinuity. Therefore, from equation (4.46) for interval $\frac{N}{2} + 1 \leq i \leq N - 1$ we have

$$\frac{2\varepsilon}{\hbar_i \psi_i} Y_{i-1} + (\frac{-2\varepsilon}{\hbar_i \Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \Psi_i} - \frac{p_i}{h_{i+1}}) Y_i + (\frac{2\varepsilon}{\hbar_i \psi_{i+1}} + \frac{p_i}{h_{i+1}}) Y_{i+1} = f_i, \text{ for } \frac{N}{2} + 1 \leq i \leq \frac{3N}{4} - 1 \tag{4.51}$$

$$\frac{2\varepsilon}{\hbar_i \psi_i} Y_{i-1} + (\frac{-2\varepsilon}{\hbar_i \psi_{i+1}} - \frac{2\varepsilon}{\hbar_i \psi_i} - \frac{p_{i+\frac{1}{2}}}{h_{i+1}}) Y_i + (\frac{2\varepsilon}{\hbar_i \psi_{i+1}} + \frac{p_{i+\frac{1}{2}}}{h_{i+1}}) Y_{i+1} = f_{i+\frac{1}{2}}, \text{ for } \frac{3N}{4} \leq i \leq N - 1 \tag{4.52}$$

For the exact solution $v(x_i)$ the local truncation error of (4.52) is defined as

$$\mathcal{L}_{hyb}^{N}(v(x_i) - V_{R_i}) = (\mathcal{L} - \mathcal{L}^N) v(x_i) = \varepsilon(\frac{d^2}{dx^2} - D^+ D_{\Psi}^-) v(x_i) + p_{i+\frac{1}{2}}(\frac{d}{dx} - D^+) v(x_i)$$

Let e_i is a local truncation error of (4.52) at point (x_i). Then,

$$|e_i| = |\varepsilon(\frac{d^2}{dx^2} - D^+ D_{\Psi}^-) v(x_i) + p_{i+\frac{1}{2}}(\frac{d}{dx} - D^+) v(x_i)| \tag{4.53}$$

Now, the Taylor series expansion of $V_{x_{i-1}}$ at x_i is

$$V_{(x_{i-1})} = V_i - h_i V_i' + \frac{h_i^2}{2} V_i'' - \frac{h^3}{6} V_i''' + \frac{h_i^4}{24} V_i^{(4)}$$

The Taylor series expansion of $V_{x_{i+1}}$ at x_i is

$$V_{(x_{i+1})} = V_i + h_i V_i' + \frac{h_i^2}{2} V_i'' + \frac{h_i^3}{6} V_i''' + \frac{h_i^4}{24} V_i^{(4)}$$

Let $h_i = h_{i+1} = h$ for $x_i \in \bar{\Omega}^N \cap \{[r+\tau_2, 1]\}$.

$$D^+D^-_\Psi v(x_i) = \frac{2}{\hbar_i}(\frac{V_{i+1}-V_i}{\Psi_{i+1}} - \frac{V_i - V_{i-1}}{\Psi_i}) = \frac{2}{2h}(\frac{V_{i+1}-2V_i+V_{i-1}}{\Psi_i}) = \frac{1}{h^2}(h^2V_i'' + \frac{h^4}{12}V_i^{(4)})$$

$$D^+v(x_i) = \frac{1}{h_i}(V_{i+1}-V_i) = V_i' + \frac{h_i}{2}V_i'' + \frac{h_i^2}{6}V_i'''$$

By substituting the above equations into (4.53), we get

$$\begin{aligned}
|e_i| &= |\varepsilon(V_i'' - (V_i'' + \frac{h^2}{12}V_i^{(4)})) + p_{i+\frac{1}{2}}(V_i' - (V_i' + \frac{h_i}{2}V_i'' + \frac{h_i^2}{6}V_i'''))| \\
&\le C\varepsilon N^{-2}|V_i^{(4)}| + CN^{-1}|V_i''| \text{ since, } h \le N^{-1} \text{ for } \frac{3N}{4} \le i \le N \\
&\text{by using the bound } \ |v_R^{(k)}(x)| \le C(1+\varepsilon^{2-k}),\ x \in \Omega^+ \text{ from Lemma 1 ,} \\
&\le C\varepsilon N^{-2}(C(1+\varepsilon^{-2})) + CN^{-1}(C(1+\varepsilon^0)) \\
&= C\varepsilon N^{-2}(1+\varepsilon^{-2}) + CN^{-1}
\end{aligned}$$

$$|e_i| \le CN^{-1}.$$

Case 2: For $p_i < 0$, we have $1 \le i \le \frac{N}{2} - 1$, and it is on the left of the discontinuity point. Therefore, from equation (4.46) for the interval $1 \le i \le \frac{N}{2} - 1$, we have

$$(\frac{2\varepsilon}{\hbar_i\psi_i} - \frac{p_i}{h_i})Y_{i-1} + (\frac{-2\varepsilon}{\hbar_i\Psi_{i+1}} - \frac{2\varepsilon}{\hbar_i\Psi_i} + \frac{p_i}{h_i})Y_i + \frac{2\varepsilon}{\hbar_i\psi_{i+1}}Y_{i+1} = f_i, \text{ for } i = \frac{N}{4}+1 \le i \le \frac{N}{2} - 1, \tag{4.54}$$

$$(\frac{2\varepsilon}{\hbar_i\psi_i} - \frac{p_{i-\frac{1}{2}}}{\hbar_i})Y_{i-1} + (\frac{-2\varepsilon}{\hbar_i\psi_{i+1}} - \frac{2\varepsilon}{\hbar_i\psi_i} + \frac{p_{i-\frac{1}{2}}}{h_i})Y_i + \frac{2\varepsilon}{\hbar_i\psi_{i+1}}Y_{i+1} = f_{i-\frac{1}{2}}, \text{ for } i = 1 \le i \le \frac{N}{4} \tag{4.55}$$

For the exact solution $v(x_i)$ the local truncation error of (4.55) is defined as

$$\mathcal{L}^N_{hyb}(V_{L_i} - v(x_i)) = (\mathcal{L} - \mathcal{L}^N)v(x_i) = \varepsilon(\frac{d^2}{dx^2} - D^+D^-_\Psi)v(x_i) + p_{i-\frac{1}{2}}(\frac{d}{dx} - D^-)v(x_i)$$

let e_i is a local truncation error of (4.55) at point (x_i). Then,

$$|e_i| = |\varepsilon(\frac{d^2}{dx^2} - D^+D^-_\Psi)v(x_i) + p_{i-\frac{1}{2}}(\frac{d}{dx} - D^-)v(x_i)| \tag{4.56}$$

Now, by using the Taylor series expansion of $V_{(x_{i-1})}$ and $V_{(x_{i+1})}$ at x_i similarly in case 1,

$$\begin{aligned}
D^+D^-_\Psi v(x_i) &= V_i'' + \frac{h^4}{12}V_i^{(4)} \\
D^-v(x_i) &= \frac{1}{h}(V_i - V_{x_{i-1}}) = V_i' - \frac{h}{2}V_i'' + \frac{h^2}{6}V_i'''
\end{aligned}$$

Substituting these expressions in 4.56,

$$
\begin{aligned}
|e_i| &= |\varepsilon(V'' - (V_i'' + \frac{h^2}{12}V_i^{(4)})) + p_{i-\frac{1}{2}}(V'_{x_i} - (V'_{x_i} - \frac{h_i}{2}V''_{x_i} + \frac{h_i^2}{6}V'''_{x_i}))| \\
&\leq C\varepsilon N^{-2}|V_i^{(4)}| + CN^{-1}|V''_{x_i}| \text{ since, } h \leq 4N^{-1} \text{ for } 1 \leq i \leq \frac{N}{4} \\
&\text{by using the bound } \ |v_L^{(k)}(x)| \leq C(1+\varepsilon^{2-k}),\ x \in \Omega^- \text{ from Lemma 1} \\
&\leq C\varepsilon N^{-2}(C(1+\varepsilon^{-2})) + CN^{-1}(C(1+\varepsilon^{0})) \\
&= C\varepsilon N^{-2}(1+\varepsilon^{-2}) + CN^{-1}
\end{aligned}
$$

$$|e_i| \leq CN^{-1}.$$

We proceed in the same way for the remaining intervals.
Hence, this completes the proof.

Lemma 3 *The local truncation error associated with the hybrid scheme in (4.47) and (4.48), satisfies*

$$|Y_i - y(x_i)| \leq CN^{-1} \textit{ for } 1 \leq i \leq \frac{N}{4} \textit{ and } \frac{3N}{4} \leq i \leq N-1.$$

Proof. The proof of this lemma is immediately follows from the separate proof of Lemma 2. As observed from Lemma 2, the outer region local truncation error is bounded and is first-order locally.

Next, we consider the local truncation error associated with the nonsmooth components.

Lemma 4 *The local truncation errors associated with the non-smooth components satisfy the following estimates*

$$
\begin{cases}
|W_{L_i} - w(x_i)| \leq CN^{-1}(lnN)^2, \textit{ for } 1 \leq i \leq \frac{N}{2} - 1, \\
|W_{R_i} - w(x_i)| \leq CN^{-1}(lnN)^2, \textit{ for } \frac{N}{2} + 1 \leq i \leq N-1.
\end{cases}
$$

where, $w(x_i)$ assumed as an exact solution.

Proof. In this proof, similar to Lemma 2, we consider the following two cases depending on the value p_i

Case 1: For $p_i < 0$, we have $1 \leq i \leq \frac{N}{2} - 1$, and it is on the left of the point of discontinuity.

Therefore, from equation (4.46) for the interval $1 \leq i \leq \frac{N}{2} - 1$, we have

$$(\frac{2\varepsilon}{\hbar_i\psi_i} - \frac{p_i}{h_i})Y_{i-1} + (\frac{-2\varepsilon}{\hbar_i\psi_{i+1}} - \frac{2\varepsilon}{\hbar_i\psi_i} + \frac{p_i}{h_i})Y_i + (\frac{2\varepsilon}{\hbar_i\psi_{i+1}})Y_{i+1} = f_i, \text{ for } \frac{N}{4} + 1 \leq i \leq \frac{N}{2} - 1, \tag{4.57}$$

$$(\frac{2\varepsilon}{\hbar_i\psi_i} - \frac{p_{i-\frac{1}{2}}}{\hbar_i})Y_{i-1} + (\frac{-2\varepsilon}{\hbar_i\psi_{i+1}} - \frac{2\varepsilon}{\hbar_i\psi_i} + \frac{p_{i-\frac{1}{2}}}{h_i})Y_i + \frac{2\varepsilon}{\hbar_i\psi_{i+1}}Y_{i+1} = f_{i-\frac{1}{2}}, \text{ for } 1 \leq i \leq \frac{N}{4} \tag{4.58}$$

For the exact solution $w(x_i)$, Let e_i be a local truncation error of (4.57) at point x_i, then

$$|e_i| = |\varepsilon(\frac{d^2}{dx^2} - D^+D^-_\Psi)w(x_i) + p_i(\frac{d}{dx} - D^-)w(x_i)| \tag{4.59}$$

Then, we want to find the Taylor series expansion of $W_{x_{i-1}}$ and $W_{x_{i+1}}$ at x_i

$$W_{(x_{i-1})} = W_{x_i} - h_iW'_{x_i} + \frac{h^2}{2}W''_{x_i} - \frac{h^3}{6}W'''_{x_i} + \frac{h^4}{24}W^{(4)}_{x_i}$$
$$W_{(x_{i+1})} = W_{x_i} + h_iW'_{x_i} + \frac{h^2}{2}W''_{x_i} + \frac{h^3}{6}W'''_{x_i} + \frac{h^4}{24}W^{(4)}_{x_i}$$

$$(\frac{d^2}{dx^2} - D^+D^-_\Psi)w(x_i) = -\frac{h^2}{12}W^{(4)}_i$$
$$(\frac{d}{dx} - D^-)w(x_i) = \frac{h_i}{2}W''_i - \frac{h_i^2}{6}W'''_i + \frac{h_i^3}{24}W^{(4)}_i$$

Substituting the above expressions into the truncation error (4.59):

$|e_i| \leq \varepsilon(\frac{h^2}{12}|W^{(4)}_i|) + p_i(\frac{h}{2}|W''_i|)$

$= C\varepsilon N^{-2}|W^{(4)}_i| + CN^{-1}|W''_i|$

$\leq C\varepsilon N^{-2}[C\varepsilon^{-4}exp(-(r-x)\frac{\alpha_1}{\varepsilon})] + CN^{-1}[C\varepsilon^{-2}exp(-(r-x)\frac{\alpha_1}{\varepsilon})]$, since, from Lemma 1 we have

$|w^{(k)}_L(x)| \leq C\varepsilon^{-k}exp(-(r-x)\frac{\alpha_1}{\varepsilon}),\ x \in \Omega^-$

$\leq C\varepsilon N^{-2}(lnN)^4 + CN^{-1}(lnN)^2$, since $\frac{1}{\varepsilon} \leq ClnN$

$$|e_i| \leq CN^{-1}(lnN)^2.$$

Hence, the local truncation errors of the left of discontinuity is complete.

Case 2: For $p_i > 0$, we have $\frac{N}{2} + 1 \leq i \leq N - 1$, and it is on the left of the point of discontinuity. Then, similarly in case 1, for $\frac{N}{2} + 1 \leq i \leq \frac{3N}{4} - 1$ we have

$$|W_{R_i} - w(x_i)| \leq CN^{-1}(lnN)^2.$$

We proceed in the same way for the remaining intervals.

Hence, the proof is complete.

Error in the interior layer region

In this subsection, we analyze the errors estimate in the interior layer region (for $x_i \in \bar{\Omega}^N \cap (r - \tau_1, r + \tau_2)$. For this, we have the following lemma.

Lemma 5 *The error associated with the hybrid scheme (4.47) and (4.48) satisfies*

$$|W_i - w(x_i)| \leq CN^{-1}(lnN)^2,\ \textit{for}\ \frac{N}{4} + 1 \leq i \leq \frac{3N}{4} - 1.$$

Proof. Here, we have $\frac{N}{4} + 1 \leq i \leq \frac{N}{2} - 1$, $i = \frac{N}{2}$, and $\frac{N}{2} + 1 \leq i \leq \frac{3N}{4} - 1$

For this, first, we determine the associated local truncation error for $\frac{N}{4}+1 \leq i \leq \frac{N}{2}-1$, and $\frac{N}{2}+1 \leq i \leq \frac{3N}{4}-1$. Since the interval $\frac{N}{4}+1 \leq i \leq \frac{N}{2}-1$, is a subset of an interval $1 \leq i \leq \frac{N}{2}-1$, from the first part of Lemma 4,

$$|W_i - w(x_i)| \leq CN^{-1}(lnN)^2, \text{ for } \frac{N}{4}+1 \leq i \leq \frac{N}{2}-1$$

Similarly, from second part of Lemma 4,

$$|W_i - w(x_i)| \leq CN^{-1}(lnN)^2, \text{ for } \frac{N}{2}+1 \leq i \leq \frac{3N}{4}-1$$

Hence, except $i = N/2$, in all interior region, we have

$$|W_i - w(x_i)| \leq CN^{-1}(lnN)^2$$

Next, we consider the mesh point at $i = N/2$,$x_i = r$, which is the point of discontinuity. At this point we have

$$\begin{aligned}
&D_x^+ w(x_i) - D_x^- w(x_i), \text{ for } i = \frac{N}{2} \\
&D_x^+ w(x_i) = W_i' + \frac{h}{2}W_i'' + \frac{h^2}{6}W_i''' + \frac{h^3}{24}W_i^{(4)} \\
&D_x^- w(x_i) = W_i' - \frac{h}{2}W_i'' + \frac{h^2}{6}W_i''' - \frac{h^3}{24}W_i^{(4)}
\end{aligned}$$

$$\begin{aligned}
&|D_x^+ w(x_i) - D_x^- w(x_i)| = |(\frac{d}{dx} - D^+)w(x_i) + (\frac{d}{dx} - D^-)y(x_i)| \leq |hW_i''| \\
&\leq N^{-1}[C\varepsilon^{-2}exp(-(r-x)\frac{\alpha_1}{\varepsilon})], \text{ since, from Lemma 1 we have} \\
&|w_L^{(k)}(x)| \leq C\varepsilon^{-k}exp(-(r-x)\frac{\alpha_1}{\varepsilon}), \ x \in \Omega^- \\
&\leq CN^{-1}\varepsilon^{-2}, \\
&\leq CN^{-1}(lnN)^2, \text{ since } \frac{1}{\varepsilon} \leq ClnN
\end{aligned}$$

Hence, the proof is complete.

Convergence analysis

To verify the convergence analysis of the proposed method we state the following theorem using the concept in stability analysis and error estimation.

Theorem 2 *The global error associated with the hybrid scheme in (4.47) and (4.48) satisfies*

$$|Y_i - y_i| \leq CN^{-1}(lnN)^2,$$

where Y_i is an approximate solution and y_i is the exact solution for every i in the domain.

Proof. The proof of this theorem follows directly from Lemmas 3, 4 and 5.

CHAPTER 5

NUMERICAL RESULTS AND DISCUSSION

To validate the theoretical results presented in this thesis, we perform experiments using the developed numerical scheme. We consider two model examples and investigate the numerical results.

Example 1 *Consider the test problem (P. A. Farrell et al., 2004),*

$$\varepsilon y''(x) + p(x)y'(x) = f(x) \tag{5.1}$$

on the domain $\Omega = (0, 1)$*, with the boundary conditions*

$$y(0) = 0, y(1) = 1 \tag{5.2}$$

$$p(x) = \begin{cases} -1, 0 \leq x \leq 0.4, \\ 1, 0.4 < x \leq 1 \end{cases} \qquad f(x) = \begin{cases} -4x, 0 \leq x \leq 0.25, \\ -1, 0.25 < x \leq 0.4, \\ 1, 0.4 < x \leq 0.5, \\ 2 - 2x, 0.5 < x \leq 1 \end{cases}$$

Example 2 *Consider the test problem (Priyadharshini & Ramanujam, 2009),*

$$\varepsilon y''(x) + p(x)y'(x) = f(x) \tag{5.3}$$

on the domain $\Omega = (0, 1)$*, with the boundary conditions*

$$y(0) = 0, y(1) = 1 \tag{5.4}$$

$$p(x) = \begin{cases} -1, 0 \leq x \leq 0.5, \\ 1, 0.5 < x \leq 1 \end{cases} \qquad f(x) = \begin{cases} 0, 0 \leq x \leq 0.5, \\ 1, 0.5 < x \leq 1 \end{cases}$$

The exact solution for the examples considered is not available. So that, to estimate maximum point wise absolute error and rate of convergence of the method , we use the double mesh principle, which is followed in the literature (Woldaregay, 2022), (Mai et al., 2023).

Maximum absolute error is given by:

$$E_{\varepsilon}^{N} = \max_{0 \leq i \leq N} |Y_{x_i}^{N} - Y_{x_{2i}}^{2N}|.$$

where, $Y_{x_i}^{N}$ and $Y_{x_{2i}}^{2N}$ are respectively the numerical solution on the mesh number N at the nodal point x_i, and on the mesh number $2N$ at the nodal point x_{2i} . The ε- uniform error estimate is given by

$$E^N = \max_{\varepsilon}(E^N).$$

The rate of convergence is computed as

$$R_{\varepsilon}^N = log_2(\frac{E_{\varepsilon}^N}{E_{\varepsilon}^{2N}}).$$

The ε-uniform rate of convergence is computed using the formula

$$R^N = log_2(\frac{E^N}{E^{2N}}).$$

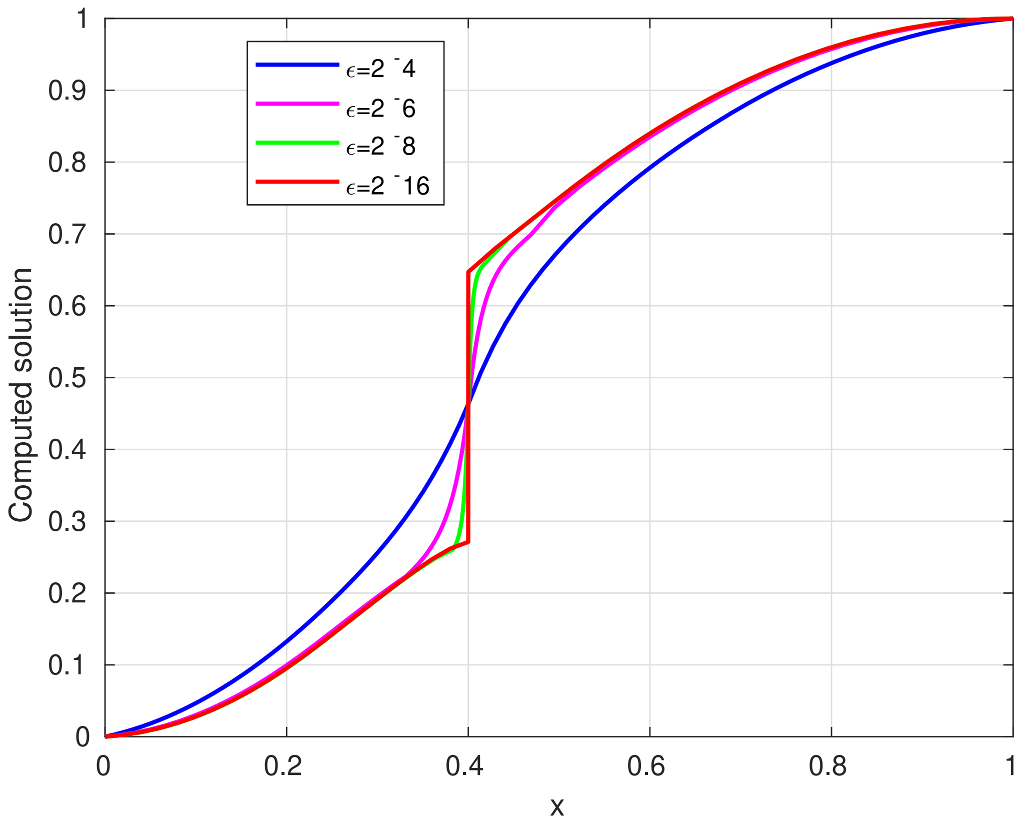

Figure 5.1: Solutions profile for Example 1 with interior layer formation as ε goes small with mesh size N = 80.

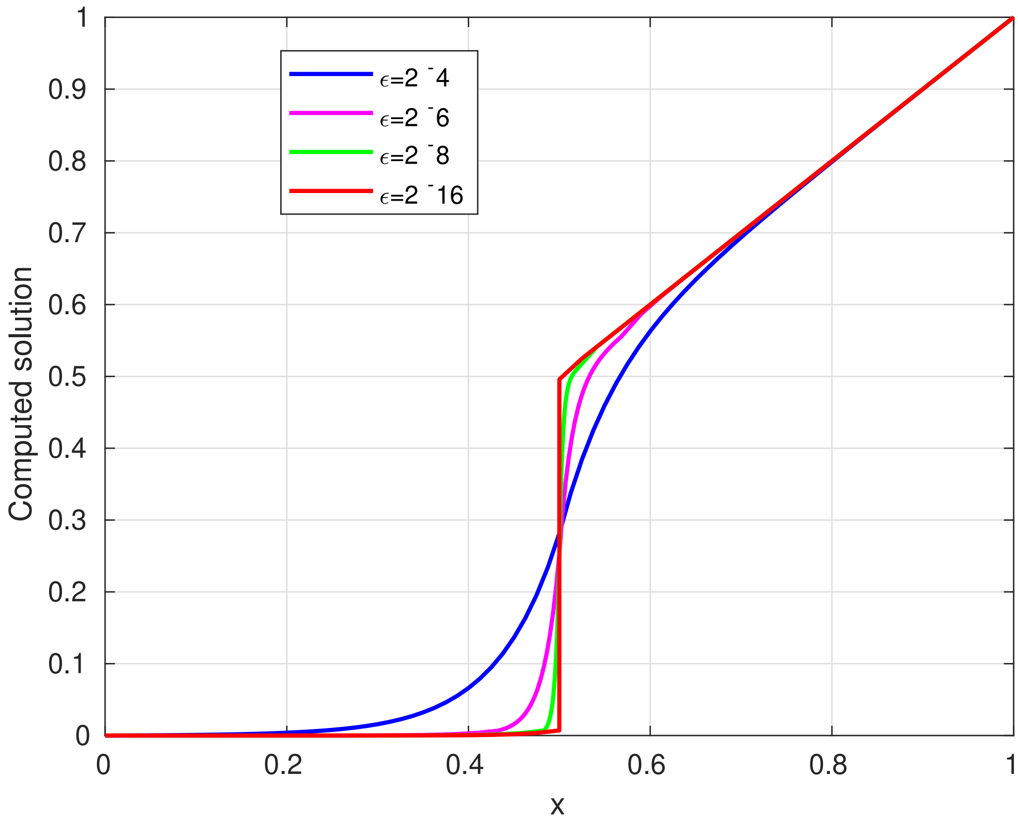

Figure 5.2: Solutions profile for Example 2 with interior layer formation as ε goes small with mesh size N = 80.

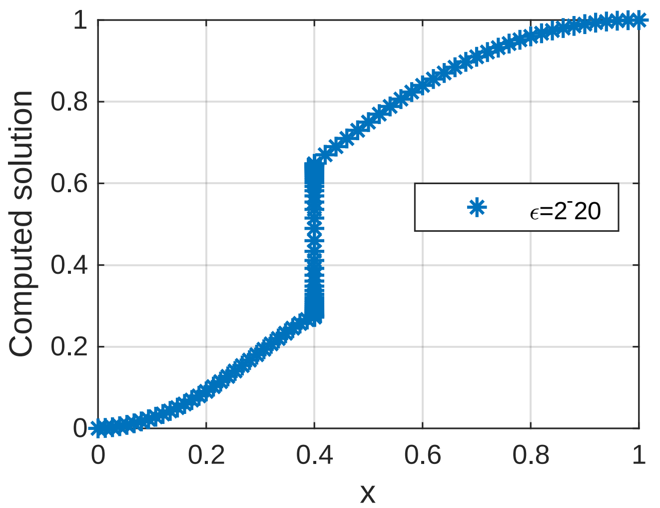

Figure 5.3: Solutions profile for Example 1 with $\varepsilon = 2^{-20}$, N = 120 on Shishkin mesh.

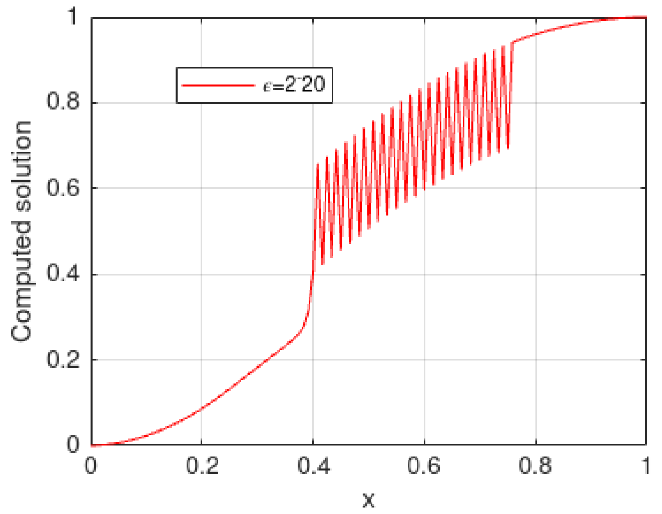

Figure 5.4: Solutions profile for Example 1 with $\varepsilon = 2^{-20}$, N = 120 on uniform mesh.

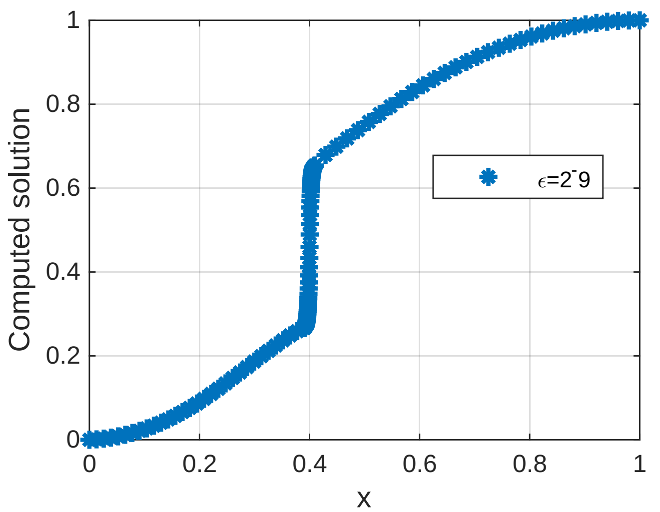

Figure 5.5: Solutions profile for Example 1 with $\varepsilon = 2^{-9}$, N = 120 on Shishkin mesh.

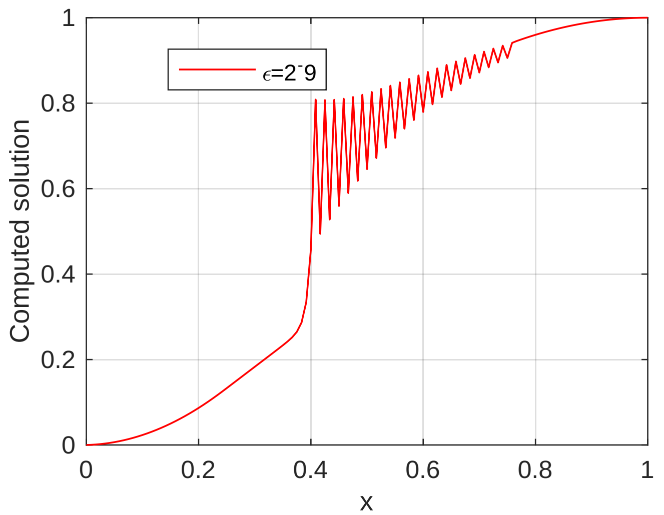

Figure 5.6: Solutions profile for Example 1 with $\varepsilon = 2^{-9}$, N = 120 on uniform mesh.

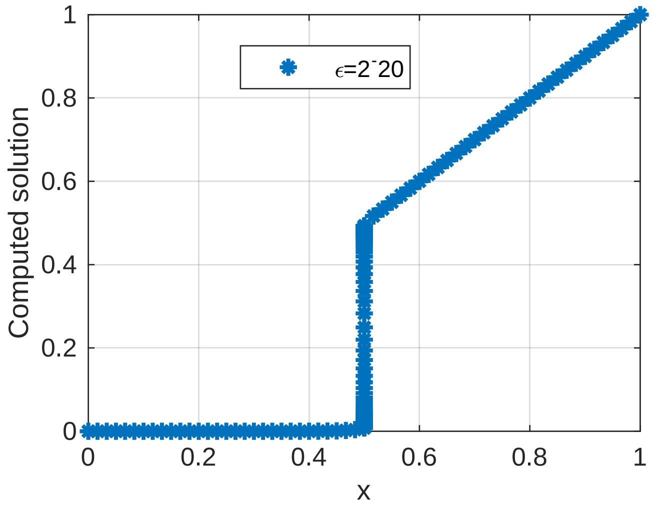

Figure 5.7: Solutions profile for Example 2 with $\varepsilon = 2^{-20}$, N = 120 on Shishkin mesh.

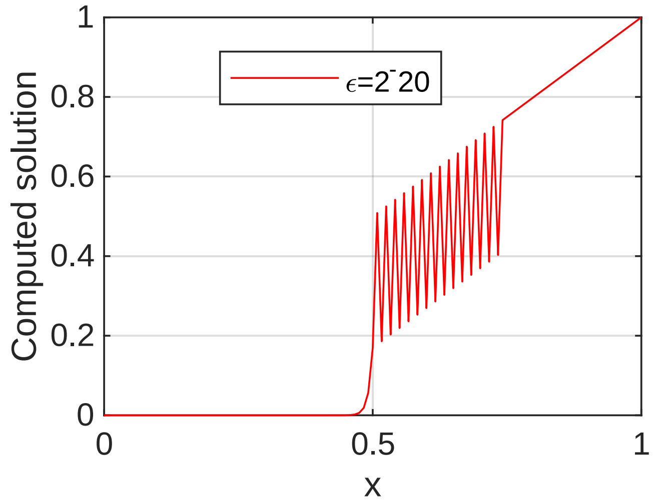

Figure 5.8: Solutions profile for Example 2 with $\varepsilon = 2^{-20}$, N = 120 on uniform mesh.

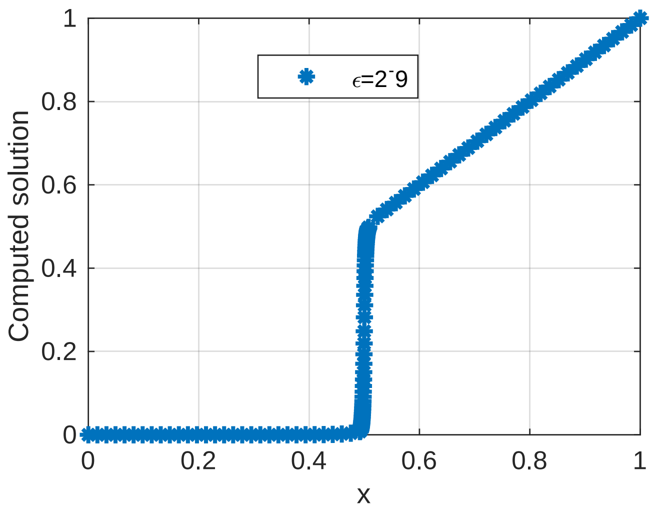

Figure 5.9: Solutions profile for Example 2 with $\varepsilon = 2^{-9}$, N = 120 on Shishkin mesh.

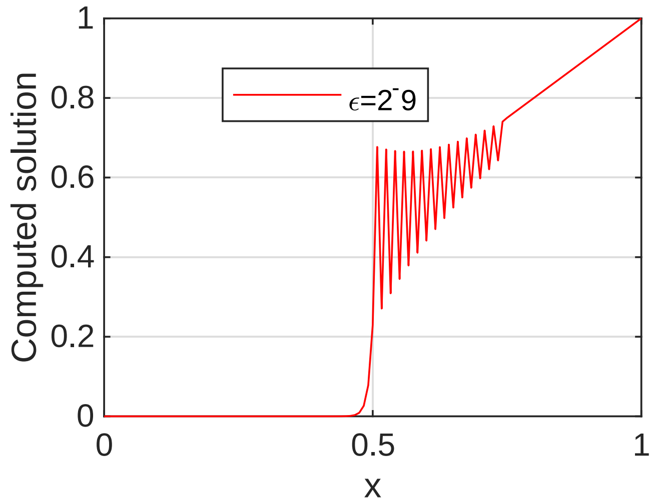

Figure 5.10: Solutions profile for Example 2 with $\varepsilon = 2^{-9}$, N = 120 on uniform mesh.

Table 5.1: Maximum absolute error and rate of convergence for Example 1

$\varepsilon \downarrow$	N$\rightarrow$				
	16	32	64	128	256
2^{-0}	1.7485e-03	8.1134e-04	3.9088e-04	1.9161e-04	9.4860e-05
	1.1077	1.0536	1.0286	1.0143	
2^{-1}	3.2650e-03	1.5569e-03	7.6243e-04	3.7664e-04	1.7722e-04
	1.0684	1.0300	1.0174	1.0876	
2^{-2}	6.5890e-03	3.2529e-03	1.6201e-03	8.0650e-04	4.0244e-04
	1.0183	1.0056	1.0063	1.0029	
2^{-3}	1.6118e-02	7.5361e-03	3.6868e-03	1.8259e-03	9.0908e-04
	1.0968	1.0314	1.0138	1.0061	
2^{-4}	4.3754e-02	1.6696e-02	7.6066e-03	3.6725e-03	1.8118e-03
	1.3899	1.1342	1.0505	1.0193	
2^{-5}	4.9977e-02	3.1622e-02	1.8090e-02	9.7690e-03	5.1723e-03
	0.66034	0.80574	0.88891	0.91740	
2^{-6}	4.3349e-02	2.1632e-02	1.5222e-02	9.2636e-03	5.1365e-03
	1.0028	0.50701	0.71651	0.85079	
E^N	4.9977e-02	3.1622e-02	1.8090e-02	9.7690e-03	5.1723e-03
R^N	0.66034	0.80574	0.88891	0.91740	

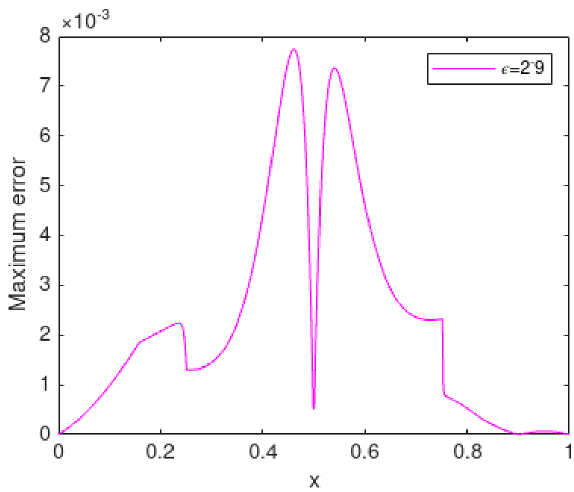

Figure 5.11: Graph of point-wise absolute error on Shishkin mesh for Example 1 with $\varepsilon = 2^{-9}$, N = 256.

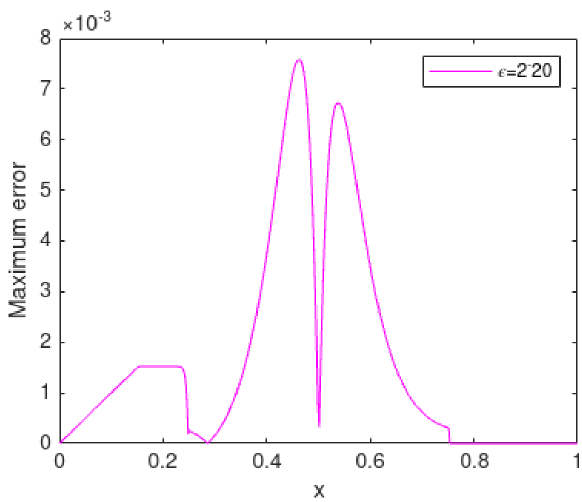

Figure 5.12: Graph of point-wise absolute error on Shishkin mesh for Example 1 with $\varepsilon = 2^{-20}$, N = 256.

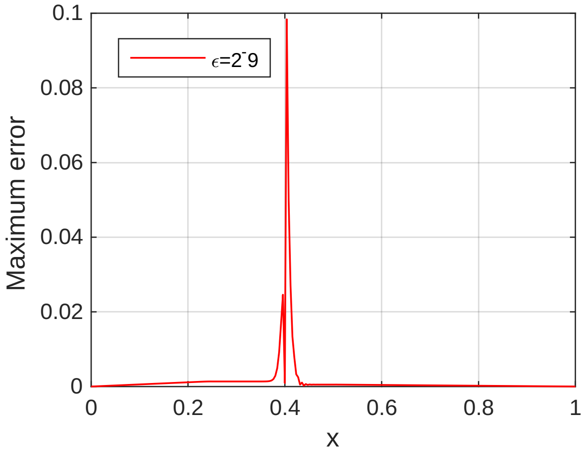

Figure 5.13: Graph of absolute error on uniform mesh for Example 1 with $\varepsilon = 2^{-9}$, N = 256.

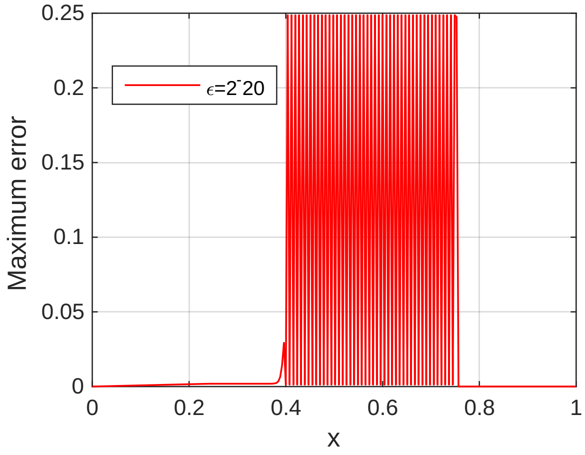

Figure 5.14: Graph of absolute error on uniform mesh for Example 1 with $\varepsilon = 2^{-20}$, N = 256.

Table 5.2: Maximum absolute error and rate of convergence for Example 2

$\varepsilon\downarrow$	N→				
	16	32	64	128	256
2^{-0}	2.7714e-03	1.4045e-03	7.0720e-04	3.5488e-04	1.7777e-04
	0.98056	0.98987	0.99479	0.99732	
$2^{-}1$	3.9757e-03	2.0202e-03	1.0195e-03	5.1228e-04	2.5679e-04
	1.0684	1.0300	1.0174	1.0876	
2^{-2}	4.4410e-03	2.3892e-03	1.2378e-03	6.2989e-04	3.1775e-04
	1.0183	1.0056	1.0063	1.0029	
2^{-3}	1.2086e-02	6.5420e-03	3.4063e-03	1.7368e-03	8.7713e-04
	1.0968	1.0314	1.0138	1.0061	
2^{-4}	5.1536e-02	2.8037e-02	1.5037e-02	7.9162e-03	4.1308e-03
	0.87825	0.89882	0.92564	0.93839	
2^{-5}	4.9899e-02	2.8459e-02	1.6741e-02	7.8031e-03	3.9302e-03
	0.81013	0.76550	0.76334	0.98944	
2^{-6}	5.0308e-02	2.8513e-02	1.8995e-02	9.8222e-03	5.0745e-03
	0.81917	0.58600	0.95150	0.95278	
E^N	5.0308e-02	2.8513e-02	1.8995e-02	9.8222e-03	5.0745e-03
R^N	0.81917	0.58600	0.95150	0.95278	

Table 5.3: Comparison of maximum absolute error of Example 1 with results in (Farrell et al., 2004)

$\varepsilon \downarrow$	N→			
	8		16	
	P. A. Farrell et al. (2004)	Our method	P. A. Farrell et al. (2004)	Our method
2^{-9}	1.3566E-1	2.3545e-02	8.4256E-2	1.7400e-02
2^{-10}	1.3638E-1	2.4121e-02	8.5151E-2	1.8572e-02
2^{-11}	1.3673E-1	2.4410e-02	8.5600E-2	1.9161e-02
2^{-12}	1.3690E-1	2.4554e-02	8.5822E-2	1.9457e-02
2^{-13}	1.3699E-1	2.4627e-02	8.5930E-2	1.9604e-02
2^{-14}	1.3703E-1	2.4663e-02	8.5982E-2	1.9679e-02
2^{-15}	1.3705E-1	2.4681e-02	8.6008E-2	1.9716e-02
2^{-16}	1.3706E-1	2.4690e-02	8.6020E-2	1.9734e-02
2^{-17}	1.3706E-1	2.4695e-02	8.6026E-2	1.9743e-02
2^{-18}	1.3706E-1	2.4697e-02	8.6029E-2	1.9748e-02
2^{-19}	1.3707E-1	2.4698e-02	8.6031E-2	1.9750e-02
E^N	1.3707E-1	2.4698e-02	8.6031E-2	1.9750e-02

Table 5.4: Comparison of maximum absolute error of Example 1 with results in (Farrell et al., 2004)

$\varepsilon \downarrow$	N$\rightarrow$ 32		64	
	P. A. Farrell et al. (2004)	Our method	P. A. Farrell et al. (2004)	Our method
2^{-9}	4.6360E-2	1.1819e-02	2.4324E-2	9.9997e-03
2^{-10}	4.6846E-2	1.1509e-02	2.4706E-2	9.8258e-03
2^{-11}	4.7095E-2	1.1351e-02	2.4907E-2	9.7337e-03
2^{-12}	4.7217E-2	1.1272e-02	2.5007E-2	9.6870e-03
2^{-13}	4.7276E-2	1.1232e-02	2.5055E-2	9.6635e-03
2^{-14}	4.7304E-2	1.1212e-02	2.5077E-2	9.6517e-03
2^{-15}	4.7317E-2	1.1202e-02	2.5088E-2	9.6458e-03
2^{-16}	4.7323E-2	1.1197e-02	2.5093E-2	9.6428e-03
2^{-17}	4.7326E-2	1.1195e-02	2.5096E-2	9.6413e-03
2^{-18}	4.7328E-2	1.1193e-02	2.5097E-2	9.6406e-03
2^{-19}	4.7328E-2	1.1193e-02	2.5097E-2	9.6402e-03
E^N	4.7328E-2	1.1819e-02	2.5097E-2	9.997e-03

Table 5.5: Comparison of maximum absolute error of Example 2 with results in (Priyadharshini & Ramanujam, 2009)

$\varepsilon \downarrow$	N$\rightarrow$						
2^-2	32	64	128	256	512	1024	E^N
A	7.3953e-03	4.3986e-03	2.0918e-03	8.4695e-04	3.0685e-04	1.0305e-04	7.3953e-03
B	2.3892e-03	1.2378e-03	6.2989e-04	3.1775e-04	1.5959e-04	7.9972e-05	2.3892e-03

where, A is the maximum absolute error in (Priyadharshini & Ramanujam, 2009), and B is the maximum absolute error of the proposed hybrid scheme.

Discussion

The solution to the problem considered exhibited an interior layer, which arises due to the discontinuity in the convection coefficient and source terms of considered problem. As we can see in Figures 5.1 and 5.2 , as the perturbation parameter ε decreased, the formation of the interior layer became more visible, demonstrating the ability of the method to capture the fine details of the problem.

Figures 5.3 - Figures 5.10 depict the solution profile for Examples 1 and 2, employing both schemes (4.33) and (4.46). In Figures 5.3, 5.5, 5.7, 5.9 the solution obtained using the Shishkin mesh revealed layer resolution properties, as they featured a significant number of mesh points that accurately captured the behavior in the interior layer. In contrast, in Figures 5.4, 5.6, 5.8, 5.10 the solution obtained using uniform meshes exhibited fewer mesh points in the interior layer, potentially leading to less accurate representations of the solution in that region. This implies that the Shishkin mesh more accurately depicts the behavior of the interior layer.

The error graphs displayed by the Shishkin mesh (Figures 5.11 and 5.12) show relatively small errors that are not significantly affected by the perturbation parameters , in contrast to the error graphs displayed by the uniform meshes (Figures (5.13) and (5.14)).

The maximum absolute error and rate of convergence of the proposed method was analyzed in Tables 5.1 and 5.2.

Moreover, the proposed method was compared with existing methods by analyzing the generated errors. The results indicate that the proposed method yields smaller errors compared to the other methods (see Tables 5.3, 5.4 and 5.5). Furthermore, it was observed that, regardless of the perturbation parameter ε, the maximum absolute errors decrease as the mesh is refined. This finding implies that the proposed method is parameter-uniform, it remains accurate as perturbation parameter ε becomes small.

CONCLUSION AND RECOMMENDATION

Conclusion

In this study, we focused on singularly perturbed convection diffusion boundary value problems (SPCDVBPs) with discontinuous convection and source terms. A parameter-uniform hybrid numerical scheme developed utilizing non-standard finite difference method (NSFDM) on Shishkin mesh and mid-point upwind finite difference scheme employing both Uniform mesh and Shishkin mesh. The presence of interior layer in the solutions was effectively handled by the proposed methods. Stability and uniform convergence analysis of proposed method was established. Two test examples were considered, and the graphical representation of the solutions profile was presented using both uniform mesh and Shishkin mesh. It was observed that the Shishkin mesh provided a layer-resolving representation with an adequate number of mesh points in the interior layer. Furthermore, the maximum absolute errors and convergence rates of the methods was tabulated, and the results indicated that the scheme is uniformly convergent with the rate of convergence almost one.

Recommendation

Based on the thesis findings, we recommend for the future researchers, that the applicability of the non-standard finite difference method (NSFDM) on Shishkin mesh and mid-point upwind finite difference scheme can also be extended to more complex problems in different scenarios. Hence, we recommend that future researchers can extend the non-standard finite difference method on Shishkin mesh and midpoint upwind finite difference scheme for semi-linear and higher-dimensional singularly perturbed differential equations (SPDEs) with non-smooth data.

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References

Boffi, D. (2020). *Convection-diffusion problems. an introduction to their analysis and numerical solution.*

Cen, Z. (2005). A hybrid difference scheme for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient. *Applied mathematics and computation*, *169*(1), 689–699.

Chandru, M., Prabha, T., & Shanthi, V. (2017). A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data. *Journal of Computational and Applied Mathematics*, *309*, 11–27.

Farrell, P., Hegarty, A., Miller, J., O'Riordan, E., & Shishkin, G. (2004). Singularly perturbed convection–diffusion problems with boundary and weak interior layers. *Journal of Computational and Applied Mathematics*, *166*(1), 133–151.

Farrell, P. A., Hegarty, A. F., Miller, J. J., O'Riordan, E., & Shishkin, G. I. (2004). Global maximum norm parameter-uniform numerical method for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient. *Mathematical and Computer Modelling*, *40*(11-12), 1375–1392.

Gregory, D. T., & Wickramasinghe, C. D. (2023). An upwind finite difference method to singularly perturbed convection diffusion problems on a shishkin mesh. *arXiv preprint arXiv:2306.03181*.

Gupta, V., Sahoo, S. K., & Dubey, R. K. (2021). Robust higher order finite difference scheme for singularly perturbed turning point problem with two outflow boundary layers. *Computational and Applied Mathematics*, *40*, 1–23.

Kadalbajoo, M. K., & Gupta, V. (2009). Numerical solution of singularly perturbed convection–diffusion problem using parameter uniform b-spline collocation method. *Journal of Mathematical Analysis and Applications*, *355*(1), 439–452.

Kadalbajoo, M. K., & Kumar, D. (2008). A non-linear single step explicit scheme for non-linear two-point singularly perturbed boundary value problems via initial value technique. *Applied mathematics and computation*, *202*(2), 738–746.

Kaushik, A., Kumar, V., & Vashishth, A. K. (2012). An efficient mixed asymptotic-numerical scheme for singularly perturbed convection diffusion problems. *Applied Mathematics and Computation*, *218*(17), 8645–8658.

Khandelwal, P., & Khan, A. (2017). Singularly perturbed convection-diffusion boundary value problems with two small parameters using nonpolynomial spline technique. *Mathematical Sciences*, *11*(2), 119–126.

Kumar, D., & Kadalbajoo, M. K. (2011). A parameter-uniform numerical method for time-dependent singularly perturbed differential-difference equations. *Applied Mathematical*

Modelling, *35*(6), 2805–2819.
Kumar, M., et al. (2011). Methods for solving singular perturbation problems arising in science and engineering. *Mathematical and Computer Modelling*, *54*(1-2), 556–575.
Mai, X., Zhu, C., & Liu, L. (2023). An adaptive grid method for a singularly perturbed convection-diffusion equation with a discontinuous convection coefficient. *Networks and Heterogeneous Media*, *18*(4), 1528–1538.
Melesse, W. G., Tiruneh, A. A., & Derese, G. A. (2020). Solving systems of singularly perturbed convection diffusion problems via initial value method. *Journal of Applied Mathematics*, *2020*, 1–8.
Mickens, R. E. (2005). *Advances in the applications of nonstandard finite difference schemes*. World Scientific.
Morton, K., & Kellogg, R. B. (1997). Numerical solution of convection-diffusion problems. *SIAM Review*, *39*(3), 535.
Mukherjee, K., & Natesan, S. (2011). ε-uniform error estimate of hybrid numerical scheme for singularly perturbed parabolic problems with nterior layers. *Numerical Algorithms*, *58*, 103–141.
Prabha, T., Chandru, M., & Shanthi, V. (2017). Hybrid difference scheme for singularly perturbed reaction-convection-diffusion problem with boundary and interior layers. *Applied Mathematics and Computation*, *314*, 237–256.
Priyadharshini, R. M., & Ramanujam, N. (2009). Approximation of derivative to a singularly perturbed second-order ordinary differential equation with discontinuous convection coefficient using hybrid difference scheme. *International Journal of Computer Mathematics*, *86*(8), 1355–1365.
Priyadharshini, R. M., Ramanujam, N., & Valanarasu, T. (2010). Hybrid difference schemes for singularly perturbed problem of mixed type with discontinuous source term. *Journal of applied mathematics & informatics*, *28*(5_6), 1035–1054.
Ranjan, K. R., & Gowrisankar, S. (2022). Nipg method on shishkin mesh for singularly perturbed convection-diffusion problem with discontinuous convection coefficient. In *International workshop of mathematical modelling, applied analysis and computation* (pp. 195–208).
Ranjan, R., & Prasad, H. S. (2022). A novel exponentially fitted finite difference method for a class of 2nd order singularly perturbed boundary value problems with a simple turning point exhibiting twin boundary layers. *Journal of Ambient Intelligence and Humanized Computing*, *13*(9), 4207–4221.
Rao, S. C. S., Chawla, S., & Chaturvedi, A. K. (2022). Numerical analysis for a class of coupled system of singularly perturbed time-dependent convection-diffusion equations with a discontinuous source term. *Numerical Methods for Partial Differential*

Equations, *38*(5), 1437–1467.

Roos, H.-G. (2008). *Robust numerical methods for singularly perturbed differential equations*. Springer.

Sahoo, S. K., & Gupta, V. (2022). Higher order robust numerical computation for singularly perturbed problem involving discontinuous convective and source term. *Mathematical Methods in the Applied Sciences*, *45*(8), 4876–4898.

Shishkin, G. I., & Shishkina, L. P. (2008). *Difference methods for singular perturbation problems*. CRC Press.

Siraj, M. K., Duressa, G. F., & Bullo, T. A. (2019). Fourth-order stable central difference with richardson extrapolation method for second-order self-adjoint singularly perturbed boundary value problems. *Journal of the Egyptian Mathematical Society*, *27*(1), 1–14.

Woldaregay, M. M. (2022). Solving singularly perturbed delay differential equations via fitted mesh and exact difference method. *Research in Mathematics*, *9*(1), 2109301.

Woldaregay, M. M., & Duressa, G. F. (2021). Robust mid-point upwind scheme for singularly perturbed delay differential equations. *Computational and Applied Mathematics*, *40*(5), 178.

Woldaregay, M. M., & Duressa, G. F. (2022). Fitted numerical scheme for singularly perturbed differential equations having two small delays. *Caspian Journal of Mathematical Sciences (CJMS)*, *11*(1), 98–114.

Woldaregay, M. M., Duressa, G. F., et al. (2022). Boundary layer resolving exact difference scheme for solving singularly perturbed convection-diffusion-reaction equation. *Mathematical Problems in Engineering*, *2022*.