Direct Boundary Integral Equation Method for Helmholtz Equations with Dirichlet Boundary Conditions

By

Gudissa Kusse

A Thesis Submitted to
The program of Applied Mathematics
School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics(Specialization in Differential Equations)

Office of Graduate Studies
Adama Science and Technology University

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Declaration

I, Gudissa Kusse Ashino, declare that this thesis is my original work and has not been presented for a degree in any other university, and all sources of material used for this thesis have been duly acknowledged.

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APPROVAL PAGE

This is certify that the thesis prepared by Gudissa Kusse entitled “Direct Boundary Integral Equation Method for Helmholtz Equation with Dirichlet boundary condition” submitted in partial fulfillment of the requirement for the degree of Master of Science in Applied Mathematics with the regulation of the University and meets the accepted standards with respect to originality.

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Acknowledgment

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Abstract

In this thesis, we analyze the Boundary Integral Equation(BIE) methods for the Helmholtz equation with Dirichlet Boundary Conditions in three-dimensions where the boundary is smooth. We present a formulation of solving the BIEs reformulated from the Helmholtz boundary value problems by a direct method. BIE is particularly attractive in developing integral equation methods, it reduces the dimensions of the problem and often transforms a problem in an infinite domain to integrals on the finite boundary in which the far field radiation condition is satisfy automatically. A Boundary value problem for Partial Differential Equation with a constant coefficient can be reduced to a BIE by using the fundamental solution and Green's representation formula, we reduced the boundary value problem into Direct Boundary Integral Equations. The integral equation formulation has a unique solution at all wave numbers by proving equivalence of the BVP and the integral equation formulation and proving uniqueness of solution for the BVP. The equivalence of the boundary value problem with the reduced integral equation in appropriate Sobolev space are proved.

Key words:- Boundary Integral Equation Method, The Helmholtz Equations, Uniqueness.

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Some of the Notations

Ω	the space of the domain.
$\Gamma = \partial\Omega$	boundary of Ω.
$\tilde{V}_k$	Single-layer potential operator
W_k	double-layer potential operator
K	adjoint double-layer operator
V_k	Single layer BIO
K'	Adjoint double-layer BIO
L	Linear operators
$\nabla^2 + k^2$	the Helmholtz operator.
$\gamma_0^{int,\ ext}(\tilde{V}_k t)$	interior or exterior Dirichlet trace of the single layer potential $\tilde{V}_k t$
$\gamma_0^{int,\ ext}(W_k g)$	interior or exterior Dirichlet trace of the double layer potential $W_k g$
$\gamma_1^{int,\ ext}(\tilde{V}_k t)$	interior or exterior conormal derivative of the single layer potential $\tilde{V}_k t$
$\gamma_1^{int,\ ext}(W_k g)$	interior or exterior Conormal derivative of the double layer potential $W_k g$
$[\gamma_0 \tilde{V}_k]$	jump relation of trace of Single layer potential $\tilde{V}_k$
$[\gamma_0 W_k]$	jump relation of trace of double layer potential W_k

$\overline{\Omega}$	closure of domain Ω.
$\frac{\partial}{\partial n} = \widehat{n}$	outward normal derivative
BVP	Boundary Value Problem.
PDE	Partial Differential Equation.
BIE	Boundary Integral Equation.
BIO	Boundary integral operator
∇	Gradient
$\Delta = \nabla^2$	Laplacian.
k	Wave number.
2D	two dimension
3D	three dimension
i	imaginary part.
δ	Dirac delta function.
int, ext	interior, exterior
$G_k(x, y)$	Fundamental solution G_k
$[\gamma_1 \tilde{V}_k]$	jump relation of conormal derivative of single layer potential
$[\gamma_1 W_k]$	jump relation of conormal derivative of double layer potential

Chapter 1

Introduction and Background

1.1 Background of the study

In this thesis, we present a direct method formulation for solving the boundary integral equation reformulated from the Helmholtz BVP. The integral equation method is an elegant mathematical way of transforming elliptic PDE into BIE with the introduction of variational methods for PDEs. The BIE Method is a powerful tool for solving certain BVPs. It is particularly attractive in developing integral equation methods. Since, when applied, it reduces the dimensions of the problem and often transforms a problem in an infinite domain to integrals on the finite boundary in which the far field radiation condition is satisfy automatically. A BVP for PDE with a constant coefficient can be reduced to a BIE [2, 31].

There are two distinct methods for reducing constant coefficients BVP to a BIEM: direct and indirect ones. The construction of these two types is based on different approaches. The former start from the representation formula, where the solution at any point in the domain is expressed in terms of its boundary value and its image under the associated boundary operator. If the problem models a physical situation, then the unknown functions have a direct physical significance, see [15]. The first integral formulation is often derived through the application of the second Green's identity, while the second integral formulation is founded on single or double layer potentials. It is from the assumption that the solution can be expressed in terms of a source density function defined on the problems boundary.

The method of BIEs has always had two important applications in the theory of BVPs for PDEs: as a theoretical tool for proving the existence of solutions and as a practical tool for the construction of solutions [26]. Both methods yield a single integral equation for a single unknown function.

The Helmholtz equation, which represents a time-independent form of the wave equation, results the complexity of the analysis. The solution in time will be a linear combination of sine and cosine functions , with angular frequency of ω, while the form of the solution in space will depend on the Boundary Conditions. The Helmholtz equation is elliptic, and the use of Green's formula's is particularly appropriate for solving this type of partial differential equation [29].

The mathematical formulation consists of the Helmholtz equation in the exterior domain accompanied by the Sommerfeld radiation condition and this method is to replace the Helmholtz equation and the Sommerfeld radiation condition with an equivalent boundary integral equation, giving rise later on to an appropriate variational formulation. The boundary equation can be formulated directly on the surface of the scatterer or smooth surface surrounding the object.

In this thesis, the problem shown that among the infinitely many solutions of the corresponding integral equations there is only one solution which has a physical meaning and corresponds either to the boundary trace of the unique solution to the exterior problem or to the boundary trace of its normal derivative. However, the density function obtained from the indirect method, in general, has no physical meaning. We give conditions for the unique-Solvability of the one equations and for the subsequent construction of the solution to the transmission problem. Given that the transmission problem itself has at most one solution in chapter four, we prove that its solution can be obtained by solving any one of the two equations, except for certain irregular values of the exterior wave number: one equations lose unique-solvability when squared exterior wave number is an eigenvalue of the interior Dirichlet problem, these two equations.

We derive two different Fredholm integral equations of the second kind, which is uniquely solvable for all values of squared exterior wave number, and the combined Green's formula integral equation can be used to solve the exterior Dirichlet problem for all values of squared exterior wave number.

Many problems of physics are formulated as second-order PDEs e.g. wave equation, diffusion (heat) equation, Helmholtz equation, Schrödinger equations. These PDEs can describe a wide variety of phenomena such as sound, heat. From this we can study about a problem solving for Helmholtz equation in sound wave of air pressure. We will take a Green's third identity(Representation formula) by using explicit knowledge of a fundamental solution of the differential equation, BVPs can be reduced to BIEs. However, the fundamental solution is only available for linear PDEs with constant coefficients.

BIEs are the most important tool for BVPs in PDEs. Many BVPs of mathematical physics and engineering can be reduced to integral equations over the boundary of the domain of interest. According to the BIE method, the BVP is replaced by an integral equation posed on the boundary (or on a portion of it) of the considered domain. The key tool in applying the BIE method is the construction of an appropriate fundamental solution $G_k(x,y)$ of the problem, which satisfies some of the BCs(e.g., Dirichlet Boundary Condition and when the domain is unbounded, also the Sommerfeld radiation condition, which can be viewed as a Boundary Condition at infinity). Then, by applying Green's theorem with respect to $G_k(x,y)$ and the sought solution $u(x)$. Finally, we obtain an integral representation of $u(x)$.

This thesis divided into six chapters. In chapter 2, the literature review, the Preliminary mathematical concepts including some of the definition and theorems under the consideration of the problem included under chapter 3. The Boundary Integral Equations method for Helmholtz equation are presented in chapter 4 and Conclusions in chapter 5.

1.2 Statement of Problem

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth and connected boundary $\partial\Omega$ and $\Omega_e = \bar{\Omega}\backslash\mathbb{R}^3$ be the exterior of Ω and we consider a Dirichlet boundary value problem. Find u a complex valued function in locally integrable Sobolev space H^1 of $\Omega_e(H^1_{loc}(\Omega_e))$ satisfying:

$$\Delta u(x) + k^2 u(x) = 0 \quad in \quad \Omega$$

$$u(x) = g_D \qquad on \quad \partial\Omega$$

and $u(x)$ at exterior condition satisfying the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0$$

where $r = |x|$, $x \in \Omega$, $g_D \in H^{\frac{1}{2}}(\Gamma)$ is a given function, k is the wave number.

The thesis answers the following main questions:

1. Is the Dirichlet Boundary value problem has unique solution in $H^1_{loc}(\Omega_e)$?
2. Can we reduce the Boundary value problem into equivalent Boundary integral equations?

1.3 Objective of the study

1.3.1 General objective

The general objective of the study is to analyze the Boundary Integral Equation for the Helmholtz Boundary value problem with Dirichlet Boundary Conditions.

1.3.2 Specific objectives

The specific objectives of this study are:

1. To show the reduction of BVP to a BIEs in a whole space (interior and exterior domain).
2. To show the equivalence of Boundary value problems with the reduced BIEs.
3. To proof the uniqueness, existence of the solution.

1.4 Scope of the study

This thesis is based on formulating a mathematical method to study the Acoustic scattering to describe the Helmholtz equations. The work attends some of the definitions and theorems behind Distribution, Sobolev space, fundamental solution, trace theorems. This study targeted to assess the analysis of the solution for the Helmholtz problem in different ways for a direct methods, wavenumber and the reduction of Boundary Value Problems to the integral equation in Partial Differential Equation. This study does include all whole space as in interior and exterior problem which have the solution to the problem inside or outside of the boundary domain.

1.5 Significance of the study

The motivation behind this study is that, how we can use the Direct method approaches of BIE to reduces the BVP, to develop knowledge rapidly increasing interest in integral equation methods, to develop a certain range of applicability and several strengths for BIEs and provide a theoretical tool for proving the existence of solutions and a practical tool for the construction of solutions. Further, may provide useful result and information for the advancement of a PDEs, the advancement of research and further guide for Engineering, Physics, Mathematics students will be doing on such types of topics. Moreover, it may serve as references material for those needs to conduct research on this area.

Chapter 2

Literature Review

The BIE method is a method for the solution of BVPs. The main features are that the dimensionality of the problem is reduced by one, although the integrals that need to be computed are of a more complicated nature. The integral equation method constitutes a general tool for the exact solution of PDEs in engineering and applied science.

In different time, many researchers develop different integral equation methods for Helmholtz equations. In this chapter, we review some of the methods developed by different researchers to study the existence, uniqueness and solutions that are related to the objectives of the study. The direct BIE method is better to solve a BVP with an infinitely unbounded domain by the condition of the radiation [2].

The study on BIE for Helmholtz equation contributed by the physicist Hermann Ludwig Ferdinand Von-Helmholtz(1821-1894) to Mathematical acoustic. Well know that by the direct approach the uniquely solvable exterior BVPs for the Helmholtz equation can not be reduced to the BVPs which are uniquely solvable for arbitrary value of the frequency parameter. The formulation of exterior problems using the Helmholtz representation is shown to lead to a system of BIEs which have a unique solution. The supplementary conditions are given in the form of BIEs of the first kind rather than volume integrals. Moreover, existence as well as uniqueness of the solution of the BIE formulation of exterior problems is established on the basis of the fact that exterior problems have at most one solution [2, 9].

A direct problem involves the determination of the unknown function u for a given boundary and the corresponding BCs, whereas an eigenvalue problem involves the determination of the eigen frequencies of the system. On the other hand, in an inverse problem some kinds of inhomogeneities in the medium (for example, for given BCs, find the shape of the inhomogeneous medium, or for a given shape of the inhomogeneous medium, find the BCs, etc.) are to be determined under some given far field conditions [3].

The basic Characteristics of Boundary integral methods are that the fundamental equations which describe the problems, including the BCs are solved for a fixed domain. The development of integral equations has focused largely on elliptic PDE's. Usually, the most studied problems on the Helmholtz equation [30]. Moreover, extending these methods to more general PDEs is difficult or impossible. In different time; many literatures develop BIEs for Helmholtz equations. In this, we review the Direct BIE for Helmholtz equation which introduced in different literatures to study the integral equation directly with Dirichlet BC.

The uniqueness of the BIE for exterior Helmholtz problem has been discussed by many authors. Author Schenek (1968) proved the failure of uniqueness by using Fredholm's integral equation theory. He proposed the Helmholtz integral equation combined with the interior integral equation to overcome the difficulty. He guaranteed the uniqueness by forming a linear combination of the original boundary integral equation and its normal derivative with respect to the field point. The boundary integral equation of the exterior problem by using a new Green's function, which expands the infinite series and satisfies the infinite equations (called null-field equations) induced by the interior integral equation, has unique solution see in reference [26].

In [15] Kleinman(1974) described the problem from mathematical theory systematically. The work in this field has been continued in recent years. Though much work has been done in this field, the physical meaning of the failure of uniqueness has not been understood. But, some authors even thought that there is no physical meaning in this problem see [5].

From the point of view of energy analysis, the cause that the uniqueness of the BIE induced from the exterior Helmholtz problem does not hold is investigated, i.e., the Sommerfeld's condition is strengthened at the infinity. Therefore, it satisfies not only the radiative wave but also the absorptive wave, and the total energy of the system is equal to a constant.

In this work, we shall present the improvement methods for the orthogonality of the eigenfunction associated with eigenvalues for the Laplacian operator as a particular Helmholtz problem, the uniqueness, derivation of fundamental solution and make observations for general Helmholtz problems. We present a result of a more careful treatment of the integrals in Helmholtz problem, in comparison to this see a References [11, 14].

Chapter 3

Preliminary Mathematical Concepts

In this chapter, we pick out the main idea to our work for some mathematical concepts, definitions, notations and useful formulas should be familiar to the reader in order to feel confident about their subsequent use and required for the development and understanding of the Boundary Integral Equation Method (BIEM). Although these relations could placed here to show the reader their important role in the theoretical foundation and development of the BIE. They will be used many time and often in the book and particularly for the transformation of the differential equations, which govern the response of physical systems within a domain, into integral equations on the boundary. The concepts are found in references [1,6,13,32,33]. Since we work in three dimension, we emphasize the Theories in $\mathbb{R}^3$.

Dirac delta function

Mathematically, the Dirac delta function is defined by how it behaves inside an integral in fact the first operation where Dirac used the delta function is the integration in distribution sense:

Definition 3.1. The symbol $\delta(x)$ is the Dirac delta function of x in 3D for $x \in \mathbb{R}^3$ is defined by:

$$\delta(x) = \begin{cases} 0, & for \quad x \neq 0 \\ \infty, & for \quad x = 0 \end{cases} \quad , \qquad \int_\Omega \delta(x) dx = \begin{cases} 1, & for \quad x \in \Omega \\ 0, & for \quad x \notin \Omega \end{cases}$$

Definition 3.2. (**Locally integrable**)

i) Let Ω be open set in $\mathbb{R}^3$ and function $u : \Omega \to \mathbb{C}$ is said to be locally integrable if

$$\int_K |u| dx < \infty \ for\ all\ compact\ subset\ K\ of\ \Omega.$$

Any continuous function is locally integrable.
ii) A locally integrable function defines a distribution in the following:

$$if\ v \in \mathcal{D}(\Omega),\ then\ u(v) := \langle u, v\rangle = \int_\Omega uvdx,\ x \in \Omega$$

or

$$\begin{aligned}
\langle \Delta u + k^2 u, v\rangle_\Omega &= \langle \Delta u, v\rangle_\Omega + \langle k^2 u, v\rangle_\Omega = \int_\Omega v\Delta u dx + \int_\Omega k^2 uvdx \\
&= -\langle \nabla u, \nabla v\rangle_\Gamma + k^2\langle u, v\rangle_\Omega = -\int_\Gamma \nabla u \nabla v dx + \int_\Omega k^2 uvdx \\
&= \langle u, \Delta v\rangle_\Omega + k^2\langle u, v\rangle_\Omega = \int_\Omega u\Delta v dx + \int_\Omega k^2 uvdx \\
&= \langle u, \Delta v + k^2 v\rangle = \int_\Omega u\left(\Delta v + k^2 v\right) dx
\end{aligned}$$

Thus,

$$\langle \Delta u + k^2 u, v\rangle_\Omega = \int_\Omega u\left(\Delta v + k^2 v\right) dx$$

Test Functions

In order to compute the locally averaged values of a distribution, the notion of a test function is required. A function $v(x)$ defined on an open set $\Omega \subset \mathbb{R}^3$ is said to have **compact support** if $v(x) = 0$ for x in the complement of a compact subset of Ω. In particular, if $\Omega = \mathbb{R}^3$, then v has compact support if there is a positive constant, C such that $v(x) = 0$ for $|x| > C$. We say that $v(x)$ is a test function if v has compact support and, in addition, v is infinitely differentiable on Ω.

Definition 3.3. (**Test function**) Let Ω be a domain in $\mathbb{R}^3$. Then, the set

$$\mathcal{D}(\Omega) = \{v \in C^\infty(\Omega) \colon\ supp(v) \text{ is compact in } \Omega\}$$

is called the set of **test functions** . The set $\mathcal{D}(\Omega)$ is also denoted by, $C_0^\infty(\Omega)$.

For instance, the function given by

$$v(x, y, z) = \begin{cases} e^{\frac{1}{(x^2+y^2+z^2)-1}}, & \text{if } x^2 + y^2 + z^2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

is a **test function** in $\mathbb{R}^3$. The support of v is the closed unit ball, i.e.,
Supp$v = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ and it is infinitely differentiable function.

Note that, if $v_1(x), v_2(x), v_3(x)$ are test functions on Ω, so is $c_1v_1 + c_2v_2 + c_3v_3$ for any $c_1, c_2, c_3 \in \mathbb{R}$. The space $\mathcal{D}(\Omega)$ is therefore a real linear space.

Uniformly convergent the entire function converges at the same rate; not just at a comparable point wise convergence, but at a specific point of function.

Convergence in $\mathcal{D}$. A sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{D}(\Omega)$ is said to be convergent in $\mathcal{D}(\Omega)$ to v in $\mathcal{D}(\Omega)$, we shall write $v_n \to v$ as $n \to \infty$, if there is a compact subset K of Ω with $supp(v_n) \subset K$, $n \in \mathbb{N}$ and

$$\partial^\alpha v_n \to \partial^\alpha v \text{ uniformly on } K \quad \forall \alpha \in \mathbb{N}_0^2.$$

That means, *uniform convergence* for all derivatives, $||v_n - v||_{C^m(\Omega)} \to 0$ if $n \to \infty$ for all $m \in \mathbb{N}_0$.

Distributions

A generalizations of the classical concept of functions which used to formulate solutions of partial differential equations is known as Distributions. A distribution is a linear functional on a space of test functions. Distributions include all locally integrable functions and have derivatives of all orders (great for linear problems) but cannot be multiplied in any natural way (not so great for nonlinear problems).

Definition 3.4. (**Distribution**) Let Ω be a domain in $\mathbb{R}^3$ and $\mathcal{D}(\Omega)$ be the space of test function. The set of distributions (or generalized functions) denoted by $\mathcal{D}'(\Omega)$ is the collection of all complex-valued linear continuous functionals u over $\mathcal{D}(\Omega)$, i.e.,

$$u : \mathcal{D}(\Omega) \to \mathbb{C}$$

where the value of u acting on v denoted by

$$u(v) := uv := \langle u, v \rangle$$

satisfies the following linearity and continuity criteria

For any $\alpha, \beta, \tau \in \mathbb{C}$ and $v_1, v_2, v_3 \in \mathcal{D}(\Omega)$

1. $\langle u, \alpha v_1 + \beta v_2 + \tau v_3 \rangle = \alpha \langle u, v_1 \rangle + \beta \langle u, v_2 \rangle + \tau \langle u, v_3 \rangle$,
2. $\langle u, v_n \rangle \to \langle u, v \rangle$ for $n \to \infty$ in $\mathbb{C}$, whenever $v_n \to v$ in $\mathcal{D}(\Omega)$.

Convergence in $\mathcal{D}(\Omega)$. A sequence of distributions $\{u_n\}$ is said to be convergent in $\mathcal{D}(\Omega)$, we shall write $u_n \to u$ in $\mathcal{D}(\Omega)$, if

$$\langle u_n, v \rangle \to \langle u, v \rangle \text{ in } \mathbb{C} \text{ as } n \to \infty \text{ for all } v \in \mathcal{D}(\Omega)$$

The simplest example of a distribution is the functional generated by a locally integrable function, $u(x) \in L_1^{loc}(\Omega)$ and is given by

$$\langle u, v\rangle = \int_\Omega u(x)v(x)dx, \qquad \text{for all } v \in \mathcal{D}(\Omega).$$

Generalized functions, which are definable in terms of functions locally integrable are said to be *regular generalized functions.* The remaining generalized functions are called *singular generalized functions.*

Example 3.1. The Dirac delta distribution δ, which is defined by, $\langle \delta, v\rangle = v(0)$, for all $v \in \mathcal{D}(\Omega)$ is singular generalized function.

Sobolev spaces

The analysis of BVPs naturally involves function spaces that are not only defined in terms of the properties of the function itself, but also in terms of the properties of its derivatives. Sobolev spaces are useful tools in the analysis of boundary value problem.

Definition 3.5. (**Sobolev-Space**) For any order $s \in \mathbb{R}$ and radius r the corresponding Locally integrable are denoted by the symbols L_{loc}, H^s_{loc}. The notation $W^{n,p}(\Omega)$ is the Sobolev space of differentiability n and integrability p.

Definition 3.6. For $p \in [1, \infty]$, $n \in \mathbb{N}$ and Ω an open subset of $\mathbb{R}^3$, the Sobolev space

$$W^{n,p}_{loc}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p_{loc}(\Omega)\ \forall |\alpha| \leq n; \quad W^{n,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega)\ \forall |\alpha| \leq n,$$
$$||u||_{W^{n,p}(\Omega)} := (\sum_{|\alpha \leq n|} ||\partial^\alpha u||^p_{L^p(\Omega)})^{1/p}, \quad p < \infty; \quad ||u||_{W^{n,p}(\Omega)} := \sum_{|\alpha \leq n|} ||\partial^\alpha u||_{L^\infty(\Omega)}, \quad p = \infty$$

Variational formulation

One of the decisive disadvantages of the strong formulation of boundary value problems is that questions concerning existence and uniqueness cannot be answered in a satisfactory way. We can overcome these difficulties by choosing a variational formulation in appropriate function spaces instead. In order to do this we multiply the differential equation by a test function and then integrate over Ω. If we then use integration by parts, the boundary conditions can directly be incorporated in the variational formulation. The solution of the variational problem is called a weak solution. The variational formulation was derived from the strong formulation by multiplying it by test functions, integrating and then integrating by parts.

The derivation of Helmholtz equation

The phenomenon of wave propagation is frequently encountered in a variety of engineering discipline. For instance, it is important to know the interaction with electromagnetic wave problems of radiation and scattering of water waves are common features in fluid mechanics where they appear in the design of break waters, offshore structures, etc.

Let $\Omega \subset \mathbb{R}^3$ and for a wave number $k \in \mathbb{R}$ such that $U : \Omega \to \mathbb{R}$ be **a scalar function** which satisfies the **wave equation**

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}U(x,t) = \Delta U(x,t), \quad for \quad t > 0, \quad x \in \Omega. \tag{3.1}$$

where c is the speed of propagation for sound wave. The equation (3.1) is valid for the wave propagation in a homogeneous, friction free medium having the constant speed of sound c. The most important examples are the acoustic scattering and the sound radiation. The time harmonic acoustic waves are of the form

$$U(x,t) = Re\{u(x)e^{-i\omega t}\} = u(x)cos(\omega t) \tag{3.2}$$

where i is the imaginary unit, $\omega > 0$ denotes the frequency of wave propagation. In equation (3.2), $u : \Omega \to \mathbb{C}$ is a **scalar, complex valued function**. By inserting (3.2) into the wave equation (3.1), we obtain the reduced wave equation or the Helmholtz equation as follow:

$$-\omega^2 u(x)e^{-i\omega t} = c^2 \Delta u(x) e^{-i\omega t}$$

After dividing this by $c^2 e^{-i\omega t}$ and reordering the terms, as the interior Helmholtz problem

$$\Delta u(x) + k^2 u(x) = 0, \text{for } k = \frac{\omega}{c}, \quad x \in \Omega. \tag{3.3}$$

or the exterior Helmholtz problem

$$\Delta u(x) + k^2 u(x) = 0, \text{for} \quad x \in \Omega^c := \mathbb{R}^3 \setminus \Omega. \tag{3.4}$$

where for the exterior problem we have to include the Sommerfeld radiation condition automatically satisfied [7]

$$\begin{aligned} &u(x) = O(r^{-1}), \text{as} |x| \to \infty \\ &\lim_{r\to\infty} \int_{|x|=r} |\frac{\partial u}{\partial n_x}(x) - iku(x)|^2 ds_x = o(r^{-1}) \to 0, \ r \to \infty \end{aligned} \tag{3.5}$$

where n_x is the exterior normal vector.

The interpretation of the unknown $u(x)$ and the parameters k depends on what the equation models. The most common areas are wave propagation problems in which case $u(x)$ is the Amplitude of a time-harmonic wave. In equation (3.2) we see that the solution $u(x)$ represents the Amplitude of time-harmonic solutions to the Pressure of wave propagation $U(x,t)$ in equation (3.1) at frequency ω.

The big 'O' and little 'o' notations in (3.5) we have the following meanings: equation (3.5) the first equation says that the pressure $u(x)$ must decrease, as we go to infinity, at least as fast as r^{-1} ; the second one says that the left hand side of this equation, namely $\frac{\partial u}{\partial r} - iku$ must decrease faster than r^{-1}. The first condition implies that $|u|^2$ decreases like r^{-2} in 3D. But this is exactly what one expects from energy considerations: the energy is spread over the surface of a sphere of radius $4\pi r^2$ in 3D. The second condition says that $\frac{\partial u}{\partial r} - iku$ should be much smaller than r^{-1}, and so much smaller than u, when r is large or $r \to \infty$.

Fundamental solution for Helmholtz equation

The fundamental solutions are the solutions in the whole space and the scalar wave in the 3-dimensional space applied in appendix section (A.1) satisfies the following Helmholtz equation [25, 27]:

$$\left(\Delta_k + k^2\right) G_k(x,y) = -\delta(x,y), \ \ (x,y) \in \mathbb{R}^3 \tag{3.6}$$

is

$$G_k(x) = \frac{e^{\pm ik|x|}}{4\pi|x|} \tag{3.7}$$

where $\pm$ indicates exterior or interior domain respectively, k is the wave number.

Fredholm integral equation

The integral form

$$\mathcal{K}u = \int_a^b K(x,y)u(y)dy = f(x) \tag{3.8}$$

which is known as a *Fredholm integral equation of the first kind*. A slightly more general form, and one that often arises in applications is an integral equation of the form

$$u - \mathcal{K}u = u(y) - \lambda \int_a^b K(x,y)u(y)dy = f(x) \tag{3.9}$$

where $\lambda \neq 0$ and $K(x,y) \neq 0$. This equation is known as a *Fredholm integral equation of the second kind* see e.g., [33].

Theorem 3.1. *The Fredholm Alternative*

If the homogeneous equation

$$\psi + \lambda \int_\Gamma K(x,y)\psi(y)ds_y = 0. \tag{3.10}$$

for boundary Γ *and scalar* λ*, only possesses the trivial solution* $\psi = 0$*, then the inhomogeneous equation*

$$\psi + \lambda \int_\Gamma K(x,y)\psi(y)ds_y = f. \tag{3.11}$$

will have a unique solution, for all square integrable functions f *and kernels* $K(x,y)$*.*

Proof. Suppose ψ_1 and ψ_2 are two linearly independent solutions of (3.11), so we have

$$\begin{aligned} \psi_1 + \lambda \int_\Gamma K(x,y)\psi_1(y)ds_y &= f, \\ \psi_2 + \lambda \int_\Gamma K(x,y)\psi_2(y)ds_y &= f. \end{aligned} \tag{3.12}$$

Then by taking the 1^{st} from the 2^{nd} results in

$$\begin{aligned} (\psi_2 - \psi_1) + \lambda \int_\Gamma K(x,y)\{\psi_2(y) - \psi_1(y)\}ds_y &= f - f \\ &= 0. \end{aligned} \tag{3.13}$$

where $\lambda \neq 0$ and $K(x,y) \neq 0$ which form (3.10) for $\psi_2 - \psi_1$. Thus, if there is only the trivial solution to (3.10), we conclude that $\psi_2 = \psi_1$: the solution to (3.11) is unique. ☐

Theorem 3.2. *If the homogeneous equation*

$$\frac{1}{2}\phi + \lambda \int_\Gamma \phi(y)\frac{\partial}{\partial n_x}G_k(x,y)ds_y = 0. \tag{3.14}$$

with boundary Γ *and* $G_k(x,y)$ *given by (3.7), has a non trivial solution* G_k*, then the transposed equation*

$$\frac{1}{2}\psi + \int_\Gamma \psi(y)\frac{\partial}{\partial n_x}G_k(x,y)ds_y = 0. \tag{3.15}$$

will also possess a non-trivial solution ψ*, and conversely.*

Chapter 4

Boundary Integral Equation for Helmholtz Equation

4.1 Green's formula and Representation formula

In mathematics, Green formula may refer to Green's Theorem in integral calculus and Green's identity in vector calculus. $\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v\nabla^2 u$ where $\nabla^2 = \nabla \cdot \nabla = \Delta$, moreover see, section (A.1.1).

4.1.1 Green's first formula

Multiplying the Helmholtz equation (3.3) with a test function v, integrating over Ω, and applying integration by parts, this gives Green's first formula

$$\begin{aligned}\int_\Omega \left[\left(\Delta u(y) + k^2 u(y)\right)\right] v(y) dy &= \int_\Omega \left(-\nabla v(y) \cdot \nabla u(y) - k^2 u(y)v(y)\right) dy + \int_{\partial\Omega} v(y)\frac{\partial u(y)}{\partial n} ds_y \\ &= \int_\Omega \underbrace{\left(-\nabla v(y) \cdot \nabla u(y) - k^2 u(y)v(y)\right)}_{(symmetry)} dy + \int_\Gamma \gamma_1^{int} u(y) \gamma_0^{int} v(y) ds_y.\end{aligned} \tag{4.1}$$

where

$$\textit{trace operator } \gamma_0^{int} v(y) = \lim_{\tilde{y}\in\Omega, \tilde{y}\to y\in\Gamma} v(\tilde{y}),$$

$$\textit{Conormal derivative } \gamma_1^{int} u(y) = \lim_{\tilde{y}\in\Omega, \tilde{y}\to y\in\Gamma} \left(n(y)\cdot \nabla_{\tilde{y}} u(\tilde{y})\right),$$

$$\textit{symmetry} \quad -\nabla v(y)\cdot\nabla u(y) + k^2 u(y)v(y) = -\nabla u(y)\cdot\nabla v(y) + k^2 v(y)u(y)$$

4.1.2 Green's second formula

A simple consequence of Green's first formula (4.1) and interchanging u by v, we deduce Green's second formula:

$$\begin{aligned}\int_\Omega \left(\Delta u(y) + k^2 u(y)\right) v(y)dy + \int_\Gamma \left[\gamma_1^{int} v(y)\gamma_0^{int} u(y)\right] ds = \int_\Omega \left(\Delta v(y) + k^2 v(y)\right) u(y)dy \\ + \int_\Gamma \gamma_1^{int} u(y)\gamma_0^{int} v(y) ds_y.\end{aligned} \tag{4.2}$$

4.1.3 Green's Representation formula

In terms of the fundamental solution (3.7), we choose a test function $G_k : \Omega \to \mathbb{C}$ satisfying

$$\int_\Omega \left(\Delta G_k(x,y) + k^2 G_k(x,y)\right) u(y)dy = u(x), \quad \int_\Omega \left(\underbrace{\Delta u(y) + k^2 u(y)}_{0}\right) G_k(x,y)dy = 0$$

the solution to Helmholtz equation (3.3) is given by the representation formula see [15]:

$$\begin{aligned}u(x) &= \pm \int_{y\in\Gamma} \left(G_k(x,y)\frac{\partial}{\partial n_y} u(y) - u(y)\frac{\partial}{\partial n_y} G_k(x,y)\right) ds_y, \\ &= \pm \int_{y\in\Gamma} \left(G_k(x,y)\gamma_1^{\pm} u(y) - \gamma_{1,y}^{\pm} G_k(x,y)\gamma_0^{\pm} u(y)\right) ds_y, \qquad x \in \Omega \ \ or\ \Omega^c.\end{aligned} \tag{4.3}$$

where the sign $\pm$ means interior or exterior.

we can derive this as follow in both inside and outside of the domain.

Integral equation for interior domain

We consider a 3D interior Helmholtz problem, the problem of computing the 3D acoustic field generated in a bounded region Ω due to a source of sound somewhere in the region Ω. Thus we seek to compute the acoustic field u in a bounded domain Ω due to a point source located at the point x_0 in the domain. The acoustic field u is assumed to satisfy:
the Helmholtz BVP

$$\Delta u + k^2 u = 0 \qquad \text{in } \Omega \setminus \{x_0\};$$

$$u = 0 \qquad \text{on } \partial\Omega;$$

that near x_0 the field is close to that in the free field case, precisely that the difference,

$$u_{int}(x) := u(x) - G(x, x_0),$$

between u and the free field solution $G(x, x_0)$, is continuous together with its gradient ∇u_{int} in a neighborhood of x_0 ; except at x_0, u and its partial derivatives up to second order are continuous in Ω and up to the boundary $\partial\Omega$.

In physical terms this problem models the acoustic field some interior space generated by a source at the point x_0. To obtain an integral equation formulation an application of Green's second theorem is made. Let x denote any point in Ω or on its boundary (except x_0). For the moment we are going to treat both x and x_0 as fixed positions and, keeping these points fixed, we are going to integrate over Ω_ε with respect to another position vector y which will serve as our variable of integration.

We apply Green's second identity (4.2) to the functions u and v defined by

$$u(y) := u(y), \quad v(y) := G(y, x),$$

on

$$\Omega_\varepsilon := \{y \in \Omega : |y - x_0| \geq \varepsilon \quad and \quad |y - x| \geq \varepsilon\},$$

i.e., Ω_ε is Ω with a small balls in 3D of radius ε cut out from around x and x_0. So that u is just the solution u to the problem that we are considering and ψ is the acoustic field due to a source at x in the free field case. Note that $u(y)$ is singular at $y = x_0$ (the source position) but is continuous with continuous partial derivatives up to second order in Ω_ε and up to its boundary because we have excluded x_0 and a little ball around x_0 from the domain Ω_ε . Similarly, $v(y)$ is singular at $y = x$ but is smooth elsewhere and so smooth in Ω_ε because we have excluded x and a little ball around x from the domain Ω_ε. Since the conditions of Green's theorem are satisfied equation (4.2) holds. To proceed further we note that u and v are both solutions to the Helmholtz equation in Ω, except at x_0 and x, respectively. Since these points are excluded from Ω_ε we have that, in Ω_ε,

$$\Delta u + k^2 u = 0 \quad and \quad \Delta v + k^2 v = 0$$

Thus,

$$u\Delta v - v\Delta u = -uk^2 v + vk^2 u \quad in \quad \Omega_\varepsilon$$

and so the integral on the right hand side of (4.2) vanishes. Hence

$$\int_{\partial\Omega_\varepsilon} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} ds = 0.$$

For the time being we focus on the case when $x \in \Omega$ and note that, as long as ε is small enough, $\partial\Omega_\varepsilon$ the boundary of Ω_ε, consists of three separate pieces: the boundary $\partial\Omega$ of Ω; the boundary γ_ε of the ball of radius ε centered on x; the boundary Γ_ε of the ball of radius ε centered on x_0. Thus the integral over $\partial\Omega_\varepsilon$ is really the sum of three integrals over these separate parts of the boundary. Thus the previous equation implies that

$$\int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds + \int_{\gamma_\varepsilon} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds + \int_{\Gamma_\varepsilon} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds = 0 \tag{4.4}$$

Consider the second integral in (4.4) for y on γ_ε, the limit as $\varepsilon \to 0$

$$v(y) = G(y,x) = \frac{e^{ik\varepsilon}}{4\pi\varepsilon}.$$

Further more, ∇_y denotes the gradient with respect to the components of y and $R := |y - x|$, it follows by the chain rule that

$$\nabla v(y) = \nabla_y G(y,x) = \frac{1}{4\pi}\frac{d}{dR}(\frac{e^{ikR}}{R})\nabla_y R = \frac{1}{4\pi}\frac{e^{ikR}(ikR-1)}{R^3}(y-x)$$

So that, on γ_ε,

$$\frac{\partial v}{\partial n}(y) = n(y)\cdot\nabla v(y) = \frac{y-x}{R}\cdot\nabla v(y) = -\frac{1}{4\pi}\frac{e^{ikR}(ik\varepsilon-1)}{R^3},$$

Thus,

$$\int_{\gamma_\varepsilon}[v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}]ds = -\frac{1}{4\pi}[\frac{e^{ik\varepsilon}}{\varepsilon}\int_{\gamma_\varepsilon}\frac{\partial u}{\partial n}ds - \frac{e^{ik\varepsilon}(ik\varepsilon-1)}{\varepsilon^2}\int_{\gamma_\varepsilon}uds].$$

Now, in the limit as $\varepsilon \to 0$, as γ_ε becomes on the center of the ball x, we see that

$$\int_{\gamma_\varepsilon}uds \sim u(x)\int_{\gamma_\varepsilon}ds = 4\pi\varepsilon^2 u(x)$$

and, similarly, since

$$|\frac{\partial u}{\partial n}| = |n\cdot\nabla u| \le |\nabla u|,$$

which is bounded in a neighborhood of x, we have that

$$\int_{\gamma_\varepsilon}\frac{\partial u}{\partial n}ds = O(\varepsilon^2) \quad as\ \varepsilon \to 0.$$

Thus, in the limit $\varepsilon \to 0$,

$$\frac{e^{ik\varepsilon}}{\varepsilon}\int_{\gamma_\varepsilon}\frac{\partial u}{\partial n}ds = O(\varepsilon) \to 0$$

while

$$\frac{e^{ik\varepsilon(ik\varepsilon-1)}}{\varepsilon^2}\int_{\gamma_\varepsilon}uds \to -4\pi u(x)$$

so that

$$\int_{\gamma_\varepsilon}[v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}]ds \to u(x).$$

A similar but slightly the calculation leads to that

$$\int_{\gamma_\varepsilon}[v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}]ds \to -v(x_0)$$

as $\varepsilon \to 0$. Thus, taking the limit $\varepsilon \to 0$ in (4.4) we get that

$$\int_{\partial\Omega}[v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}]ds + u(x) - v(x_0) = 0$$

Remembering the definitions of u and v we see that this equation can be rearranged slightly as an explicit formula for $u(x) = u(x)$, that

$$u(x) = G_k(x_0, x) + \int_{\partial\Omega} [G_k(y, x)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial G_k(y, x)}{\partial n_y}]ds_y, \tag{4.5}$$

this equation valid for all $x \in \Omega$ with $x \neq x_0$.

Remark 4.1. First of all the $G_k(y, x)$ in the normal derivative $\frac{\partial}{\partial n(y)}$ is necessary to indicate that we are computing partial derivatives with respect to the y (and not with respect to the x) variables. Thus, explicitly, $\frac{\partial G(y,x)}{\partial n(y)}$ denotes the rate of increase of $G(y, x)$ as y moves off the boundary in the direction $n(y)$. Similarly, the y in $ds(y)$ is there to indicate that the integration is taking place with respect to the y variables; the integrand is a function of two variables, of x and y, but the integration is only with respect to y (with x fixed).

Secondly, what does this formula give us? This formula, sometimes called a Green's representation formula, is an explicit representation for the solution u in terms of the explicitly known Green's function G_k and in terms of Cauchy data values of u and its normal derivative $\dfrac{\partial u}{\partial n}$ on the boundary. Thus, once we know the values of u and $\dfrac{\partial u}{\partial n}$ on $\partial\Omega$, equation (4.5) will give us an explicit formula for computing the solution throughout the domain Ω.

An important consequence is that (4.5) must hold irrespective of what boundary conditions apply on $\partial\Omega$. But first we consider how the derivation of (4.5) differs when x is on the boundary $\partial\Omega$ instead of inside Ω. A version of equation (4.4) holds in this case, namely

$$\int_{\partial\Omega'} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_{\gamma'_\varepsilon} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_{\Gamma_\varepsilon} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds = 0 \tag{4.6}$$

the last term in this equation as before, the primes added to the first two terms indicating that they have changed somewhat. Precisely, $\int_{\partial\Omega'}$ denotes the integral over $\partial\Omega$ but with the part of $\partial\Omega$ within distance ε of x omitted. In the limit as $\varepsilon \to 0$ this makes no difference. The second term is affected more significantly: γ'_ε denotes that part of the surface of the ball of radius ε centered on x that lies within Ω. In the limit as $\varepsilon \to 0$ we find that

$$\int_{\gamma'_\varepsilon} uds \sim u(x) \int_{\gamma'_\varepsilon} ds \sim A(x)\varepsilon^2 u(x)$$

where

$$A(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\gamma'_\varepsilon} ds$$

is the solid angle (in the range 0 to 4π) subtended at x by the domain Ω.

At most points on $\partial\Omega$ (all points except edges and corners) it holds that

$$A(x) = 2\pi,$$

the area of a hemisphere of unit radius. Thus, taking the limit as $\varepsilon \to 0$ in (4.6), we arrive at a modified version of (4.5) that holds when $x \in \partial\Omega$ that

$$\frac{A(x)}{4\pi}u(x) = G(x_0, x) + \int_{\partial\Omega}[G(y,x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y,x)}{\partial n(y)}]ds(y), \tag{4.7}$$

with $\frac{A(x)}{4\pi} = \frac{1}{2}$ at most points on $\partial\Omega$. Since we have not used the boundary condition in deriving (4.7), this equation also holds irrespective of the boundary condition.

Making use now of the Dirichlet boundary condition to replace $\partial\Omega$ in (4.5) and (4.7), we arrive at versions of these equations which are satisfied by the solution u to our particular acoustics problem, that

$$u(x) = G(x_0, x) - \int_{\partial\Omega}[G(y,x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y,x)}{\partial n(y)}]ds(y), \quad x \in \Omega, \tag{4.8}$$

and

$$\frac{A(x)}{4\pi}u(x) = \frac{1}{2}u(x) = G(x_0, x) - \int_{\partial\Omega}[G(y,x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y,x)}{\partial n(y)}]ds(y), \quad x \in \partial\Omega. \tag{4.9}$$

This last equation is our first integral equation for an acoustics problem, the unknown function to be found being u (the pressure) on the boundary $\partial\Omega$. Since the integration is a surface integral, over the boundary of the region Ω, we call this integral equation a BIE. The BIE method for this problem consists of two ways: first solve the integral equation (4.9) to find u on $\partial\Omega$; after this first step everything on the right hand side of (4.8) is known and so this equation can be used to compute u anywhere in Ω.

An integral equation for an exterior domain

Suppose first of all that we wish to solve the same problem as in above section but with the simpler BC that $u = g_D \quad on \quad \partial\Omega$. Equations (4.5) and (4.7) hold irrespective of the boundary condition. Substituting $u = 0$ on $\partial\Omega$ in these equations we get that

$$u(x) = G(x_0, x) + \int_{\partial\Omega} G(y,x)\frac{\partial u}{\partial n}(y)ds(y), \quad x \in \Omega, \tag{4.10}$$

and that

$$0 = G(x_0, x) + \int_{\partial\Omega} G(y,x)\frac{\partial u}{\partial n}(y)ds(y), \quad x \in \partial\Omega. \tag{4.11}$$

This last equation is our second boundary integral equation: the unknown function is $\frac{\partial u}{\partial n}$ on the boundary $\partial\Omega$. This equation is a first kind integral equation as the unknown function only appears under the integral sign. Once (4.11) has been solved to find $\dfrac{\partial u}{\partial n}$ then (4.10) can be used to compute u throughout Ω.

We seek to compute the acoustic field u in the unbounded domain $\Omega = \mathbb{R}^3 \setminus S$ due to a point source located at the point x_0 in Ω. The acoustic field u is assumed to satisfy: the Helmholtz equation

$$(\Delta + k^2)u = 0 \ in \ \Omega$$

(except at x_0); the Dirichlet boundary condition

$$u = g$$

on $\partial\Omega$; near x_0, the field is closed to the free field case $u_{scat}(x) := u(x) - G(x, x_0)$ between u and the free field solution $G(x, x_0)$ is continuous together with its gradient ∇u_{scat} in a neighborhood of x_0; except at x_0, u and its partial derivatives up to 2^{nd} order are continuous in Ω and up to the boundary and the Sommerfeld radiation conditions(3.5).

To derive a boundary integral equation formulation for this problem we will make use of our results for the case considered in the previous section where Ω is bounded. In order to do this we introduce, for $R > 0$, the set Ω_R which is the part of the unbounded set Ω which lies within distance R of the origin, in symbols $\Omega_R := \{y \in \Omega : |y| < R\}$. We assume that R is chosen large enough so that both x_0 and S the scattering are inside Ω_R. Then, because the set Ω_R is bounded, and because equations (4.5) and (4.7) hold irrespective of the boundary conditions on $\partial\Omega$, they hold for this problem, but not, directly, for the unbounded region Ω but for the region Ω_R. Thus equations (4.5) and (4.7) hold for this problem, as long as we replace Ω by Ω_R, with R sufficiently large. Now $\partial\Omega_R$ (the boundary of Ω_R) consists of two disconnected pieces, namely Ω_R, the boundary of Ω and Γ_R, the surface of the sphere of radius R centered on the origin. Thus equation (4.5) applied to this problem tells us that, for $x \in \Omega_R$,

$$u(x) = G(x_0, x) + \int_{\partial\Omega} [G(y, x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y, x)}{\partial n(y)}]ds(y) + \int_{\Gamma_R} [G(y, x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y, x)}{\partial n(y)}]ds(y), \tag{4.12}$$

this equation holding for all sufficiently large R.

We finish the derivation of the representation we want for u by taking the limit as $R \to \infty$. We note that, since $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ on Γ_R, it holds that

$$\begin{aligned}\int_{\Gamma_R} G(y, x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y, x)}{\partial n(y)}]ds(y) = &-\int_{\Gamma_R} G(y, x)\left(\frac{\partial u}{\partial r}(y) - iku(y)\right) ds_y \\ &- \int_{\Gamma_R} u(y)\left(\frac{\partial G(y, x)}{\partial r}(y) - ikG(x, y)\right) ds(y).\end{aligned}$$

It follows from the Sommerfeld radiation conditions, which are satisfied both by u and by G, that the integrand on Γ_R decreases at the rate $o(R^{-2})$ as $R \to \infty$. Since the area of Γ_R increases only proportional to R^2 it follows that the integral over Γ_R vanishes in the limit as $R \to \infty$. Thus, taking the limit as $R \to \infty$ in (4.12), it follows that

$$u(x) = G(x_0, x) + \int_{\partial\Omega} [G(y,x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y,x)}{\partial n(y)}]ds(y), \quad for \quad x \in \Omega.$$

Similarly, applying (4.7) to the region Ω_R and then taking the limit as $R \to \infty$, we get that

$$\frac{A(x)}{4\pi}u(x) = G(x_0, x) + \int_{\partial\Omega} [G(y,x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial G(y,x)}{\partial n(y)}]ds(y), \quad for \quad x \in \Omega.$$

Remark 4.2. **(i)** these equations are identical to equations (4.5) and (4.7) (though now Ω denotes an unbounded rather than a bounded region); as we have not used the boundary condition on $\partial\Omega$ yet, these equations hold irrespective of the boundary condition.

Finally, representations of solutions of the Helmholtz equation obtained by applying Green's theorem or the Helmholtz representation have the following forms. For radiating wave functions u_e satisfying

$$\begin{aligned} &(\Delta + k^2)u_e(x) = 0, \quad for \quad x \in \Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}, \\ &|(\frac{\partial u_e}{\partial n} - iku_e(x)| = O(\frac{1}{|x|}) \quad as \quad |x| \to \infty. \end{aligned} \tag{4.13}$$

we obtain

$$\int_{\partial\Omega} \left[G_k(x,y)\frac{\partial u_e(y)}{\partial n_y} - u_e(y)\frac{\partial G_k}{\partial n_y}(x,y) \right] dS_y = \begin{cases} u_e(x) \\ \frac{1}{2}u_e(x) \\ 0 \end{cases}, \quad x \in \begin{cases} \Omega_e \\ \partial\Omega \\ \Omega_i \end{cases} \tag{4.14}$$

and

$$\int_{\partial\Omega} \left[u_i(y)\frac{\partial G_k}{\partial n_y}(x,y) - G_k(x,y)\frac{\partial u_i(y)}{\partial n_y} \right] dS_y = \begin{cases} 0 \\ \frac{-1}{2}u_i(x) \\ -u_i(x) \end{cases}, \quad x \in \begin{cases} \Omega_e \\ \partial\Omega \\ \Omega_i \end{cases} \tag{4.15}$$

where Ω_e is the exterior domain, Ω_i is the interior domain surrounded by boundary Γ and from (4.14) and (4.15)

$$\int_{\partial\Omega} \left[G_k(x,y)\frac{\partial}{\partial n_y}(u_e + u_i)(y) - (u_e + u_i)(y)\frac{\partial G_k}{\partial n_y}(x,y) \right] dS_y = \begin{cases} u_e(x) \\ \frac{1}{2}(u_e + u_i)(x) \\ u_i(x) \end{cases}, x \in \begin{cases} \Omega_e \\ \partial\Omega \\ \Omega_i \end{cases}$$

Consider (4.14) and (4.15), and apply a Dirichlet boundary condition $u = 0$ (as we shall be considering later), by taking the normal derivative with respect to x and letting x tend to the boundary, we have

$$\frac{1}{2}\frac{\partial u_e(x)}{\partial n_y} = -\int_{\partial\Omega} \frac{\partial G_k(x,y)}{\partial n_x}\frac{\partial u_e(y)}{\partial n_y} dS_y \quad x \in \partial\Omega \tag{4.16}$$

where the $\frac{1}{2}\frac{\partial u_e}{\partial n}$ term comes from a jump condition as x tends to Γ see, e.g., [12]. Alternatively from (4.14), again applying a Dirichlet boundary condition $u = 0$ we have

$$-\int_{\partial\Omega} G_k(x,y)\frac{\partial u_e(y)}{\partial n_y} ds_y = 0, \quad x \in \partial\Omega \tag{4.17}$$

Equations (4.16) and (4.17) are each boundary integral equations that we could use to solve for $\frac{\partial u}{\partial n}$. Both are inhomogeneous Fredholm equations: (4.16) is of the second kind with kernel $\frac{\partial}{\partial n}G_k(x,y)$, whereas (4.17) is of the first kind with kernel $G_k(x,y)$. Once $\frac{\partial u}{\partial n}$ has been computed from (4.16) or (4.17), the first equation in (4.14) then gives a formula for $u(x)$ at any point $x \in \partial\Omega$.

We remark that the form of the radiation condition given in eq.(4.13) is satisfied by $G_k(x,y)$, which thus represents a point of spherical waves which are outgoing at infinity. If k is replaced by $(-k)$, then $\bar{G}_k(x,y)$ would play the role of fundamental solution. We notice that (4.15) remains valid if $\bar{G}_k(x,y)$ replaces $G_k(x,y)$.

4.2 Boundary integral operators and their properties

In representation formula (4.3), the integral operator is given by

$$(\tilde{V}_k w)(x) := \int_\Gamma G_k(x,y)w(y)ds_y,$$
$$(W_k v)(x) := \int_\Gamma \frac{\partial G_k}{\partial n_y}(x,y)v(y)ds_y,$$

on $x \in \mathbb{R}^3\backslash\Gamma$ and unknown data $w \in H^{-\frac{1}{2}}(\Gamma)$ and a given $v \in H^{\frac{1}{2}}(\Gamma)$ is called Single-layer and double-layer potential respectively.

The direct boundary integral operator values of the single and double layer potential for $x \in \Gamma$ are denoted as follow respectively.

$$
\begin{aligned}
(V_k w)(x) &:= \int_\Gamma G_k(x,y) w(y) ds_y, \quad x \in \Gamma, \\
(K_k v)(x) &:= \int_\Gamma \frac{\partial G_k}{\partial n_y}(x,y) v(y) ds_y, \quad x \in \Gamma, \\
(K_k' v)(x) &:= \int_\Gamma \frac{\partial G_k}{\partial n_x}(x,y) v(y) ds_y, \quad x \in \Gamma.
\end{aligned}
$$

Both the single and double layer potential satisfy the Helmholtz equation $\Delta u + k^2 u = 0$ on $\mathbb{R}^3 \setminus \Gamma$, i.e., it satisfies the following,

$$
\Delta(\tilde{V}_k w)(x) + k^2(\tilde{V}_k w)(x) = 0, \quad \Delta(W_k v)(x) + k^2(W_k v)(x) = 0
$$

Theorem 4.1. *[4, 9, 28] The following operators are linear and bounded*

$$
\begin{aligned}
&\tilde{V}_k : H^{-1/2}(\Gamma) \to H^1(\Omega) \\
&W_k : H^{1/2}(\Gamma) \to H^1(\Omega) \\
&V_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \\
&K_k : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \\
&K_k' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\
&D_k : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)
\end{aligned} \tag{4.18}
$$

where D_k be the conormal derivative of the double layer potential defines the hyper-singular BIO.

The single layer potential is continuous on the boundary, but its normal derivative jumps by a specified amount as we cross the boundary. This property used to solve the prescribed normal derivative on the boundary ($\frac{\partial u}{\partial n}$). The double layer potential is discontinuous on the boundary, its value jumps by a specified amount as we cross the boundary. This property used to solve the Dirichlet problem $u = g$ (prescribed boundary values).

Trace operator

In mathematics, the concept of Trace operator plays an important role in studying the existence and uniqueness of solutions to BVPs The trace operator makes it possible to extend the notion of restriction of a function to the boundary of its domain to "generalized" functions in a Sobolev space.

Definition 4.1. Let Ω be a bounded open set in $\mathbb{R}^3$ whose boundary $\partial\Omega$ is at least C^1 smooth and let

$$T : C^1(\bar{\Omega}) \to L^p(\partial\Omega)$$

be a linear operator defined by

$$T(u) = u|\partial\Omega$$

on the collection of all real-valued compactly-supported C^1 functions with domain in the topological closure $\bar{\Omega}$ of Ω. In studying BVPs, we shall need to make sense of the restriction $u|\Gamma$ as an element of a Sobolev space on Γ when u belongs to a Sobolev space on Ω.

Lemma 4.2. *[19, 20]Define the trace operator $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ by*

$$\gamma u(x') = u(x') \quad for \quad x' \in \Gamma.$$

and Conormal derivative

$$\gamma_1^{int} : H^1(\Omega, \Delta + k^2) \to H^{\frac{1}{2}}(\Gamma)$$

If $s > \dfrac{1}{2}$, then γ has a unique extension to a bounded linear operator

$$\gamma : H^s(\mathbb{R}^3) \to H^{s-1/2}(\mathbb{R}^2).$$

Theorem 4.3. *[9, 19] Let $\Omega \subset \mathbb{R}^d$ be a $C^{n-1,1}$ domain. For $1/2 < s \leq n$ the interior trace operator*

$$\gamma_0^{int} : H^s(\Omega) \to H^{s-1/2}(\Gamma)$$

is bounded satisfying

$$||\gamma_0^{int} v||_{H^{s-1/2}(\Gamma)} \leq C_T ||v||_{H^s(\Omega)} \; for \; all \; v \in H^s(\Omega).$$

For a Lipschitz domain Ω we can apply Theorem 4.3 with $n = 1$ to obtain the boundedness of the trace operator $\gamma_0^{int} : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ for $s \in (\frac{1}{2}, 1]$. This remains for $s \in (\frac{1}{2}, \frac{3}{2})$.

Theorem 4.4. *[4]The operators*

$$\begin{aligned}
&\gamma_0^{int}\tilde{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Omega), \quad \gamma_0^{ext}\tilde{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Omega),\\
&\gamma_1^{int}\tilde{V} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Omega), \quad \gamma_1^{ext}\tilde{V} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Omega),\\
&\gamma_0^{int}W : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad \gamma_0^{ext}W : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma),\\
&D_k = -\gamma_1^{int}W_k : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \qquad -\gamma_1^{ext}W : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).
\end{aligned} \tag{4.19}$$

are linear and continuous.

For the interior and exterior Dirichlet trace, Conormal derivative of the single and double layer potential $\tilde{V}_k w$ and $W_k v$, we obtain in sense of $v \in H^{1/2}(\Gamma)$ and $w \in H^{-1/2}(\Gamma)$ see [4], sec 3.3.1.

$$\begin{aligned}
&\gamma_0^{int}(\tilde{V}_k w)(x) = \gamma_0^{ext}(\tilde{V}_k w)(x) = (V_k w)(x), \\
&\gamma_0^{int}(W_k v)(x) = -\frac{1}{2}v(x) + (K_k v)(x), \quad \gamma_0^{ext}(W_k v)(x) = \frac{1}{2}v(x) + (K_k v)(x), \\
&\langle \gamma_1^{int}\tilde{V}_k w, v\rangle_\Gamma = \langle \frac{1}{2}w + K_k' w, v\rangle_\Gamma, \qquad \langle \gamma_1^{ext}\tilde{V}_k w, v\rangle_\Gamma = \langle -\frac{1}{2}w + K_k' w, v\rangle_\Gamma, \\
&\langle \gamma_1^{int}W_k v, w\rangle_\Gamma = \langle \frac{1}{2}v - K_k' v, w\rangle_\Gamma, \qquad \langle \gamma_1^{ext}W_k v, w\rangle_\Gamma = \langle \frac{1}{2}v - K_k' v, w\rangle_\Gamma.
\end{aligned} \tag{4.20}$$

Also, factors $\pm\frac{1}{2}$ multiplying I come classically only for smooth boundaries and should be modified according to the solid angle described at the point where the trace is taken. However, since all our domains are assumed to be piecewise smooth, the change of values only occurs in sets of zero measure and can be discarded.

In particular, from equation (4.18), (4.19) and (4.20) we now conclude the jump relation of the single and double-layer potential $\tilde{V}_k w$ and $\tilde{W}_k v$, i.e.,

$$\begin{aligned}
[\gamma_0 \tilde{V}_k w] &:= \gamma_0^{ext}(\tilde{V}_k w) - \gamma_0^{int}(\tilde{V}_k w) = 0, \quad for \quad x \in \Gamma, \\
[\gamma_1 \tilde{V}_k w] &:= \gamma_1^{ext}(V_k w) - \gamma_1^{int}(V_k w) = -w, \quad for \quad x \in \Gamma, \\
[\gamma_0 W_k v] &:= \gamma_0^{ext}(W_k v) - \gamma_0^{int}(W_k v) = v, \quad for \quad x \in \Gamma, \\
[\gamma_1 W_k v] &:= \gamma_0^{ext}(W_k v) - \gamma_0^{int}(W_k v) = 0, \quad for \quad x \in \Gamma.
\end{aligned}$$

4.2.1 The Interior Dirichlet BVP for Helmholtz equation

We consider the interior Dirichlet BVP for the Helmholtz equation:

$$\begin{aligned}
\Delta u(y) + k^2 u(y) = 0, \quad for \quad x \in \Omega, \\
\gamma_0^{int} u(x) = g(x), \quad for \quad x \in \Gamma.
\end{aligned} \tag{4.21}$$

This formulation of the problem is posed for almost all values of k. By using the representation formula (4.3), the solution of equation (4.21) is given by

$$u(x) = \int_\Gamma G_k(x,y)t(y)ds_y - \int_\Gamma \gamma_{1,y}^{int} G_k(x,y)g(y)ds_y \quad for \quad x \in \Omega.$$

where $t = \gamma_1^{int} u$ is the unknown conormal derivative of u on Γ which has to be determined from some appropriate boundary integral equation.

Applying the interior trace operator γ_0^{int} to the representation formula, this gives a boundary integral equation to find $t \in H^{-1/2}$ such that

$$(V_k t)(x) = \frac{1}{2}g(x) + (K_k g)(x) = (\frac{1}{2}I + K_k)g(x) \quad for \quad x \in \Gamma. \tag{4.22}$$

For $t \in H^{-1/2}(\Gamma)$ is the solution of the variational problem

$$\langle V_k t, w\rangle_\Gamma = \langle(\frac{1}{2}I + K_k)g, w\rangle \quad \forall w \in H^{-1/2}(\Gamma). \tag{4.23}$$

When applying the interior normal derivative γ_1^{int} to the representation formula, this gives a second kind BIE to find $t \in H^{-1/2}(\Gamma)$ such that

$$\frac{1}{2}t(x) - (K_k^{'} t)(x) = (\frac{1}{2}I - K_k^{\prime})t(x) \quad for \quad x \in \Gamma. \tag{4.24}$$

To investigate the unique solvability of the variational problem (4.23), and, therefore, of the boundary integral equation (4.22) as well as of the BIE (4.24).

We first consider the Dirichlet eigenvalue problem for the Laplace operator,

$$-\Delta u(x) = \lambda u(x) \quad for \quad x \in \Omega, \quad \gamma_0^{int} u(x) = 0 \quad for \quad x \in \Gamma. \tag{4.25}$$

Let $\lambda \in \mathbb{R}$ be a certain eigenvalue, and let u_λ be the corresponding eigenfunction. Since the eigenvalue problem (4.25) can be seen as the Helmholtz equation with the wave number satisfying $k^2 = \lambda$, a variational problem in (4.22) and (4.24), for the conormal derivative $t_\lambda = \gamma_1^{int} u_\lambda$ the boundary integral equations, we obtain

$$\begin{aligned}
&(V_k t_\lambda)(x) = \frac{1}{2}\gamma_0^{int} u_\lambda(x) + (K_k \gamma_0^{int} u_\lambda)(x) = (\frac{1}{2}I + K_k)\gamma_0^{int} u_\lambda(x) = 0 \quad for \quad x \in \Gamma \\
&\frac{1}{2}t_\lambda(x) - (K_k^{'} t_\lambda)(x) = (\frac{1}{2}I - K_k^{\prime})t_\lambda(x) = 0 \quad for \quad x \in \Gamma.
\end{aligned} \tag{4.26}$$

In particular, the single-layer boundary integral operator V_k and the adjoint double- layer boundary integral operator $\frac{1}{2}I - K_k^{'}$ are singular, and, therefore, not invertible, if $k^2 = \lambda$ is an eigenvalue of the Dirichlet eigenvalue problem (4.25). On the other hand, if k^2 is not an eigenvalue of the Dirichlet eigenvalue problem (4.25), the single layer potential V is injective and hence, since V is coercive, also invertible. This shows the unique solvability of the variational problem (4.23) and of the boundary integral equation (4.22) in this case. Note that also in this case the second kind BIE(4.24) is uniquely solvable.

4.2.2 The Exterior Dirichlet BVP for Helmholtz equation

In the case when the domain Ω_e is unbounded, we consider the exterior boundary value problem

$$\begin{aligned} -\Delta u(x) - k^2 u(x) = 0, \quad x \in \Omega_e = \mathbb{R}^3 \setminus \Omega, \\ \gamma_0^{ext} u(x) = g(x) \quad x \in \Gamma, \end{aligned} \tag{4.27}$$

and additionally Sommerfel condition in equation (3.5).

Note that the exterior Dirichlet boundary value problem is uniquely solvable due to the radiation condition. The solution of the above problem is given by the representation formula(4.3),

$$u(x) = -\int_\Gamma G_k(x,y)\gamma_1^{ext} u(y) ds_y + \int_\Gamma \gamma_{1,y}^{ext} G_k(x,y) g(y) ds_y \quad for \quad x \in \Omega_e.$$

To find the yet unknown Neumann datum $t = \gamma_1^{ext} u \in H^{-\frac{1}{2}}(\Gamma)$ by considering the boundary integral equation which results from the representation formula when applying the exterior trace operator γ_0^{ext},

$$(V_k t)(x) = -\frac{1}{2} g(x) + (K_k g)(x) \quad for \quad x \in \Gamma.$$

This boundary integral equation is equivalent to a variational problem to find $t \in H^{-1/2}\Gamma$ such that

$$\langle V_k t, w \rangle_\Gamma = \langle (-\frac{1}{2} I + K_k) g, w \rangle_\Gamma \quad \forall w \in H^{-1/2}(\Gamma) \tag{4.28}$$

Since the single layer potential V_k of the exterior Dirichlet boundary value problem coincides with the single layer potential of the interior Dirichlet boundary value problem, V_k is not invertible when $k^2 = \lambda$ is an eigenvalue of the Dirichlet eigenvalue problem (4.25). However, we have

$$\langle (-1/2 I + K_k) g, t_\lambda \rangle_\Gamma = -\langle g, (1/2 I - K'_{-k}) t_\lambda \rangle_\Gamma = 0,$$

and, therefore,

$$(-1/2 I + K_k) g \in Im V_k.$$

The variational problem (4.28) of the direct approach is solvable, but the solution is not unique.

Instead of a direct approach, we may also consider an indirect single layer potential approach

$$\begin{aligned} u(x) &= \left(\tilde{V}_k w\right)(x) \\ &= \int_\Gamma G_k(x,y) w(y) ds_y \ for \ x \in \Omega_e. \end{aligned}$$

And applying the exterior trace operator leads to the boundary integral equation to find $w \in H^{\frac{1}{2}}(\Gamma)$ such that

$$(V_k w)(x) = g(x) \ for \ x \in \Gamma. \tag{4.29}$$

Again, we have unique solvability of the boundary integral equation (4.29) only for those wave numbers k^2 which are not eigenvalues of the interior Dirichlet eigenvalue problem (4.21).

4.2.3 The Coercivity of boundary integral operators

The mapping properties of all boundary integral operators as introduced above are well established, see [7], [9], [19], [24]. Recall that we consider the general case when $\Gamma = \partial\Omega$ is a Lipschitz boundary.

Lemma 4.5. *The single-layer boundary integral operator $V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is coercive, i.e., there is a compact operator $C_V : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ such that*

$$\langle (V_k + C_V)w, w\rangle_\Gamma \geq C_1^V ||w||^2_{H^{-\frac{1}{2}}(\Gamma)}, \textit{for all } w \in H^{-\frac{1}{2}}(\Gamma). \tag{4.30}$$

Proof. Let $V = V_0$ denote the Laplace single-layer boundary integral operator which is $H^{-\frac{1}{2}}(\Gamma)$ elliptic, see, e.g. ([24], Section 6.6.1).

Moreover, $C_V := V - V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$is compact, see, for example, ([24], Section 6.9). Now the coercivity follows from

$$\langle (V_k + C_V)w, w\rangle_\Gamma = \langle Vw, w\rangle_\Gamma \geq C_1^V ||w||^2_{H^{-\frac{1}{2}}(\Gamma)}, \text{for all } w \in H^{-\frac{1}{2}}(\Gamma).$$

□

For $t, w \in H^{-\frac{1}{2}}$ we obtain

$$\begin{aligned}
\langle (V_k t, w\rangle_\Gamma &= \int_\Gamma (V_k t)(x) w\bar{(x)} ds_x \\
&= \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{e^{ik|x-y|}}{|x-y|} t(y) ds_y w\bar{(x)} ds_x \\
&= \frac{1}{4\pi} \int_\Gamma t(y) \int_\Gamma \frac{e^{ik|x-\bar{y}|}}{|x-y|} w(x) ds_x ds_y \\
&= \int_\Gamma t(y) (V_k \bar{w})(y) ds_y = \langle t, V_k w\rangle_\Gamma.
\end{aligned} \tag{4.31}$$

Moreover, for $k \in \mathbb{R}$ we find that;

$$Im(\langle V_k w, w\rangle_\Gamma) \geq 0 \text{for all } w \in H^{-\frac{1}{2}}(\Gamma).$$

Lemma 4.6. *The boundary integral operator $\frac{1}{2}I - K_k : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is coercive, i.e. there is a compact operator $C_K : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ such that*

$$\langle \left[\left(\frac{1}{2} I - K_k \right) + C_K \right] v, v\rangle_{V^{-1}} \geq (1 - C_K) ||v||^2_{V^{-1}} \textit{for all } v \in H^{\frac{1}{2}}(\Gamma). \tag{4.32}$$

Here, $C_K < 1$ is the contraction constant of $\frac{1}{2}I + K_0 : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$.

Proof. Let $K = K_0$ denote the Laplace double-layer boundary integral operator for which we can use the contraction property [22]

$$||\left(\frac{1}{2}I + K\right)v||_{V^{-1}} \leq C_K||v||_{V^{-1}}\text{for all } v \in H^{\frac{1}{2}}(\Gamma), \quad C_K < 1.$$

Moreover, $C_K := K_k - K : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is compact, and hence we obtain

$$\langle\left[\left(\frac{1}{2}I - K_k\right) + C_K\right]v, v\rangle_{V^{-1}} = \langle\left(\frac{1}{2}I - K\right)v, v\rangle_{V^{-1}} \geq (1 - C_K)||v||^2_{V^{-1}}\text{for all } v \in H^{\frac{1}{2}}(\Gamma)$$

□

In a similar way, we conclude that $\frac{1}{2}IK'_k : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is coercive, i.e.,

$$\langle\left[\left(\frac{1}{2}I - K'_k\right) + C_{K'}\right]w, w\rangle_V = \langle\left(\frac{1}{2}I - K'\right)w, w\rangle_V \geq (1 - C_K)||w||^2_V\text{for all } w \in H^{-\frac{1}{2}}(\Gamma)$$

where $||\cdot||_V = \sqrt{\langle V\cdot, \cdot\rangle_\Gamma}$ is an equivalent norm in $H^{-\frac{1}{2}}(\Gamma)$ which is induced by the Laplace single-layer boundary integral operator V.

Note that $C'_K := K'_{-k} - K' : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is compact.

Moreover, as for the single-layer boundary integral operator V , see (4.31), we conclude

$$\langle K_k v, w\rangle = \langle v, K'_{-k}w\rangle_\Gamma\text{for all } v \in H^{\frac{1}{2}}(\Gamma).$$

4.2.4 The non-invertibility of layer potentials for special k

From the representation formula for the interior problems we can conclude that

Lemma 4.7. *If the single-layer boundary integral operator V_k is not invertible then k^2 is an eigenvalue of the interior Dirichlet problem and if $\frac{1}{2}I + K_k$ is not invertible then k^2 is an eigenvalue of the interior Neumann problem.*

Proof. Suppose k^2 is an eigenvalue of the interior Dirichlet problem: there exists $u \neq 0$ such that

$$-\Delta u = k^2 u, \quad in\ \Omega,$$

$$u = 0, \quad on\ \ \Gamma.$$

From the interior representation formula, we know that

$$u = V_k\gamma_1^{int}u - W_k\gamma_1^{int}u = V_k\gamma_1^{int}u.$$

Hence if we write $t = \gamma_1^{int} u$ then from the above representation formula and the fact that $u \neq 0$ we deduce that

$$t \neq 0, \quad V_k t = 0$$

and hence V_k cannot be invertible. $\square$

Similarly we can show that $\frac{1}{2}I + K_k$ is not invertible if there exists $u \neq 0$ such that

$$-\Delta u = k^2 u, \quad in\ \Omega,$$

$$\gamma_1^{int} u = 0, \quad on\ \ \Gamma.$$

Remark 4.3. However if $Im\ k > 0$ then V_k is invertible. This is because eigenvalues of an elliptic operator are positive real numbers.

Theorem 4.8. *[19] If $k^2 = \lambda$ is an eigenvalue of the Dirichlet eigenvalue problem of the Helmholtz equation,*

$$\Delta u_\lambda(x) + \lambda u_\lambda = 0\ for\ x \in \Omega,\ \gamma_0^{int} u_\lambda(x) = 0\ for\ x \in \Gamma,$$

then the single layer potential $V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is not injective, i.e.,

$$(V_k \gamma_1^{int} u_\lambda)(x) = 0\ for\ x \in \Gamma$$

Moreover,

$$(\frac{1}{2}I - K_k')\gamma_1^{int} u_{\lambda_0}(x) = 0\ for\ x \in \Gamma$$

Proof. The assertion immediately follows from the direct boundary integral equations

$$(V_k \gamma_i^{int} u_\lambda)(x) = (\frac{1}{2}I - K_k')\gamma_0^{int} u_{\lambda 0}(x) = 0\ for\ x \in \Gamma$$

and

$$(\frac{1}{2}I - K_k')\gamma_i^{int} u_{\lambda 0}(x) = 0.$$

Hence we conclude, that if k^2 is not an eigenvalue of the Dirichlet eigenvalue problem of the Helmholtz equation, the Helmholtz single layer potential $V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is coercive and injective, i.e. the boundary integral equation (4.26) admits a unique solution. $\square$

4.2.5 The Injectivity of boundary integral operators

Since all boundary integral operators are coercive, we can apply Fredholm's alternative to investigate the unique solvability of boundary integral equations arising in certain transmission and boundary value problems for the Helmholtz equation. It remains to consider the injectivity of the involved boundary integral operators. This is related to interior eigenvalue problems of the Laplace operator.

The next two results are direct consequences of the boundary integral equations which hold for any solution of the interior Helmholtz equation.

Proposition 4.9. *Let $(\lambda, u_\lambda) \in \mathbb{R} \times H^1(\Omega)$, $\lambda > 0$, be a solution of the interior Dirichlet eigenvalue problem*

$$\Delta u_\lambda = \lambda u_\lambda(x) \quad for \quad x \in \Gamma. \tag{4.33}$$

$t_\lambda := \frac{\partial}{\partial n_x} u_\lambda \in H^{-\frac{1}{2}}(\Gamma)$ we then conclude for $x \in \Gamma$

$$\begin{aligned} (V_k t_\lambda)(x) &= (V_{-k} t_\lambda)(x) = 0, \\ \left(\frac{1}{2}I - K_k'\right) t_\lambda(x) &= \left(\frac{1}{2}I - K_{-k}'\right) t_\lambda(x) = 0. \end{aligned} \tag{4.34}$$

In particular, the single-layer boundary integral operator V and the adjoint double layer boundary integral operator $\frac{1}{2}I - K_k'$ are not injective if k^2 corresponds to an eigenvalue λ of the interior Dirichlet eigenvalue problem (4.33) for $0 < k := \sqrt{\lambda} \in \mathbb{R}$.

Proposition 4.10. *Let $(k, t) \in \mathbb{R} \times H^{-\frac{1}{2}}(\Gamma)$ be an eigensolution of the single-layer boundary integral operator eigenvalue problem $V_k t = 0$. Then,*

$$\lambda := k^2, u_\lambda(x) := \int_\Gamma G_k(x,y) t(y) ds_y \quad for \quad x \in \Omega$$

define an eigensolution of the interior Dirichlet eigenvalue problem (4.33). Moreover,

$$t = \frac{\partial}{\partial n_x} u_\lambda, \ on \ \Gamma.$$

Proof. Since u_λ is defined as a single-layer potential, by definition, u_λ is a solution of the interior Helmholtz equation, i.e.,

$$\Delta u_\lambda(x) + k^2 u_\lambda(x) = 0 \quad for \quad x \in \Omega.$$

Moreover,

$$u_\lambda(x) = \int_\Gamma G_k(x,y) t(y) ds_y = (V_k t)(x) = 0 \quad for \quad x \in \Gamma.$$

Hence, u_λ is a solution of the interior Dirichlet eigenvalue problem (4.25). Since the single-layer potential $u_\lambda = \tilde{V}_k t$ is also a solution of the exterior Helmholtz problem (3.3) satisfying the radiation condition (3.5), the homogeneous Dirichlet boundary condition u_λ on Γ implies u_λ in Ω_e and the jump relation in (4.20) of the single-layer potential gives

$$\gamma_1^{ext}\tilde{V}_k t - \gamma_1^{int}\tilde{V}_k t = -t, \quad \frac{\partial}{\partial n_x}u_\lambda = \gamma_1^{int}\tilde{V}_k t = t \ \ on \ \ \Gamma.$$

□

A similar result as given in Proposition 4.10 for the single-layer boundary integral operator V_k also holds for the hypersingular boundary integral operator D_k.

Proposition 4.11. *Let $(k,u) \in \mathbb{R} \times H^{\frac{1}{2}}(\Gamma)$be an eigensolution of the hypersingular boundary integral operator eigenvalue problem $D_k u = 0$. Then,*

$$\mu := k^2, \quad u_\mu(x) := -\int_\Gamma \frac{\partial}{\partial n_y} G_k(x,y)u(y)ds_y \quad for \quad x \in \Omega$$

define an eigensolution of the interior Neumann eigenvalue problem $t_\mu(x) := \frac{\partial}{\partial n_x}u_\mu(x) = 0$ satisfying $u_\mu = u$ on Γ. Moreover, we also have

$$D_{-k}u = 0, \quad \left(\frac{1}{2}I - K_k\right)u = 0, \quad \left(\frac{1}{2}I + K_{-k}\right)u = 0.$$

More involved is the consideration of the adjoint double-layer boundary integral operator eigenvalue problem $\left(\frac{1}{2}I - K_k'\right)t = 0$.

Lemma 4.12. *Let $(k,t) \in \mathbb{R} \times H^{\frac{1}{2}}(\Gamma)$ be an eigensolution of the adjoint boundary integral operator eigenvalue problem $\left(\frac{1}{2}I - K_k'\right)t = 0$. Then,*

$$\lambda := k^2, \quad u_\lambda(x) := \int_\Gamma G_k(x,y)t(y)ds_y \quad for \quad x \in \Omega$$

define an eigensolution of the interior Dirichlet eigenvalue problem (4.33) satisfying

$$t = \frac{\partial}{\partial n_x}u_\lambda \ \ on \ \Gamma.$$

Moreover, we have

$$\left(\frac{1}{2}I - K_k'\right)t = 0, \quad V_k t = 0, \quad V_{-k}t = 0.$$

Proof. By definition, u_λ is a solution of the interior Helmholtz equation. For the Dirichlet trace of u_λ we obtain

$$u_\lambda(x) = (V_k t)(x) \quad for \ x \in \Gamma,$$

and therefore,

$$\left(\frac{1}{2}I - K_k\right)u_\lambda(x) = \left(\frac{1}{2}I - K_k\right)(V_k t)(x) = V_k\left(\frac{1}{2}I - K_k\right)t(x) = 0 \ for \ x \in \Gamma$$

follows.

On the other hand, by considering the Neumann trace of u_λ we obtain

$$\frac{\partial}{\partial n_x} u_\lambda(x) = \left(\frac{1}{2}I + K_k'\right) t(x) = t(x) - \left(\frac{1}{2}I - K_k'\right) t(x) = t(x) \quad for \quad x \in \Gamma, \tag{4.35}$$

and by using the second boundary integral equation of (4.35)

$$\left(D_k u_\lambda\right)(x) = \left(\frac{1}{2}I - K_k'\right) t(x) = 0 \quad for \quad x \in \Gamma$$

follows.

Since u_λ implies a solution of the Neumann eigenvalue problem,

$$\left(\frac{1}{2}I + K_k\right) u_\lambda(x) = 0 \; for \quad x \in \Gamma,$$

and therefore

$$u_\lambda(x) = (V_k t)(x) = 0 \quad for \quad x \in \Gamma$$

follows. □

4.3 The equivalence and Uniqueness Theorems

In this section we will give necessary and sufficient conditions for the existence of a unique solution of the integral equations for the interior problems of the Helmholtz equation. For the exterior problem the radiation conditions guarantee that the EDP for the Helmholtz equation has a unique solution for every k. Some of the integral equations that appear during the boundary reduction of the interior and exterior problems are identical. It follows that, although the boundary value problem has a unique solution, the integral operators of the exterior problem are not invertible in the natural Sobolev spaces on Γ for every wave number.

Theorem 4.13. *[**Equivalence of BVP with BIE**] Let $u \in H^1_{loc}$ be a solution to the original Dirichlet Helmholtz boundary value problem, then equation (4.3) be the solution of BIEs system.*

Proof. As similar to reference [8], *we choose an arbitrary fixed point* $x \in \Omega$ and a sphere $B(x,r) := \{x, y \in \mathbb{R}^3 : |x-y| = r\}$. We assume the radius $r > 0$ to be small enough such that $B(x,r) \subseteq \Omega$ and the unit normal $\tilde{n}(y) = \frac{x-y}{|y-x|} = \frac{x-y}{r}$ into the interior of $B(x,r)$.

Now we apply the second Green's formula to the functions $u(y)$ and $G_k(x,y)$ in the region $\Omega_r := \Omega_r \setminus B(x,r)$ to obtain follow

$$\begin{aligned}
&\int_{\partial\Omega+\partial B(x,r)}\{u(y)\frac{\partial G_k(x,y)}{\partial n(y)}-\frac{\partial u}{\partial n}(y)G_k(x,y)\}ds_y\\
&=\int_{\partial\Omega}\{u(y)\frac{\partial G_k(x,y)}{\partial n(y)}-\frac{\partial u}{\partial n}(y)G_k(x,y)\}ds_y+\int_{\partial B(x,r)}\{u(y)\frac{\partial G_k(x,y)}{\partial n(y)}-\frac{\partial u}{\partial n}(y)G_k(x,y)\}ds_y\\
&=\int_{\Omega_r}\{u(y)\Delta_y G_k(x,y)-G_k(x,y)\Delta u(y)\}ds_y=u(x)-\int_{\Omega_r}G_k(x,y)\left[\Delta u(y)+k^2u(y)\right]ds_y
\end{aligned} \tag{4.36}$$

Since on $B(x,r)$ we have

$$G_k(x,y)=\frac{e^{ikr}}{4\pi r},\qquad \nabla_y G_k(x,y)=(\frac{1}{r}-ik)\frac{e^{ikr}}{4\pi r}n(y)$$

a straightforward calculation, obtain that

$$\begin{aligned}
\int_{\partial B(x,r)}\{u(y)\frac{\partial G_k(x,y)}{\partial n(y)}-\frac{\partial u}{\partial n}(y)G_k(x,y)\}ds_y&=\frac{e^{ikr}}{4\pi r}\int_{|y-x|=r}\{u(y)\left(\frac{1}{r}-ik\right)-\frac{\partial u}{\partial\tilde{n}}(y)\}ds_y\\
&=\frac{e^{ikr}}{4\pi r^2}\int_{|y-x|=r}u(y)ds-\frac{e^{ikr}}{4\pi r}\int_{|y-x|=r}\{iku(y)+\frac{\partial u}{\partial\tilde{n}}(y)\}ds_y
\end{aligned}$$

For $r\to 0$ the first term tends to $u(x)$, because the surface area of $\partial B(x,r)$ is just $4\pi r^2$ and

$$\frac{e^{ikr}}{4\pi r^2}\int_{|y-x|=r}u(y)ds=\frac{e^{ikr}}{4\pi r^2}\int_{|y-x|=r}\left[u(y)-u(x)\right]ds+e^{ikr}u(x),$$

where we obtain by continuity

$$\frac{e^{ikr}}{4\pi r^2}\int_{|y-x|=r}\left[u(y)-u(x)\right]ds\le\sup_{y\in B(x,r)}|u(y)-u(x)|\to 0,\ r\to 0.$$

Similarly the second term tends to zero. Therefore, also the limit of the volume integral exists as $r\to 0$ and yields the desired formula for $x\in\Omega$.

$$\lim_{r\to 0}\int_{\Omega x,r}\{u(y)\frac{\partial G_k(x,y)}{\partial n(y)}-\frac{\partial u}{\partial n}(y)G_k(x,y)\}ds_y=-u(x),\quad x\in\Omega.$$

Let now $x\in\partial\Omega$. Then we proceed in the same way. The domains of integration in (4.36) have to be replaced by $\partial\Omega\setminus B(x,r)$ and $\partial B(x,r)\cap\Omega$, respectively. In the computation the region $\{y\in\mathbb{R}^3:|y-x|=r\}$ has to be replaced by $\{y\in\Omega:|y-x|=r\}$. The surface area is $2\pi r^2+O(r^3)$ which gives the factor $\frac{1}{2}$ of $u(x)$.

For $x\in\Omega^c$ the functions u and $v=G_k(x,)$ are both solutions of the Helmholtz equation in all of Ω. Application of Green's second identity yields the assertion to the function $u(y)$ and $G_k(x,y)$ in the region Ω, from (4.36) the representation formula is established for $x\in\Omega$. $\square$

To explain the uniqueness we start with two well known theorems of Fredholm integral equations in the previous chapter. Thus, the Fredholm alternative tells us then that for (3.3), either k^2 is not an eigenvalue and there is a unique solution to the equation, k^2 is an eigenvalue and the corresponding eigenfunctions are solutions of equation with a zero right hand side.

Similarly, the implication of Theorems 3.1 and 3.2 which has a solution $\dfrac{\partial u}{\partial n}$ where $u(x)$ solves (4.27), and can be rearranged into

$$\frac{1}{2}\frac{\partial u_e(x)}{\partial n} + \int_{\partial\Omega} \frac{\partial G_k(x,y)}{\partial n_x}\frac{\partial u_e(y)}{\partial n} dS_y = \frac{\partial u_i(x)}{\partial n} \quad x \in \Omega_e,$$

an inhomogeneous equation in the form (3.11), will have non unique solutions if the homogeneous equations in the form of (3.14) and (3.15) have non trivial solutions for some ϕ and ψ see [16] The result of the problem depends on some restriction of wave number(k_i and k_e). Such constraints of wave numbers are

Theorem 4.14. *[7][**Uniqueness theorem**] Let k_e be such that k_i is real and positive, or $im(k_e) > 0$. Let k_i and ρ be such that $\rho \neq 0$ and $im(\bar{\rho}k_e) \geqq 0$ and $im(\rho\bar{k}_e)(k_i)^2 \geqq 0$. Then the problem has at most one solution.*

Proof. Let v_i and v_e solve the homogeneous Helmholtz problem in which $u_{inc} \equiv 0$. Two applications of Green's theorem and use of Dirichlet BC give

$$\int_{S_R} v_e \frac{\partial \bar{v}_e}{\partial n} dS = \int_{B_{e,R}} \{|\nabla v_e|^2 - \bar{k}_e^2 |v_e|^2\} dV + \int_{B_i} \bar{\rho}\{|\nabla v_i|^2 - \bar{k}_i^2|v_i|^2\}dV,$$

Where $S_R = \{x \in B_e : r_x = R\}, B_{e,R} = \{x \in B_e : r_x < R\}$ and the over bar denotes the complex conjugate. multiply by k_e and take the imaginary part to give

$$\begin{aligned} Im\{k_e \int_{S_R} v_e \frac{\partial \bar{v}_e}{\partial n} dS\} = Im(k_e) &\int_{B_{e,R}} \{|\nabla v_e|^2 + |k_e v_e|^2\} dV \\ + Im(\bar{\rho}k_e) &\int_{B_i} |\nabla v_i|^2 dV + Im(\rho \bar{k}_e {k_i}^2) \int_{B_i} |v_i|^2 dV \end{aligned} \tag{4.37}$$

If $Im(k_e) > 0$, the left-hand side of 4.37 tends to zero as $R \to \infty$ (since V_e decays exponentially), hence this theorem implies that

$$\int_{B_i} |\nabla v_e|^2 dV = 0,$$

and the result follows by standard arguments [7]. If k_e is real and positive, the right-hand side of 4.37 is non-negative, where as the Sommerfeld radiation condition gives

$$k_e \int_{S_R} |v_e|^2 ds + Im \int_{S_R} v_e \frac{\partial \bar{v}_e}{\partial n} ds = o(1) \quad R \to \infty.$$

The result follows by appealing to Rellich's lemma [7], Lemma 3.11].

If we have uniqueness (i.e., if the homogeneous problem, with parameters k_e, k_i and ρ, has only the trivial solution), we shall say that $G_{k_{i,e}}(k_e; k_i; \rho)$ holds. Later, we shall also require uniqueness with k_e and k_i interchanged, and ρ replaced by σ, say; thus, we would then require that $G_{k_{i,e}}(k_i; k_e; \sigma)$ holds. We shall also use the following notation: if $G_{k_{i,e}}(k_e; k_i; \rho)$ holds and k_i satisfies the problem in the theorem, we say that $G'_{k_{i,e}}(k_e; k_i; \rho)$ holds. □

Theorem 4.15. *[7]Suppose that $u \in H^1_{loc}(\Omega_e)$ be a radiating solution of the Helmholtz equation (4.27) and satisfying the radiation condition (3.5). If*

$$Im\left(k\int_\Gamma (\frac{\partial^{ext}\overline{u}}{\partial n})u ds_y\right) \geq 0,$$

then $u = 0$ on Ω_e

Proof. We considering Green's first formula (4.1) with respect to the bounded domain $\Omega_R = B_R(y_0) \setminus \overline{\Omega}$, for a fixed $y_0 \in \Omega$, let $B_R(y_0)$ be a ball of radius R with center y_0 and including Ω and choosing $v = \overline{u}$ as test function, we obtain

$$\begin{aligned}\int_{\Omega_R} |\nabla u(y)|^2 dy - k^2 \int_{\Omega_R} |u(y)|^2 dy &= \int_{\partial\Omega_R} \gamma_1^{int} u(y)\overline{\gamma_0^{int} u(y)} ds_y \\ &= \int_{\partial B_{R(y_0)}} \gamma_1^{int} u(y)\overline{\gamma_0^{int} u(y)} ds_y - \int_\Gamma \gamma_1^{ext} u(y)\overline{\gamma_0^{ext} u(y)} ds_y,\end{aligned}$$

when taking into account the opposite direction of the normal vector $n(x)$ for $x \in \Gamma$. Since the left hand side of the above equation is real, we conclude

$$Im\int_{\partial\Omega_{R(y_0)}} \gamma_1^{int} u(y)\overline{\gamma_0^{int} u(y)} ds_y = Im\int_\Gamma \gamma_1^{ext} u(y)\overline{\gamma_0^{ext} u(y)} ds_y$$

By the use of this property, the Sommerfeld radiation condition (3.5) implies

$$\begin{aligned}&\lim_{R\to 0}\left[\int_{\partial B_{R(y_0)}} |\gamma_1^{int} u(y) - ik\gamma_0^{int} u(y)|^2 ds_y\right] \\ &= \lim_{R\to 0}\left[\int_{\partial B_{R(y_0)}} |\gamma_1^{int} u(y)|^2 ds_y + k^2\int_{\partial B_{R(y_0)}} |\gamma_0^{int} u(y)|^2 ds_y - 2kIm\int_{\partial B_{R(y_0)}} \gamma_1^{int} u(y)\overline{\gamma_0^{int} u(y)} ds_y\right] \\ &= \lim_{R\to 0}\left[\int_{\partial B_{R(y_0)}} |\gamma_1^{int} u(y)|^2 ds_y + k^2\int_{\partial B_{R(y_0)}} |\gamma_0^{int} u(y)|^2 ds_y - 2kIm\int_\Gamma \gamma_1^{ext} u(y)\overline{\gamma_0^{ext} u(y)} ds_y\right] = 0\end{aligned}$$

since,
$2kIm\int_\Gamma \gamma_1^{ext} u(y)\overline{\gamma_0^{ext} u(y)} ds_y = \lim_{R\to\infty}\left[\int_{\partial B_{R(y_0)}} |\gamma_1^{int} u(y)|^2 ds_y + k^2\int_{\partial B_{R(y_0)}} |\gamma_0^{int} u(y)|^2 ds_y\right] \geq 0.$

In particular, this gives
$\lim_{R\to\infty}\left[\int_{\partial B_{R(y_0)}} |u(y)|^2 ds_y\right] = O(1) \to 0,$ *and* $|u(x)| = O(\frac{1}{|x|})$ *as* $|x| \to \infty.$ □

Chapter 5

Conclusions

In this work, several boundary integral formulations for the solution of the interior and exterior Dirichlet boundary value problem, and for the problems of the Helmholtz equation with a constant wave numbers were presented. Although the focus of this work was based on the existence and uniqueness of solutions and the equivalence with the solution of the boundary value problem and boundary integral equation, almost all formulations allow for the use of standard arguments, see [23].

The operators $(\Delta + k^2)$ that adding to Laplace operator Δ with a constant factor k^2 give the so-called Helmholtz-operator. For the unknown function $u(x)$ is the sound pressure distribution when considering a time-harmonic processes as in equation (3.2), wave number k with frequency ω and the sound speed c in the considered medium (air, water). The associated boundary conditions may involve a fixed given sound pressure (so, called Dirichlet boundary condition) at a part of the boundary Γ. If the boundary condition $u = 0$, then the problem be soft sound (free surface), e.g., the free surface of a water domain when gravity wave are neglected.

The basic idea in case of interior and exterior BVPs for the Helmholtz equation by different methods are studied in the spaces for smooth domain. By the direct BIE approach the uniquely solvable exterior boundary value problems for the Helmholtz equation can be reduced to the equivalent BIEs which are uniquely solvable for arbitrary value of the frequency parameter. Many solutions of the corresponding integral equations obtained by the direct approach there is only one solution which has a physical meaning and corresponds either to the boundary trace of the unique solution to the exterior problem or to the boundary trace of its normal derivative.

In the direct formulations, the standard argument shows that the integral equation formulation must have a unique solution, we haven't presented any argument as to why there should be only one solution. For the eigenvalue problem (4.25) can be seen as the Helmholtz equation with the wave number k satisfying $k^2 = \lambda$, we obtain for the conormal derivative $t_\lambda = \gamma_1^{int} u_\lambda$ the boundary integral equations (4.26). Thus, the boundary integral operators V_k and $\frac{1}{2}I - K_k^{'}$ are singular, and, therefore, not invertible, if $k^2 = \lambda$ is an eigenvalue of the Dirichlet eigenvalue problem (4.25). On the other hand, if k^2 is not an eigenvalue of the Dirichlet eigenvalue problem (4.25), the single layer potential V_k is injective and hence, since V_k is coercive(sign definite), also invertible. This shows the unique solvability of the variational problem (4.23) and of the boundary integral equation (4.22) in this case, the second kind BIE (4.24) is uniquely solvable.

Recommendation

As we seen in the previous chapters, I recommend that the problem in Helmholtz BVP does not solved as an exception of the Sommerfeld radiation condition and again they ensure by the weak Rellich's condition. Finally, for a direct method we can use the an indirect method(single and double layer potential) to investigate the unique solvability of the variational problem form.

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Appendix A

Appendix

A.1 The Derivation of the Fundamental solution

A.1.1 Descriptions of Coordinates

Cartesian coordinates

With $x = (x, y, z)$, with $x, y, z \in \mathbb{R}$ we have

$$\nabla u = \frac{\partial u}{\partial x} e_x + \frac{\partial u}{\partial y} e_y + \frac{\partial u}{\partial z} e_z$$

where e_x, e_y, e_z are unit vectors pointing in the x, y, z directions. Finally

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Volume and surface elements are

$$dV = dxdydz, \quad dS = dxdy$$

for surfaces in the xy plane for example (constant z).

Definition A.1. (**Divergence**). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. For the divergence of a function $u(x, y, z)$ is given by:

$$\text{div } u = \nabla \cdot u = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})u$$

We call the divergence of u because it is a measure of the degree to which the vector field u diverges or flows outward from any position.

Cylindrical Polar Coordinates

With $x = (x, y, z)$, cylindrical polar coordinates r, θ, z are defined

$$x = rcos\theta, \quad y = rsin\theta, \quad z = z$$
$$r = \sqrt{x^2 + y^2}, \; R = \sqrt{r^2 + z^2} \quad \theta = tan^{-1}(\frac{y}{x}), \quad z = z.$$

with $r \geq 0, \theta \in [0, 2\pi), z \in (-\infty, \infty)$. We also have

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Volume and surface elements are

$$dV = rdrd\theta dz, \quad dS = rd\theta dz$$

for surfaces parallel to the z-axis (constant r).

Spherical Polar Coordinates

the coordinate mapping $(r, \theta, \phi) \rightarrowtail (x, y, z)$ which is defined as:

$$x = rcos\theta sin\phi, \quad y = rsin\theta sin\phi, \quad z = rcos\phi, \; r = \rho sin\phi$$
$$\rho = \sqrt{r^2 + z^2}, \quad \theta = cos^{-1}(\frac{z}{r}), \quad \phi = tan^{-1}(\frac{y}{x})$$

with $r, \rho \geq 0, \theta \in [0, 2\pi), \phi \in [0, \pi)$. We also have

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\frac{\partial u}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2 sin^2\phi}\frac{\partial^2 u}{\partial \phi^2}.$$

Volume and surface elements are

$$dV = \rho^2 sin\phi d\rho d\theta d\phi, \quad dS = \rho^2 sin\phi d\theta d\phi$$

for surfaces with constant r.

A.1.2 Solution in rectangular coordinate

1) If $u = \frac{e^{ikr}}{4\pi r}$, then $\Delta u + k^2 u = 0$ in three-dimensions.

Proof:- We consider that $r = \sqrt{x^2 + y^2 + z^2}$. We replace in to the equation $u = \frac{e^{ikr}}{4\pi r}$. Then we get the result $u = \dfrac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$. When We take the first and second derivative of u with respect to the independent variables x, y, z.

$$\begin{aligned}
u_x &= \frac{(e^{ik\sqrt{x^2+y^2+z^2}})'(4\pi\sqrt{x^2+y^2+z^2}) - (4\pi\sqrt{x^2+y^2+z^2})'(e^{ik\sqrt{x^2+y^2+z^2}})}{(4\pi\sqrt{x^2+y^2+z^2})^2} \\
&= \frac{(\dfrac{ikxe^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}})(4\pi\sqrt{x^2+y^2+z^2}) - (\dfrac{4\pi xe^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}})}{(4\pi\sqrt{x^2+y^2+z^2})^2} \\
&= \frac{(4\pi ikx - \dfrac{4\pi x}{\sqrt{x^2+y^2+z^2}})e^{ik\sqrt{x^2+y^2+z^2}}}{(4\pi\sqrt{x^2+y^2+z^2})^2} = \frac{4\pi x(ik - \dfrac{1}{\sqrt{x^2+y^2+z^2}})e^{ik\sqrt{x^2+y^2+z^2}}}{(4\pi)^2(\sqrt{x^2+y^2+z^2})^2} \\
&= \frac{x(ik - \dfrac{1}{\sqrt{x^2+y^2+z^2}})e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)} = \frac{ikxe^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)} - \frac{xe^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)^{3/2}}
\end{aligned}$$

$$u_y = \frac{ikye^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)} - \frac{ye^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)^{3/2}}, \quad u_z = \frac{ikze^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)} - \frac{ze^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)^{3/2}}$$

$$\begin{aligned}
u_{xx} &= (\frac{(ik - \dfrac{k^2x^2}{\sqrt{x^2+y^2+z^2}})(4\pi(x^2+y^2+z^2)) - (8\pi ikx^2)}{(4\pi(x^2+y^2+z^2))^2})e^{ik\sqrt{x^2+y^2+z^2}} \\
&- \left\{\frac{4\pi(x^2+y^2+z^2)^{3/2} + 4\pi ikx^2(x^2+y^2+z^2) - 12\pi x^2(x^2+y^2+z^2)^{1/2}}{(4\pi(x^2+y^2+z^2)^{3/2})^2}\right\} e^{ik\sqrt{x^2+y^2+z^2}} \\
&= 4\pi e^{ik\sqrt{x^2+y^2+z^2}}(\frac{ik(x^2+y^2+z^2) - k^2x^2(x^2+y^2+z^2)^{1/2} - 2ikx^2}{(4\pi)^2(x^2+y^2+z^2)^2}) \\
&- 4\pi e^{ik\sqrt{x^2+y^2+z^2}}(\frac{(x^2+y^2+z^2)^{3/2} + ikx^2(x^2+y^2+z^2) - 3x^2(x^2+y^2+z^2)^{1/2}}{(4\pi)^2(x^2+y^2+z^2)^3}) \\
&= \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi}(\frac{ik}{(x^2+y^2+z^2)} - \frac{k^2x^2}{(x^2+y^2+z^2)^{3/2}} \\
&- \frac{3ikx^2}{(x^2+y^2+z^2)^2} - \frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}) \\
&= \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi}(\frac{ik(x^2+y^2+z^2)^{3/2} - (k^2x^2+1)(x^2+y^2+z^2) - 3ikx^2(x^2+y^2+z^2)^{1/2} + 3x^2}{(x^2+y^2+z^2)^{5/2}})
\end{aligned}$$

$$u_{yy} = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi}(\frac{ik(x^2+y^2+z^2)^{3/2} - (k^2y^2+1)(x^2+y^2+z^2) - 3iky^2(x^2+y^2+z^2)^{1/2} + 3y^2}{(x^2+y^2+z^2)^{5/2}})$$

$$u_{zz} = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi}(\frac{ik(x^2+y^2+z^2)^{3/2} - (k^2z^2+1)(x^2+y^2+z^2) - 3ikz^2(x^2+y^2+z^2)^{1/2} + 3z^2}{(x^2+y^2+z^2)^{5/2}})$$

Then we need to show that:

$$(\Delta + k^2)u = (\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + k^2)u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + k^2 \right) u = 0,$$

First we consider that $x_1 = x, \quad x_2 = y, \quad x_3 = z$ and again the derivatives are:

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial^2 u}{\partial x^2} = u_{xx}; \quad \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial y^2} = u_{yy}; \quad \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial^2 u}{\partial z^2} = u_{zz}$$

which implies

$$(\Delta + k^2)u = \textstyle\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} + k^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u$$

$$= u_{xx} + u_{yy} + u_{zz} + k^2 u$$

$$= \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi} \left(\frac{ik(x^2+y^2+z^2)^{3/2} - (k^2x^2+1)(x^2+y^2+z^2) - 3ikx^2(x^2+y^2+z^2)^{1/2} + 3x^2}{(x^2+y^2+z^2)^{5/2}} \right)$$

$$+ \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi} \left(\frac{ik(x^2+y^2+z^2)^{3/2} - (k^2y^2+1)(x^2+y^2+z^2) - 3iky^2(x^2+y^2+z^2)^{1/2} + 3y^2}{(x^2+y^2+z^2)^{5/2}} \right)$$

$$+ \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi} \left(\frac{ik(x^2+y^2+z^2)^{3/2} - (k^2z^2+1)(x^2+y^2+z^2) - 3ikz^2(x^2+y^2+z^2)^{1/2} + 3z^2}{(x^2+y^2+z^2)^{5/2}} \right)$$

$$+ k^2 \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}$$

$$= \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi} \frac{3ik(x^2+y^2+z^2)^{3/2} - k^2(x^2+y^2+z^2+3)(x^2+y^2+z^2) + 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}$$

$$- \frac{3ik(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^{5/2}} + k^2 \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}$$

$$= \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi} \left(\frac{(3ik - 3ik)(x^2+y^2+z^2)^{3/2} - k^2(x^2+y^2+z^2)^2 + (3-3)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} \right.$$

$$+ k^2 \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}$$

$$= \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi} \left(-\frac{k^2(x^2+y^2+z^2)^2}{(x^2+y^2+z^2)^{5/2}} \right) + k^2 \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}$$

$$= -\frac{k^2 e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi(x^2+y^2+z^2)^{1/2}} + k^2 \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}$$

When We replace back for $r = \sqrt{x^2+y^2+z^2}$ with $\Delta u := \frac{-k^2 e^{ikr}}{4\pi r}$, now by substituting

$$\Delta u + k^2 u = \frac{-k^2 e^{ikr}}{4\pi r} + \frac{k^2 e^{ikr}}{4\pi r} = 0$$

Therefore, $u = \frac{e^{ikr}}{4\pi r}$ is the fundamental solution of $\Delta u + k^2 u = 0$ for the outgoing wave radiation condition. $\blacksquare$

A.1.3 Solution in Fourier transform

If $G_k(x,y)$ be a fundamental solution for Helmholtz problem, then the integral transformation

$$(\mathcal{F}G_k)(\rho) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-ix\cdot\rho}G_k(x,\rho)d\rho, \quad \rho\in\mathbb{R}^3,$$

is called the Fourier transform of G_k, and

$$G_k(|x|) = (\mathcal{F}^{-1}G_k)(\rho) = (2\pi)^{-3}\int_{\mathbb{R}^2} e^{ix\cdot\rho}G_k(x,\rho)d\rho, \quad \rho\in\mathbb{R}^3$$

is called the inverse Fourier transform of $G_k(\rho)$.

By applying Fourier transform on both sides to Helmholtz problem (3.6) we can use the spherical coordinate description as follow :

$$\mathcal{F}_x[(\Delta+k^2)G_k] = \mathcal{F}_x[-\delta], \quad \mathcal{F}_x[\Delta_x G_k] + \mathcal{F}_x[k^2G_k] = \mathcal{F}_x[-\delta], \quad (-|\rho|^2+k^2)\widehat{G_k} = -1$$

which implies that

$$\mathcal{F}_x[G_k(\rho)] = \widehat{G_k}(\rho) = \frac{-1}{k^2-|\rho|^2} = \frac{1}{|\rho|^2-k^2}$$

The problem now arises is that we went to find the inverse Fourier transform of $\widehat{G_k}(|\rho|)$ to obtain the solution. But, $\widehat{G_k}$ has singularities on the sphere of radius k. A way that to handle the singularities at $\rho=\pm k$ is to let k have a "small", non-zero imaginary part. We assume that Im(k)<0 and Re$(\rho)>0$. Then we write :

$$\begin{aligned}
G_k(r) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{i\rho z}}{\rho^2-k^2}dV;\ z=\rho cos\theta, dV=\rho^2 sin\theta d\rho d\theta d\phi \\
&= \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{e^{i\rho r cos\theta}}{\rho^2-k^2}\rho^2 sin\theta d\rho d\theta d\phi = \frac{1}{(2\pi)^3}\int_0^{2\pi}d\phi\int_0^{\pi}\int_0^{\infty}\frac{e^{i\rho r cos\theta}}{\rho^2-k^2}\rho^2 sin\theta d\rho d\theta \\
&= \frac{1}{(2\pi)^3}\phi|_0^{2\pi}\int_0^{\infty}\left[\frac{\rho^2}{\rho^2-k^2}\int_0^{\pi}e^{i\rho r cos\theta}sin\theta d\theta\right]d\rho \\
&= \frac{1}{(2\pi)^2}\int_0^{\infty}\left[\frac{\rho^2}{\rho^2-k^2}\int_0^{\pi}e^{i\rho r cos\theta}sin\theta d\theta\right]d\rho;\ u=cos\theta,\ du=-sin\theta d\theta\ for\ \theta\in[0,\pi], u\in[1,-1] \\
&= \frac{1}{(2\pi)^2}\int_0^{\infty}\left[\frac{\rho^2}{\rho^2-k^2}\int_{-1}^{1}e^{i\rho r u}du = \left(\frac{\rho^2}{\rho^2-k^2}\right)\frac{1}{i\rho r}e^{i\rho r u}|_-^1\right]d\rho \\
&= \frac{1}{(2\pi)^2 r}\int_0^{\infty}\frac{2\rho sin\rho r}{\rho^2-k^2}d\rho = \frac{1}{2\pi^2 r}\left[\int_{-\infty}^{0}\frac{\rho sin(\rho r)}{\rho^2-k^2}d\rho + \underbrace{\int_0^{\infty}\frac{\rho sin(\rho r)}{\rho^2-k^2}d\rho}_{0}\right] \\
&= \frac{1}{2\pi^2 r}\int_{-\infty}^{\infty}\frac{\rho sin(\rho r)}{\rho^2-k^2}d\rho = \frac{1}{2\pi^2 r}\left(\frac{\pi}{2}e^{-ikr}\right) = \frac{1}{4\pi}e^{-ikr}
\end{aligned}$$

Again, we assumed that Im$(k>0)$, we would have recovered the solution $\dfrac{e^{ikr}}{4\pi r}$. Finally, we can let the imaginary part of k approach 0 and recover the result for real k.

A.1.4 Solution in distributional sense

1) We can solve (3.6) without the use of integral transformation. Instead of $r \neq 0$, we write

$$\nabla^2 u(r) + k^2 u(r) = 0$$

Exploiting the spherical symmetry of the problem, we have

$$\nabla^2 u + k^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + k^2 u = 0$$

which can be rearranged as

$$\frac{\partial^2}{\partial r^2}(ru) + k^2(ru) = 0 \tag{A.1}$$

solution to (A.1) are trivially to be

$$u = C_{\pm}\frac{e^{\pm ikr}}{r}$$

we find the constant $C_{\pm}$ by enforcing the condition

$$\int_{|r|\leq\epsilon} \nabla^2 u(r) dV = -1.$$

Applying the divergence theorem in the sense of Distribution sense

$$\oint_{r=\epsilon} (\frac{\partial u(r)}{\partial r})|_{r=\epsilon}\epsilon^2 sin\theta d\theta d\phi = -4\pi C_{\pm} = -1,$$

thus, $C_{\pm} = \frac{1}{4\pi}$.

Finally, we obtain the fundamental solution for (3.6)

$$u(r) = \frac{1}{4\pi}\frac{e^{\pm ikr}}{r}.$$

2) Three-dimensional Helmholtz equation for $r \to 0$ and wave number $k \neq 0$

$$(\Delta + k^2)u^*(x, \xi) = -\delta(x, \xi),$$

the fundamental solution is

$$u^*(x - \xi) = u^*(r) = \frac{1}{4\pi r}e^{-ikr}$$

Proof. Really[1] the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$ for $x = \xi$ is straight forward by applying the Leibniz formula. $\nabla^2(uv) = v\nabla^2(u) + 2\nabla(u)\cdot\nabla(v) + u\nabla^2(v)$, i.e., here for more see ([27],in §9, [25]) for more of this:

$$\begin{aligned}\Delta(\frac{1}{r}e^{-ikr}) &= \nabla^2(\frac{1}{r}e^{-ikr}) \\ &= e^{-ikr}\Delta(\frac{1}{r}) + 2\nabla(\frac{1}{r})\nabla(e^{-ikr}) + \frac{1}{r}\Delta(e^{-ikr})\end{aligned}$$

[1] **Note that** $\lim_{r\to\infty} u^*(r)$ is zero or ∞, these fundamental solutions represent wave forms that convergence or diverge to infinity.

then

$$(\Delta + k^2)u^* = \frac{1}{4\pi}\{e^{-ikr}\Delta(\frac{1}{r}) + 2\nabla(\frac{1}{r})\cdot\nabla(e^{-ikr}) + \frac{1}{r}\Delta(e^{-ikr}) + k^2\frac{1}{r}e^{-ikr}\}$$

Since differentiations give

$$\begin{aligned}
\nabla(\frac{1}{r}) &= \frac{\partial}{\partial x_i}(\frac{1}{r}) \\
&= -\frac{1}{r^2}(\frac{x_i - \xi_i}{r}) \\
&= -\frac{(x_i - \xi_i)}{r^3}, \\
\nabla(e^{-ikr}) &= \frac{\partial(e^{-ikr})}{\partial x_i} = -ik(\frac{(x_i - \xi_i)}{r})e^{-ikr}, \\
\Delta(e^{-ikr}) &= \frac{-ik(\frac{(x_i-\xi_i)}{r})e^{-ikr}}{\partial x_i} \\
&= -(\frac{2ik}{r} + k^2)e^{-ikr},
\end{aligned}$$

then we obtain

$$\begin{aligned}
(\Delta + k^2)u^* =& \frac{1}{4\pi}\{-4\pi\delta(x,\xi) + 2\frac{(x_i - \xi_i)}{r^2}ik\frac{(x_i - \xi_i)}{r} - \frac{1}{r}(\frac{2ik}{r} + k^2) + \frac{k^2}{r}\}e^{-ikr} \\
&= \frac{1}{4\pi}\{-4\pi\delta(x,\xi) + 2\frac{(x_i - \xi_i)^2}{r^4}ik - \frac{1}{r}(\frac{2ik}{r} + k^2) + \frac{k^2}{r}\}e^{-ikr} \\
&= \frac{1}{4\pi}\{-4\pi\delta(x,\xi) + 2\frac{(x_i - \xi_i)^2}{r^4}ik - \frac{2ik}{r^2} - \frac{k^2}{r} + \frac{k^2}{r}\}e^{-ikr} \\
&= \frac{1}{4\pi}\{-4\pi\delta(x,\xi) + \frac{2ik}{r^2} - \frac{2ik}{r^2} - \frac{k^2}{r} + \frac{k^2}{r}\}e^{-ikr} \\
&= -\delta(x,\xi)e^{-ikr} \ as\ r \to 0,\ e^{-ikr} \to 1 \\
&= -\delta(x,\xi).
\end{aligned}$$

Therefore, $(\Delta + k^2)u^* = -\delta(x,\xi)$ □

A.2 Properties of the eigenfunctions and eigenvalues of the Dirichlet problem

1. Symmetry: In wave problems we often use the complex exponential and therefore we have to consider all functions as complex. Use Green's second identity with $v(x) = G(x, x_1)$ and $u(x) = G(x, x_2)$ so that

$$\int_\Omega (\overline{G(x,x_1)})(\nabla^2 + k^2)G(x,x_2) - G(x,x_2)\overline{(\nabla^2 + k^2)G(x,x_1)}dx = 0.$$

We use Dirac delta function on the right hand side as: $(\nabla^2 + k^2)G(x, x_1) = \delta(x - x_j), j = 1, 2, 3$ we exploit the property of the Dirac delta function to find

$$\overline{G(x_2), x_1} = G(x_1, x_2) \tag{A.2}$$

in both two and three dimensions. So, for the Helmholtz operator, the Greens function is Hermitian symmetric. So, in case of using an integration by parts (Greens formula) we see that for any two functions satisfying the Dirichlet boundary conditions

$$\int_\Omega (-\Delta u - k^2 u)v dx = -(-\int_\Omega \nabla v.\nabla u)dx - k^2 \int_\Omega v.udx = -\int_\Omega u\Delta v dx - k^2 \int_\Omega u.v.$$

This verifies the symmetry of the Laplace operator.

$$\int_\Omega v\Delta u dx = \int_\Omega u\Delta v dx.$$

2. Orthogonality:

Proposition A.1. *Eigenfunctions associated with different eigenvalues are orthogonal to each other.*

Proof. Let v_n, v_m be two eigenfunctions associated with the eigenvalues $\lambda_n \neq \lambda_m$, Respectively; namely,

$$\begin{aligned} -\Delta v_n &= \lambda_n v_n, \\ -\Delta v_m &= \lambda_m v_m \end{aligned}$$

The inner product of $\langle \Delta v_n, v_m \rangle$ and $\langle v_n, \Delta v_m \rangle$ be formed as

$$\begin{aligned} \langle -\Delta v_n, v_m \rangle &= -\int_\Omega \Delta v_n v_m dx, \\ &= -\int_\Omega \Delta(v_n v_m) dx \\ &= \int_\Omega \lambda_n v_n v_m dx \\ &= \lambda_n \int_\Omega v_n v_m dx \end{aligned} \tag{A.3}$$

$$\begin{aligned} \langle v_n, -\Delta v_m \rangle &= -\int_\Omega \Delta v_m v_n dx \\ &= -\int_\Omega \Delta(v_m v_n) dx \\ &= \int_\Omega \lambda_m v_m v_n dx \\ &= \lambda_m \int_\Omega v_m v_n dx \end{aligned} \tag{A.4}$$

By Substract (A.4) from (A.3) we obtain

$$\lambda_m \int_\Omega v_n v_m dx - \lambda_n \int_\Omega v_n v_m dx = (\lambda_m - \lambda_n) \int_\Omega v_n v_m dx = 0 \tag{A.5}$$

The symmetry property implies

$$(\lambda_m - \lambda_n) \int_\Omega v_n v_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Hence the orthogonality. □