ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

SCHOOL OF APPLIED NATURAL SCIENCE

PROGRAM OF APPLIED MATHEMATICS

ANALYSIS OF MULTISTEP METHODS FOR NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH APPLICATIONS

A thesis submitted in partial fulfillment of the requirements for the degree of master’s.

By Teha Mohammed ID NO GSU/0446/06

Advisor Dr.Tekle Gemechu (PhD)

January 2018

Adama Ethiopia

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Approval Sheet

This is to certify that the thesis prepared by Teha Mohammed entitled “Analysis of multistep methods for numerical solution of ordinary differential equations with applications” submitted in partial fulfillment of the requirement for the degree of Master’s of science in Applied Mathematics with the regulation of the University and meets the accepted standards with respect to originality.

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Abstract

This thesis provides a practical overview of multistep methods for numerical solution of ordinary differential equations. A Multistep method is used for numerical solution of some ordinary differential equations. The approach is to obtain Multiple Finite Difference Methods (MFDMs) which are combined as simultaneous numerical integrators to form some block methods where, the Stability and Convergence of the block methods are investigated. We also compare their performance with some single steps. The study shows that the methods are zero-stable, consistent and convergent. These are proven by using related theorems such as stone-weierstrass theorem. The study also enables us to investigate the method with larger stability region. The block methods derived are tested to illustrate the accuracy and efficiency of the method.

Acknowledgements

The completion of this study would have not been realized without the heip of others and I would like to take this opportunity to thank everyone who helped me with this thesis. First I would like to express my thanks and appreciation to my advisor Dr. Tekle Gemechu for his guidance ,continual advice ,patience ,vast knowledge and critical review of the thesis. I would like also to thank my family members for their periodic contribution for my work.

Teha Mohammed
January 2018

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ABBREVIATIONS AND ACRONYMS

ODE..Ordinary Differential Equation

IVPs ………………………………………………….. Initial Value Problems

MFDMs ……………………………………………… *Multiple Finite Difference Methods*

SYMBOLS AND ACRONOMY

i…………………………………………………………..index

ϕ ……………………………………………………………phi variant

ψ ……………………………………………………………psi

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CHAPTER ONE

1. Introduction

1.1 BACKGROUND OF THE STUDY

A great many physical problems give rise to differential equations. Traditionally, solutions to these differential equations can be obtained using analytical methods. However, solutions to certain differential equations are very difficult by any means other than an approximate solution by the application of numerical methods. These methods are classified into two, One-step and Multistep methods.

For the first order ordinary differential equation of the form Y'=f(t,y) and

the second order ordinary differential equation of the form Y''=f(t,y,y')

where f is a continuous function, the second order ordinary differential equation is conventionally solved by first reducing it to a system of first order differential equations and then applying the various methods available for solving systems of first order Initial Value Problems (IVPs).

This approach was extensively discussed in the literature and a few notable ones are Lambert (Lambert, J. D ,1973,1976,1991),(Fatunla, S.O,1982,1984,1985,1986), (Jennings ,1987) and (Jator, S. N. 2007). Although there has been tremendous success with this approach, it has certain draw backs. For instance, computer programs associated with the methods are often complicated especially when incorporating subroutines to supply the starting values for the methods resulting in longer computer time and more computational work. In addition (Vigo-Aguiar, J. and Ramos, H. 2003,2006) stated that these methods do not utilize additional information associated with a specific ordinary differential equation, such as the oscillatory nature of the solution.

A lot of efforts have been devoted to the development of various methods for solving

y′=f(t,y). To mention but few are (Twizell, E. H. and Khaliq, A. Q. M.1984, Yusuph, Y. and Onumanyi, P.2005,(Simos, T. E. 2002, Fatunla, S.O,1982,1984,1985,1986,1987), (Henrici, P. 1962) and (Lambert, J. D ,1973,1976,1991). Several methods have also been proposed in the literature for solving y′=f(t,y).. For instance, (Hairer, E.and Wanner, G. 1976) proposed Nystrom type methods and stated order conditions for determining the parameters of the methods. Multi-step Methods have been considered by (Vigo-Aguiar, J. and Ramos, H. 2003,2006) and (Awoyemi, D. O. (1996,199,2001). In (Awoyemi, D. O. 1996) multi-steps were proposed and implemented in a predictor-corrector scheme using the Taylor series algorithm to supply the starting values. Although, the implementation of the methods yielded good accuracy but the procedure is more costly to implement.

In this thesis, we propose multi-step collocation for numerical solution of initial value problems of first order ordinary differential equations and, we also show that this method is zero-stable and consistent and hence convergent.

1.2 Statement of the problem

The study of numerical methods for solving ODE was considered in most of the literature reviews .For example in (Awoyemi, D. O. 1999) a class of continuous methods for general first order initial value problems of ordinary differential equations was discussed. In (Awoyemi, D. O. and Kayode, S. J. 2005b) an Implicit Collocation Method for numerical Solution of First order Ordinary Differential Equations was presented, In (Brown, R.L.1974),Some characteristics of implicitly multistep multi-derivative integration formulas was discussed. But most of the reviews lack clear discussion of stability; convergence determination of stability region of multistep method and the accuracy of the solutions of some ordinary differential equations.

In this study we intended to answer the following.

1. What determines the amount of numerical error in an approximation?
2. How is the accuracy of multi-step method?
3. Are some deferential equations more difficult to approximate numerically than others? If So, can this be predicted without doing numerical experiments?
4. Which scheme has less truncation error in solving the problems?
5. What is the stability region of the methods?

1.3 Objective of the study

1.3.1 1.3.1. General objective

The general objective of this thesis is to analysis of multistep methods for solving some ordinary differential equations.

1.3.2 Specific objective

This research work will have the following specific objectives to

- study what determines the amount of numerical error.
- study the accuracy of multi-step methods.
- determine which scheme is having less truncation error in solving the problems
- Numerical computations for initial value problem of first order ODEs
- compare multi step methods one with the other
- determine the stability and the convergence of multistep method.

1.4 Significance of the study

The main purpose of this study are :-

1. Application of multistep methods to solve ordinary differential equations.
2. Progress in the study of the research area.
3. provide to the researcher's own contribution to all natural science students especially for (mathematics and physics) researchers (for post graduate students) who may use as additional reference on the case (i.e. application of multi-step methods on initial value problems of first order ODE) for similar works in the university.

1.5 Delimitation/scope/ of the study

This study is mainly delimitated on stability analysis and accuracy of multistep method

1.6 1.6. limitation of the study

The limitation of the study are due to ;-

1. Time constraint/shortage of time.

2. Shortage of materials.

1.7.Organization of the thesis

This thesis is organized in five chapters. In first chapter background of the study is presented ; In chapter two the related literature of the thesis is presented; In chapter three the methodology to organize this study is presented; In chapter four the main body of the thesis is discussed briefly with examples and figure presentation and chapter five summarizes the result of the study and recommendation.

2.Review of literature

In this chapter We trie to present some related literatures from different sources such as published journals and some reference books which are cited by number according to the lists on the last page. The reviews of related literature presented in this chapter focus on multistep methods for solving problems of ODEs.

2.1. Ordinary differential equations (ODEs)

An ordinary differential equation is an equation that contains an independent variable, dependent variable and the derivatives of dependent variable. Or a relation between a function, its derivatives and the variable up on which they depend.(Awoyemi, D. O. and Kayode, S. J. 2005b) First order ordinary differential equations may be written in the form

$$y'(x)=f(x,y(x)) \tag{2.1}$$

$$y(x_o) = y_o \tag{2.2}$$

where $y`(x) = \frac{dy}{dx}$, x is an independent variable and y is dependent variable. A function y(x) is called a solution of equation (2.1) and the initial value (2.2) is given. A second order ordinary differential equation for y is, under mild assumption for equation (2.1) together with equ (2.2) in the form;

$$y'' = f(x,y,y') \tag{2.3}$$

With two free parameters which represents two uniquely determined initial values

$$y(x_o) = y_o$$
$$y'(x_o) = y'_o$$

Generally an order n ordinary differential equation in x with $y^{(n)}$ has the **explicit** form

$$y^{(n)} = f(x,y,y` ,y^{(n)}) \tag{2.4}$$

There is a unique solution with n initial values;

$$Y(x_o) = y_o$$
$$Y`(x_o) = y`_o$$
$$. \qquad .$$

2.2.Multistep methods

In Runge-Kutta methods, the solution at Y_{n+1} is based on the solution at Y_n. That is Y_{n+1} is calculated from Y_n.

$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 4k_2 + k_3)$ Ki, i=1,2,3

In multi-step methods, Y_{n+1} is calculated from the solutions Yn, Y_{n-1}, etc. Multi-step methods are more difficult to program, but they are more efficient than the Runge-Kutta methods.(Fatunla, S. O. 1988):. To present the multi-step methods, we need the following notation:

$f_k = f(t_k,y_k)$, for any integer k.

Many physical problems give rise to differential equations. Traditionally, solutions to these differential equations can be obtained using analytical methods. However, solutions to certain differential equations are very difficult by any means other than an approximate solution by the application of numerical methods. These methods are classified into two thus: One-step and Multistep methods(Adeniyi, R. B. and Alabi, M. O. 2006)

For instance Runge-Kutta methods are a one-step method because they require only the value at one mesh point X_{n-1} to compute the value of the approximate solution at the next X_n. Notice, however, that the Runge-Kutta method does invoke a series of intermediate values during the computation. The idea of a multistep method is to use previously calculated values at a number of mesh points to aid the computation for later points much as those intermediate values used in a Runge-Kutta method(Taiwo, O.A. and Oladele, R.O. 1996).

Consider the initial value problem

$dy/dx = f(x,y)$, $y(a)=y_0$ where $x,y,f \in R$.

A method that makes use of the values of the dependent variable y(x) and its derivative $f'(x)=f(x,y)$ at k different mesh points X_{n-1} , X_{n-2} , ... , X_{n-k} is called a multistep or a k-step method. More precisely, after approximations at X_{n-k} , ... , X_{n-1} have been determined, we have values $Y_{n-k},...,Y_{n-1}$ and hy'_{n-k} , ... , $h_{Y'n-1}$ available. We want to use this information to determined Y_n and hY'_n at X_n. We define Yn := $[Y_n,Y_{n-1},...,Y_{n-k+1} ; hY'_n,Y'_{n-1},...,hY'_{n-k+1}]^T$.

The objective of a multistep method is to find a numerical approximation for Y_n from Y_{n-1} and to repeat this process.

where f is a continuous function, is conventionally solved by first reducing it to a system of first order differential equations and then applying the various methods available for solving systems of first order Initial Value Problems (IVPs).

This approach is extensively discussed in the literature and a few notable ones are Lambert (Lambert, J. D ,1973,1976,1991),(Fatunla, S.O,1982,1984,1985,1986), (Jennings ,1987) and (Jator, S. N. 2007):. Although there has been tremendous success with this approach, it has certain draw backs. For instance, computer programs associated with the methods are often complicated especially when incorporating subroutines to supply the starting values for the methods resulting in longer computer time and more computational work. In addition (Vigo-Aguiar, J. and Ramos, H. 2003,2006) stated that these methods do not utilize additional information associated with a specific ordinary differential equation, such as the oscillatory nature of the solution.

A lot of efforts have been devoted to the development of various methods for solving $y'=f(t,y)$ To mention but few are (Twizell, E. H. and Khaliq, A. Q. M.1984), (Yusuph, Y. and Onumanyi, P.2005), (Simos, T. E. 2002),(Fatunla, S.O,1982,1984,1985,1986,1987), (Henrici, P. 1962) and (Lambert, J. D ,1973,1976,1991). Several methods have also been proposed in the literature for solving (1.1). For instance, (Hairer, E.and Wanner, G. 1976) proposed Nystrom type methods and stated order conditions for determining the parameters of the methods. Multi-step methods have been considered by (Vigo-Aguiar, J. and Ramos, H. 2003,2006) and (Awoyemi, D. O. (1996,199,2001). In (Awoyemi, D. O. 1996) multi-steps were proposed and implemented in a predictor-corrector scheme using the Taylor series algorithm to supply the starting values. Although, the implementation of the methods yielded good accuracy but the procedure is more costly to implement.

In this thesis, we propose multi-step for numerical solution of initial value problems of first order ordinary differential equations and, We also show that this method is zero-stable and consistent, and hence convergent.

As opposed to one-step methods, which only utilize one previous value of the numerical solution to approximate the subsequent value, (Taiwo, O.A. and Oladele, R.O. (1996) multistep methods approximate numerical values of the solution by referring to more than one previous value. Accordingly, multistep methods may often achieve greater accuracy than one-step methods that use the same number of function evaluations, since they utilize more information about the known portion of the solution than one-step methods do. A special category of multistep methods are the multi-step methods, where the numerical solution to the ODE at a specific location is expressed as a linear combination of the numerical solution's values and the function's values at previous points (Sirisena, U. W. and Onumanyi, P. and Chollon, J.P.2001). For the standard system of ODEs, y′ = f(t,y), a multistep method with k-steps would have the form:

$$\boldsymbol{y_n = -\sum_{j=1}^{k} \alpha_j y_{n-j} + h \sum_{j=0}^{k} \beta_j f_{n-j}} \qquad (2.5)$$

where αj,βj are constants, y_n is the numerical solution at $t = t_0$, and $f_n = f(t_n,y_n)$. For the rest of this discussion, we will make the assumption that f is differentiable as many times as needed, and we will consider the scalar ODE y′ = f(t,y) for simplicity in notation. The generalization to systems of ODEs is presented(Twizell, E. H. and Khaliq, A. Q. M.1984):. It is important to note that in the above expression, all of the previous integration steps are assumed to be equally spaced, although it is possible to generalize these schemes to have variable step-sizes. Also, note that if $\beta_0 = 0$, the scheme is explicit (because it does not depend on f_n), and otherwise the scheme is implicit. We are now ready to examine some of the most popular multistep methods**.**

2.3 Stiff differential equation..

A differential equation of the form y` = f(t; y) is said to be stiff if its exact solution y(t) includes a term that decays exponentially to zero as t increases, but whose derivatives are much greater in magnitude than the term itself. An example of such a term is e^{-ct}, where c is a large, positive constant, because its k^{th} derivative is $c^k e^{-ct}$. Because of the factor of c^k, this derivative decays to zero much more slowly than e^{-ct} as t increases. Because the error includes a term of this form, evaluated at a time less than t, the error can be quite large if h is not chosen sufficiently small to o_set this large derivative. Furthermore, the larger c is, the smaller h must be to maintain accuracy(Lambert, J. D. 1973).

Non-stiff initial value ordinary differential equations are problems for which all of the components evolve simultaneously on comparable time-scales. Non-stiff problems are often solved by using explicit methods usually with some error control (Jim Lambers, 2010).

The Runge-Kutta method is most accurate and stable method. Fourth Order Runge-Kutta method intends to increase accuracy to get better approximated solution. This means that the aim of this method is to achieve higher accuracy and to find explicit method of higher order. Euler method is the first order accurate; in addition it requires only a single evaluation of f (x_n, y_n) to obtain y_{n+1} from y_n. In contrast, Runge-Kutta method has higher accuracy. It re-evaluates the function *f* at two consecutive points (x_n, y_n) and (x_{n+1}, y_{n+1}). It requires four evaluations per step. Due to this, Runge-Kutta method is quite accurate, and it has faster rates of convergence (Simruy

Hürol,2013). Runge-Kutta method has the same property of stability. Order of a method assures the accuracy of the method. For RK4 method, the local truncation error is appeared $O(h^5)$. This means that the order of RK4 method is four. In case that RK4 is consistent and zero stable, RK4 converges to analytical solution. Euler's method is the simplest of all linear single-step method to obtain the approximated solution of the specified initial value problem. But it generates large error in each successive step during the computation which is the accumulated error. In order to avoid the formation of larger error, step-size should be taken excessively small. Therefore; it needs high computation of time. Additively, approximated solution converges slower to analytical solution. This means that the order of method is 1 and the error is observable $O(h^2)$. It is a slow rate of convergence (Van der Houwen, P. J. 1979). The approximated solution converges faster to exact solution and the order of RK4 is 4 and the truncation error is $O(h5)$. Method is re-evaluating the function f at each time to obtain the predictable solution. It requires four evaluations per step. So, the computation of function may take long time. The Euler's method excessively small step size converges to analytical solution. Therefore, large number of computation is needed. In contrast, Runge- Kutta method gives better results and it converge faster to analytical solution and has less iteration to get accuracy solution. Explicit Fourth Order Runge-Kutta method is more accurate than the Explicit Euler method (Simruy Hürol,2013).The stability region of explicit Runge-kutta methods is a bounded region s in the complex plane, and the step-size h must be chosen such that if the vector $h\lambda$ is in the left half of the complex plane, then it lies in s (Justin Steven Calder Prentice, 2011).

2.3.1 Truncation (or discretization) Error:

It is caused when approximations are used to estimate some quantity. Truncation error is composed of two parts , local and global truncation errors.

Truncation error is defined by

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n); h)\ .$$

(a) Local Truncation Error: It is defined by T_{n+p}, and introduces the local error at x_{n+p}.

It is shown as;

$$T_{n+p} = y(x_{n+p}) - y(x_n) - h\Phi(x_n, y(x_n); h)$$

Local Truncation error arises when a numerical method is used to solve initial value problem. The error occurs after the first step and form in` each step.

$$T_{n+1} = y(x_{n+1}) - y_{n+1}$$

To obtain the local truncation error, take difference between left hand side and right hand side of method, and expand by using Taylor series. The remaining term is called Local Truncation error. Finally, divide the result to step size h. It is shown as;

$$\tau_n(y) = \frac{1}{h} T_n$$

If $y(x)$ is assumed to be continuously differentiable, then the local truncation error for both explicit and implicit method can be written in the form,

$$y(x_{n+p}) - y_{n+p} = C_{p+1} h^{p+1} y_{p+1}(x_n) + O(h^{p+2})$$

$$T_{n+p} = C_{p+1} h^{p+1} y_{p+1}(x_n) + O(h^{p+2})$$

where C^{p+1} states the error constant, p indicates the order of the method, and $C_{p+1} hy^{p+1}(x)$ is the principal local truncation error. Thus, if the local truncation error is $O(h^{p+1})$, we can say that p represent the order of the method. This means that $|Tn+p| < Cp+1h^{p+1}$. If p is larger, then the method is more accurate. Generally the error that is committed at a particular step is called the local truncation error[20].

(b) Global (or Accumulated) Truncation Error: It is denoted by e_n which is expressed as $e_n = y(x_n) - y_n$ where $y(x_n)$ is exact solution and y_n is approximate solution. Global Truncation error is caused by the accumulation of the local error in all of the iterations. The combined effect of all the local errors is called the global error.

We define the global error, e_n, by $e_n = y(x_n) \quad y_n$.

2.3.2. Round-off Error:

It originates due to the operations of computer that takes limited additional number of digit. After calculating the approximations of methods, the result is dropped in specific location. It is denoted by R_{n+k} which is also committed at the n^{th} application of the methods.

Chapter 3

3.Methods and procedures

In this part the researcher tries to consider the study design, period and source of information, study procedure, instrumentation and administration.

3.1 Design

Documentary and experimental design

3.2 Source of information

The researcher used previous works on similar study and used different reference books, down loaded documents from internet about stability and convergence of multistep methods on initial value problems of ODE

3.3 Instrumentation

To get the numerical solution of initial value problems of first order ODE ,the researcher shows how to use the multistep methods . Also use a table to compare and contrast the exact and approximate solution.

3.4 Data Collection

Here the researcher plans to collect the necessary data (information) from different references;

- Articles /jornals/ and Research papers
- Reference books
- Download materials from internet etc. on his study.

3.5 Procedure of Data Analysis

After the collection of necessary information the researcher used the procedures to analysis.

1. Problem description
2. Discretization
3. Define and derive the formulae of multi-step methods.
4. Defining the necessary terms like consistence, stability and convergence.
5. Discuss the stability region of Adam Bashforth and Adam Mouton methods
6. Discussing the schemes from tables and figures.
7. Numerical experiments by tables with different size-step.
8. Summarize the result in conclusion.

CHAPTER FOUR

4 Result and Discussions

Before we begin with multi step methods, let us present some single step methods for comparisons.

4.1 .Third-Order Improved Runge-Kutta Method (IRK3)

To present multistep methods let us first present some single step methods.Explicit third-order improved Runge-Kutta (IRK) methods is the method used in two and three stage which indicated as the required number of function evaluations per step. The third-order Runge kutta method in two-stage has a lower number of function evaluations than the classical third-order RK method while maintaining the same order of local accuracy. In three-stages, the new method is more accurate compared to the classical third-order RK method. The stability region of methods is given and numerical examples are presented to illustrate the efficiency and accuracy of the new methods.

Consider the numerical solution of the initial value problem for the system of ordinary differential equation.

$$y'(x) = f(x, y(x)), \qquad x \in [x_o, x],$$
$$y(x_o) = y_o \qquad (4.1)$$

Butcher table for Runge-kutta methods

0	0		
C_2	a_{21}	0	
C_3	a_{31}	a_{32}	0
	b_1	b_2	b_3

In two-stages (s=2), the general form of IRK3 can be written as:

$$y_{n+1} = y_n + h(b_1k_1 - b_{-1}k_{-1} - b_{2}(k_2\, k_{-2}))$$

$$k_1 = f\,(x\,,y)$$

$$k_{-1} = f\,(x_{-1},\, y_{-1})$$

$$k_2 = f\,(x_n + c_2h\,, y_n + ha_{21}\, k_1)$$

$$k_{-2} = f(\, x_{n-1} + c_2h,_{yn-1} + ha_{21}k_{-1}) \qquad (4.2)$$

where $0 \leq c_2 \leq 1$. In the derivation of the method we will use

$$c_i = \Sigma\sum_{j}^{i-1} aij$$

Which is called the row sum condition of RK method, so here we have $c_2 = a_{21}$.

0			
1/2	1/2		
1	-1	2	
	1/6	4/6	1/6

The coefficient tables of special Runge-Kutta methods of order 3

$$y_{n+1} = yn + \frac{1}{6}h(k_1 + 4k_2 + k_3)$$

The aim of Runge–Kutta methods is to eliminate the need for repeated differentiation of the differential equations.

Consider the ordinary differential equations (ODE): with initial condition

$$y' = f(x, y(x)),\ \ y(x_0) = y_0.$$

We can write

$$y(x_{n+1}) = y(x_n) + \int_{tn}^{xn+1} f\ (\tau, y(\tau))d\,\tau$$

The value of the integral is now approximated by a quadrature formula for a number of support abscissas and corresponding weights.

Here, we choose to take three support abscissas x_1, x_2 and x_3 of the interval $[x_n, x_{n+1}]$, and corresponding weights b_1, b_2, b_3.

For the approximation y_{n+1}, we obtain

$$y_{n+1} = y_n + h[b_1f(x_1, y(x_1)) + b_2f(x_2, y(x_2)) + b_3f(x_3, y(x_3))],$$

where $h = x_{n+1} - x_n$. However, we do not know the value of $y(x_i)$, and we need to approximate. Before we continue, let us consider the well known second order

Heun's method as an illustration :x

$$y_{n+1} = y_n + \frac{h}{2}\,[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n)]$$

Here, we only have used two support abscissas, and we can see a method based on the trapezoidal rule, where yn+1 is approximated with one Euler step from yn. Higher order Runge-Kutta methods are however more complicated, and take quite some work to derive.

Let us now continue with the three abscissas, and write $x_1 = t_n$, $x_2 = t_n + c_2h$, $x_x = t_n + c_3$

i.e with the choice of ξ_1 s.t. $y(\xi_1) = y(\xi_n)$ as approximated by y_n.

We further approximate

$$y(\xi_2) : y\star_2 = y_n + ha_{21}f(x_n, y_n)$$

$$y(\xi_3) : y\star 3 = y_n + ha_{31}f(t_n, y_n) + ha_{32}f(t_n + c2h, y\star 2)$$

Together, this gives us the method

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + c_2h, y_n + ha_{21}k_1)$$

$$k_3 = f(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))$$

$$y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3). \qquad (4.3)$$

Eight parameters: c_2, c_3, a_{21}, a_{31}, a_{32}, b_1, b_2, b_3.

We wish to determine c_2, c_3, a_{21}, a_{31}, a_{32}, b_1, b_2, b_3 in such a way that this method has an order as high as possible. At the minimum, the approximations y_2 and y_3 should be correct for the equation $y' = 1$, which gives us the conditions

$$c_2 = a_{21},\ c_3 = a_{31} + a_{32}.$$

The local discretization error of method (1) is given by

$$d_{n+1} = y(t_{n+1}) - y(x_n) - h(b_1k_1 + b_2k_2 + b_3k_3),$$

where k_i is like k_i with y_n replaced by $y(x_n)$.

The Runge-Kutta methods are so called one-step methods, with $y_{n+1} = \Phi(x_n, y_n, h)$, for some function Φ. From the error analysis of one-step methods done earlier, we know that with a local error of $O(h^{p+1})$ we have a global error of $O(h^p)$.

If the method is to be of order three, at least, then we must have

$$b_1 + b_2 + b_3 = 1, \qquad c_2b_2 + c_3b = \frac{1}{2}, \qquad c_2b_3a_{32} = 1/6, \qquad c_2^2b_2 + cc_3^2b3 = \frac{1}{3}$$

Four nonlinear equations, six unknowns: $b_1, b_2, b_3, c_2, c_3, a_{32}$.

The solution is not unique, and hence there are several third order schemes on this form. It would however not be possible to obtain fourth order: the coefficient of the h^4 term contains a term independent of the six parameters. One solution yields Heun's third order method,

$$k_1 = f(x_n, y_n),$$

$$k_2 = f(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1),$$

$$k_3 = f(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_2),$$

$$y_{n+1} = y_n + \frac{1}{4}h(k_1 + 3k_3) \qquad .(4.4)$$

and another Kutta's third order method,

$$k_1 = f(x_n, y_n),$$

$$k2 = f(x_n + \frac{1}{2}h, yn + \frac{1}{2}hk_1)$$

$$k_3 = f(x_n + h, y_n - hk_1 + 2hk_2)$$

$$y_{n+1} = yn + \frac{1}{6}h(k_1 + 4k_2 + k_3) \qquad (4.5)$$

4.2 Fourth-Order Runge–Kutta Method

Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y(x)), y_0 = y(x_0)$$

The fourth-order Runge–Kutta method is obtained from the Taylor series along the same lines as the second-order method. Since the derivation is rather long and not very instructive, we skip it. The final form of the integration formula again depends on the choice of the parameters; that is, there is no unique Runge–Kutta fourth order formula. The most popular version, which is known simply as the Runge–Kutta method, entails the following sequence of operations:

$$K_1 = hF(x, y)$$

$$K_2 = hF\_(x + \frac{h}{2}\, y + \frac{K_1}{2})$$

$$K_3 = hF(x + \frac{h}{2}, y + \frac{K_2}{2})$$

$$K_4 = hF(x + h, y + K_3)$$

$$y(x + h) = y(x) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \qquad .(4.6)$$

The coefficient tables of special Runge-Kutta methods of order 4 are:

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	2/6	2/6	1/6

The main drawback of this method is it does not lend itself to an estimate of the truncation error. Therefore, we must guess the integration step size h, or determine it by trial and error. The derivation of higher order explicit Runge-Kutta (ERK) methods requires a technique based on graph theory, which we will not cover. One could do this also for a four-stage method, i.e. start the derivation with four support abscissas. This time the maximal order is four, and there are several different fourth order methods. One is the classical fourth order Runge-Kutta method

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \quad .(4.7)$$

The maximum order *p* of a general Runge-Kutta method depends on the stages *r*:

stage *r*	1	2	3	4	5	6	7	8	9
Order *p*	1	2	3	4	4	5	6	6	7

4.3 Multistep methods for Ordinary Differential Equations.

Numerical methods are commonly used for solving mathematical problems in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically. Numerical methods, on the other hand, can give an approximate solution. In this chapter we present Adams-Bashforth and Adams-Moulton methods.

4.4 The Adams Family

4.4.1 The Adams-Bashforth [AB] Methods (Explicit Adams Methods)

The most widely utilized multistep methods used for nonstiff problems are the Adams-Bashforth methods, which are members of the Adams Family that are explicit. The spirit of the Adams-Bashforth technique is rooted in the StoneWeierstrass Theorem.

Theorem - Stone-Weierstrass Theorem - Let f(t) : R → C be continuous functionon t ∈ [a,b]. For all $\epsilon > 0, \exists$ a polynomial $\varphi(t) \ni \|f(t) - \varphi(t)\| < \epsilon$.

In other words, any continuous function can be approximated to an arbitrary accuracy by a polynomial; generally, the more demanding the accuracy of the approximation, the higher the order needed of such a polynomial. With the Stone-Weierstrass Theorem in mind, we start with the ODE in question:

$y' = f(t,y)$, and we integrate both sides to obtain:

$$y(t_n) = y(t_{n-1}) + \int_{t_{n-1}}^{t_n} f\big(t, f(t)\big)dt \quad (4.8)$$

If we could integrate f(t,y(t)) analytically, we (likely) would not need to resort to numerical methods to determine the solution to the ODE. If we cannot integrate f(t,y(t)) analytically, according to the Stone-

Weierstrass Theorem above, we can approximate it with arbitrary accuracy by a polynomial φ(t), and since all polynomials can be integrated analytically, we obtained, fair approximation of the solution to the ODE:

$$y(t_n) \approx y(t_{n-1}) + \int_{t_{n-1}}^{t_n} \emptyset(t)dt \tag{4.9}$$

Now to ensure that our approximation is reasonable, we insist that $\emptyset(t_{n-i}) = f(t_{n-i})$ for a reasonable number of integer values i. For example, setting $\emptyset(t_{n-1})$ to be the constant $f(t_{n-1})$ will result in the scheme:

$$y(t_n) \approx y(t_{n-1}) + ht_{n-1} \tag{4.10}$$

Note that this scheme, which is also known as the 1-step Adams-Bashforth Method, is simply the classic Forward Euler (FE) method, this scheme has $\alpha_1 = -1, \beta_0 = 0, \beta_1 = 1$ and $\beta_j, \alpha_j = 0$ for $j > 1$. Let us now construct the 2-step Adams-Bashforth scheme. We first need an interpolation polynomial φ(t) such that

$(t_n - i) = f(t_n - 1)$ for i = 1,2. The desired linear function is displayed below:

$$\boldsymbol{f(t, y) \approx \emptyset\varphi(t) = f(t_{n-2}) + \frac{f(t_{n-1}) - f(t_{n-2})}{t_{n-1} - t_{n-2}}(t - t_{n-2})} \tag{4.11}$$

Together with (3), we have that

$$y(t_n) \approx y(t_{n-1}) \left[f(t_{n-2)}\, t + \frac{f(t_{n-1}) - f(t_{n-2})}{t_{n-1} - t_{n-2}} \frac{(t - t_{n-2})^2}{2} \right]_{t_{n-1}}^{t_n} \tag{4.12}$$

$$y(t_n) \approx y(t_{n-1}) + h\left(\frac{3}{2} f(t_{n-1}) - \frac{1}{2} f(t_{n-2})\right) \tag{4.13}$$

So in accordance with the line above, we can define our 2-step Adams-Bashforth scheme to be:

$$y_n = y_{n-1} + h\left(\frac{3}{2} f_{n-1} - \frac{1}{2} f_{n-2}\right). \tag{4.14}$$

This can also be expressed in terms of (1) with $\alpha_1 = -1, \beta_0 = 0, \beta_1 = 3/2$,

$\beta_2 = -1/2$ and $\alpha_j = 0, \beta_{j+1}$ for all $j > 1$.

Continuing in this manner, we can construct k-step Adams-Bashforth methods by interpolating f through k previous points: $t = t_{n-1}, t_{n-2}, \ldots t_{n-k}$. Such a scheme could be derived by constructing a degree $\leq k-1$ polynomial φ(t) such that $\varphi(t_{n-i}) = f(t_{n-i})$ for i = 1,2,...k, and integrating it as in (4.9), then replacing $y(t_n), y(t_{n-1})$ and $f(t_{n-i})$ with y_n, y_{n-1} and f_{n-i} respectively. As shown below, the resultant k-step Adams-Bashforth method can be expressed in the form of (1) with $\alpha_1 = {}_{-1}, \beta_0 = 0, \beta_j$ defined as displayed below for $1 \leq j \leq k$, and $\alpha_j = 0, \beta_{j+k-1} = 0$ for $j > 1$:

$$y_n = y_{n-1} + h \sum_{j=1}^{k} \beta_1 f_{n-1}, \quad (4.15)$$

where

$$\beta_j = (-1)^{j-1} \sum_{t=j-1}^{k-1} \binom{i}{j-1} (-1)^i \int_0^1 \binom{-s}{i} ds \quad (4.16)$$

It is important to mention that for such schemes, k starting values must be given. If only the initial condition is provided, the other k − 1 points can be determined by a different scheme (for example, a Runge-Kutta method of the same order). Adams-Bashforth methods also tend to have small regions of absolute stability (to be discussed later), and this inspired the construction of implicit Adams methods (called Adams-Moulton methods) which are the topic of the following discussion.

4.4.2 The Adams-Moulton [AM] Methods (Implicit Adams Methods)

The difference between Adams-Moulton and Adams-Bashforth methods is that Adams-Moulton methods use an interpolating polynomial of degree ≤ k rather than ≤ k−1 , and it includes f at the unknown value t_n as well. A k-step Adams-Moulton scheme can be expressed in the form of (1) as follows:

$$y_n = y_{n-1} + h \sum_{j=0}^{k} \beta_1 f_{n-1} \quad (4.17)$$

It is apparent that when k = 1 and $\beta_1 = 0$ we have the classic Backward Euler (BE) method. Likewise, if k = 1 and $\beta_1 \neq 0$ we have the implicit trapezoidal method.

4.4.3 Derivation of multistep methods

Adams Method: /3rd order

Reformulate ODE solving as a numerical quadrature problem

$$dx = f(x,t)$$

$$\int_{t_n}^{t_{n+1}} dx = x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f(x,t)dt$$

Apply method of undetermined coefficients

$$\int_{t_n}^{t_{n+1}} f(x,t)dt \approx c_0 f(x_{n-2}, t_{n-2}) + c_1 f(x_{n-1}, t_{n-1}) + c_2 f(x_n, t_n)$$

x is a function of t

$\int_{t_n}^{t_{n+1}} f(x,t)dt \approx c_0 f_{n-2} + c_1 f_{n-1} + c_2 f_n$

Apply method of undetermined coefficients

$$\int_{t_n}^{t_{n+1}} f(x,t)dt \approx c_0 f_{n-2} + c_1 f_{n-1} + c_2 f_n$$

$$\int_{t_n}^{t_{n+1}} f(x,t)dt \approx c_0 f_{n-2} + c_1 f_{n-1} + c_2 f_n$$

Interpolate 3 points $(0, f_n), (-h, f_{n-1}), (-2h, f_{n-2})$

$f(t) = 1 \qquad \int_0^h 1dt = h = c_0(1) + c_1(1) + c_2(1)$

$f(t) = t \qquad \int_0^h tdt = \frac{h^2}{2} = c_0(-2h) + c_1(-h) + c_2(0)$

$f(t) = t^2 \qquad \int_0^h t^2dt = \frac{h^3}{3} = c_0(-2h)^2 + c_1(-h)^2 + c_2(0)$

$$\begin{matrix} f(t) = 1 \\ f(t) = t \\ f(t) = t^2 \end{matrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ -2h & -h & 0 \\ 4h^2 & h^2 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ h^2/2 \\ h^3/3 \end{bmatrix}$$

Solving linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ -2h & -h & 0 \\ 4h^2 & h^2 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ h^2/2 \\ h^3/3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5h/12 \\ -4h/3 \\ 23h/12 \end{bmatrix}$$

there fore 3rd-order Adams method will be

$$\int_{t_n}^{t_{n+1}} dx = x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f(x,t)dt$$

$$x_{n+1} - x_n \approx c_0 f_{n-2} + c_1 f_{n-1} + c_2 f_n$$

$$= \frac{5h}{12} f_{n-2} + \frac{-4h}{3} f_{n-1} + \frac{23h}{12} f_n$$

$$= \frac{h}{12} [5f_{n-2} - 16f_{n-1} + 23f_n]$$

Adams Method: Fourth-Order Formula

the same way but now using 4 previous steps

$\int_{t_n}^{t_{n+1}} dx = x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f(x,t)dt$

$\int_{t_n}^{t_{n+1}} f(x,t)dt \approx c_0 f_{n-3} + c_1 f_{n-2} + c_2 f_{n-1} + c_3 f_n$

$\int_{t_n}^{t_{n+1}} f(t)dt \approx c_0 f_{n-3} + c_1 f_{n-2} + c_2 f_{n-1} + c_3 f_n$

Interpolate 4 points $(0, f_n), (-h, f_{n-1}), (-2h, f_{n-2}), (-3h, f_{n-3})$

$f(t) = 1 \qquad \int_0^h 1dt = h = c_0(1) + c_1(1) + c_2(1) + c_3(1)$

$f(t) = t \qquad \int_0^h tdt = \frac{h^2}{2} = c_0(-3h) + c_1(-2h) + c_2(-h) + c_3(0)$

$f(t) = t^2 \quad \int_0^h t^2 dt = \frac{h^3}{3} = c_0(-3h)^2 + c_1(-2h)^2 + c_2(-h)^2 + c_3(0)$

$f(t) = t^3 \quad \int_0^h t^2 dt = \frac{h^4}{4} = c_0(-3h)^3 + c_1(-2h)^3 + c_2(-h)^3 + c_3(0)$

Solving linear system

$$\begin{array}{l} f(t)=1 \\ f(t)=t \\ f(t)=t^2 \\ f(t)=t^3 \end{array} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3h & -2h & -h & 0 \\ 9h^2 & 4h^2 & h^2 & 0 \\ -27h^3 & -8h^3 & -h^3 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ h^2/2 \\ h^3/3 \\ h^4/4 \end{bmatrix}$$

Solving for the coefficients, we obtain

$x_{n+1} = x_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] + 0(h^5)$

where the error term can be computed by integrating the error of cubic interpolating polynomial

$$Error = \frac{251}{720} h^5 x^{(5)}(\varepsilon)$$

Adam-Moulton Method

Improvement over the Adams method using a predictor-corrector scheme

First compute X_{n+1} using predictor

$$x_{n+1} = x_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] + \frac{251}{720} h^5 x^{(5)}(\varepsilon_1)$$

Compute derivative $f_{n+1} = (x_{n+1}, t_{n+1})$

Recompute X_{n+1} using corrector

$$x_{n+1} = x_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] - \frac{19}{720} h^5 x^{(5)}(\varepsilon_2)$$

(Convergence Theorem)

we have:

$|y_n - u_n| \leq (|y_0 - u_0| + nh\tau(h))\, e^{nh\Lambda},\ 1 \leq n \leq N_h$. ………………(4.5)

Therefore, if the consistency assumption

$\lim_{h \to 0} \Phi(tn, yn, f(tn, yn);\ h) = f(t_n, y_n),\ \forall t_n \geq t_0$

holds and $|y_0 - u_0| \to 0$ as $h \to 0$, then the method is convergent.

Moreover, if $|y_0 - u_0| = O(h^p)$ and the method has order p, then it is also convergent with order p.

Proof. Setting $w_j = y_j - u_j$, subtracting

$u_{n+1} = u_n + h\Phi(t_n, u_n, f_n; h),\ 0 \leq n \leq N_h - 1,\ u_0 = y_0,$

from $y_{n+1} = y_n + h\Phi(t_n, y_n, f(t_n, y_n); h) + \varepsilon_n+1,\ 0 \leq n \leq N_h - 1$

yields inequality

$|wn\ (h)| \leq |w0| + h\sum_{j=0}^{n-1} |\delta j + 1| + h\Lambda\sum_{j=0}^{n-1} |wj\ (h)|,\ 1 \leq n \leq N_h.$

with the understanding that

$w_0 = y_0 - u_0$, and $\delta_{j+1} = \tau_{j+1}(h)$.

The estimate $|y_n - u_n| \leq (|y_0 - u_0| + nh\tau(h))\, e^{nh\Lambda},\ 1 \leq n \leq N_h$

From the fact that $nh \le T$ and $\tau(h) = O(h^p)$, we can conclude that $|y_n - u_n| \le Ch^p$ with C depending on T and Λ but not on h.

A consistent and zero-stable method is thus convergent. This property is known as the Lax-Richtmyer theorem or equivalence theorem (the converse:“a convergent method is zero-stable” being obviously true)[19].

4.4.4. STABILITY ANALYSIS OF Third-Order Runge-Kutta Method

Generally, to define the stability region of the method we applied the test problem $y_0 = \lambda y$, where λ is a complex number. Here, by applying the test problem to (2) we have

$$y' = \lambda y,$$

$$k_1 = \lambda y_n$$

$$k_{-1} = \lambda y_{n-1}$$

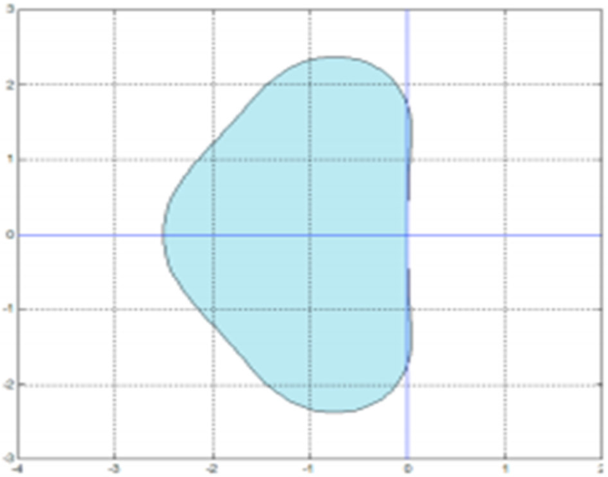
$$k_2 = \lambda(1+\lambda h a_{21})y_n$$

$$k_{-2} = \lambda (1+\lambda h a_{21})y_{n-1}$$

Substituting all the above values in (4.3.4) we have

$$y_{n+1} = (1+ \lambda h(b_{1+} b_2)(1+\lambda h a_{21})))y_n - (b-1+ b_2(1+\lambda h\, a_{21})y_{n-1}$$

Stability region of the methods is the set of the values of λh such that all the roots of the stability polynomial are inside a unit circle. Here, the stability region of IRK3 which is plotted in Figure.



RK 3

Figure 1.tability region of third- order Runge-kutta method.

4.4.5 Stability analysis of fourth order Runge-Kutta's method

Example.4.5.1 .Consider the following system of equations,

$$du/dx=8u(x)-5v(x)+10w(x)$$

$$dv/dx=2u(x)+1v(x)+2w(x)$$

$$dw/dx=-4u(x)+4v(x)+6w(x)$$

where initial conditions are $u(0) = 2, v(0) = 2$, $w(0) = -3$. It can be shown in general form as

$y(0) = [2,2,-3]T$. Its theoretical solution is given

$$u(x) = 6e^{2x} -4e^{3x},$$

$$v(x) = -4e^{3x} +6e^{2x},$$

$$w(x) = -e^{-2x} +3e^{2x}$$

where $\mathbf{y}(x) = [u(x),v(x),w(x)]T$

The computation of Explicit Runge Kutta of fourth order method in same stiff system is given at different step sizes in Table for given systems. Step size is started from 0.1 and computation progress is proceed of half. Evaluation of analytical solution is same and the approximated solution is computed by using the step sizes 0.1, 0.05, respectively. In addition, local error of numerical method is generated by taking modulus. Error tables can be found below. As it is seen in table, at the step size $h = 0.1$, errors in system lie in the region of absolute stability. Thus, absolute error is started to behave as approximate solutions. But, the approximation of u(x),v(x),w(x) is not close to analytical solution. It is also appreciable in absolute error which is not small absolute error. Again, when step sizes are taken 0.05, we can see in the table that absolute error is started to behave as approximate solutions. These results indicate that the acceptable approximations, and step sizes are in the region of the absolute stability. It is also noticeable from the error evaluation that has excessively small difference. This shows that approximated solution turn into the exact solution at step-size h decreasing.

Table .1.Fourth order Runge-kutta(RK4) method with h=0.1 for n=10

I	H	u_{app} (x)	v_{app} (x)	w_{app} (x)	u(x)	v(x)	w(x)	v(x)−v_{ap}(x)	v(x)−v_{app}(x)	w(x)−w_{app} (x)
0	0	2	2	-3						
1	0.1	−0.48695	1.92905	−1.2482	−0.48705	1.929	−1.24818	0.000101	6.87E −05	2.38E −05
2	0.2	−3.2663	1.662663	0.453508	−3.26656	1.6625	0.453554	0.000255	0.00019	4.56E −05
3	0.3	−6.54505	1.094692	2.173418	−6.54554	1.0943	2.173487	0.000497	0.000392	6.82E −05
4	0.4	−10.5836	0.073496	3.980555	−10.5845	0.07277	3.980649	0.000872	0.000718	9.43E −05
5	0.5	−15.718	−1.61584	5.947442	−15.7195	−1.6171	5.947569	0.00145	0.001231	0.000127
6	0.6	−22.3891	−4.27587	8.153016	−22.3914	−4.2779	8.153185	0.002326	0.002022	0.000169
7	0.7	−31.1815	−8.33026	10.68579	−31.1851	−8.3335	10.68602	0.003642	0.003225	0.000225
8	0.8	−42.8757	−14.3695	13.64742	−42.8813	−14.375	13.64772	0.005598	0.005031	0.000299
9	0.9	−58.5187	−23.2133	17.15675	−58.5271	−23.221	17.15715	0.008483	0.007717	0.000397
10	1	−79.5174	−35.9961	21.35463	−79.5301	−36.008	21.35516	0.012707	0.01168	0.000526

Table.2 Fourth order Runge-Kutaa(RK4) method with h=0.05 n=20

I	H	u_{app}(x)	v_{app}(x)	w_{app}(x)	u(x)	v(x)	w(x)	u(x)−u_{app}(x)	v(x)−v_{app}(x)	w(x)−w_{app}(x)
0	**0**	**2**	**2**	**-3**						
2	**0.1**	−0.4870	1.92898	−1.24817	−0.48705	1.929	−1.24817	6.92E −06	4.91E −06	1.45E −06
4	**0.2**	−3.2665	1.66248	0.453551	−3.26655	1.6625	0.453553	1.77E −05	1.35E −05	2.83E −06
6	**0.3**	−6.5455	1.09432	2.173482	−6.54554	1.0943	2.17348	3.48E −05	2.79E −05	4.30E −06
8	**0.4**	−10.584	0.072829	3.980643	−10.5844	0.07277	3.98064	6.13E −05	5.12E −05	6.05E −06
10	**0.5**	−15.719	−1.61697	5.947561	−15.7194	−1.6171	5.94756	0.0001021	8.76E −05	8.25E −06
12	**0.6**	−22.391	−4.2777	8.153174	−22.3914	−4.2779	8.15318	0.0001641	0.000144	1.11E −05
14	**0.7**	−31.1848	−8.33325	10.686	−31.1851	−8.3335	10.6860	0.000257	0.000229	1.49E −05
16	**0.8**	−42.8809	−14.3741	13.6477	−42.8813	−14.375	13.6477	0.000395	0.000357	2.00E −05
18	**0.9**	−58.5265	−23.2204	17.15712	−58.5271	−23.221	17.15715	0.000600	0.000548	2.67E −05
20	**1**	−79.5292	−36.0069	21.35512	−79.5301	−36.008	21.35516	0.000899018	0.000829558	3.55E −05

Example.4.5.3. Apply RK-4 method to solve the initial value problem y`=2xy+1, y(0) = 3

X	h = 0.2	h = 0.1	h = 0.05	``Exact``
0.0	3.0000000	3.0000000	3.0000000	3.0000000
0.2	3.3278464	3.3278516	3.3278519	3.3278519
0.4	3.9660449	3.9660585	3.9660593	3.9660593
0.6	5.0669967	5.0670371	5.0670394	5.0670395
0.8	6.9365341	6.9366906	6.9367003	6.9367009
1.0	10.1842322	10.1848777	10.1849209	10.1849239

As step-size h decreased the difference between approximate value and exact solution becomes very small.

When we compare numerical solution of using RK-4 in table 4.5.3 with numerical solutions in table 4.2.3 and table 4.3.3 using Euler`s method and Improved Euler method respectively, the error in RK-4 is very small than error in the others.

Applying the classical Runge-Kutta's method of order 4 to the test differential equation

$y' = \lambda y$ yields the stability function

$$Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{3!}(h\lambda)^3 + \frac{1}{4!}(h\lambda)^4 \ .$$

The boundary of the region of stability is defined as $|g(z)| = 1$. Since all complex points on theunit circle are of the form $e^{i\phi}$, we get

$$1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{3!}(h\lambda)^3 + \frac{1}{4!}(h\lambda)^4 = e^{i\phi}$$

This equation can be solved either numerically of with a formula for the roots of cubic polynomials. With $\phi \in [0, 2\pi]$ the boundary of the region of stability describes a closed curve in the complex plane. The regions of stability are bounded and cover only small parts of the left-half plane.

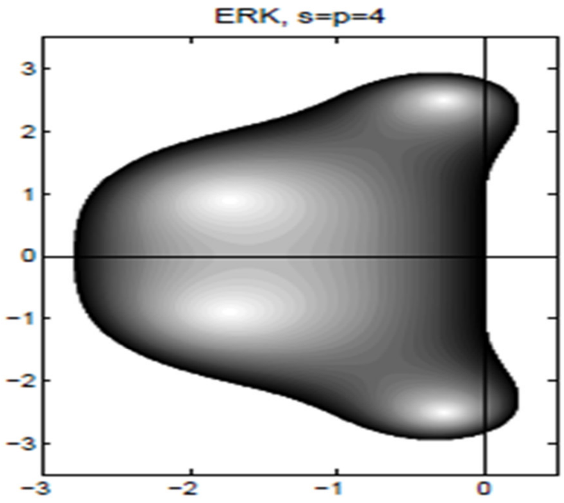


Figure 2. Region of absolute Stability for RK-4.

This figure tells us a huge amount. Consider first a scalar differential equation $dx/dt = \lambda x$, with possibly complex λ. We know that for the differential equation, the origin is stable for λ lying in the left half plane, or, if we think of the map $\Phi h = e^{h\lambda}$ as defining a discrete dynamics, the origin is stable independent of h if λ is in the left half plane. Figure 2 **Table .1.Fourth order Runge-kutta(RK4) method with h=0.1 for n=10**

shows stability regions of fourth order Runge-Kutta method. The shading in the figure indicates the magnitude $|R(z)|$ within the stability region. The stability limit for this method is;

$$\left|1+z+z^2/2+z^3/6+z^4/24\right| = 1$$

Where , $z = h\lambda$

4.5 Order and consistency of multistep Methods

Investigating convergence of multistep methods is quite different from that of non-linear one-step methods (such as the Runge-Kutta) methods. For Runge-Kutta methods, -stability is automatic, and investigating the order can be cumbersome. For multistep methods, zero-stability is not necessarily automatic, and needs to be confirmed for each scheme. Contrarily, and again unlike the Runge-Kutta methods, investigating the order of multistep methods is . We will start this section by investigating the order of multistep methods.

4.5.1 Order

To begin, let us define the linear operator Lh[y(t)], where y(t) is an arbitrarily continuously differentiable funciton on [0,b]:

$$Lh[y(t)] = \sum_{j=0}^{k}\left[\alpha_j y(t-jh) - h\beta_j y^t(t-jh)\right] \qquad (4.18)$$

This expression is based on Eq. (1). Recalling that $y' = f(t,y(t))$, we can write the above expression in the following way:

$$l_h[y(t)] = \sum_{j=0}^{k}\left[\alpha_j y(t-jh) - h\beta_j f(t-jh, y(t-jh))\right] \qquad (4.19)$$

which becomes, after expanding $y(t - jh)$ and $f(t - jh,y(t - jh))$ in a Taylor series about t and simplifying:

$$Lh[y(t)] = C_0y(t) + C_1hy'(t) + ... + C_qh^qh^q(t) + ..., \qquad (4.20)$$

where,

$$C_0 = \sum_{j=0}^{k} \alpha_j \text{ , and} \qquad (4.21)$$

$$c_i = (-1)^i\left[\frac{1}{i!}\sum_{j=1}^{k} j^i\alpha_j + \frac{1}{(i-1)!}\sum_{j=0}^{k} j^{i-1}\beta_{j\ ,i=1,2,3,}\right] \qquad (4.22)$$

Now, the order of the method is p if the local truncation (or discretization) error is $d_n = O(h^p)$, which is given by:

$$d_n = \frac{l_h[y(t_n)]}{h} \quad , \qquad (4.23)$$

where $y(t_n)$ is the exact solution at $t = t_n$. So using the version of the expression given above we get that the method is order p if:

$$C_0 = C_1 = ... = C_p = 0, C_{p+1} \neq 0. \qquad (4.24)$$

Combining this result, we get that

$$d_n = C_{p+1}h^py^{(p+1)}(t_n) + O(h^{p+1}), \qquad (4.25)$$

where C_{p+1} is the error constant of the scheme.

It can then be shown that Adams-Bashforth and BDF methods are of order k (where k is the number of steps), while Adams-Moulton methods are of order k + 1 (with the exception of the case where the scheme is completed in a single step with $\beta_1 = 0$, as in Backward Euler, which is order k = 1).

Calculating each value of C_q in (14) can be tedious though. The easiest way to check for consistency ($p \geq 1$) is to use the fact that a method is consistent if and only if

$$\sum_{j=0}^{k}\alpha_j = 0 \text{ and } \sum_{j=1}^{k} j\alpha_j + \sum_{j=0}^{k}\beta_j = 0 \qquad (4.26)$$

This result can also be demonstrated in terms of the characteristic polynomials of the recurrence relations arising from the expression for the numerical scheme:

$$p(\varepsilon) = \sum_{j=0}^{k}\alpha_j\varepsilon^{k-j} \text{ , } \alpha_0 = 1 \qquad (4.27)$$

$$\sigma(\varepsilon) = \sum_{j=0}^{k}\beta_j\varepsilon^{k-j} \qquad (4.28)$$

In particular, the scheme is consistent if and only if $\rho(1) = 0$, $\rho'(1) = \sigma(1)$.

4.6 Stability analysis Multistep Methods

4.6.1 Zero-Stability

Here we are looking at Eq. (1) in the limit as h → 0, rendering this equation into the form:

$$\alpha_k y_{n-k} + \alpha_{k-1} y_{n-k+1} + \ldots \alpha_0 y_n = 0. \qquad (4.29)$$

This recurrence relation has the characteristic polynomial $\rho(\varepsilon)$ defined earlier. Due to consistency (see previous section), we know that $\xi = 1$ is a root of Eq. (4.12). If the rest of the roots of the equation are distinct, then we have a solution of the form:

$$y_n = \sum_{t=1}^{k-1} c_i + \varepsilon_i^n c_k (1)^n \qquad (4.30)$$

If $\varepsilon_1 = \varepsilon_2$ is a double root, then the solution will have the form:

$$y_n = \sum_{t=3}^{k-1} c_i + \varepsilon_i^n c_k (1)^n + c_1 \varepsilon_1^n + c_2 \varepsilon_2^n \qquad (4.31)$$

Similarly, if $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ is a triple root, then the solution will have the form:

$$y_n = \sum_{t=4}^{k-1} c_i + \varepsilon_i^n c_k (1)^n + c_1 \varepsilon_1^n + c_2 n \varepsilon_2^n + c_3 n(n-1) \varepsilon_3^n \qquad (4.32)$$

We can see that if $|\ \varepsilon_i\ | > 1$, then our solution will diverge as n gets large. Likewise, if $|\ \varepsilon_i| = 1$ is not a simple root, we will again have divergence. Therefore, a multistep method is 0-stable if and only if all roots of the equation $\rho(\varepsilon) = 0$ satisfy $|\ \varepsilon_i| \le 1$, where if $|\ \varepsilon_i| = 1$, then ε_i is a simple root, for $1 \le i \le k$. Now by a theorem sometimes referred to as the Dahlquist Theorem, if this root condition is satisfied, and the method is accurate to order p, and the initial values are accurate to order p, then the method is convergent to the order p. We can further specify the strength of the stability of the scheme by defining strongly stable as meaning all roots of $\rho(\varepsilon) = 0$ have the property $|\ \varepsilon\ | < 1$ with the exception of the one root ε_i which equals 1. A scheme can then be defined as weakly stable if it is 0-stable, but not strongly stable.

4.6.2 Absolute Stability, A-stability and L-stability by applying Eq. (1) to the test equation $y' = \lambda y$ and letting $y_n = \xi^n$, we find that ε must satisfy $\rho(\varepsilon) - h\lambda\sigma(\varepsilon)$. Now to address absolute stability, we define the stability polynomial as $\varepsilon = \rho(\varepsilon) - h\lambda\sigma(\varepsilon)$ and note that the absolute stability region is the region of values of λh such that $|y_n|$ does not grow with increasing values of n. For this condition to be met, all roots ξi of $\varepsilon = 0$ must satisfy $|\xi i| \le 1$. As for A-stability, recall that a numerical scheme is A-stable if its absolute stability region contains the entire left half of the complex plane (i.e. it contains $\mathrm{Re}(\lambda h < 0)$). It turns out that explicit multistep methods cannot be A-stable, and that A-stable multistep methods with order greater than 2 do not exist. The most accurate LM method (method with smallest error constant) is the second-order implicit trapezoidal method (with error constant $c_3 = \frac{1}{12}$. We conclude that A-stability is rare in multi-step methods. After applying our numerical scheme to the test equation $y' = \lambda y$, we can rearrange our expression to obtain an expression of the form $y_n = R(z) y_{n-1}$, where $z = \lambda h$. If $R(z) \to 0$ as

Re(z) → −∞, we say that the scheme is L-stable (or has stiff decay. This explains why the Backward Euler method works better than the Trapezoidal Rule for some problems, even though the Trapezoidal Rule is of higher order!

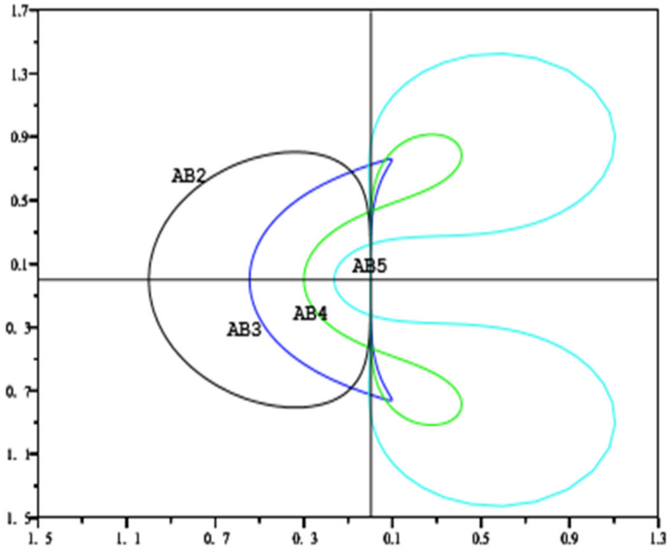


Fig. 3– Adams-Basforth region of stability

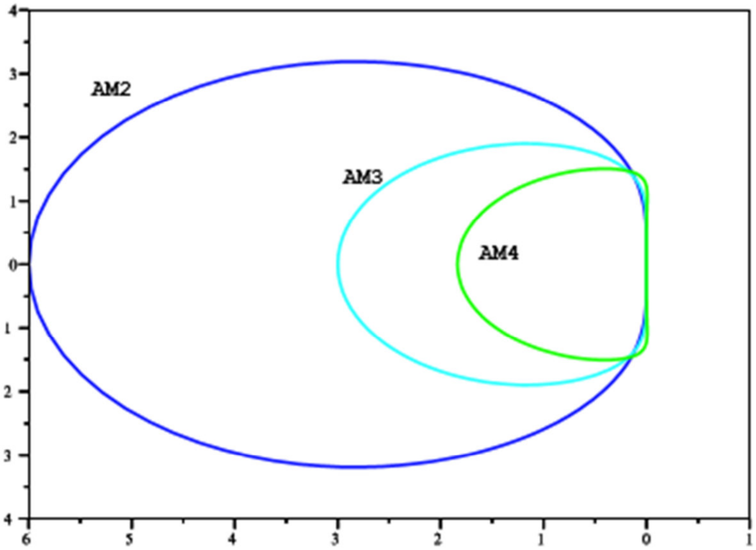


Fig. 4 – Adams-Moulton region of stability

4.7 Tested Examples and result

Example Solve the equation

$y' = x + y \qquad y(0) = 0$

from $x = 0$ to $x = 1$, using the Adams-Bashforth method

The exact solution of this problem is $y = e^x - 1 - x$. The first four starting values are computed, using this solution. The first column of the results gives the values of *xn* with $h = 1/8$, the second column gives *yn* as computed by formula

$$y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] + 0(h^5)$$

the third column gives the value *y(xn)* as computed from the solution, and the fourth column gives the error *en =y n - y(x n)*

Table 3 result from Adams-Bashforth method

N	XN	YN	$Y(XN)$	ERROR
0	0.	0.	0.	0.
1	0.31250000E-01	0.49340725E-03	0.49340725E-03	0.
2	0.6250000E-01	0.19944459E-02	0.19944459E-02	0.
3	0.93750000E-01	0.45351386E-02	0.4351386E-02	0.
4	0.1250000E-00	0.81484411E-02	0.81484467E-02	-0.55879354E-08
5	0.1562500E-00	0.12868421E-01	0.12868434E-01	-0.12922101E-07
6	0.18750000E-00	0.18730211E-01	0.18730238E-01	-0.26309863E-07
7	0.21875000E-00	0.25770056E-01	0.25770098E-01	-0.41676685E-07
8	0.25000000E-00	0.34025350E-01	0.34025416E-01	-0.65192580E-07

Example:- Solve the equation

y´ = *x* + *y*   *y*(0) = 0

from *x* = 0 to *x* = 1, using the Adams-Moulton predictor-corrector formulas.

Table 4 result from Adams-Moulton method

N	XN	YN	$Y(XN)$	ERROR
0	0.	0.	0.	0.
1	0.31250000E-01	0.49340725E-03	0.	0.
2	0.62500000E-01	0.19944459E-02	0.	0.
3	0.93750000E-01	0.45351386E-02	0.	0.
4	0.12500000E-01	0.81484520E-02	0.78164571E-09	0.53551048E-08
5	0.15625000E-00	0.12868445E-01	0.90637643E-09	0.11408702E-07
6	0.18750000E-00	0.18730249E-01	0.88143028E-09	0.11175871E-07
7	0.21875000E-00	0.25770108E-01	0.91469178E-09	0.10011718E-07
8	0.25000000E-00	0.34025417E-01	0.93132257E-09	0.13969839E-08

Unit 5

5 Discussion and conclusion

5.1 Discussion

For many years Runge-Kutta methods were used almost exclusively for general-purpose work, but recently predictor-corrector methods have been gaining in popularity. In the past few years much more sophisticated general-purpose methods using both variable orders and variable steps have been developed. The Adams methods described previously are the most widely used in variable-order-variable-step methods. The objective of these methods is to automatically select the proper order and the proper step which will minimize the amount of work required to achieve a specified accuracy for a given problem. Other important advantages of these methods are that they are self-starting since a low-order method can be used at the start, and they can easily be adjusted to supply missing values when the step size is changed. A complete description of a subroutine called DIFSUB based on an Adams vari able-order-variable-step method is given in Gear [30, pp. 158-167]. A subroutine called DVOGER, also based on Gear’s method, is available in the IMSL programs and has been adapted to run on most modern computers.

5.2 Conclussion

The difference between Adams-Moulton and Adams-Bashforth methods is that Adams-Moulton methods use an interpolating polynomial of degree $\leq k$ rather than $\leq k-1$, and it includes f at the unknown value t_n as well.It is apparent that when $k = 1$ and $\beta_1 = 0$ we have the classic Backward Euler (BE) method. Likewise, if $k = 1$ and $\beta_1 \neq 0$ we have the implicit trapezoidal method.

Adams-Moulton methods have smaller error constants, use less steps, and have larger stability regions than their Adams-Bashforth counterparts (of the same order). However, AM methods using more than one step tend to have smaller regions of absolute stability than other implicit methods such as Runge-Kutta methods (in fact, they tend to be bounded, which often defeats the purpose of using an implicit scheme).

5.3. RECOMMENDATION

We recommended that for future work we will be solving Ordinary Differential Equation by using other numerical methods such as Cubic spline ,Finite difference Method and finite element make comparison between them

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