ANALYSIS OF TRAFFIC FLOW AT MERGING JUNCTION

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DEJENE FEYISA

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance of a thesis entitled ANALYSIS OF TRAFFIC FLOW AT MERGING JUNCTION" by DEJENE FEYISA in partial fulfilment of the requirements for the degree of Masters of Science, MSc. Degree

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Examining Committee

External Examiner:

Supervisors:

Internal Examiner:

Chair Person:

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Author: DEJENE FEYISA DEGGEFA

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Authors Signature

Approved by the Examination Committee ______ ______ ______

(Advisor)	Signature	Date
(Internal Examiner)	Signature	Date
(External Examiner)	Signature	Date
(Chair Person)	Signature	Date
(Dep't. Head)	Signature	Date
(Sch. Dean)	Signature	Date
(Sch. Grad. Studies Dean)	Signature	Date

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Abbreviation

ODE- Ordinary Differential Equation

PDE- Partial Differential Equation

LWR- Lighthill -Whitham-Richards

CFL- Courant Friedrich-Lewy

Abstract

Most highways merge at some section specially when entering a city or a town. Some also with several number of lanes, narrows down due to infrastructure or land topology. This lead to vehicles slowing down and as a result traffic congestion situation is created.In this thesis, we attempted to analyze traffic flow at merging junction. To this aim,we have considered a merging junction with two incoming and one outgoing roads. The evolution of traffic on each segment of the merging junction is governed by non-linear hyperbolic partial differential equation.
Keywords: Merging Junction,Conservation Law,

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Chapter 1

Introduction

The origins of traffic flow theory dates back to the 1950's with the seminal works of Lighthill and Whitham 1955, and independently Richards 1956, who proposed a macroscopic model for traffic based on conservation of vehicles. The model, now known as the Lighthill -Whitham-Richards (LWR), is a nonlinear hyperbolic conservation law. The main mathematical challenge to the LWR, and more generally systems of conservation laws, is the development of discontinuities, which can occur in finite time even from smooth initial conditions. Recently,several authors have been extended the LWR model to traffic flow on road networks. The scalar LWR traffic flow model is inefficient to capture all the properties of traffic phenomenon. The modelling of traffic flow in highway networks is of vital importance for pavement management, providing information for traffic control, traffic flow prediction, traffic diversion, and the implementation of roadwork.

Typically, traffic flow models are broadly categorized into two main groups: macroscopic models and microscopic models. Macroscopic traffic models are used to identify the aggregate behavior of sets of vehicles, easy to validate and ensure a good real-time quality, such as the fluid-dynamic traffic models Lighthill, and Whitham 1955. Microscopic models are applied to model the travel behavior of an individual vehicle which is recognized as a function of the traffic conditions in its environment Cremer and Lundwig 1986. As drivers behavior in real traffic is difficult to observe and measure, microscopic models are difficult to validate accurately Daganzo 1994. In addition, the computational effort required by microscopic models is significantly higher than that for macroscopic models and the data required by microscopic models are harder to get. As for macroscopic models, they do not distinguish their components flows by origins and destinations; therefore when the traffic stream arrives at junctions, macroscopic models usually assign fixed turning ratios for the traffic stream Daganzo 1994. In this thesis,we focus on macroscopic models.

As an extension to LWR model, provided a pioneering flow-dynamic model based on first-order differential equations, termed the LWR model, which was the first model used to describe unidirectional traffic flow on highway networks. Payne 1971 developed a second-order model Wagner , Hoffman 1996, based on the car-following model Rothery 1997 that considered the driver's reaction time. In real traffic, heavy congestion frequently occurs at the junction on a highway. Vehicles change their direction at the junction by changing to the other lane. The merging traffic flow is called weaving. When vehicles change their lane at the weaving section of junction, they slow down. The lane changing becomes easy by the slowdown because the speed is slower, the headway necessary for changing the lane is shorter. Also, the slowdown is stronger, the traffic is more congested at the junction. When we mean a junction, it can be represents an intersection of different roads such as a merge or a divergence or T-intersection. Thus, traffic congestion is induced by the slowdown when vehicles move aside on the other lane for changing their direction.

1.1 Statement Of The Problem

The fluctuation of traffic dynamics is frequently observed at merging junction on the road network. It is highly observed in the congested traffic state. Such phenomena may lead to capacity drop on the road network or on the highway. To precisely present these phenomena developing model analysis that describe the relation between flow and merging junction is very important. To this end, the leading research questions are:

1. can we describe the fluctuation of traffic at merging junction has mathematical model?

2. What are the impact of merging junction on the dynamical properties of traffic evolutions?

1.2 Objectives Of The Study

1.2.1 General Objective

The general objective of this study is to investigate the impact of merging junction on the traffic dynamics on the road network.

1.2.2 Specific Objective

The specific objectives of the study are:

1. Formulate a mathematical model that describe the fluctuation of traffic at merging junction.

2. Examine the impact of merging junction on the traffic dynamics.

3. Study the solution nature of the governing equations.

1.3 Methodolgy

In this study we develop a mathematical model that describe the fluctuation of traffic at merging junction based on a system of non-linear hyperbolic partial differential equation.We study the governing equation analytically.

1.4 Significance and Beneficiaries

This study provides a reliable information on traffic flux. The significance of this study:

(a) To maximize the traffic flux at junction.

(b) To minimize the traffic congestion , accidents , pollution etc at merging junction.

The outcome of the study benefits all the policy makers transportation to develop awareness about merging junction on the road network or highway.

1.5 Expected outcomes

The expected outcomes of the proposed study are:

1. The mathematical model that precisely describe the impact of merging junction on traffic flux on the road network.
2. Maximise the traffic flux at merging junction.
3. The headway between delayed vehicle to be specified.

Chapter 2

Literature Review

Traffic flow theory entered the scientific mind in the 1930's with studies performed by Greenshields 1955 and Adams 1936. These early attempts at traffic modeling made use of probability theory to describe traffic flow and tried to find connections between speed and traffic density. This followed by the development of different approaches to modeling in the 1950 introducing car-following, wave and queueing theory approaches which resulted among other things in the much dominant work by Lighthill and Whitham 1955. The LWR model describes the evolution of traffic flow on a road segment with uniform topology. The change in road segment characteristics (crossing, number of lanes, speed limit, curvature, etc.) can be modeled using a junction . The treatment of junctions requires specific efforts for physical consistency and mathematical compatibility with the link model. For uniqueness of the solution of the junction problem, different conditions have been used, for instance, maximizing the incoming flow through the junction was sug- gested by Daganzo 1995 and Coclite, Garavello and Piccoli 2005. Holden and Risebro 1995 consider maximizing a concave function of the incoming flow.

When attempting to model traffic flow there are three main perspectives to choose from: Microscopic , Mesoscopic and Macroscopic. As one might suspect, Microscopic models concentrates on the behavior of each individual vehicle (look at a single vehicle in a chain of others and how it responds to the traffic around it) whereas Macroscopic models, looks all vehicles as a stream, treating them as a single unit. Mesoscopic models, also known as Kinetic models, are somewhere in the border country between the other two. In going research macroscopic outlooks are clearly dominating, followed by the microscopic ones whereas mesoscopic models are rarely seen at all.

Macroscopic models describe the traffic flow by continuous aggregate functions like average density, velocity and flow in the space-time domain. The dynamics of traffic flow is modelled by a nonlinear system of Partial Differential Equations. Typically, a macroscopic model defines a relation between the traffic density, the average velocity and the traffic flow. (Gerlough and Huber, 1975; Pensaud and Hurdle, 1991; Ross, 1991; Hall, Hurdle and Banks, 1992; Gilchrist and Hall, 1992; Disbro and Frame, 1992). Within the class of macroscopic models, a classification based on the order of the models can be made. The oldest model was proposed by Lighthill and Whitham in 1955 and independently by Richards in 1956 and is of first order.
Lighthill-Whitham-Richards model is given below:

$$\rho_t + (\rho u(\rho))_x = 0, u(\rho) = u_{max}(1 - \frac{\rho}{\rho_{max}}), 0 \leq \rho \leq \rho_{max}$$

The only state variable of this model is the traffic density. The Lighthill, Whitham, Richards model was extended later on in order to be able to cope with shock waves and stop-and-go traffic in congested traffic situations (Newell, 1993). The model described by Payne (1971) is of second order since it has two state variables: traffic density and average velocity.

Since macroscopic traffic models only work with aggregate variables and do not describe the traffic situation on the level of independent vehicles, they are less computationally intensive than microscopic models. Due to the fact that a macroscopic traffic model has fewer parameters to estimate than a microscopic model, it is easier to identify and to tune a macroscopic model.

Chapter 3

Hyperbolic conservation laws

The works presented in this thesis is based on hyperbolic conservation laws that are nonlinear partial differential equations where the unknown variable is a conserved quantity. In this chapter, we introduce some basic notions about scalar conservation laws.

3.1 Derivation of Conservation Law

Conservation law states that the change in the total amount of a physical quantity contained in any region of space must be equal to the net flux of that quantity across the boundary of that region. With this in mind, assume that a road is a line with infinite length and size of a vehicle is assumed to be negligible compared to the road length. Furthermore, assume that vehicles do not enter or leave the section of the road at any of its points. Fix any two distinct points x_1 and x_2 on the section of the road. Then the number of vehicles between x_1 and x_2 varies according to the rule

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(t,x)\,dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho(t,x)\,dx \tag{3.1}$$

This rate of change must be equal to the net flux across x_1 and x_2 which is given by

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho(t,x)\,dx = f(\rho(t,x_1)) - f(\rho(t,x_2)) \tag{3.2}$$

Equation (3.2) measures the flow of vehicles entering the road segment at x_1 and leaving the segment at x_2. In other words, ρ is neither created nor destroyed but conserved. Thus the total amount of ρ contained inside any given interval $[x_1,\, x_2]$ can be changed only due to the flow of ρ across boundary points.

Integrating both sides of equation (3.2) on the time interval $[t_1, t_2]$ yields

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial t}\rho(t,x)\,dxdt = \int_{t_1}^{t_2} f(\rho(t,x_1)) - f(\rho(t,x_2))\,dt \tag{3.3}$$

For continuously differentiable functions equation (3.3) can be written as

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\rho_t(t,x)\,dxdt = -\int_{t_1}^{t_2}\int_{x_1}^{x_2} f(\rho(t,x))_x\,dxdt \tag{3.4}$$

Rearranging and simplifying equation (3.4) gives

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\rho_t(t,x) + f(\rho(t,x))_x\right]\,dxdt = 0 \tag{3.5}$$

Since this equation holds for arbitrary limits of integration, the integrand must be zero. Consequently equation (3.5) holds for all x_1 and x_2.

Hence, a scalar conservation law in one space-dimension is a first order partial differential equation of the form:

$$\rho_t + f(\rho)_x = 0 \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{3.6}$$

where $\rho : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is the conserved quantity and $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is the flux with t being the time variable and x the one-dimensional space variable.

3.1.1 Weak solution

It is known that for non-linear conservation laws, classical solutions may not exist even for very smooth initial data because discontinuities develop in finite time. For example consider the scalar Cauchy problem

Example 3.1.1 *Consider the problem*

$$\begin{cases} \rho_t + \rho\rho_x = 0 \\ \rho(0,x) = \rho_0(x) = \dfrac{1}{1+x^2}. \end{cases} \tag{3.7}$$

For small $t > 0$, the solution can be found by the method of characteristics as follows:

$$\begin{cases} \frac{dx}{dt} = \rho, \\ \frac{d\rho}{dt} = 0 \end{cases} \tag{3.8}$$

Solving this system of ordinary differential equation we have
$\rho(t,x) = c_1$ and $x(t) = c_1 t + c_2$, where c_1 and c_2 are arbitrary constants. Applying

initial condition one can obtain that

$$x(t) = x + \frac{t}{1+x^2}.$$

The function $\rho(t,x)$ is constant along the characteristic lines in the x-t plane. Now the main goal is to find the positive time t_s(breaking time) and the location x_s of the first appearance of the shock. To achieve this we consider, $f(\rho) = \frac{\rho^2}{2}$, $f'(\rho) = \rho$, $f''(\rho) = 1$, and $\rho_0(\xi) = \frac{1}{1+\xi^2}$, where $\rho_0'(\xi) = \frac{-2\xi}{(1+\xi^2)^2}$.
The non-negative function

$$z(\xi) = -f''(\rho_0(\xi))\rho_0'(\xi) = \frac{2\xi}{(1+\xi^2)^2}$$

has a maximum at ξ_M, where $z'(\xi_M) = 0$ which on solving gives $\xi_M = \frac{1}{\sqrt{3}}$ and $z(\xi_M) = \frac{\sqrt{27}}{8}$. Thus, the breaking time t_s and the location of the first appearance of the shock is at

$$x_s = f'(\rho(\xi_M))t + \xi_M = \sqrt{3}.$$

For t sufficiently small $0 \leq t < \frac{8}{\sqrt{27}}$, the characteristic lines do not intersect each other and the solution is unique. The solution to our Cauchy problem is thus given implicitly by

$$\rho(x + \frac{t}{1+x^2}, t) = \frac{1}{1+x^2}.$$

On the other hand, when $t > \frac{8}{\sqrt{27}}$, the characteristic lines start to intersect each other. As a result, the map

$$x \longmapsto x + \frac{t}{1+x^2}$$

is not one-to-one and the map $(x + \frac{t}{1+x^2}, t) = \frac{1}{1+x^2}$ no longer defines a single valued solution of our Cauchy problem. Hence, in order to construct solutions globally in time, we must work in a space of discontinuous functions and consider the conservation law in a distributional sense. Now we give the definition of weak solution[1].

Definition 3.1.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field. A measurable function $\rho : \mathbb{R} \times [0,\infty) \to \mathbb{R}^n$ is a weak solution of the system of conservation laws*

$$\begin{cases} \rho_t + (f(\rho))_x = 0 \\ \rho(0,x) = \phi(x) \end{cases} \tag{3.9}$$

if

$$\int_0^\infty \int_{-\infty}^\infty (\rho\varphi_t + f(\rho)\varphi_x)dxdt + \int_{-\infty}^\infty \rho(0,x)\varphi(0,x)dx = 0 \tag{3.10}$$

for all smooth function $\varphi : \Omega \longrightarrow \mathbb{R}^n$ *with compact support.*

The proof of the following theorem can be found in [1] and other literature in compact form. Here it is proved in detail.

Theorem 3.1.1 *Suppose* $\rho^-, \rho^+ \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$*. If the function*

$$\rho(t,x) = \begin{cases} \rho^+ & \text{if } x > \lambda t \\ \rho^- & \text{if } x < \lambda t \end{cases} \tag{3.11}$$

is a weak solution of the system of conservation laws (3.9), then

$$\lambda(\rho^+ - \rho^-) = f(\rho^+) - f(\rho^-). \tag{3.12}$$

proof 3.1.2 *Let* $\varphi = \varphi(t,x)$ *be any continuously differentiable function with compact support* $supp\varphi \subset \Omega$*. Let* Γ *be a smooth curve that divides* Ω *into two parts.* $\Omega^- := \Omega \cap \{x < \lambda t\}$, $\quad \Omega^+ := \Omega \cap \{x > \lambda t\}$*. Let* $\vec{n} = (n_1, n_2)$ *be the unit normal*

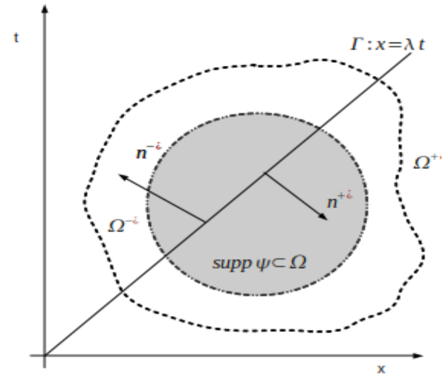


Figure 3.1: Region where the divergence theorem is applied

to the curve Γ *pointing from* Ω^- *to* Ω^+ *and* $\vec{n^-} = -\vec{n^+}$*. Suppose that the curve* Γ *is represented parametrically as* $\{(t,x) : x = s(t) = \lambda t\}$ *for some smooth function* $s : [0,\infty) \to \mathbb{R}$*.*

Then $n^+ = (n_1^+, n_2^+) = \dfrac{1}{\sqrt{1+(\dot{s}(t))^2}}(1, -\dot{s}(t))$ *where* $\dot{s}(t) = \dfrac{d}{dt}\lambda t = \lambda$*. Substituting we get* $\quad n^+ = \dfrac{1}{\sqrt{1+\lambda^2}}(1, -\lambda)$*.*

Define the vector field $g := (\rho(t,x)\cdot\varphi(t,x), f(\rho(t,x))\cdot\varphi(t,x))$*. From definition of weak solution we know that*

$$\iint_\Omega (\rho\varphi_t + f(\rho)\varphi_x)dxdt = 0 \tag{3.12}$$

Equation(3.1.2) can be written as

$$0 = \iint_{\Omega^+\cup\Omega^-} divg \quad dxdt = \int_{\partial\Omega^-}(n^-\cdot g)ds + \int_{\partial\Omega^+}(n^+\cdot g)ds \tag{3.12}$$

where ds denotes the differential of the arc length along the curve Γ. *Since supp*$\varphi \subset \Omega$, *then* $\varphi = 0$ *on the boundary* $\partial\Omega$. *That is*

$$\iint_{\Omega}(\rho\varphi_t + f(\rho)\varphi_x)dxdt = \iint_{\Omega^-}(\rho\varphi_t + f(\rho)\varphi_x)dxdt + \iint_{\Omega^+}(\rho\varphi_t + f(\rho)\varphi_x)dxdt = 0 \tag{3.12}$$

Applying integration by parts on Ω^- *we get*

$$\begin{aligned}\iint_{\Omega^-}(\rho\varphi_t + f(\rho)\varphi_x)dxdt &= -\iint_{\Omega^-}(\rho_t + f(\rho)_x)\varphi dxdt + \int_{\Gamma=\partial\Omega^-} g\cdot\vec{n}^- ds \\ &= -\int_{\partial\Omega^-} n^-\cdot g ds = -\int_{\partial\Omega^-}(\rho^-\cdot n_2 + f(\rho^-)\cdot n_1)\varphi ds\end{aligned} \tag{3.12}$$

Similarly, applying integration by parts on Ω^+ *with* n^+ *gives*

$$\iint_{\Omega^+}(\rho\varphi_t + f(\rho)\varphi_x)dxdt = \int_{\partial\Omega^+}(\rho^+ n_2 + f(\rho^+)n_1)\varphi ds \tag{3.12}$$

Adding equation(3.1.2) and (3.1.2) and comparing with equation (3.1.2) we obtain

$$-\int_{\Gamma}(\rho^- n_2 + f(\rho^- n_1))\varphi ds + \int_{\Gamma}(\rho^+ n_2 + f(\rho^+ n_1))\varphi ds = 0 \tag{3.12}$$

or which on rearranging gives

$$-\int_{\Gamma}((\rho^+ - \rho^-)n_2 + (f(\rho^+) - f(\rho^-)n_1))\varphi ds = 0 \tag{3.12}$$

Since $\varphi \in C_c^1(\Omega)$ *is arbitrary, we must have*

$$(\rho^+ - \rho^-)n_2 + (f(\rho^+) - f(\rho^-))n_1 = 0. \tag{3.12}$$

After rearranging and simplifying equation (3.1.2) we obtain the famous Rankine-Hugoniot jump condition

$$\lambda(\rho^+ - \rho^-) = f(\rho^+) - f(\rho^-). \tag{3.12}$$

Here we observe that equation (3.1.2) form a set of n scalar equations relating the right and left states of $\rho^-, \rho^+ \in \mathbb{R}^n$ and the speed λ of the discontinuity.

Remark 3.1.1 *In the scalar case, one can arbitrarily assign the left and right states* $\rho^-, \rho^+ \in \mathbb{R}$ *and determine the shock speed as*

$$\lambda = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-} \tag{3.13}$$

Geometrically this can be interpreted as the slop of the secant line through the points $(\rho^-, f(\rho^-))$ and $(\rho^+, f(\rho^+))$ on the graph of the flux function f.

The definition of weak solution alone does not guarantee uniqueness, since it is possible to construct infinitely many weak solutions starting from an initial datum. Therefore, it is necessary to introduce some admissibility conditions, motivated by physical consideration.

3.2 Entropy Admissible Condition

In the previous section, we have seen that the weak formulation is not enough to guarantee uniqueness of the solution for initial value problems. In this section we shall see how to guarantee unique solution by supplementing the weak formulation with admissibility condition. One way is using the entropy condition. The entropy acts as an indicator of discontinuities, and can be used to isolate a unique solution. This leads to the following definition.

Definition 3.2.1 *A C^1 function $\eta : \mathbb{R}^n \to \mathbb{R}$ is an entropy for (3.6) if it is convex and there exists a C^1 function $q : \mathbb{R}^n \to \mathbb{R}$ such that*

$$D\eta(\rho) \cdot Df(\rho) = Dq(\rho) \tag{3.14}$$

for every $\rho \in \mathbb{R}^n$. The function q is said to be an entropy flux for η. The pair (η, q) is called entropy-entropy flux pair for (3.6).

Definition 3.2.2 *A weak solution $\rho = \rho(t, x)$ to the Cauchy problem*

$$\begin{cases} \rho_t + f(\rho)_x & = 0 \\ \rho(0, x) & = \rho_0 \end{cases} \tag{3.15}$$

is said to be entropy admissible if, for every C^1 function $\varphi \geq 0$ with compact support in $[0, T] \times \mathbb{R}$ and for every entropy-entropy flux pair (η, q), it holds

$$\int_0^T \int_{\mathbb{R}} (\eta(\rho)\varphi_t + q(\rho)\varphi_x) dx dt \geq 0. \tag{3.16}$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a smooth flux function. Suppose that the system of conservation laws

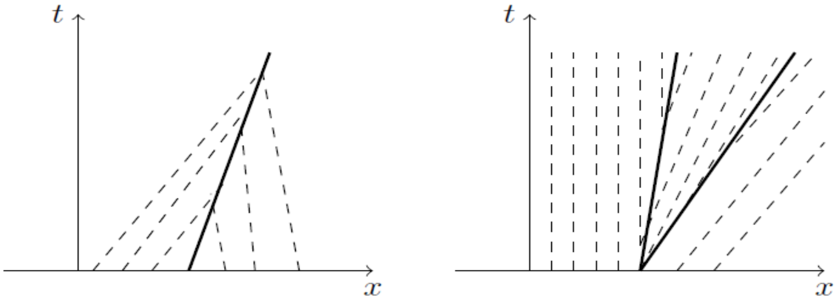
$$\rho_t + f(\rho)_x = 0 \tag{3.17}$$

is strictly hyperbolic.

Remark 3.2.1 (Lax condition) *A discontinuity connecting two states ρ^- and ρ^+ and traveling with speed λ given by* (3.13) *is entropy if and only if*

$$f'(\rho^-) \geq \lambda \geq f'(\rho^+).$$

The geometric meaning of this condition is given in Figure 3.2. In particular, this condition requires that characteristics run into the jumps and that jumps where new characteristics are "created" are not allowed.



(a) Entropy admissible solution. (b) Not entropic solution.

Figure 3.2: Geometric explanation for the Lax admissibility condition.

3.3 The Riemann Problem

A Riemann problem is a Cauchy problem with Heaviside type initial datum. Let $\Omega \subseteq \mathbb{R}$ be an open set and $f : \Omega \to \mathbb{R}$ be a smooth and strictly concave flux.

Definition 3.3.1 *A Riemann problem for equation* (3.6) *is a Cauchy problem with the initial datum of the form*

$$\rho_0(x) = \begin{cases} \rho^- & \text{if } x < 0, \\ \rho^+ & \text{if } x > 0, \end{cases} \tag{3.18}$$

where $\rho^-, \rho^+ \in \Omega$.

The Riemann problem provides the building block for the construction of the solution of Cauchy problems with more general initial data as well as some numerical approximation.

Chapter 4

Traffic Distribution at Junction

4.1 Introduction

In the previous chapters, different types of traffic flow models and the evolution of the traffic state on a single road has been presented in the literature. In this chapter we shall introduce traffic distribution at junction with special emphasize on junctions with two incoming and one outgoing roads. When we mean a junction, it can be represents an intersection of different roads such as a merge or a diverge or T-intersection. If a road network is viewed as a graph where each arc is a road segment, then the junctions are correspond to the nodes. The origin or end nodes are also the particular forms of junctions.

Lighthill and Whitham 1955 and, independently Richards 1956 introduced a hydrodynamic model for traffic flow on a single infinite road. They thought of traffic as a fluid and used fluid dynamics equations to describe its behaviour. The proposed model is based on the conservation of cars and it consists of a single conservation law, which describes the traffic evolution in terms of macroscopic variables (density, average speed of cars). This type of models is referred to as macroscopic models in traffic literature.

4.2 The model

Let us consider a unidirectional stretch of road which is modelled by an interval $\mathrm{I} = [a, b]$ with $a < b, a, b \in \mathbb{R}$ and the possibility of either a and b equal to ∞. The model is based on the equation for the conservation of mass:

$$\rho_t + f(\rho)_x = 0 \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{4.1}$$

where $\rho = \rho(t, x) \in [0, \rho_{\max}]$ is the conserved quantity representing the density (number of cars per unit length), $\rho_{\max}$ being the maximal density allowed in the car. The flow f : $[0, \rho_{\max}] \to \mathbb{R}$ is a smooth flux function that is usually given by $f(\rho) = \rho v$ where $v = v(\rho)$ is the average speed of cars.

The following hypotheses are made on the flux:

(A_1) f is a C^2 function;

(A_2) f is a strictly concave function:$f''(\rho) > 0$;

(A_3) f(0) = f($\rho_{\max}$) = 0,

The main assumption for the LWR model is that the velocity depends only on the density of cars. A reasonable supposition is that v is a decreasing function of the density. In the transportation literature, the graph that links the flux and the density is called fundamental diagram. According to the choice of the velocity function we can have a variety of fundamental diagrams. The simplest choice is a linear function of the density,

$$v(\rho) = v_{max}(1 - \frac{\rho}{\rho_{max}})$$

See Figure 4.1. The corresponding fundamental diagram is obtained by multiplying the density by the speed.

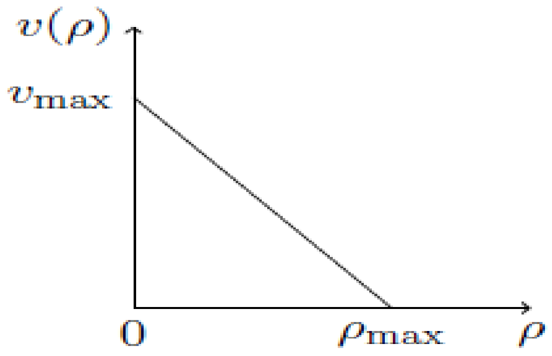


Figure 4.1: Speed of cars: linear decreasing function.

This gives a C^2 concave function like the one in Figure 4.2 This flow-density relation was introduced by Greenshields 1935 and it is one of the most used in the mathematical community in transportation.

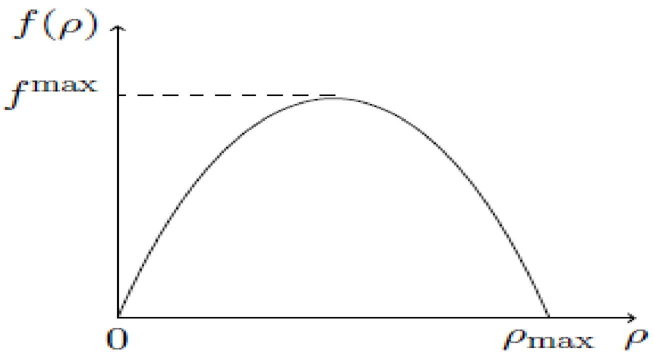


Figure 4.2: Fundamental diagram.

Consider the Riemann problem at junction

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ \rho(0,x) = \begin{cases} \rho^- & \text{if } x < 0; \\ \rho^+ & \text{if } x > 0. \end{cases} \end{cases} \tag{4.2}$$

Now we can define the Riemann Solver for the LWR model as follows:

Definition 4.2.1 *The Riemann solver $\mathcal{RS}$ for the problem (4.2) is the (right continuous) map $\rho(t,x) \to \mathcal{RS}(\rho^-, \rho^+)(t,x)$given by the standard weak entropy solution. It is defined as follows.*

a) If $\rho^- < \rho^+$, then the entropy-admissible solution is given by the shock wave

$$\rho(t,x) = \begin{cases} \rho^- & \text{if } x < \lambda t; \\ \rho^+ & \text{if } x > \lambda t. \end{cases} \tag{4.3}$$

where, by the Rankine-Hugoniot condition, we get $\lambda = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-}$

b) If, instead, $\rho^- > \rho^+$ the entropy-admissible solution to the Riemann problem is given by the rarefaction wave

$$\rho(t,x) = \begin{cases} \rho^- & \text{if } x < f'(\rho^-)t; \\ (f')^{-1}(\frac{x}{t}) & \text{if } f'(\rho^-)t < x < f'(\rho^+)t \\ \rho^+ & \text{if } x > f'(\rho^+)t. \end{cases} \tag{4.4}$$

To distribute the traffic at the junction the following assumptions are made:

- The drivers have some prescribed preferences that means that there are some fixed coefficients which distributes the traffic from the incoming roads to the outgoing ones.
- The drivers choose to maximize the flux through the junction, respecting the prescribed preferences.

4.3 Traffic evolution at merging junction

We fix a junction with two incoming roads $[a_i, b_i], i = 1, 2$ and one outgoing road $[a_3, b_3]$ and assume that a right way of parameter $p \in (0, 1)$ is given. When the number of cars are too many to pass all of them through the junction, there is a yielding rule that describes the percentage of cars, going through the junction, that comes from the first road. That is, if N be the quantity that can pass through the outgoing road. Then pN cars come from first incoming road and $(1-p)N$ cars from the seconding road.

The solution to the Riemann problem $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$ is formed by a single wave on each road connecting the initial states to $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ determined in the following way. For each incoming road I_i we define

$$\gamma_i^{max}(\rho_{i,0}) = \begin{cases} f(\rho_i), & \text{if } \rho_{i,0} \in [0, \rho_c], \\ f(\rho_c), & \text{if } \rho_{i,0} \in (\rho_c, 1] \end{cases} \tag{4.5}$$

while for each outgoing road I_j we define

$$\gamma_j^{max}(\rho_{j,0}) = \begin{cases} f(\rho_c), & \text{if } \rho_{j,0} \in [0, \rho_c], \\ f(\rho_j), & \text{if } \rho_{j,0} \in (\rho_c, 1] \end{cases} \tag{4.6}$$

Since our aim is to maximize traffic flow at junction we set:

$$\hat{\gamma}_3 = \min\left\{\gamma_1^{max}(\rho_{1,0}) + \gamma_2^{max}(\rho_{2,0}), \gamma_3^{max}(\rho_{3,0})\right\}, \tag{4.7}$$

where the function $\gamma_i^{max}, i = 1, 2, 3$, are defined in(4.5)and (4.6).

In fact, $\hat{\gamma}_3$ is the maximum flux which can respect the Rankine-Hugoniot condition at the junction. In this case, the matrix A is simply given by the column vector $(1, 1)$ and thus it gives no additional restriction. This is due to the fact that there is a single outgoing road.

Consider now the space (γ_1, γ_2) and the line:

$$\gamma_2 = \frac{1-p}{p}\gamma_1 \tag{4.8}$$

p as defined above. Let Q be the point of intersection of the line (4.8) with the line $\gamma_1 + \gamma_2 = \hat{\gamma}_3$. The final fluxes must belong to the region:

$$\Omega = \left\{(\gamma_1, \gamma_2) : 0 \le \gamma_i \le \gamma_i^{max}(\rho_{i,0}), 0 \le \gamma_1 + \gamma_2 \le \gamma_3\right\}.$$

There are two different cases:

- Q belongs to Ω,
- Q is outside Ω.

In the first case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = \mathrm{Q}$, while in the second we set $(\hat{\gamma}_1, \hat{\gamma}_2) = \mathrm{R}$, where R is the point of the segment $\Omega \cap \{(\gamma_1, \gamma_2) : \gamma_1 + \gamma_2 = \hat{\gamma}_3\}$ closest to the line (4.8).

4.3.1 Numerical scheme

The first step is then to discretize the junction model. We define a numerical grid in $(0, T) \times \mathbb{R}$ using the following notation.

- Δx is the fixed space grid size;
- Δt is the time step given by the CFL condition;
- $(t^n, x_j) = (n\Delta t, j\Delta x)$ for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ are the grid points.

Each road is divided in N + 1 cells numbered from 0 to N. The first and last cell of an edge are always a junction and we assume that these cells are ghost cells.

4.3.2 Godunov Scheme

The Godunov scheme as introduced in [19,20] is based on exact solutions to Riemann problems. The main idea of this method is to approximate the initial datum by a piecewise constant function, then the corresponding Riemann problems are solved exactly and a global solution is simply obtained by piecing them together. Finally, one takes the mean on the cell and proceeds by induction. Under the CFL condition

$$\Delta t \max_{j \in \mathbb{Z}} |\lambda^n_{j+\frac{1}{2}}| \leq \Delta x, \tag{4.9}$$

the waves generated by different Riemann problems do not interact. In the above inequality, $\lambda^n_{j+\frac{1}{2}}$ is the wave speed of the Riemann problem solution at the interface $x_{j+\frac{1}{2}}$ at time t^n. Under the condition (4.9) the scheme can be written as

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}(F^G(\rho_i^n, \rho_{i+1}^n) - F^G(\rho_{i-1}^n, \rho_i^n)), i = 2, 3, \ldots, \mathrm{N} - 1 \ \forall n \tag{4.10}$$

where the numerical flux F^G takes the following expression:

$$F^G(u, v) = \begin{cases} \min(f(u), f(v)) & \text{if } u \leq v, \\ \max(f(u), f(v)) & \text{if } v < u \ < \rho_c \vee \rho_c < v < u, \\ f(\rho_c) & \text{if } v < \rho < u. \end{cases} \tag{4.11}$$

for concave flux f. Consider a junction with two incoming and one outgoing road. We assume that the network is initially empty and set the number of grid points on each road segment N = 31. Then the simulation result give the following. Figure

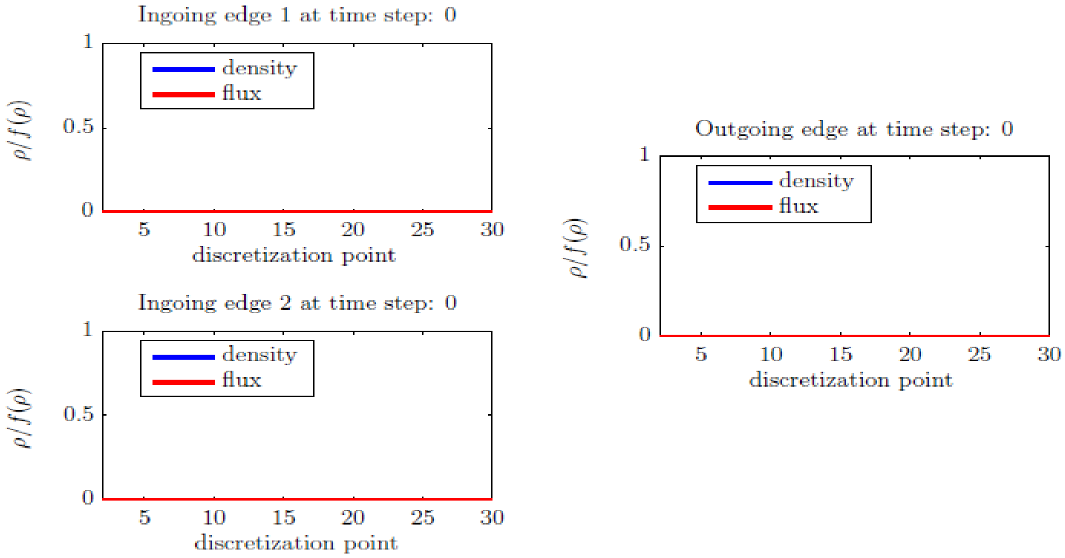


Figure 4.3: Initial the network is empty.

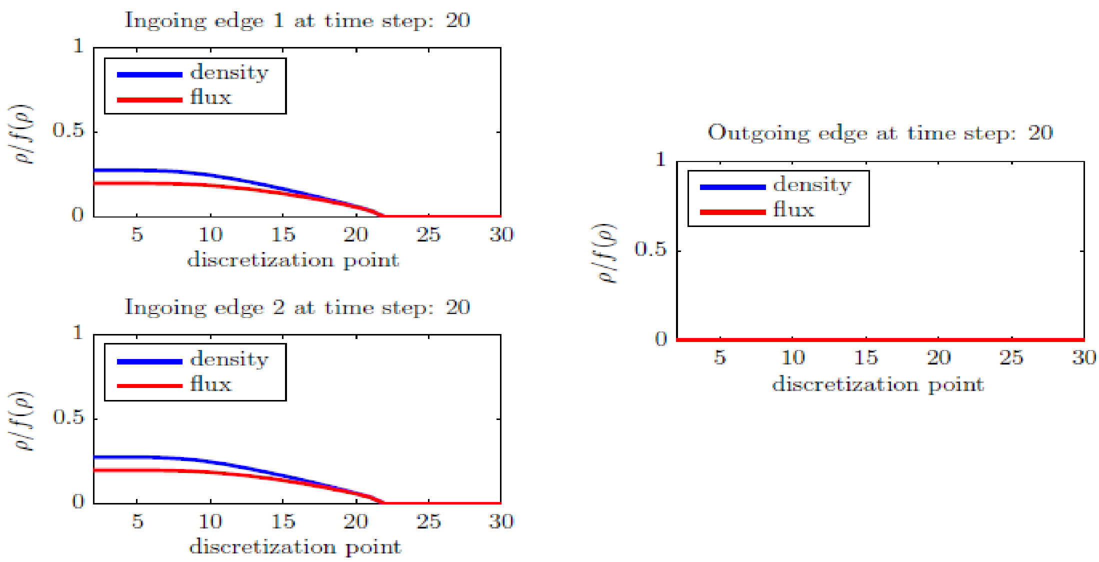


Figure 4.4: Traffic evolution on the incoming road at time step 20.

4.3 shows the network is empty. Figure 4.4 depicts traffic evolution is smooth for the first time step 20 with free flow speed on the incoming roads while the outgoing road stay empty. The simulation result shown in Figure 4.5 reveals that shock wave

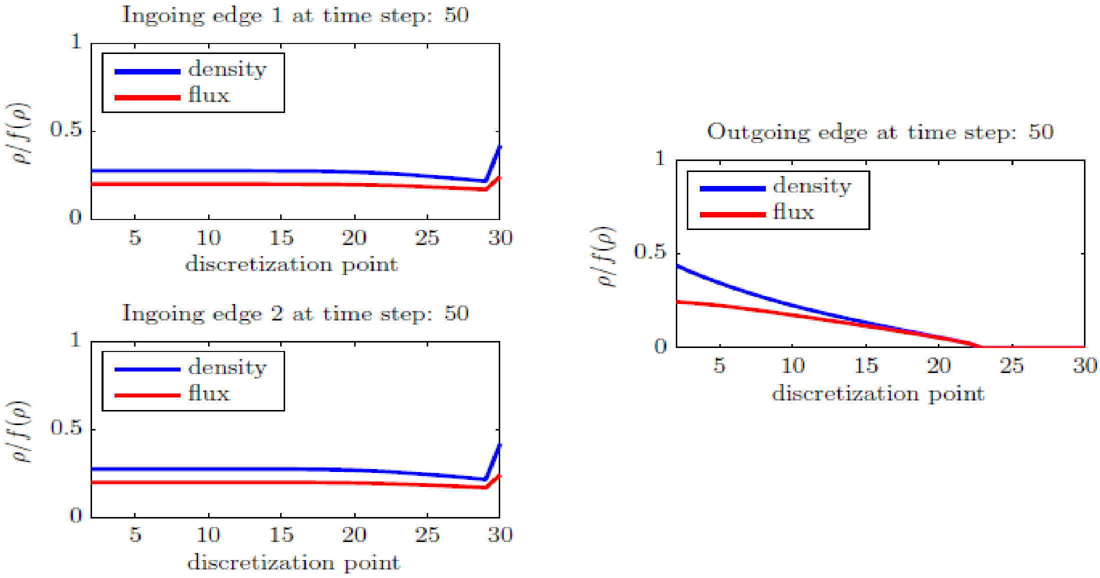


Figure 4.5: Traffic evolution on the incoming and outgoing roads at time step 50 with priority parameter p = 0.5.

propagates back on the incoming roads while rarefaction wave on the out going main road.

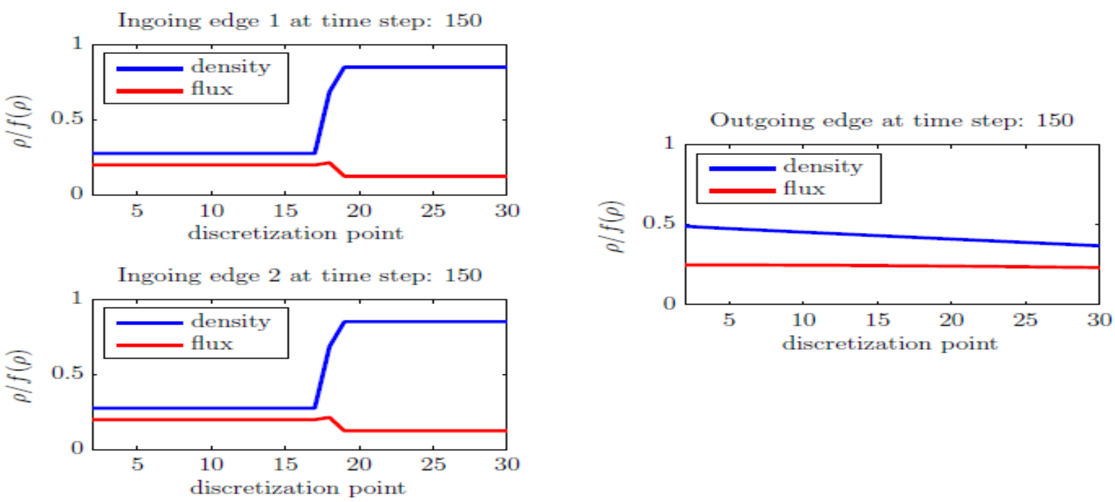


Figure 4.6: Traffic evolution on the incoming and outgoing roads at time step 150 with priority parameter p = 0.5.

Chapter 5

Conclusion

Macroscopic traffic models describe the evolution of vehicles position in terms of macro- scopic variables in terms of density and average speed of cars. The LWR is one of the first fluid dynamic type of traffic flow model proposed in 1955. It is a non-linear hyperbolic system of conservation law. The LWR model has the ability to capture the behavior of traffic flow phenomena such as shock formation and propagation. The LWR model is also used to study traffic flow phenomena at junction under chapter four.

In this section, the issue of traffic congestion and its socio-economic impact is presented briefly under chapter one. The advantage and disadvantages of modelling approaches were recalled before giving detail analysis. In chapter two we reviewed some of the existing related literature. In chapter three we have briefly derived hyperbolic conservation laws. The main results of this thesis can be summarized as follows. In section 4.3 traffic evolution at merging junction, by Numerical and Godunov scheme has been modeled as a 2x1 type junction. The evolution of traffic flow on the whole road networks of the traffic evolution at merging junction was described by nonlinear scalar hyperbolic partial differential equation (PDE). At each junction, the Riemann problem was uniquely solved using a right way of parameter in the case of merging junctions and in the other case we have used traffic distribution . The model validation with real data will be considered in the future work.

Bibliography

[1] A. Bressan *Hyperbolic Systems of Conservation Laws - The One-dimensional Cauchy Problem, Oxford Univ. Press, 2000.*

[2] B.D. Greenshields *,"A Study in Highway Capacity", Highway Research Board Proceedings Vol. 14, p. 458, 1935*

[3] *, B. De Schutter, Bellemans T., and B. De Moor (2002). "Models for traffic control," Journal A, vol. 43, no. 3-4, pp. 13-22, 2002.*

[4] G.M. Coclite, M. Garavello, *B. Piccoli, "Traffic Flow on a Road Net- work", SIAM J. Math. Anal., Vol. 36, pp. 1862-1886, 2005*

[5] C. F. Daganzo, *The cell transmission model: A simple dynamic representation of highway traffic consistent with the hydrodynamic theory. Transportation Research, 28B(4), 269–287, 1994.*

[6] H. Payne. *"Models of freeway traffic and control.", Simulation Council Proceedings, 1:5161, 1971.*

[7] C. F. Daganzo. *C. F. Daganzo. "Requiem for second-order fluid approximations of traffic flow.", Transportation Research Part B, 29(4):277 286, 1995.*

[8] *M.J. Lighthill, G.B. Whitham, "On kinematic waves: II. A theory of traffic flow on long crowded roads", Proc. R. S. A229, pp. 317-345, 1955*

[9] *G.F. Newell, "A simplified theory if kinematic waves in highway traffic, Part I: General theory, Part II: Queueing at freeway bottlenecks, Part III: Multi-destimation flows", Transpn. Res., 27B, pp. 281-313, 1993*

[10] *R. W. Rothery, "Car-following models."In Traffic flow theory: A state of the art report - revised monograph on traffic flow theory, N. H. Gartner, C. Messer,1997.*

[11] *Banks, James H. (1992). "Freeway Speed-Flow-Concentration Relationships: More Evidence and Interpretations." Transportation Research Record 1225, pp. 53-60.*

[12] *P.I. Richards, "Shock Waves on the Highway", Operations Research, 4, pp. 42-51, 1956*

[13] *W.F. Adams, "Road Traffic Considered as a Random Series", J. Inst. Civil Engineers 4, pp. 121-130, 1936*

[14] *B D Greenshield. A study in highway capacity. Highway Research Board, Proceedings, 14:458, 1935.*

[15] *H. Holden and N. H. Risebro, A Mathematical Model of Traffic Flow on a Network of Unidirectional Roads, SIAM J. Math. Anal., 26 (1995), pp. 999-1017.*

[16] *John, Fritz (1991), Partial differential equations (4th ed.), Springer, ISBN 978-0387906096*

[17] *B.N. Pensaud, and Hurdle, V. F. (1991). "Some New Data That Challenge Some Old Ideas About Speed-Flow Relationships." Transportation Research Record. 1194: 191- 8.*

[18] *Gerlough, Daniel L. And Huber, Matthew J. (1975). Traffic Flow Theory: A Monograph TRB Special Report 165. Transportation Research Board, Washington DC.*

[19] *E.Godlewski,and P.A.Raviart. Hyperbolic systems of Conservation laws,Mathematics and Application, Elipses, Paris, (1991).*

[20] *S.K. Godunov. A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics,(1959) pp.271-290.*