THE ANALYSE OF BOUNDARY INTEGRAL EQUATION METHODS IN LOW FREQUENCY ACOUSTICS IN 3D

M.Sc THESIS

By

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Abstract

In this thesis we consider the direct integral formulations for boundary value problems of the Helmholtz equation. We reduce the boundary value problems to boundary integral equation and analyses the boundary potentials associated with low frequency. From boundary potential we introduce the Calderon projectors and give the basic mapping properties. We discuss unique solvability for the corresponding boundary integral equations (BIE) and its relations to the interior eigenvalue problems of the Laplacian. Based on the integral representations, we study the asymptotic behaviors of the solutions to the boundary value problems (BVP) when the wave number tends to zero in $\mathbb{R}^3$*. We arrive at the asymptotic expansions for the solutions, and show that in all the cases, the leading terms in the expansions are always the corresponding potentials for the Laplacian and our integral equation procedures developed here are general enough and can be adapted for treating similar low frequency scattering problems.*

Keywords: integral equation, Calderon projector, asymptotic expansion, scattering.

Notation and Abbreviations

Ω	A simply connected boundary region in $\mathbb{R}^3$.
$\bar{\Omega}$	The closure of the set Ω.
Γ	smooth boundary.
$C(\bar{\Omega})$	A set of all continuous functions on $\bar{\Omega}$.
$C^1(\bar{\Omega})$	The set of all continuous functions and differentiable on $\bar{\Omega}$.
$C^2(\Omega)$	The set of all twice continuously differentiable functions on Ω.
$C^\infty(\Gamma)$	The set of all infinitely differentiable function on Γ.
∇	The gradient operators.
$\triangle$	Laplace's operator.
δ	Dirac's delata function.
$\mathbb{C}$	Complex number.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^n$	n-dimensional Eculidean space.
F	Fourier transform operator.
F^{-1}	Inverse Fourier transform operator.
$W_p^k(\Omega)$	Sobolev spaces of order k.
BVP	Boundary Value Problem.
BIE	Boundary Integral Equation .
PDE	Partial defferential equation.

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Chapter 1

Introduction

1.1 Background of the study

The boundary integral equation (BIE) method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility of reducing a boundary value problem (BVP) for a linear partial differential equation in a domain to an integral equation on the domain boundary, that is, to reduce the problem dimensionality by one. The main thing necessary for the reduction of a boundary value problem (BVP) to a boundary integral equation (BIE) is a fundamental solution of the original partial differential equation, available in an analytical form and cheaply calculated. After the fundamental solution is used in the corresponding Green formula, one can reduce the problem to a boundary integral equation (BIE). The boundary integral equation (BIE) has found application in such diverse topics as stress analysis, potential flow, fracture mechanics, acoustics and electromagnetic.

Acoustics: Acoustics is the science of sound that study about the small pressure waves in air which can be detected by human ear and sound. The field of acoustic can be subdivided into several special topics on [**17**], such as:

- **Theoretical acoustics** is analytical and numerical methods for sound field calculations.
- **Non linear acoustics** is non linear effects that occur at events of extremely high sound pressure such as explosions or sonic booms of objects that move faster than the speed of sound.

- **Under water acoustic** is sound propagation in water, sonar systems, seismic explorations.
- **Ultrasound** is non destructive test procedures for materials, medical applications.
- **Vibrations** is vibrational behavior of bodies, sound radiation of vibrating structures.
- **Noise control** is description and modeling of noise sources, investigations on noise protection measures.
- **Room acoustics** is assessment, planing and prediction of sound fields in rooms.
- **Building acoustics** is noise control in buildings, transmission loss of building structures.
- **Electroacoustics** is transducers(microphones, loud speakers), recording devices, public address systems, signal processing in acoustics.
- **Acoustics of the ear** is structure of the ear, characteristics of the ear, perception and subjective evaluation of noise.

Low frequency: A range of radio frequencies spanning from 30 to 300kHz. These correspond to wave lengths of 10km to 1km, respectively, which are kilometric waves. Also pertaining to this interval of frequencies.
Linear acoustic or low frequency obviously give rise to a range of integral equation formulations, for example depending on whether the acoustic field lies in an interior or exterior domain. However, the integral equations that arise in all problems in the same spatial dimensions contain similar integral operators. The problem of acoustic scattering by various objects has always been a subject of interest for researchers in various disciplines. As it is indicated in [**19**] acoustic problems can be categorized in to: the solution region of the problem (interior or exterior), the nature of the equation describing the problem(differential or integral or both) and the associated boundary conditions (Dirichlet or Neumann). Describe the physical problem is interesting to derive an integral representation formula for solutions of the Laplace Beltrami operator on the sphere, and introduce the single and double layer

potentials. We investigate their jump properties, and use these to derive an integral equation for the solution of a Dirichlet problem.

1.2 Mathematical Preliminaries

Basic terminologies and theorems that are defined and stated respectively below here can help us to reduced boundary value problem (BVP) to boundary integral equation (BIE) and also can help us to explain fundamental solution of Helmholtz equation and boundary integral operator (BIO).

Definition 1.2.1. (Linear functional, [2]). Suppose that B is a Banach space over $\mathbb{R}$ equipped with a norm $||.||$. A linear function is a linear mapping l from B to $\mathbb{R}$, that is, $l : B \longmapsto \mathbb{R}$ which satisfies

$$l(af + bg) = al(f) + bl(g) \tag{1.1}$$

for all $a, b \in \mathbb{R}$ and $f, g \in B$

1.2.1 Integral Equation

An integral equation in which an unknown function appears under an integral sign.

Fredholm integral equation of first type is given by

$$f(x) = \int_a^b K(x,t)\psi(t)dt \tag{1.2}$$

where ψ is unknown function, f is known function, K is another known function of two variables often called the kernel function.

Fredholm integral equation of second type is given by

$$\psi(x) = f(x) + \lambda \int_a^b K(x,t)\psi(t)dt \tag{1.3}$$

where the parameter λ is unknown factor.

Definition 1.2.2. (Distribution, [2]). A distribution on the open set $\Omega \subseteq \mathbb{R}^n$ is a linear function (1.1), $\Upsilon : C_c^\infty \longmapsto \mathbb{R}$ such that the following boundedness property holds: For every compact $k \subset \Omega$ there exist an integral $N \geq 0$ and a constant C such that

$$|\Upsilon(\phi)| \leq C\,||\phi||_{C^N}\,, \forall \phi \in C^\infty \;\; with \;\; supp(\phi) \subseteq k$$

where, N is the order of the distribution if it is finite distribution and ϕ is any test function.

Definition 1.2.3. (Direct Product of Distributions,[2]). Let $f \in D'(\mathbb{R}^p)$, $g \in D'(\mathbb{R}^q)$. Then the direct product f(x)g(y) is the distribution on $\mathbb{R}^{p+q}$ given by

$$\langle f(x)g(y), \phi(x,y)\rangle = \langle f(x), \langle g(y), \phi(x,y)\rangle\rangle \tag{1.4}$$

Definition 1.2.4. (Convolutions,[11]). The classical definition of the convolution of two functions defined on $\mathbb{R}^n$ is:

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy \tag{1.5}$$

Definition 1.2.5. (Convolution Of Distributions). Let $f, g \in D'(\mathbb{R}^n)$. Then the convolution of f and g is defined by:

$$\langle f * g, \phi\rangle = \langle f(x)g(y), \phi(x+y)\rangle \tag{1.6}$$

From the corresponding properties of the direct product, it follows that:

$$\begin{aligned}\langle \delta * f, \phi\rangle &= \langle\langle \delta(x)f(y), \phi(x+y)\rangle\rangle \\ &= \langle f(y), \langle \delta(x), \phi(x+y)\rangle\rangle \\ &= \langle f(y), \phi(y)\rangle \\ &= \langle f, \phi\rangle\end{aligned}$$

i.e., $\delta * f = f$

Definition 1.2.6. (Fourier Transform,[20]). Let $f \in C_0^s(\mathbb{R}^m)$, s = 0, 1, ...Then, we define the Fourier transform of f by:

$$\hat{f}(\xi) := F[f](\xi) := (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{-ix\xi}f(x)dx \tag{1.7}$$

where,

$$\xi \,\varepsilon\, \mathbb{R}^m,\ \xi = (\xi_1, ..., \xi_m),\ \ x = (x_1, ..., x_m)\ and\ \xi\,.x = \xi_1x_1 + ... + \xi_mx_m = \sum_{j=1}^{m}\xi_jx_j$$

Definition 1.2.7. (Inverse Forier Transform,[20]). Knowing the Fourier transform $g(\xi) = \hat{f}(\xi)$, the function f can be reconstructed with the aid of the inverse Fourier transformation.

$$F^{-1}[g](x) := (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{ix\xi}g(\xi)d\xi \tag{1.8}$$

Indeed, it holds that $F[\hat{f}] = F^{-1}(F[f]) = f$ (inversion formula), provided the integrals on the right hand side exist.

1.2.2 Fundamental Solutions

. Fundamental solutions are essential for the solution of boundary integral equation. They describe the reaction of an infinite homogeneous domain do to a single load. Consider the following linear differential operator L with constant coefficients a_α[11]:

$$L(D) = \sum_{|\alpha|=0}^{n} a_\alpha D^\alpha \tag{1.9}$$

Definition 1.2.8. A distribution F in $D'(\mathbb{R}^n)$ is said to be the fundamental solution of the differential operator (1.9), if it satisfies the relation:

$$L(D)F = \delta(x) \quad in \quad \mathbb{R}^n \tag{1.10}$$

1.2.3 Green's Identities

Let $u, v \in C^2(\Omega)$, Ω be a domain with smooth boundary Γ, $\vec{n}$ be the out ward unit normal vector to Γ. Recall the following notations of field theory

- $gradu = \nabla u = (u_x, u_y, u_z)$
- $div\vec{F} = \nabla.\vec{F} = f_x + g_y + h_z$
- $\triangle\, u = div(\nabla u) = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

where $\vec{F}(f, g, h)$ is a vector field, $dv = d_x d_y d_z, ds_y$ is a surface element, ds_y is an arc length element at p on Γ. now the three Green's theorems are given below.

Theorem 1.2.1. ***(Green's First Identity,[23])***. *Let $\Omega \in \mathbb{R}^3$ be a domain bounded by the closed smooth contour Γ and let n_y denote the unit normal vector to the boundary Γ directed into the exterior or Γ and corresponding to a point $y \in \Gamma$. Then, for every function u which is once continuously differentiable in the closed domain $\bar{\Omega} = \Omega \cup \Gamma$, $u \in C^1(\bar{\Omega})$ and every function v which is twice continuously differentiable in $\bar{\Omega}, v \in C^2(\bar{\Omega})$, then:*

$$\int_\Omega (u \triangle v + \nabla u.\nabla v)dx = \int_r u\frac{\partial v}{\partial n_y} ds_y \tag{1.11}$$

Theorem 1.2.2. ***(Green's Second Identity,[23])***. *For u, $v \in C^2(\bar{\Omega})$, the Green's second identity is given by,*

$$\int_\Omega (u \triangle v - v \triangle u)dx = \int_\Gamma (u\frac{\partial v}{\partial n_y} - v\frac{\partial u}{\partial n_y})ds_y \tag{1.12}$$

Theorem 1.2.3. ***(Green's Third Identity,[23])***. *Let $D \subset \mathbb{R}^m$ be a bounded domain of class C^1 and let $u \in C^2$ be harmonic in D. Then,*

$$u(x) = \int_{\partial D} \{\frac{\partial u}{\partial n}(y)\phi(x,y) - u(y)\frac{\partial \phi}{\partial n_y}(x,y)\}ds_y \quad , x \in D \tag{1.13}$$

1.2.4 Fundamental Solution of The Helmholtz Equation

The fundamental solution for the scalar wave in the n-dimensional space satisfies the following Helmholtz equation,**[16]**

$$\{\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_j} + k^2\, \}U_{[n]}(x, y, k) = -\delta(x - y) \tag{1.14}$$

where k is the wave number and $\delta(x-y)$ is the Dirac delta function which in two dimension is defined by[**8**]

$$\delta(x-y) = \begin{cases} 0 & when\ \ x \neq y \\ \infty & when\ \ x = y \end{cases}$$

and

$$\int_{\Omega} \delta(x-y) d\Omega_y = \int_{\Omega^*} \delta(x-y) d\Omega_y = 1$$

where, $\Omega^* \subset \Omega$.

Applying the Fourier transform (1.7) of definition (3.4) to Eq.(1.14), yields the solution $\tilde{U}_{[n]}(\xi, y, k)$ in the transformed domain:

$$\tilde{U}_{[n]}(\xi, y, k) = \frac{exp(-i\xi.y)}{(|\xi|^2 - (k+i\epsilon)^2)}$$

where, ϵ is a small positive number to avoid the singularity appeared in the calculation of the inverse Fourier transform of $\tilde{U}_{[n]}$. Using the inverse Fourier transform (1.8) $\tilde{U}[n]$ may be written as:

$$U_{[n]}(x, y, k) = \lim_{\epsilon \to 0} \frac{1}{(2\pi)^n} \int \frac{1}{|\xi|^2 - (k+i\epsilon)^2} exp(i\xi.(x-y)) d\xi \tag{1.15}$$

Using (1.15) for a one dimensional problem (n=1), the fundamental solution is

$$\begin{aligned} U_1(x, y, k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\xi^2 - (k+i\epsilon)^2} \ \ e^{i\xi(x-y)}\ d\xi \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\xi^2 - (k+i\epsilon)^2} \ \ (e^{i\xi(x-y)} + e^{-i\xi(x-y)})\ d\xi \\ &= \frac{i}{2k} e^{ik|x-y|} \end{aligned} \tag{1.16}$$

For a three dimensional problem n = 3, we have:

$$\begin{aligned} U_3(x, y, k) &= \frac{1}{(2\pi)^3} \int_0^{\infty} |\xi|^2 d\,|\xi| \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} \frac{e^{i|\xi|rcos\theta}}{|\xi|^2 - (k+i\xi)^2} d\theta \\ &= \frac{-1}{8\pi^2 r} \int_{-\infty}^{\infty} \frac{s(e^{isr} - e^{-isr})}{s^2 - (k+i\epsilon)^2} ds \end{aligned} \tag{1.17}$$

where $r = |x-y|$. Comparing (1.17) with (1.16), we find the relation between $U_{[3]}$ and $U_{[1]}$ as follows:

$$U_{[3]} \ \ = \ \ \frac{-1}{2\pi r} \frac{dU_{[1]}}{dx}|_{|x-y|}$$

This relation leads to the explicit expression for a fundamental solution in a three dimensional Helmholtz problem in the following form:

$$U_{[3]}(x, y, k) \quad = \quad \frac{e^{ikr}}{4\pi r} \tag{1.18}$$

1.2.5 Boundary potentials

The two main boundary potentials mostly used are:

single layer boundary potentials u(x) is defined

$$u(x) = \int_{\Gamma} E_k(x, y)\sigma(y)ds_y \tag{1.19}$$

Double layer boundary potential u(x) is given by

$$u(x) = \int_{\Gamma} \frac{\partial E_k}{\partial n_y}(x, y)\varphi(y)ds_y \tag{1.20}$$

Where, $\sigma(y)$ is the density of single layer boundary potential, $\mu(y)$ is the density of double layer boundary potential, Γ is the smooth boundary of the region Ω, n is the out ward unit normal vector along Γ.

Definition 1.2.9. By $C^{\lambda}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$ we denote the class of Holder continuous functions defined by the semi-norm

$$[f]_{\lambda} := sup\frac{|f(x+h) - f(x)|}{|h|^{\lambda}} < \infty$$

equipped with the norm

$$||f||_{C^{\lambda}} = sup_{x\in\Omega}\,|f(x)| + [f]_{\lambda} = sup_{x\in\Omega}\,|f(x)| + sup_{x\in\Omega}\frac{|f(x+h) - f(x)|}{|h|^{\lambda}}$$

$C^{\lambda}(\Omega)$ is a Banach Space we shall deal with the case $\Omega = B_{\mathbb{R}}$ where $B_{\mathbb{R}} = B(0, \mathbb{R}) := \{x \in \mathbb{R}^n : |x| < \mathbb{R}\}$, $0 < \mathbb{R} \leq \infty$ we define the generalized Holder Space.

Definition 1.2.10. (Sobolev Space,[1]). Let Ω be an open set in $\mathbb{R}^m$. Let $k \in N \cup \{0\}$ and let $1 \leq p \leq +\infty$. The Sobolev space $W^{k.p}(\Omega)$ *or* $L^p_k(\Omega)$ is defined to be the set of all function u defined on Ω such that for every multi-index $a = (a_1, ..., a_m)$ with $|a| \leq k$, where $|a| = a_1 + ... + a_m$, the mixed partial derivative

$$D^a u = \frac{\partial^{|a|}u}{\partial x_1^{a_1}.....\partial x_m^{a_m}}$$

is both locally integrable and belongs to $L^p(\Omega)$ i.e., $||D^a u||_{L^p} < \infty$. In short, the Sobolev space on Ω is defined as

$$W^{k,p}(\Omega) = \{u \in L^P(\Omega) : D^a u \in L^p(\Omega),\ \ \forall\,|a| \leq k\} \tag{1.21}$$

$k \in N$ is called order of Sobolev space (1.21). If $p = 2$, one often uses the notation $H^k(\Omega)$ or $L^2_k(\Omega)$. If $p = 2$ and $k = 0$, we get $H^0(\Omega) = L^2(\Omega)$.
For $u \in L^p_k(\Omega)$, we define

$$||u||_{L^p_k(\Omega)} = \{\sum_{|a|\leq k} ||\partial^a u||^p_{L^p}\}^{\frac{1}{p}} = \{\sum_{|a|\leq k} \int_{\Omega} |\partial^a u|^p dx\}^{\frac{1}{p}} \quad for \;\; 1 \leq p \leq \infty$$

For $p = \infty$, we define

$$|u|_{L^\infty_k(\Omega)} = \sum_{|a|\leq k} esssup\, |\partial^a u|$$

These expressions are norms. Indeed,

$$||\lambda u||_{L^p_k} \;\; = |\lambda| \, ||u||_{L^p_k} \quad , \quad \forall \lambda \in C$$

Hence, $L^p_k(\Omega)$ is a Banach space; i.e., it is a complete normed space.

1.3 Statement of the Problems

Let Ω be a bounded domain in $\mathbb{R}^3$ and with a smooth boundary Γ. The boundary value problems (BVP) begin with the Helmholtz equation arises in connection with wave propagation in acoustics defined as follows:

$$\triangle\, u + k^2 u = 0 \;\; , \quad in \;\; \Omega \tag{1.22}$$

Where $\triangle = \nabla^2$ denotes the Laplace operator in $\mathbb{R}^3$. In acoustics, $k = \frac{\omega}{c}$ denotes the complex wave number and u corresponds to the acoustic pressure fields. Here ω and c are the frequency and the speed of sound respectively. In order to avoid resonance states, we assume that Im$k \geq 0$. We are interested in the solution of (1.22), when the wave number k is small, low frequency acoustics. The result concerning with low frequency behavior of the solution of (1.22) developed in **[15]**. These result are obtained by using boundary integral equation (BIE) methods. The solution behavior depend on the specific boundary conditions to be considered is the interior Dirichlet boundary condition.

$$u \mid_{\Gamma} = \varphi \tag{1.23}$$

In acoustic scattering, (1.23) is the conditions for modeling the situations for soft scatters. The governing acoustic equations were defined in **[3, 21]** we know that interior Helmholtz problems are well-posed. The solvabilitys of the boundary integral equations(BIE) will be essential and can serve as the mathematical foundations for the boundary element methods. Therefore, this research work attempt to address the following basic questions:

1. How can the Dirichlet boundary value problem with low frequency be described by integral equation?

2. Does the unique solution of the integral equation for Dirichlet boundary value problem of low frequency acoustic exists?

1.4 Objective of the Study

1.4.1 General Objectives

The general objective of the study is to analyse low frequency acoustics problems using boundary integral equation (BIE) method.

1.4.2 Specific Objectives

The specific objectives of the study are:

(a) To reduce the Dirichlet boundary value problems to boundary integral equation.

(b) Investigate the properties of Integral operators in the Holder spaces and Sobolove spaces.

(c) To investigate the unique solvability of boundary integral equation with low frequency acoustics.

1.5 Significance

The significance of the study are to:

- Improve our understanding of the integral equation of for low frequency acoustics.

- Understand low frequency acoustics on a bounded domain.

- Helps other researchers on this area for further work.

Chapter 2

Literature Review

Since the 1980s, the boundary integral equation (BIE) method has played an important role in the analysis of both forward and inverse scattering problems. Acoustic, elastic and electromagnetic wave scattering problems arise in many applications of mathematical, physical and engineering interest, including the modeling of radar, sonar, nose barrier design and atmospheric particle scattering. Often the scattering problem comprises the forward map in the formulation of an inverse problem, for example in non-destructive testing or in methods for detecting hydrocarbon-bearing deposits under the sea bed. While in general the scattered wave has to be found in an in-homogeneous medium, there are a substantial number of applications in which the material is either homogeneous is or piecewise homogeneous, at least sufficiently far away from the scatterer. In these cases boundary integral equation (BIE) methods are of considerable interest and form the basis for several commercial scattering codes [**5**]. This review focuses on the efficient solution of high-frequency acoustic scattering problems in homogeneous media, using integral equation methods. Hence we consider the Helmholtz equation (1.22), in a domain $\Omega^c = \mathbb{R}^n/\bar{\Omega}$, n = 2 or 3, where Ω is a bounded domain and (1.22) is to be solved subject to some suitable boundary condition on Γ and radiation condition in the far field. The Helmholtz equation is of course derived from the linear wave equation under the assumption that all quantities vary harmonically ($e^{-i\omega t}$) in time. Here ω is the angular frequency and $k := \frac{\omega}{c} > 0$ is the wavenumber, where c is the speed of sound . The problem (1.22) has solutions which oscillate in space with wavelength $\lambda = \frac{2\pi}{k}$. For example the plane waves $u(x) = exp(ikx.\hat{a})$, where $\hat{a} \in \mathbb{R}^n$ is a unit vector, are solutions. The number of oscillations grows linearly in k and so the application of conventional (piecewise polynomial) boundary elements to this problem leads to full matrices of dimension at least $N = O(k^{n-1})$ and a solution time of this order or worse as $k \longmapsto \infty$. Domain finite elements lead to sparse matrices but require even large N. Since this lack of robustness with respect to increasing values of k puts high-frequency problems beyond the reach of many standard algorithms, much recent research has been devoted to finding more robust methods[**4**].

The studied method is solving the transmission problem using a single bounded integral equation for a single unknown tangential vector field on the interface. The method consists of representing the solution in one domain by some combination of a single layer potential and a double layer potential, and inserting this representation into the transmission conditions and the additional descriptive relations of the other domain. Several authors including [**3**] and [**15**] solve boundary integral equation (BIE) method in low frequency acoustic on two dimension. However, the authors made an assumption that did not for three dimension. So by modify the assumption of those authors and solving the boundary integral equation (BIE) method in low frequency acoustic on three dimension as detailed in the literature survey above.

Chapter 3

BIE Methods In Low Frequency Behavior

3.1 Boundary value problems

In this chapter we are confined the interior Dirichlet boundary problem (1.22) and (1.23). In the next section;

- We consider the boundary potentials associated with (1.22) and discuss the reductions of the these boundary value problems (BVP) to boundary integral equation (BIE).
- Finding the four basic boundary integral operators (BIO) from boundary potential.
- We introduce the Calderon projectors and give the basic mapping properties.
- The solvabilities of the boundary integral equations (BIE) will also be discussed.
- Our main results concerning the low frequency behaviors of the solutions to the boundary value problems (BVP) are presented.

3.2 Boundary Integral Operators

To reduce the boundary value problems to boundary integral equations, we begin with the Green representation for the solution of Eq.(1.22)

$$\begin{aligned} u(x) &= \int_\Gamma E_k(x,y)\frac{\partial u}{\partial n}(y)ds_y - \int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)u(y)ds_y, \\ &:= V_k\frac{\partial u}{\partial n}(x) - W_k u(x) \end{aligned} \tag{3.1}$$

Here, V_k and W_k are referred to as the single and double layer potential, and E_k is the fundamental solution of the Helmholtz equation defined by:

$$E_k = \frac{e^{ikr}}{4\pi r} \ , \qquad r = |x-y| \tag{3.2}$$

The following integral expression is called **a single layer potential:**

$$V_k\sigma(x) = \int_\Gamma E_k(x,y)\sigma(y)ds_y \tag{3.3}$$

The following integral expression is called **a double layer potential:**

$$K_k\mu(x) = \int_\Gamma \frac{\partial E_k}{\partial n_x}(x,y)\mu(y)ds_y \tag{3.4}$$

Theorem 3.2.1. *The single layer potential (3.3) is continuous with respect to the variable x in $\mathbb{R}^3$ and especially when crossing the surface Γ, where its value is*

$$V_k\sigma(x) = \int_\Gamma E_k(x,y)\sigma(y)ds_y, \qquad \text{when} \quad x\in\Gamma, \sigma(x)\in C^0(\Gamma) \tag{3.5}$$

Its normal derivatives have discontinuities when crossing Γ. Their limit values on both side of Γ are

$$\frac{\partial}{\partial n}V_k\sigma(x)|_{int} = \frac{\sigma(x)}{2} + \int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)\sigma(y)ds_y \tag{3.6}$$

$$\frac{\partial}{\partial n}V_k\sigma(x)|_{ext}(x) = -\frac{\sigma(x)}{2} + \int_\Gamma \frac{\partial E_k}{\partial n_y}(x-y)\sigma(y)ds_y \tag{3.7}$$

The double layer potential (3.4) has discontinuity when crossing Γ. If $\mu\in C^0(\Gamma)$,the corresponding limit values on both side of Γ are

$$K_k\mu(x)_{int} = -\frac{\mu(x)}{2} + \int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)\mu(y)ds_y \tag{3.8}$$

$$K_k\mu(x)_{ext} = \frac{\mu(x)}{2} + \int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)\mu(y)ds_y \tag{3.9}$$

The normal derivatives of a double layer potential is continuous when crossing Γ. The associated kernel admits a strong singularity equivalent to $\dfrac{1}{|x-y|^3}$ and so is not an integrable function. Yet,we write it formally as an improper integral

$$\frac{\partial}{\partial n}K_k\mu(x) = \frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)\mu(y)ds_y \tag{3.10}$$

Consequently, any potential sum of a single layer with density of and a double layer with density μ is such that $\sigma = \dfrac{\partial u}{\partial n}|_\Gamma$ and $\mu = u|_\Gamma$.

The proof of this theorem appear on [**21**]

In the representation (3.1), the traces $\mu :=\mathrm{u}|_\Gamma$ and $\sigma := \dfrac{\partial u}{\partial n}|_\Gamma$ are the Cauchy data of the solution u on Γ. These Cauchy data are related by the boundary integral equations(BIE):

$$\begin{pmatrix}\mu\\ \sigma\end{pmatrix} = \begin{pmatrix}\frac{1}{2}I - K_k & V_k \\ D_k & \frac{1}{2}I + K'_k\end{pmatrix}\begin{pmatrix}\mu\\ \sigma\end{pmatrix} \tag{3.11}$$

Here V_k, K_k, K'_k, D_k are the four basic boundary integral operators (BIO) defined by

$$\begin{aligned}
V_k\sigma(x) &:= \int_\Gamma E_k(x,y)\sigma(y)ds_y\\
K_k\mu(x) &:= \int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)\mu(y)ds_y\\
K'_k\sigma(x) &:= \int_\Gamma \frac{\partial E_k}{\partial n_x}(x,y)\sigma(y)ds_y\\
D_k\mu(x) &:= -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)\mu(y)ds_y
\end{aligned}$$

From Eq.(3.3) we get:

$$V_k\sigma(x) = V\sigma(x) + \frac{ik}{4\pi}\int_\Gamma \sigma(y)ds_y + S_k\sigma(x) \tag{3.12}$$

where

$$S_k\sigma(x) = \frac{1}{4\pi}\int_\Gamma \{\sum_{m=2}^{\infty}(\frac{(ikr)^m}{m!})\}\frac{1}{r}\sigma(y)ds_y \tag{3.13}$$

From Eq.(3.4) we get:

$$K_k\mu(x) = K\mu(x) + R_k\mu(x) \tag{3.14}$$

Then

$$R_k\mu(x) = \frac{1}{4\pi}\int_\Gamma \{\sum_{m=2}^{\infty}(\frac{m-1}{m!})(ikr)^m\}\frac{1}{r^2}\mu(y)ds_y \tag{3.15}$$

In terms of the operator $K'_k\sigma(x)$, the **jump conditions** for the normal derivatives of a single layer distribution can be written:

$$\begin{aligned}
K'_k\sigma(x) &= \int_\Gamma \frac{\partial E_k}{\partial n_x}(x,y)\sigma(y)ds_y\\
&= K'\sigma(x) + R'_k\sigma(x)
\end{aligned} \tag{3.16}$$

where

$$R'_k\sigma(x) = \frac{1}{4\pi}\int_\Gamma \{\sum_{m=2}^{\infty}(\frac{m-1}{m!})(ikr)^m\}\frac{1}{r^2}\sigma(y)ds_y \tag{3.17}$$

In terms of the operator $D_k\mu(x)$, the **hypersingular integral operator** denote the normal derivative of a double layer distribution, that is:

$$\begin{aligned} D_k\mu(x) &= -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial E_k}{\partial n_y}(x,y)\mu(y)ds_y \\ &= D\mu(x) + \frac{\partial}{\partial n_x}R_k\mu(x) \end{aligned} \tag{3.18}$$

Then

$$\frac{\partial}{\partial n_x}R_k\mu(x) = \frac{1}{4\pi}\int_\Gamma\{\sum_{m=2}^{\infty}\frac{(m-1)(m-2)}{m!}(ikr)^m\}\frac{1}{r^3}\mu(y)ds_y. \tag{3.19}$$

Note that S_k and R_k have bounded kernels for all Γ. The matrices of boundary integral operators (BIO)

$$C = \begin{pmatrix} \frac{1}{2}I - K_k & V_k \\ D_k & \frac{1}{2}I + K_k' \end{pmatrix} \tag{3.20}$$

are referred to as the Calderon projectors with respect to the domain Ω. The Calderon projector maps the Cauchy data into itself. We note that the solution u in the domain Ω is completely determined from the representation (3.1), provided one knows its Cauchy data on the boundary Γ. In the classical Holder function space, the boundary integral operator (BIO) in (3.11) have the mapping properties as follows.

Theorem 3.2.2. *Let $\Gamma \in C^2$ and $0 < \alpha < 1$, a fixed constant. Then the boundary integral operators (BIO) in (3.20). define continuous mappings in the following spaces,*

$$\begin{aligned} &V_k : C^a(\Gamma) \longrightarrow C^{1+a}(\Gamma), \\ &K_k : C^{1+a}(\Gamma) \longrightarrow C^{2+a}(\Gamma), \\ &K_k' : C^a(\Gamma) \longrightarrow C^{1+a}(\Gamma), \\ &D_k : C^{1+a}(\Gamma) \longrightarrow C^a(\Gamma). \end{aligned}$$

*Similar mapping properties are also available in the Sobolev space **[14]***

Theorem 3.2.3. *For $\Gamma \in C^2$, the following operators are continuous for $|s| \le 1/2$:*

$$\begin{aligned} &V_k : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \\ &K_k : H^{1/2+s}(\Gamma) \longrightarrow H^{3/2+s}(\Gamma), \\ &K_k' : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \\ &D_k : H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma). \end{aligned}$$

We remark that for smooth boundary, $\Gamma \in C^\infty$,the above theorem remains valid for $s \in \mathbb{R}$, while for the Lipschitz domain, $\Gamma \in C^{0,1}$ again $|s| \le 1/2$, but K_k and K_k' are only continuous as the mappings $K_k : H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma)$ and $K_k' : H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma)$
*The proof of this theorem appear on **[18]**.*

3.3 Exceptional Or Irregular Frequencies

From (3.11) we see that the Cauchy data of a solution of (1.22) in Γ is related to each other by two boundary integral equations (BIE). As is well known,for regular elliptic boundary value problems (BVP) only half of the Cauchy data on Γ is given. For the remaining part, the two equations from (3.11) define an over determined system of boundary integral equations (BIE) which may be used for determining the complete Cauchy data. In general,any combination of them can serve as a boundary integral equation (BIE) for the missing part of the Cauchy data. Hence the reduction from a boundary value problem (BVP) to a boundary integral equation (BIE) is by no means a unique process. However, the so-called direct approach for formulating boundary integral equations (BIE) becomes rather simple, if one considers the Dirichlet problem. The boundary integral equation (BIE) in (3.11) can be employed for the interior Dirichlet problems,

$$\mu = \varphi = u|_\Gamma \qquad on \;\; \Gamma \qquad \text{is given} \qquad .$$

Here the missing Cauchy data on Γ is $\sigma = \dfrac{\partial u}{\partial n}|_\Gamma$. Thus, for instance, we may use either the boundary integral equation (BIE) of the first kind

$$V_k\sigma(x) = \frac{1}{2}\varphi(x) + K_k\varphi(x) \quad , \qquad x \in \Gamma \tag{3.21}$$

Or boundary integral equation (BIE) of the second kind

$$D_k\varphi(x) = (\frac{1}{2} - K'_k)\sigma(x) \quad , \qquad x \in \Gamma \tag{3.22}$$

The unique solvability for these boundary integral equations (BIE) is important. In particular, for $k \neq 0$ and for given $\varphi \in C^{1+a}(\Gamma)$, (3.21) is uniquely solvable with $\sigma \in C^a(\Gamma)$, except for certain values of $k \in \mathbb{C}$ which are the so-called exceptional or irregular frequencies of the boundary integral operator (BIO) V_k. For any irregular frequency k_0, the operator V_{k_0} has a non-trivial null space $kerV_{k_0} = span\{\sigma_{0j}\}$. The eigensolutions σ_{0j} are related to the eigensolutions u_{0j} of the interior Dirichlet problem for the Laplacian. Hence the eigensolution u_0 for boundary value problem (BVP) and exceptional value k_0 is:

$$\begin{aligned} -\triangle u_0 &= k_0^2 u_0 \qquad in \;\; \Omega \\ u_0|_\Gamma &= 0 \qquad on \;\; \Gamma \end{aligned} \tag{3.23}$$

And the eigensolution σ_0 for boundary integral equation (BIE) is:

$$\sigma_{0j} = \frac{\partial u_{oj}}{\partial n}|_{\Gamma}.$$

Moreover, the solutions are real-valued and

$$dimkerV_{k_0} = \text{dimension of the eigenspace of (3.23).}$$

As is known, the eigenvalue problem (3.23) admits denumerable infinitely many eigenvalue k_{0l}^2. They are all real and have at most finite multiplicity. More over, they can be ordered according to size $0 < k_{01}^2 < k_{02}^2 <$ .... and have $+\infty$ as their only limit point. When k_0 is an eigenvalue. (3.21) admits solutions in $C^a(\Gamma)$ if and only if the given boundary values $\varphi \in C^{1+a}(\Gamma)$ satisfy the orthogonality conditions.The solvability condition for given φ is:

$$\int_\Gamma \varphi\sigma_0 ds = \int_\Gamma \varphi\frac{\partial u_0}{\partial n} ds = 0 \qquad \forall\sigma_0 \in kerV_{k_0}, \tag{3.24}$$

Correspondingly, for $\varphi \in C^{1+a}(\Gamma)$, the boundary integral (3.24) has solution $\sigma \in C^a(\Gamma)$ if and only if (3.24) is satisfied. The relation between the eigensolutions of the boundary integral equations (BIE) and the interior egienvalue problem of the Laplacian are given explicitly in (3.23).

3.4 Low Frequency Behavior

Of particular interest is the case $k \longmapsto 0$ which corresponds to low frequency behavior. This case also determine the large time behavior of the solution to time dependent problem if (1.22) is obtained from the wave equation by the Fourier Laplace transformation. As will be seen, the interior Dirichlet boundary value problems will exhibit a singular behavior for $k \longmapsto 0$. The singular behavior can be obtained from the explicit asymptotic expansions of the boundary integral equations (BIE) in (3.21) and (3.22). The latter then follows directly from the series development of the fundamental solution and their derivatives. To illustrate the idea, let us consider the fundamental solution $E_k(x, y)$ in (3.2) we see that for small k.

$$E_k(x, y) = E(x, y) + \frac{ik}{4\pi} + S_k(x, y), \tag{3.25}$$

Here

$$E(x, y) = \frac{1}{4\pi r}$$

denotes the fundamental solution for the 3-dimensional Laplace equation,

and

$$S_k(x,y) = \frac{1}{4\pi r}\sum_{m=2}^{\infty}\frac{(ikr)^m}{m!} \tag{3.26}$$

Correspondingly, for the double layer potential kernel we obtain

$$\frac{\partial}{\partial n_y}E_k(x,y) = \frac{\partial}{\partial n_y}E(x,y) + R_k(x,y),$$

Where

$$R_k(x,y) = -\{\sum_{m=2}^{\infty}(\frac{m-1}{m!})(ikr)^m\}\frac{\partial}{\partial n_y}E(x,y) \tag{3.27}$$

Similarly, for the hypersingular kernel we obtain

$$\frac{\partial R_k}{\partial n_x}(x,y) = \frac{1}{4\pi r^3}\sum_{m=2}^{\infty}\frac{(m-1)(m-2)}{m!}(ikr)^m \tag{3.28}$$

The term k appears in (3.25) explicitly which shows that V_k is a singular perturbation of V (the corresponding boundary integral operator (BIO) V_k with E_k replaced by E). We have shown that the other operators are regular perturbation of the corresponding operators of the Laplacian as $k \longmapsto 0$. Let us consider first, for interior Dirichlet problem in (3.22),

$$D_k\varphi = (\frac{1}{2} - K' - R'_k)\sigma \quad on \quad \Gamma \tag{3.29}$$

for $\varphi \,\varepsilon\, C^{1+a}(\Gamma)$, the equation

$$D\varphi = (\frac{1}{2}I - K')\tilde{\sigma}$$

has a unique solution $\tilde{\sigma} \in C^a(\Gamma)$,we may rewrite (3.29) as

$$\begin{aligned} \sigma &= (\frac{1}{2}I - K')^{-1} R'_k\sigma + (\frac{1}{2} - K')^{-1} D_k\varphi \\ &= \tilde{\sigma} + (\frac{1}{2}I - K')^{-1} R'_k\sigma + (\frac{1}{2}I - K')^{-1}\frac{\partial}{\partial n_x}R_k\varphi \\ &= \tilde{\sigma} + O(k^2) \end{aligned} \tag{3.30}$$

Where the last expressions can be obtained from the expansions (3.27) and (3.28) in [18]. The analysis for the integral equation of the first kind for interior Dirichlet problem in (3.21) is more involved,

$$V\sigma + \omega + S_k\sigma = \frac{1}{2}\varphi + K\varphi + R_k\varphi \tag{3.31}$$

Where

$$\omega = \frac{ik}{4\pi}\int_{\Gamma}\sigma ds \tag{3.32}$$

Where the kernels of the integral operators S_k and R_k are given by (3.26) and (3.27) respectively. Both are of order $O(k^2)$. Hence,(3.31) is a regular perturbation of the interior Dirichlet problem of the Laplacian equation. Similar to (3.30), we seek the solution of (3.31) and (3.32) in the asymptotic form,

$$\begin{aligned} \sigma &= \tilde{\sigma} + \alpha_1(k)\tilde{\sigma_1} + \sigma_R \\ \omega &= \tilde{\omega} + \alpha_1(k)\tilde{\omega_1} + \omega_R \end{aligned} \tag{3.33}$$

Where $\tilde{\sigma}$, $\tilde{\omega}$ correspond to the solution of the interior Dirichlet problem for the Laplacian equation. If we insert (3.33) with $\alpha_1(k) = k$ then the function $\tilde{\sigma}$ is given by the solution of

$$V\tilde{\sigma} = \frac{1}{2}\varphi + K\varphi,$$

Which corresponds to the interior Dirichlet problem of the Laplacian. Hence,

$$\int_{\Gamma}\tilde{\sigma}ds = 0 \qquad and \qquad \tilde{\sigma_1} \equiv 0$$

Since it is the solution of

$$V\tilde{\sigma_1} + \frac{i}{4\pi}\int_{\Gamma}\tilde{\sigma}ds = 0$$

Therefore, the solution of (3.31) is of the form $\sigma = \tilde{\sigma} + O(k^2)$. Which was obtain with the integral equation of the second kind. With the solution of the boundary integral equations (BIE) available, we now summarize the asymptotic behavior of the solution of the boundary value problems (BVP) for small k by substituting the boundary densities into the representation (3.1). Hence, we arrive at the following asymptotic expression

$$u(x) = V\tilde{\sigma}(x) - W\tilde{\mu}(x) + C(x;k) + R(x;k) \tag{3.34}$$

Where as the missing densities are the solutions of the corresponding boundary integral equations (BIE) presented above. In (3.34), $C(x;k)$ denotes the lowest order terms of the perturbations in Ω, where as $R(x;k)$ denotes the remaining boundary potentials. The behaver of C and R for $k \longmapsto 0$ are $C(x;k) = 0$ and $R(x;k) = O(k^2)$.

Chapter 4

Discussion, Conclusion and Recommendation

4.1 Discussion

Acoustics was originally the study of small pressure wave in air which can be detected by the human ear and sound. Low frequency acoustics is a radio frequencies spanning from 30kHz to 300kHz correspond to wave length of 10km to 1km respectively. Boundary integral equation (BIE) are a classical tool for the analysis of boundary value problem (BVP)for partial differential equation (PDE).

4.2 Conclusion

In this thesis, we have developed and analyzed the behavior of the solution to the boundary value problem (BVP) for low frequency acoustic in $\mathbb{R}^3$. It was showed that reduction of boundary value problem(BVP) to boundary integral equation (BIE) by using Green representation. It also showed that the expansion for the solution always the corresponding potential for the Laplacian. Finally, we found that the unique solvability for the corresponding boundary integral equation (BIE) and its relation to the interior eigenvalue problem of low frequency.

4.3 Recommendation

In this thesis we avoid resonance state by using boundary integral method on low frequency acoustics using constant wave number k. So, this thesis can be further extended by introducing with the variable rather than taking constant wave number k.

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