Solving Linear Second Order Delay Differential Equations by Steps Method

Korsa Debele

A Thesis Submitted to the Department of Applied Mathematics

School of Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of

Master's in Applied mathematics (Differential Equations)

Office of Graduate Studies
Adama Science and Technology University

September 8, 2021

Adama, Ethiopia

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Declaration

I declare that this thesis proposal " Solving linear second order delay differential equations by steps method" is my own work. That it has not been submitted before for any degree or examination in other university. It is being submitted before for partial fulfillment for degree of MSc in Adama Science and Technology University School of Applied natural Science.

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Name of student	**Signature**	**Date**

Recommendation

We, the advisor of this thesis, hereby certify that we have read the revised version of the thesis entitled "Solving linear second order delay differential equations by steps method " prepared under our guidance by Korsa Debele submitted in partial fulfillment of the requirements for the degree of Master's of Science in Applied Mathematics.

Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	**Signature**	**Date**
Co-advisor	**Signature**	**Date**

Approval Page

We, the advisors of the thesis entitled "Solving linear second order delay differential equations by steps method" and developed by Korsa Debele, hereby certify that the recommendation and suggestion made by the board of examiners are appropriately incorporated into the final version of the thesis.

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We, the undersigned, members of the Board of Examiners of the thesis by Korsa Debele, have read and evaluated the thesis entitled "Solving linear second order delay differential equations by steps method" and examined the candidate during the open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics.

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List of Symbols and Acronyms

$\phi_Y(t)$	History function.
$\phi_{Y_{t_0}}$	History function at t_0
τ	Delay(time lag)
$Y(t-\tau)$	Solution of the delay terms.
t_0	Initial time.
t_f	Final time.
$[t-\tau, t_0]$	Pre-interval.
$[t_0, t_f]$	Time interval.
$\varphi(t)$	Fundamental matrix
A, B	square matrices
$\|.\|$	Euclidean norm
$C := C([-\tau, t_0], \mathbb{R})$	The space of continuous functions $[-\tau, 0] \to \mathbb{R}$
DDE23	Matlab solver.
PDE	Partial differential equations.
ODE	Ordinary differential equations.
DDE	Delay differential equations.
SDDDE	State dependent delay differential equations.
NDDDE	Neutral delay differential equations.
NLDDDE	Nonlinear delay differential equations.

Abstract

This thesis concentrates on steps method to solve delay differential equations (DDEs) with a single constant delay and constant coefficients. In this study the stability type of second order delay differential equation at equilibrium point is being considered where each coefficients are in $\mathbb{R}$*. First, the general stability theory for delay differential equations was highlighted before giving an in-depth stability analysis of the equation of harmonic oscillator. It turns out that a Theorem of Pontryagin (1908 -1988) is really helpful for answering these stability questions. Due to this Theorem all values for each coefficients in* $\mathbb{R}$*, are determined such that equilibrium point is asymptotically stable for damped harmonic oscillator. However, this does not cover the stability type at equilibrium point for all coefficients. So more analysis was done in order to give a full answer of the stability problem. Finally comparing of analytical solution obtained by steps method with codes from Matlab solver DDE23 are shown.*

Keywords: Delay differential equation; Linear delay differential equation; Constant delay; Steps method.

Chapter 1

Introduction

1.1 Background of the study

After the first world war, the development and use of automatic control systems resulted in studies of an entirely different class of differential equations called delay differential equations (DDEs). Any system involving a feedback control will almost certainly involve time delays (Erneux, 2009).

Time delay systems are those systems in which a significant time delay exists between the applications of input to the system and their resulting effect. Such systems arise from an inherent time delay in the components of the system or a deliberate introduction of time delay into the system for control purposes(Richard, 2003). Many of the processes, both natural and man made, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality (Kuang & Smith, 1993; Gimeno i Alquézar, 2015).

We have many examples of time delay in our life. A simple example of time delay is reforestation. A cut forest, after replanting, it will take at least 20 years before reaching any kind of maturity. For certain species of trees (redwoods, for example) it would be much longer. Hence, any mathematical model of forest harvesting and regeneration clearly must have time delays built into it. Another example occurs due to the fact that animals must take time to digest their food before further activities and responses take place (Kuang & Smith, 1993).

In mathematics delay differential equations (DDEs) are a differential equations in which the derivative of the unknown function at certain time is given in terms of the value of the function at previous times (Gimeno i Alquézar, 2015; Krasovskii, 1963; Breda, Maset, & Vermiglio, 2014). These types of equations are a large and important class of dynamical systems. They often arise in either natural or technological control problems. In these systems, a controller monitors makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action (Zulkefli & Maan, 2014).

Delay differential equations (DDEs) are differing from ordinary differential equations (ODEs), in that the derivatives at any time depends on the solution at prior times. Let us consider the ordinary differential equation

$$\dot{x}(t) = f(t, x(t)), \tag{1.1}$$

together with the initial condition

$$x(t_0) = x_0. \tag{1.2}$$

It is well known that under certain assumptions on f, equation (1.1) has a unique solution and it is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds, \quad \text{for } \ t \geq t_0.$$

But, if we consider a differential equation of the form

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau)), \ \ \tau > 0, t \geq t_0 \\ x(t) &= \phi(t), \ \ -\tau \leq t \leq t_0, \end{aligned} \tag{1.3}$$

where $\phi(t)$ is history function. Then in which the right hand side of (1.3) depends not only on the instantaneous position $x(t)$ for $t \geq t_0$. Also on $x(t-\tau)$, the position at τ units back, that is to say the equation has past memory. Such an equation is called an differential equation with delay or delay differential equation.

The following simple example and graph is taken from (Zou, Fränzle, Zhan, & Mosaad, n.d.), demonstrates the difference between DDE and their related ODE obtained by neglecting delays.

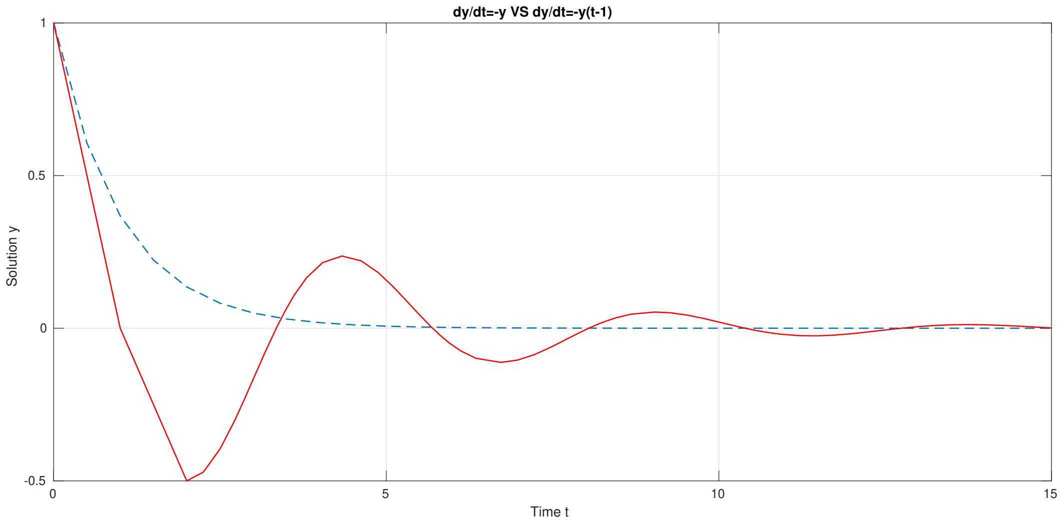

Figure 1.1: Solutions of the ODE $\dot{x}(t) = -x(t)$ (dashed graph) and the related DDE $\dot{x}(t) = -x(t-1)$ (solid line), both on similar initial conditions $x(0) = 1$ and $x([-1,0]) \equiv 1$, respectively.

In figure 1.1, the dashed and solid lines represent the solution of the ODE $\dot{x}(t) = -x(t)$ without delay and of the related DDE $\dot{x}(t) = -x(t-1)$ with 1 is delay respectively. Both are given as initial condition, where for the ODE we assume an initial condition $x(0) = 1$, which we generalize for the DDE to $x([-1,0]) \equiv 1$. Hence the dashed line (representing the ODE behavior) always stays above the horizontal axis whereas, in contrast, the solid line (representing the DDE solution) visits the negative range repeatedly. Even though the difference between the solutions of the ODE and the DDE becomes smaller when the delay turns smaller.

There are different species of delay differential equations. Such as: linear delay differential equations (LDDEs), nonlinear delay differential equations (NLDDEs), neutral delay differential equations (NDDEs) and stochastic delay differential equations (SDDEs). In this study we will focus on the analytical solution of linear second order delay differential equation of the form

$$\begin{aligned} \ddot{x}(t) = f(t, x(t), \dot{x}(t-\tau)),\;\; t_0 \le t \le t_f, \tau > 0 \\ x(t) = \phi(t),\;\; t < t_0 \end{aligned} \tag{1.4}$$

by steps method where, $f : [0,\infty] \times \mathbb{R} \times C[t_0 - \tau, t_0] \to \mathbb{R}$ is given continuous function, τ is delay and $\phi(t)$ is initial function defined on $C[t_0 - \tau, t_0]$.

1.2 Statement of the problem

Let $\tau > 0$ is constant delay and $\phi(t) : [t_0-\tau, t_0] \to \mathbb{R}$ be a known continuously differentiable function. Consider second order delay differential equations

$$\ddot{x}(t) + \xi x(t) + \eta \dot{x}(t-\tau) = 0, \;\; t_0 \leq t \leq t_f, \tag{1.5}$$

with history function

$$x(t) = \phi(t), \;\; -\tau \leq t \leq t_0, \tag{1.6}$$

where $\xi, \eta \in \mathbb{R}\backslash\{0\}$. To reduce equation (1.5) to system of first order delay differential equations, let $\dot{x}(t) = y(t)$ and $\ddot{x}(t) = \dot{y}(t)$ then $\dot{y}(t) = -\xi x(t) - \eta y(t-\tau)$. Hence equation (1.5) is equivalent to

$$\dot{Y}(t) = AY(t) + BY(t-\tau) \tag{1.7}$$

where,

$$\dot{Y}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}, \; A = \begin{bmatrix} 0 & -\xi \\ 1 & 0 \end{bmatrix}, \; Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & -\eta \end{bmatrix}, \; Y(t-\tau) = \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix}.$$

Therefore problem (1.7) with it's history function is written in the form of

$$\begin{cases} \dot{Y}(t) = AY(t) + BY(t-\tau), \;\; t_0 \leq t \leq t_f, \\ Y(t) = \phi_Y(t), \;\; -\tau \leq t \leq t_0. \end{cases} \tag{1.8}$$

Hence in this study we will look for analytical solution of delay differential equations of the form (1.8), by applying steps method over each consecutive interval for $t_0 \leq t \leq t_f$.
Thus, this study is intended to answer the following basic questions:

- How to find analytic solution of (1.5)
- How to apply steps method on (1.8) to find analytic solution?
- How to analyze the stability?

1.2.1 General objective of the study

The general objective of this study is to solve second order delay differential equations by steps method.

1.2.2 Specific objectives of the study

The specific objectives of this study are to:

- find analytic solution for linear second order DDEs by steps method.
- analyze the stability of second order delay differential equations.
- compare analytic solution obtained by steps method with approximate solution of DDE23 graphically by merging.

1.3 Significance of the study

The result of this study will help:

- to understand the concept of delay differential equations.
- to develop a skills how to apply steps method to find analytic solution.
- the researcher to conduct research in this area.

Chapter 2

Literature Review

In this chapter, we review different study about solution of delay differential equations. Many researchers give different analysis of solutions to delay differential equations.

An approach for the analytical solution, free and forced, to systems of delay differential equations (DDEs) has been developed in (Yi, Ulsoy, & Nelson, 2006; Asl & Ulsoy, 2003). The solution is expressed in the form of an infinite series of modes written in terms of the matrix Lambert function. Their study has similarity with the concept of the state transition matrix in linear ordinary differential equations (ODEs), enabling its use for general classes of linear first order delay differential equations. In their paper, they utilized the analytical solution to systems of DDEs in terms of the matrix Lambert function to present a Laplace domain solution. Also presented validation examples, and also emphasize the analogy of the solution method to systems of ODEs. However, their study has some limitations, i.e they didn't put any hint for solution of second order delay differential equations.

(Jaaffar, Abdul Majid, & Senu, 2020), Studied a fifth-order direct multistep block method is proposed for solving the second-order delay differential equations (DDEs) directly with boundary conditions using constant step size. Thus, an efficient numerical method is needed for the numerical treatment of time delay in the applications. The proposed direct block method computes the numerical solutions at two points concurrently at each computed step along the interval. Different types of delays involved in his research. The shooting technique is utilized to deal with the boundary conditions by applying a Newton-like method to guess the next initial values.

As authors in (Seong & Majid, 2017), the implementation of direct two-point fourth and fifth order multi step block method in the form of Adams–Moulton method to solve second order delay differential equations (DDEs) directly without transforming the equations into system of first order DDEs. The authors proposed this methods to compute the numerical solutions at two points simultaneously which are implemented in predictor–corrector (PECE) mode.

To solve second order DDEs, author (Rasdi & Majid, 2015) apply the method of direct extended two and three point implicit one-step block method. As this author, this method is to solve directly the second order delay differential equations using the extended two and extended three point implicit one-step block methods using constant step size. The stability polynomials for the methods are obtained and their regions of absolute stability are discussed in his paper. The efficiency of the proposed method is supported by some numerical results.

(Rasdi et al., 2013) had proposed a two point and three point one-step block method for solving second order delay differential equations (DDEs). The one-step block method will solve directly the second order DDEs without reducing to first order equations. The solutions for the DDEs at two and three points simultaneously along the interval was computed by the two point and three point one-step block method. In his work he used the method to solve the retarded type of DDE of single delay using constant step size.

From the above literature survey it is very relevant to point out that no attempt has been made to study on analytical solution of second order linear delay differential equations. Hence this thesis focus on analytical solution of second order delay differential equations by steps method.

Chapter 3

Mathematical Preliminaries

In this chapter we gather some general preliminaries that we will use throughout this thesis. We shall not go into details but cite references in which the interested reader can find further details.

3.1 Method of steps

In the method of steps we begin with the constant coefficient delay differential equation, defined for $t > t_0$ and the initial function defined on the interval $t_0 - \tau \leq t \leq t_0$ where τ is the delay. We are looking for a continuous extension into the future. We first consider the interval $t_0 \leq t \leq t_0+\tau$ on which the DDE reduces to an ODE. We find a solution valid on this interval and then use this solution as the initial function for the interval $t_0+\tau \leq t \leq t_0+2\tau$. We then find a solution on $t_0 + \tau \leq t \leq t_0 + 2\tau$ and in this way the solution is extended forward from interval to interval (Bellman & Cooke, 1963).

Example 3.1.1. *Consider*

$$\begin{aligned} \frac{dx}{dt} &= -x(t-1), 0 \leq t \leq 3 \\ x(t) &= -10, t \leq 0 \end{aligned} \tag{3.1}$$

Then to solve (3.1):

1^{st} *step: Find solution for* $0 \leq t \leq 1$*:*

$0 \leq t \leq 1 \implies -1 \leq t-1 \leq 0 \implies x(t-1) = 10$

$$\frac{dx}{dt} = -10$$

$$x(t) = -10t$$

2^{nd} *step: Find solution for* $1 \leq t \leq 2$*:*

$1 \leq t \leq 2 \implies 0 \leq t-1 \leq 1 \implies x(t-1) = 10 - 10(t-1)$

$$\frac{dx}{dt} = -(10 - 10(t-1))$$

$$x(t) = -10(t-1) + 5(t-1)^2$$

3^{rd} *step: Find solution for* $2 \leq t \leq 3$*:*

$2 \leq t \leq 3 \implies 1 \leq t-1 \leq 2 \implies x(t-1) = -10(t-2) + 5(t-2)^2$

$$\frac{dx}{dt} = -(-10(t-2) + 5(t-2)^2)$$

$$x(t) = -5 + 5(t-2)^2 + \tfrac{5}{3}(t-1)^3$$

3.2 Fundamental matrix

Definition 3.2.1. *(Layek, 2015) If* $\psi_1(t), \psi_2(t), ..., \psi_n(t)$ *are linearly independent solution of the system of differential equation*

$$\dot{\psi}(t) = P\psi(t),$$

then every solution can be written in the form

$$\psi(t) = c_1\psi_1(t) + c_2\psi_2(t) + ... + c_n\psi_n(t) = \varphi(t)C.$$

A matrix $\varphi(t)$ *is called Fundamental matrix solution of the system of* $\dot{\psi}(t) = P\psi(t)$ *, if its columns forms a set of* n *linearly independent solutions of* $\dot{\psi}(t) = P\psi(t)$ *.*

3.3 Solution of nonhomogeneous linear systems

Theorem 3.3.1. *(Hastings, 1992) Let P be* $n \times n$ *matrix and let* $\varphi(t)$ *be a fundamental matrix for the homogeneous system*

$$\psi'(t) = P\psi(t).$$

Then, for any vector valued function f *that continuous on* (α, β)*, any* t_0 *in* (α, β) $\forall \alpha, \beta \in \mathbb{R}$ *and any constant column vector* ψ_{t_0}*, the solution to the nonhomogeneous*

$$\psi'(t) = P\psi(t) + f(t), \tag{3.2}$$

with initial value problem

$$\psi(t_0) = \psi_{t_0},$$

is given by

$$\psi(t) = [\varphi(t)][\varphi(t_0)]^{-1}\psi_{t_0} + [\varphi(t)]\int_{t_0}^{t}[\varphi(s)]^{-1}f(s)ds.$$

Proof. We start by expressing the yet unknown solution $\psi(t)$ to the nonhomogeneous system as

$$\psi(t) = \varphi(t)u(t), \tag{3.3}$$

where $\varphi(t)$ is the fundamental matrix for $\psi'(t) = P\psi(t)$ and $u(t)$ is a yet to be determined vector valued function. Plugging this formula for $\psi(t)$ into our nonhomogeneous system and using the product rule of derivative

$$\begin{aligned}
(\varphi(t)u(t))' &= P[\varphi(t)u(t)] + f(t) \\
\Longrightarrow\ [P\varphi(t)]u(t) + \varphi(t)u'(t) &= [P\varphi(t)]u(t) + f(t) \\
\Longrightarrow\ \varphi(t)u'(t) &= f(t).
\end{aligned} \tag{3.4}$$

But fundamental matrices are invertible. So we can rewrite equation (3.4) as

$$u'(t) = [\varphi(t)]^{-1}f(t). \tag{3.5}$$

Here, definite integral of (3.5) becomes

$$u(t) - u(t_0) = \int_{s=t_0}^{t} u'(s)ds = \int_{s=t_0}^{t}[\varphi(s)]^{-1}f(s)ds. \tag{3.6}$$

Then

$$u(t) = u(t_0) + \int_{s=t_0}^{t}[\varphi(s)]^{-1}f(s)ds,$$

$$\psi(t) = [\varphi(t)]u(t) = \varphi(t)u(t_0) + [\varphi(t)]\int_{s=t_0}^{t}[\varphi(s)]^{-1}f(s)ds \tag{3.7}$$

and

$$\psi(t_0) = [\varphi(t_0)]u(t_0) + [\varphi(t_0)]\int_{s=t_0}^{t_0}[\varphi(s)]^{-1}g(s)ds = [\varphi(t_0)]u(t_0) + 0. \tag{3.8}$$

Solving (3.8) for $u(t_0)$ and plugging it in to equation (3.7) gives

$$\psi(t) = [\varphi(t)][\varphi(t_0)]^{-1}\psi(t_0) + \varphi(t)\int_{s=t_0}^{t}[\varphi(s)]^{-1}f(s)ds.$$

□

3.4 Uniform convergence

Definition 3.4.1. *(Aliprantis & Burkinshaw, 1998) A sequence of functions $\{x_n\}$ is said to converge uniformly on an interval $[\alpha, \beta]$ to a function x if for any $\epsilon > 0$ there exists an integer N for which*

$$|x_n(t) - x(t)| < \epsilon, \forall n \geq N.$$

Theorem 3.4.1 (Weierstrass M-Test). *(Nelson, 2019) Let $\{x_n(t)\}$ be a sequence of functions with $|x_n(t)| \leq M_n, \ \ \forall t \in [\alpha, \beta]$ with $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{n=1}^{\infty} x_n(t)$ converges uniformly in $[\alpha, \beta]$ to a unique function $x(t)$.*

Proof. Let $x(t) = \sum_{n=1}^{\infty} x_n(t)$ and $s_k(t) = \sum_{n=1}^{k} x_n(t)$ converges uniformly to $x(t)$. That is

$$|x(t) - s_k(t)| = |\sum_{n=k+1}^{\infty} x_n(t)| \leq \sum_{n=k+1}^{\infty} |x_n(t)| \leq \sum_{n=k+1}^{\infty} M_n. \tag{3.9}$$

Since $\sum_{n=1}^{\infty} M_n$ converges. For any $\epsilon > 0$ we can find an N such that $\sum_{n=1}^{\infty} M_n < \epsilon \ \ \forall n \leq N$. This implies that $|x(t) - s_k(t)| < \epsilon \implies s_k$ converges uniformly to x(t).
Hence $\sum_{n=1}^{\infty} x_n(t)$ uniformly converges to $x(t)$. □

Lemma 3.4.1. *(Gronwall's Lemma).(Holte, 2009) Suppose that f(t) and g(t) are continuous real valued functions with $f(t) \geq 0, g(t) \geq 0$ on an interval $[\alpha, \beta]$. If,*

$$f(t) \leq c + k \int_a^t g(s) f(s) ds$$

for $c, k \geq 0$, then $f(t) \leq c\, e^{\int_a^t kg(s)ds}$.

Proof. Let $G(t) = c + k \int_a^t g(s) f(s) ds$. This implies $f(t) \leq G(t)$ and $G(a) = c, \ \ G'(t) = kg(t) f(t)$

$$\begin{aligned} G'(t) = kg(t)f(t) \leq kg(t)G(t) \quad &\implies \quad G'(t) \leq kg(t)G(t) \\ &\implies \quad \frac{G'(t)}{G(t)} \leq kg(t). \\ &\qquad\ \int_a^t \left(\frac{G'(t)}{G(t)}\right) ds \leq \int_a^t kg(s)ds \\ &\implies \quad \ln(G(s))|_a^t \leq \int_a^t kg(s)ds \\ &\implies \quad \ln\left(\frac{G(t)}{c}\right) \leq \int_a^t kg(s)ds \\ &\implies \quad G(t) \leq c\, e^{\int_a^t kg(s)ds} \\ &\implies \quad f(t) \leq G(t) \leq c\, e^{\int_a^t kg(s)ds}. \end{aligned} \tag{3.10}$$

Hence from (3.10) we conclude that $f(t) \leq c\, e^{\int_a^t kg(s)ds}$. □

Definition 3.4.2 (Lipschitz). *(Erneux, 2009) Let $\mathbb{D}$ be an open subset of $\mathbb{R}^2$. A function $f : \mathbb{D} \to \mathbb{R}^2$ is said to satisfy a Lipschitz condition on $\mathbb{D}$ if there is a positive constant K such that for all $(t, x_1), (t, x_2)$ in the domain $\mathbb{D}$,*

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|.$$

3.5 Software Packages DDE23

(Shampine, Thompson, & Kierzenka, 2000) In this section we will show that how to use DDE23 Matlab solver for solving linear delay differential equations, with constant delay and constant coefficient. We illustrate the straightforward solution of a DDE by computing and plotting the solution of any system of linear delay differential equations with constant delay and coefficients. For instance, consider the equation

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = bx_2(t - \tau_1) \end{cases} \quad t_0 \leq t \leq t_f \tag{3.11}$$
$$x_1(t) = 1, x_2(t) = 1, \ \ t \leq t_0.$$

The syntax has the form

```
sol = dde23(ddefile, lags, history, tspan );.
```

The interval $[t_0, t_f]$ is the interval of integration which is denote by $(''tspan'')$, the history argument is the name of a function that evaluates the solution at the input value of τ and returns it as a column vector, the function for evaluating the DDEs. Here $exam1h.m$ can be coded as:

```
function v=exam1f(t,x,Z)
xlag1=Z(:,1);
v=zeroos(2,1);
x(1)=x(2);
x(2)=b*xlag1(2);
```

The input array Z contains approximations to the solution at all the delayed arguments. Specifically, $Z(:, j)$ approximates $x(t - \tau_j)$ for τ_j given as $lags(j)$. It is not necessary to define local vectors $xlag1$ as we have done here, but often this makes the coding of the DDEs clearer. The $ddefile$ must return a column vector, (Shampine et al., 2000). This is perhaps a good place to point out that DDE23 does not assume that terms like $x(t-\tau_j)$ actually appear in the equations. After defining the equations in $exam1f.m$, the complete program $exam1.m$ to compute and plot the solution is:

```
plot(sol.t,sol.x);
title('********');
xlabel('+++++');
ylabel('>>>>>');
```

3.6 Stability of a harmonic oscillator with delay

(Li, 2019) In the field of control engineering of mechanical systems aftereffect phenomena are very common. Hence, aftereffect phenomenon are called delay systems. For instance equation describing a harmonic oscillator with delay system is given by

$$\ddot{y}(t) + ay(t) + by(t-1) = 0, \quad a, b \in \mathbb{R}\backslash\{0\}. \tag{3.12}$$

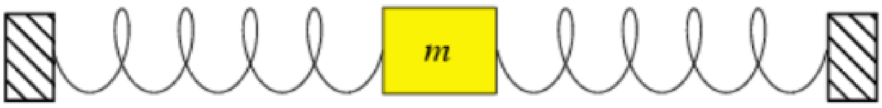

Figure 3.1: Mass spring system with delay in left spring.

From figure 3.1, the left spring is made of a material that causes a delayed response of one time unit before the spring stretches again when the spring is pushed in. So the left spring has a delayed feedback, whereas the right spring is made of a material with no such delay. The term $by(t-1)$ corresponds, therefore, with the left spring. Obviously, the term $ay(t)$ corresponds with the right spring.
To determine the stability of y = 0 for various $a, b \in \mathbb{R}\backslash\{0\}$, finding the characteristic equation is very important. Thus we seek exponential solutions of the form $y(t) = e^{\lambda t}$ with $\lambda \in \mathbb{C}$, Substituting $y(t) = e^{\lambda t}$ in equation (3.12) gives

$$\lambda^2 + a + be^{-\lambda} = 0. \tag{3.13}$$

Multiplying e^{λ} on both sides of (3.13) and define $H(\lambda)$ results, equivalently in

$$H(\lambda) = \lambda^2 e^{\lambda} + ae^{\lambda} + b = 0. \tag{3.14}$$

H is a special function, usually called an exponential polynomial or a quasi-polynomial.

The following Theorems and Lemmas are useful criterion to determine all about stability of equation (3.12).

Theorem 3.6.1. *(Smith, 2011; Hale, 1993) $y = 0$ is asymptotically stable for equation (3.12), if $\Re(\lambda) < 0$ for every characteristic root; it is unstable if there is a root satisfying $\Re(\lambda) > 0$.*

Theorem 3.6.2 (Pontryagin's). *(Pontrjagin, 1942) Let $H(\lambda) = h(\lambda, e^{\lambda})$, where $h(z, w)$ is a polynomial with a principal term. The function $H(iy)$ is now separated into real and imaginary parts; that is, we set $H(iy) = F(y) + iG(y)$. If all the zeros of the function $H(\lambda)$ lie in the open left half plane, then the zeros of the functions $F(y)$ and $G(y)$ are real, interlacing, and*

$$\Delta(y) := G'(y)F(y) - G(y)F'(y) > 0 \tag{3.15}$$

for all $y \in \mathbb{R}$. Moreover, in order that all the zeros of the function $H(\lambda)$ lie in the open left half plane, it is sufficient that one of the following conditions be satisfied:

(a) All the zeros of the functions $F(y)$ and $G(y)$ are real interlace, and the inequality (3.15) is satisfied for at least one value of y.

(b) All the zeros of the function $F(y)$ are real and for each of these zeros $y = y_0$, condition (3.15) is satisfied; that is, $F'(y_0)G(y_0) < 0$.

(c) All the zeros of the function $G(y)$ are real and for each of these zeros $y = y_0$ condition (3.15) is satisfied; that is, $G'(y_0)F(y_0) > 0$.

Lemma 3.6.1. *(Cahlon & Schmidt, 2004) If the zero solution of (3.12) is asymptotically stable, then $\Delta(0) = a(a + b) > 0$.*

Proof. We have

$$\begin{aligned} G(y) &= -y^2 \sin(y) + a \sin(y) \\ G'(y) &= -2y \sin(y) + (a - y^2) \cos(y) \\ F(y) &= -y^2 \cos(y) + a \cos(y) + b \\ F'(y) &= -2ycos(y) + y2sin(y) - asin(y). \end{aligned}$$

All the zeros of H lie in the left open half plane, because $y = 0$ is asymptotically stable. So we must have $\Delta(y) > 0$ for all $y \in \mathbb{R}$ by Theorem 3.6.2. In particular,

$$\Delta(0) = G'(0)F(0) - G(0)F'(0) = a(a + b) > 0.$$

□

Theorem 3.6.3. *If $a < 0$, then the zero solution of (3.12) is not asymptotically stable.*

Proof. $G(y) = (a - y^2)\sin(y)$, therefore $\pm i\sqrt{a}$ is a complex root of G as $a < 0$. This means that there is a root $\hat{\lambda}$ of H with $\Re(\hat{\lambda}) \geq 0$ by Theorem 3.6.2 This implies that $y = 0$ is not asymptotically stable. □

Theorem 3.6.4. *(Cahlon & Schmidt, 2004; Li, 2019) Assume $b < 0$. Then the zero solution of equation (3.12) is asymptotically stable if and only if $a > 0$ and there exists $k \in \mathbb{N} \cup \{0\}$ such that*

$$2k\pi < \sqrt{a} < (2k + 1)\pi$$

and

$$b > \max\{(2k)^2\pi^2 - a, a - (2k + 1)^2\pi^2\}.$$

Theorem 3.6.5. *(Li, 2019) Assume $b > 0$. Then the zero solution of (3.12) is asymptotically stable if and only if $a > 0$, and there exists $k \in \mathbb{N} \cup \{0\}$ such that*

$$(2k + 1)\pi < \sqrt{a} < (2k + 2)\pi$$

and

$$b < \min\{a - (2k + 1)^2\pi^2, (2k + 2)^2\pi^2 - a\}.$$

3.6.1 (Asymptotic) stability region

By considering Theorem 3.6.2, 3.6.3 and 3.6.4, all values of $a, b \in \mathbb{R}, b \neq 0$, regions of asymptotic stability are determined such that $y = 0$ is asymptotically stable for (3.12).

For every $k \in \mathbb{N} \cup \{0\}$ defines a set $A_k \subseteq \mathbb{R}^2$ such that $y = 0$ is asymptotically stable for (3.12) if $(a, b) \in A_k$ by using Theorem 3.6.3 and 3.6.4 .In this case A_k is a triangle. All values of $a \in \mathbb{R}$ and $b \neq 0$ for which $y = 0$ is asymptotically stable for (3.12) consists of infinitely many triangles A_k that alternate around $b = 0$. Denote $S = \bigcup_0^\infty A_k$ as the asymptotic stability region.

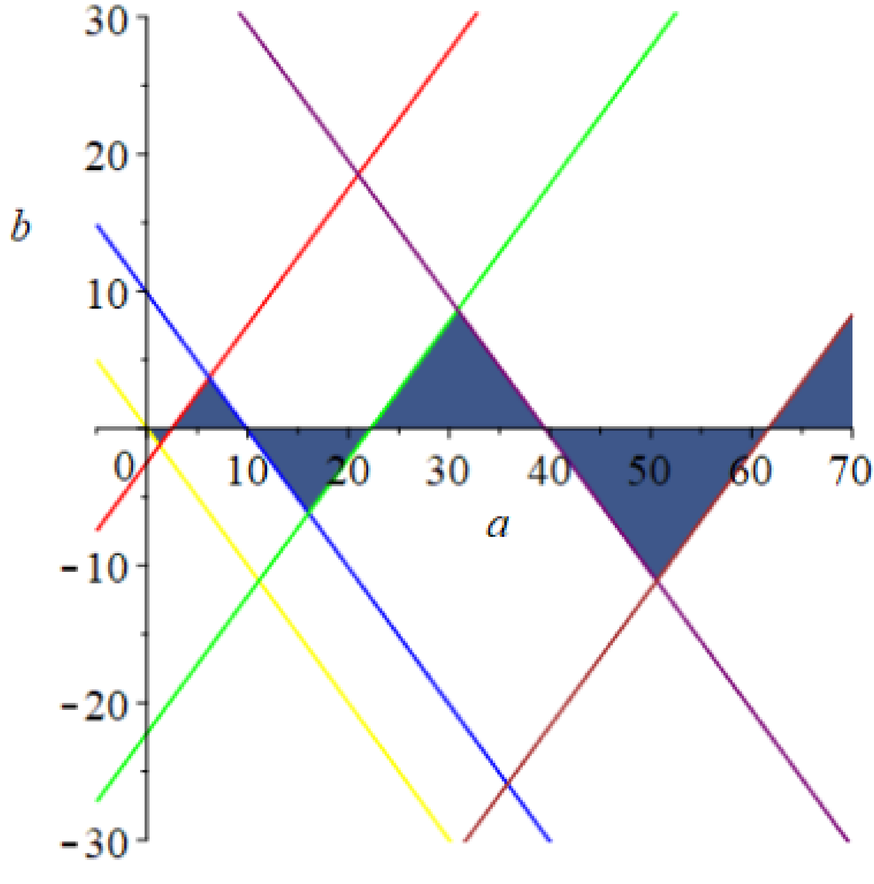

Figure 3.2: Asymptotic stability region enclosed by critical lines

Theorem 3.6.6. *(Li, 2019) If $(a, b) \notin \bigcup_0^\infty C_n \cup S$, then $y = 0$ is unstable for equation (3.12) where C_n is critical line for $n \in \mathbb{N} \cup \{0\}$.*

Theorem 3.6.7. *(Li, 2019) If $(a, b) \in \partial S \backslash \{(0, 0)\}$, then $y = 0$ is stable for equation (3.12).*

Chapter 4

Result and Discussion

4.1 Analysis of solution for second order delay differential equations

Let $\tau > 0$ is constant delay and $\phi(t) : [t_0 - \tau, t_0] \to \mathbb{R}$ be a continuously differentiable function. Consider second order delay differential equation

$$\begin{aligned} \ddot{x}(t) + \xi x(t) + \eta \dot{x}(t-\tau), \;\; t_0 \le t \le t_f, \\ x(t) = \phi(t), \;\; t_0 - \tau \le t \le t_0. \end{aligned} \tag{4.1}$$

Then after some substitution we obtain system of first order DDE

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\xi & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\eta \end{pmatrix} \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \end{pmatrix}, t_0 \le t \le t_f. \tag{4.2}$$

Now define g and Y_t respectively as, $g : [0, \infty) \times \mathbb{R} \times C := C([t_0 - \tau, t_0], \mathbb{R}^2) \to \mathbb{R}^2$ by $g(t, \Gamma) = A\Gamma(0) + B\Gamma(-\tau)$, where $A = \begin{pmatrix} 0 & 1 \\ -\xi & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & -\eta \end{pmatrix}$

and $Y_t \in C$ by $Y_t(\theta) = Y(t + \theta)$ for $\theta \in [t_0 - \tau, t_0]$ with $Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

Then $g(t, Y_t) = AY_t(0) + BY_t(-\tau) = AY(t) + BY(t-\tau) = \dot{Y}(t)$.

Therefore, problem (4.1) is equivalent to

$$\begin{aligned} \dot{Y}(t) = g(t, Y_t), \;\; t_0 \le t \le t_f, \\ Y(t) = \phi_Y(t), \;\; t \in [t_0 - \tau, t_0]. \end{aligned} \tag{4.3}$$

Our goal in this section is to establish general Theorem of existence and uniqueness of solutions of system of DDEs of retarded type (4.3).

4.1.1 Existence and Uniqueness

Before we go to state the Theorem of existence and uniqueness let's see the following lemma.

Lemma 4.1.1. *(Erneux, 2009) Let $g : [0,\infty)\times C \to \mathbb{R}^2$ be continuous vector valued function and satisfy Lipschitz condition. Then for each finite interval $[a,b] \in [0,\infty]$ and $M > 0$, there exists $L > 0$ such that*

$$\|g(t,\phi_Y)\| \leq L, t \in [a,b], ||\phi_Y|| \leq M.$$

Proof. If $\hat{0}$ denotes the zero function in $C, ||\phi_Y|| \leq M$, and K is Lipschitz constant for $t \in [a,b]$. Then,

$$\|g(t,\phi_Y)\| \leq \|g(t,\phi_Y) - g(t,\hat{0})\| + \|g(t,\hat{0})\| \leq K||\phi_Y - \hat{0}|| + \|g(t,\hat{0})\| \leq KM + P = L,$$

where $P = \max_{a\leq s\leq b} \|g(s,\hat{0})\|$. □

Theorem 4.1.1 (Existence). *Suppose $g : [0,\infty)\times\mathbb{R}\times C \to \mathbb{R}^2$ is continuous vector valued function and satisfies the Lipschitz condition (Lip), $t_0 \in [0,\infty)$ and $M > 0$. There exists $\delta > t_0$, such that if $\phi_Y \in C$ satisfies $||\phi_Y|| \leq M$, then there exists a solution $Y(t) = Y(t,\phi_Y)$ of (4.3) defined on $[t_0 - \tau, t_0 + \delta]$.*

Proof. Suppose that $||\phi_Y|| \leq M$. Let K be the Lipschitz constant for g on the set $[t_0, t_0+\delta]$ and let L be the bound on $\|g\|$ given in Lemma 4.1.1 for that set.

Let's use the method of successive approximations to solve (4.3) for $t \geq t_0$ and starting with the initial guess,

$$Y^{(0)}(t) = \phi_Y(t_0), t_0 \leq t \leq t_0 + \delta$$

and $Y^{(0)}(t) = \phi_Y(t - t_0)$ for $t \in [t_0 - \tau, t_0]$. Clearly, $\|Y^{(0)}(t)\| \leq M$ on $t_0 \leq t \leq t_0 + \delta$. Now, for $n = 1, 2, \ldots$, define

$$Y^{(n)}(t) = \phi_Y(t_0) + \int_{t_0}^{t} g(s, Y_s^{(n-1)})ds, t_0 \leq t \leq t_0 + \delta. \tag{4.4}$$

Again, $Y^{(n-1)}(t) = \phi_Y(t - t_0), t_0 - \tau \leq t \leq t_0$ so they are defined on $[t_0 - \tau, t_0 + \delta]$. For n=1, equation (4.4) becomes:

$$\|Y^{(1)}(t) - Y^{(0)}\| = \| \int_{t_0}^{t} g(s, Y_s^0)ds\| \leq L(t - t_0), t_0 \leq t \leq t_0 + \delta.$$

For n, equation (4.4) becomes

$$\begin{aligned} \|Y^{(n)}(t) - Y^{(n-1)}\| &= \| \int_{t_0}^{t} [g(s, Y_s^{(n-1)}) - g(s, Y_s^{(n-2)})]ds\| \\ &\leq K \int_{t_0}^{t} ||Y_s^{(n-1)} - Y_s^{(n-2)}||ds. \end{aligned} \tag{4.5}$$

In particular, setting n = 2

$$\|Y^{(2)}(t) - Y^{(1)}(t)\| \leq K\int_{t_0}^{t} L(s-t_0)ds = \frac{KL(t-t_0)^2}{2}$$

and

$$\|Y^{(3)}(t) - Y^{(2)}(t)\| \leq K\int_{t_0}^{t} KL\frac{(s-t_0)^2}{2}ds = \frac{L}{K}\frac{[K(t-t_0)]^3}{3!}.$$

An induction argument yields that

$$\|Y^{(n)}(t) - Y^{(n-1)}(t)\| \leq \frac{L}{K}\frac{[K(t-t_0)]^n}{(n)!}. \tag{4.6}$$

To show $Y^{(n)}(t)$ converges uniformly to a continuous function $Y(t)$ on $[t_0, t_0+\delta]$, take summation on both side of (4.6).

Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\|Y^{(n)}(t) - Y^{(n-1)}(t)\| &\leq \sum_{n=1}^{\infty}\frac{L}{K}\frac{[K(t-t_0)]^n}{(n)!} \\ &\leq \frac{L}{K}\sum_{n=1}^{\infty}\frac{(K\delta)^n}{n!}. \end{aligned} \tag{4.7}$$

Hence (4.7) is finite.

Therefore for $t_0 \leq t \leq t_0+\delta$, the infinite series

$$\sum_{n=1}^{\infty}\|Y^{n}(t) - Y^{(n-1)}(t)\| \leq \frac{L}{K}\sum_{n=1}^{\infty}\frac{(K\delta)^n}{n!}.$$

Since $\frac{L}{K}\sum\limits_{n=1}^{\infty}\frac{(K\delta)^n}{n!}$ converges, by Weierstrass M-Test, the series $\sum\limits_{n=1}^{\infty}\|Y^{(n)}(t) - Y^{(n-1)}(t)\|$ converge on $[t_0, t_0+\delta]$.

Therefore, the sequence of functions $\left\{Y^{(n)}(t)\right\}_{n=1}^{\infty}$ defined by (4.4) , converges uniformly to $Y(t)$ on $[t_0, t_0+\delta]$, i.e.

$$Y(t) = \lim_{n\to\infty} Y^{(n)}(t).$$

Now to prove that $Y(t)$ is a solution of (4.3), showing that $g(t, Y_t^{(n)}) \to g(t, Y_t)$ uniformly on $[t_0, t_0+\delta]$. If $g(t, Y_t)$ is Lipschitz continuous then we have

$$\|g(t, Y_t^{(n)}) - g(t, Y_t)\| \leq K\|Y_t^{(n)} - Y_t\|.$$

Also, by uniform convergence of $\{Y^{(n)}(t)\}$, for all $\epsilon > 0$ there exists $N(\epsilon) > 0$ such that

$$\|Y^{(n)}(t) - Y(t)\| \leq \frac{\epsilon}{K}, \quad \forall n > N$$

$$\|g(t, Y^{(n)}(t)) - g(t, Y(t))\| \leq K\|Y^{(n)}(t) - Y(t)\| \leq K\frac{\epsilon}{K}\forall n > N.$$

Therefore, $g(t, Y^{(n)}(t)) \to g(t, Y(t))$ uniformly on $[t_0, t_0 + \delta]$.

$$Y(t) = \lim_{n\to\infty} Y^{(n)}(t) = \phi_Y(t_0) + \lim_{n\to\infty}\int_{t_0}^{t} g(s, Y_s^{(n-1)})ds = \phi_Y(t_0) + \int_{t_0}^{t} g(s, Y_s)ds. \quad (4.8)$$

Hence, the function $Y(t)$ satisfies (4.3). □

Theorem 4.1.2 (Uniqueness). *Let $g : [0, \infty) \times C \times \mathbb{R} \to \mathbb{R}^2$ be continuous vector valued function and locally Lipschitz on its domain. Then given any $\phi_Y \in C$, there is at most one solution of the delay differential equation (4.3) on $[t_0 - \tau, t_0 + \delta]$ for $\delta > t_0$.*

Proof. Suppose(for contradiction) that $Z(t)$ is a second solution of (4.3). Then,

$$Y(t) = \phi(t_0) + \int_{t_0}^{t} g(s, Y(s))ds \ \ and \ \ Z(t) = \phi(t_0) + \int_{t_0}^{t} f(s, Z(s))ds. \quad (4.9)$$

Subtracting these two equations gives

$$Y(t) - Z(t) = \int_{t_0}^{t}[g(s, Y(s)) - f(s, Z(s))]ds$$

$$\|Y(t) - Z(t)\| \leq \int_{t_0}^{t} \|[g(s, Y(s)) - f(s, Z(s))]\|ds \leq K\int_{t_0}^{t} \|Y(s) - Z(s)\|ds,$$

where K is Lipschitz constant. Then, we obtain

$$\|Y(t) - Z(t)\| \leq K\int_{t_0}^{t} \|Y(s) - Z(s)\|ds.$$

By Gronwall's Lemma,

$$\|Y(t) - Z(t)\| = 0$$

$$\implies Y(t) - Z(t) = 0 \implies Y(t) = Z(t) \quad \forall t \in [t_0, t_0 + \delta],$$

contradicting. Therefore there cannot be two different solution on its domain. □

4.1.2 Stability Analysis

A key tool for determining the stability of delay differential equations are the corresponding characteristic equations. Hence consider

$$\dot{Y}(t) = AY(t) + BY(t - \tau). \quad (4.10)$$

We seek exponentially growing solutions of (4.10) of the form

$$Y(t) = e^{\lambda t}V, V \neq 0,$$

where λ is complex and $V \in \mathbb{R}^2$. Then $\dot{Y}(t) = \lambda e^{\lambda t}V = AY(t) + BY(t-\tau) = Ae^{\lambda t}V + Be^{\lambda t}e^{-\lambda\tau}V$ which is equivalent to $AV + Be^{-\lambda\tau}V = \lambda V$, where A and B are matrices obtained after reduction of $\ddot{x}(t) + \xi x(t) + \eta\dot{x}(t-\tau)$ to system of first order delay differential equations.

For simplicity, let $\tau = 1$, then equation (4.10) becomes:

$$\dot{Y}(t) = AY(t) + BY(t-1). \tag{4.11}$$

Here the corresponding characteristic equation of (4.11) is

$$\det(A + Be^{-\lambda} - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 \\ -\xi & -\eta e^{-\lambda} - \lambda \end{vmatrix} = 0$$

$$\iff \quad \lambda^2 + \xi + \lambda\eta e^{-\lambda} = 0. \tag{4.12}$$

The stability of a solution is determined by the largest real part of all the solutions of the characteristic equation(Smith, 2011). Let $\lambda_k, k \in \{0 \cup \mathbb{N}\}$ be all the solutions of the characteristic equation (4.12) and define

$$\beta = \max_{k\in\{0\cup\mathbb{N}\}} \Re(\lambda_k).$$

If $\beta > 0$, then the solution of the DDE is unstable, if $\beta < 0$, then the solution is asymptotically stable, and if $\beta = 0$, then we have to continue our investigation. We wish to know the curves on which $\beta = 0$. To find these curves, we write $\lambda = i\omega$ with $\omega \in \mathbb{R}$ and substitute in equation (4.12), we obtain the following equation for ω.

$$-\omega^2 + \xi + \eta i\omega(\cos\omega - i\sin\omega) = 0. \tag{4.13}$$

By separating the real and imaginary part,we get

$$\begin{cases} -\omega^2 + \xi + \eta\omega\sin\omega = 0 \\ \eta\omega\cos\omega = 0. \end{cases} \tag{4.14}$$

Now, $\cos(\omega) = 0$ if and only if $\omega = (2n+1)\frac{\pi}{2}$ with $n \in \mathbb{Z}$.

Therefore

$$\eta_n(\xi) = \frac{2(-1)^n}{\pi(2n+1)}\left(((2n+1)\frac{\pi}{2})^2 - \xi\right). \tag{4.15}$$

Hence equation (4.15) is useful to create critical line.

4.1.3 (Asymptotic) stability region in our case

The following Theorem 4.1.3 and Theorem 4.1.4 are basically Theorem 3.4 and Theorem 3.5 respectively in in ((Li, 2019)) what I have stated in mathematical preliminaries on Theorem 3.6.3 and Theorem 3.6.4, but it has been translated in our parameters ξ and η and their proofs also slightly been adjusted to fit our context.

Theorem 4.1.3. *Assume $\eta < 0$. Then the zero solution of (4.11) is asymptotically stable if and only if $\xi > 0$ and there exists $k \in \mathbb{N} \cup \{0\}$ such that*

$$\frac{(4k+1)\pi}{2} < \sqrt{\xi} < \frac{(4k+3)\pi}{2}$$

and

$$\eta > \max\left\{\left(\frac{(4k+1)\pi}{2}\right)^2 - \xi, \xi - \left(\frac{(4k+3)\pi}{2}\right)^2\right\}.$$

Proof. By multiplying e^{λ} on both side of (4.12) and define:

$$H(\lambda) = \lambda^2 e^{\lambda} + \xi e^{\lambda} + \lambda\eta.$$

Hence from Pontryagin's Theorem we have:

$$\begin{aligned} F(x) &= -x^2\cos(x) + \xi\cos(x) \\ F'(x) &= -2x\cos(x) + x^2\sin(x) - \xi sin(x) \\ G(x) &= -x^2\sin(x) + \xi\sin(x) + \eta x \\ G'(x) &= -2x\sin(x) - (x^2 - \xi)\cos(x) + \xi. \end{aligned}$$

Assume that $\xi > 0$. The zeros of F are $x = \pm\sqrt{\xi}$ and $x = (2n+1)\frac{\pi}{2}$ with $n \in \mathbb{Z}$. If x_0 is a zero of F, then

$$\Delta(x_0) = -F'(x_0)G(x_0) = -\left(-2x\cos(x) + x^2\sin(x) - \xi\sin(x)\right)\left(-x^2\sin(x) + \xi\sin(x) + \eta x\right).$$

All the zeros of F and G are real and $\Delta(x) > 0$ for all $x \in \mathbb{R}$ by Theorem 3.6.2. So as $\eta < 0$ and $x = \pm\sqrt{\xi}$ are zeros of F, the following holds

$$\begin{aligned} \Delta(-\sqrt{\xi}) = \Delta(\sqrt{\xi}) > 0 &\iff 2\eta\xi\cos(\sqrt{\xi}) > 0 \\ &\iff \cos(\sqrt{\xi}) < 0 \\ &\iff \frac{(4k+1)\pi}{2} < \sqrt{\xi} < \frac{(4k+3)\pi}{2} \quad for\ \ some\ \ k = 0, 1, 2, ... \end{aligned}$$

So there exists $k \in \mathbb{N} \cup \{0\}$ such that $\frac{(4k+1)\pi}{2} < \sqrt{\xi} < \frac{(4k+3)\pi}{2}$.
At the points $x = (2n+1)\frac{\pi}{2}, (n \in \mathbb{Z})$ we have

$$\Delta((2n+1)\frac{\pi}{2}) = \left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)^2 - \eta(2n+1)\frac{\pi}{2}\left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)(-1)^n.$$

Thus

$$\begin{aligned}\Delta((2n+1)\frac{\pi}{2}) > 0 \iff & \left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)^2 + \eta(2n+1)\frac{\pi}{2}\left(\left(\xi - (2n+1)\frac{\pi}{2}\right)^2\right)(-1)^n > 0 \\ \iff & \left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)^2 > \eta(2n+1)\frac{\pi}{2}\left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)(-1)^n.\end{aligned} \tag{4.16}$$

Inequality (4.16) must hold. Hence we distinguish the following two cases for $n \in \mathbb{Z}$.

Case 1 : Let $2n > 4k \implies 2n+1 > 4k+1$. Thus $(2n+1)\frac{\pi}{2} > \frac{\pi}{2}(4k+1)$ and $0 < \frac{\pi}{2}(4k+1) < \sqrt{\xi}$, so that $\left(\frac{\pi}{2}(4k+1)\right)^2 < \xi < \left(\frac{\pi}{2}(4k+3)\right)^2 \leq \left((2n+1)\frac{\pi}{2}\right)^2$ and therefore $\left((2n+1)\frac{\pi}{2}\right)^2 - \xi > 0$. For n is even, the right hand side of (4.16) is negative, and inequality (4.16) is satisfied.
For n is odd, (4.16) is equivalent to

$$\xi - \left((2n+1)\frac{\pi}{2}\right)^2 < \eta(2n+1)\frac{\pi}{2}.$$

Observe that $\xi - \left((2n+1)\frac{\pi}{2}\right)^2 < \eta(2n+1)\frac{\pi}{2}$ holds for all odd n with $2n+1 > 4k+1 \implies n > 2k$, if and only if it holds for $2n+1 = 4k+3 \implies n = 2k+1$.

Case 2 : Let $0 \leq 2n+1 \leq 4k+1$.
Thus $0 < (2n+1)\frac{\pi}{2} < \frac{\pi}{2}(4k+1) < \sqrt{\xi}$ and $\left((2n+1)\frac{\pi}{2}\right)^2 - \xi < 0$. For n odd, the right hand side of (4.16) is negative, and inequality (4.16) is satisfied.
For n even, (4.16) is equivalent to

$$\left((2n+1)\frac{\pi}{2}\right)^2 - \xi < \eta\left((2n+1)\frac{\pi}{2}\right)$$

. Observe that $\left((2n+1)\frac{\pi}{2}\right)^2 - \xi < \eta\left((2n+1)\frac{\pi}{2}\right)$ holds for all even n with $0 \leq 2n+1 \leq 2k+1$ if and only if it holds for $n = 2k$.

Combining these two cases results in $\eta > \max\left\{\left(\frac{(4k+1)\pi}{2}\right)^2 - \xi, \xi - \left(\frac{(4k+3)\pi}{2}\right)^2\right\}$.
Now suppose that $\xi > 0$ and there exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$\frac{(4k+1)\pi}{2} < \sqrt{\xi} < \frac{(4k+3)\pi}{2}$$

and $\eta > \max\left\{\left(\frac{(4k+1)\pi}{2}\right)^2 - \xi, \xi - \left(\frac{(4k+3)\pi}{2}\right)^2\right\}$. Then all the zeros of F are real and for each zero $x_0 \in \{\sqrt{\xi}, -\sqrt{\xi}, (2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$ of F we have $\Delta(x_0) = -F'(x_0)G(x_0) > 0$. Therefore, every zero of H has negative real part by Theorem 3.6.2.(b). □

Theorem 4.1.4. *Assume $\eta > 0$. Then the zero solution of (4.11) is asymptotically stable if and only if $\xi > 0$ and there exists $k \in \mathbb{N} \cup \{0\}$ such that*

$$\frac{(4k+3)\pi}{2} < \sqrt{\xi} < \frac{(4k+5)\pi}{2}$$

and

$$\eta > \min\left\{\xi - \left(\frac{(4k+3)\pi}{2}\right)^2, \left(\frac{(4k+5)\pi}{2}\right)^2 - \xi\right\}.$$

Proof. We have

$$\begin{aligned} F'(x) &= -2x\cos(x) + x^2\sin(x) - \xi\sin(x) \\ G(x) &= -x^2\sin(x) + \xi\sin(x) + \eta x. \end{aligned}$$

Assume that that $\xi > 0$. The zeros of F are $x = \pm\sqrt{\xi}$ and $x = (2n+1)\frac{\pi}{2}$ with $n \in \mathbb{Z}$. If x_0 is a zero of F, then

$$\Delta(x_0) = -F'(x_0)G(x_0) = -\left(-2x\cos(x) + x^2\sin(x) - \xi\sin(x)\right)\left(-x^2\sin(x) + \xi\sin(x) + \eta x\right).$$

All the zeros of F and G are real and $\Delta(x) > 0$ for all $x \in \mathbb{R}$ by Theorem 3.6.2. So as $\eta > 0$ and $x = \pm\sqrt{\xi}$ are zeros of F, the following holds

$$\begin{aligned} \Delta(-\sqrt{\xi}) = \Delta(\sqrt{\xi}) > 0 &\iff 2\eta\xi\cos(\sqrt{\xi}) > 0 \\ &\iff \cos(\sqrt{\xi}) > 0 \\ &\iff \frac{(4k+3)\pi}{2} < \sqrt{\xi} < \frac{(4k+5)\pi}{2} \quad for\ \ some\ \ k = 0, 1, 2, ... \end{aligned}$$

So there exists $k \in \mathbb{N} \cup \{0\}$ such that $\frac{(4k+3)\pi}{2} < \sqrt{\xi} < \frac{(4k+5)\pi}{2}$.

At the points $x = (2n+1)\frac{\pi}{2}, (n \in \mathbb{Z})$ we have

$$\Delta((2n+1)\frac{\pi}{2}) = \left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)^2 - \eta(2n+1)\frac{\pi}{2}\left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)(-1)^n.$$

Thus

$$\begin{aligned} \Delta((2n+1)\frac{\pi}{2}) > 0 &\iff \left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)^2 + \eta(2n+1)\frac{\pi}{2}\left(\left(\xi - (2n+1)\frac{\pi}{2}\right)^2\right)(-1)^n > 0 \\ &\iff \left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)^2 > \eta(2n+1)\frac{\pi}{2}\left(\left((2n+1)\frac{\pi}{2}\right)^2 - \xi\right)(-1)^n. \end{aligned} \tag{4.17}$$

Inequality (4.17) must hold. Thus we distinguish the following two cases for $n \in \mathbb{Z}$.

Case 1 : Let $2n+1 > 4k+3$. Thus $(2n+1)\frac{\pi}{2} > \frac{\pi}{2}(4k+3)$ and $0 < \frac{\pi}{2}(4k+3) < \sqrt{\xi}$, so that $\left(\frac{\pi}{2}(4k+3)\right)^2 < \xi < \left(\frac{\pi}{2}(4k+5)\right)^2 \leq \left((2n+1)\frac{\pi}{2}\right)^2$ and therefore $\left((2n+1)\frac{\pi}{2}\right)^2 - \xi > 0$. For n is odd, the right hand side of (4.17) is negative, and inequality [4.17] is satisfied. For n is even, (4.17) is equivalent to $\left((2n+1)\frac{\pi}{2}\right)^2 - \xi > \eta$. Observe that $\left((2n+1)\frac{\pi}{2}\right)^2 - \xi > \eta$ holds for all even n with $2n+1 > 4k+3 \implies n > 2k+1$ if and only if it holds for $n = 2k+2$.

Case 2 : Let $0 \leq 2n+1 \leq 4k+3$.
Thus $0 < (2n+1)\frac{\pi}{2} < \frac{\pi}{2}(4k+3) < \sqrt{\xi}$ and $\left((2n+1)\frac{\pi}{2}\right)^2 - \xi < 0$. For n even, the right hand side of (4.17) is negative, and inequality (4.17) is satisfied. For n odd, (4.17) is equivalent to $\xi - \left((2n+1)\frac{\pi}{2}\right)^2 > \eta\left((2n+1)\frac{\pi}{2}\right)$. Observe that $\xi - \left((2n+1)\frac{\pi}{2}\right)^2 > \eta$ holds for all odd n with $0 \leq 2n+1 \leq 2k+3$ if and only if it holds for $n = 2k+1$.

Combining these two cases results in $\eta > \min\left\{\xi - \left(\frac{(4k+3)\pi}{2}\right)^2, \left(\frac{(4k+5)\pi}{2}\right)^2 - \xi\right\}$.
Now suppose that $\xi > 0$ and there exists a $k \in \mathbb{N} \cup \{0\}$ such that

$$\frac{(4k+1)\pi}{2} < \sqrt{\xi} < \frac{(4k+3)\pi}{2}$$

and $\eta > \min\left\{\left(\frac{(4k+1)\pi}{2}\right)^2 - \xi, \xi - \left(\frac{(4k+3)\pi}{2}\right)^2\right\}$. Then all the zeros of F are real and for each zero $x_0 \in \{\sqrt{\xi}, -\sqrt{\xi}, (2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$ of F we have $\Delta(x_0) = -F'(x_0)G(x_0) > 0$. Therefore, every zero of H has negative real part by Theorem 3.6.2.(b). □

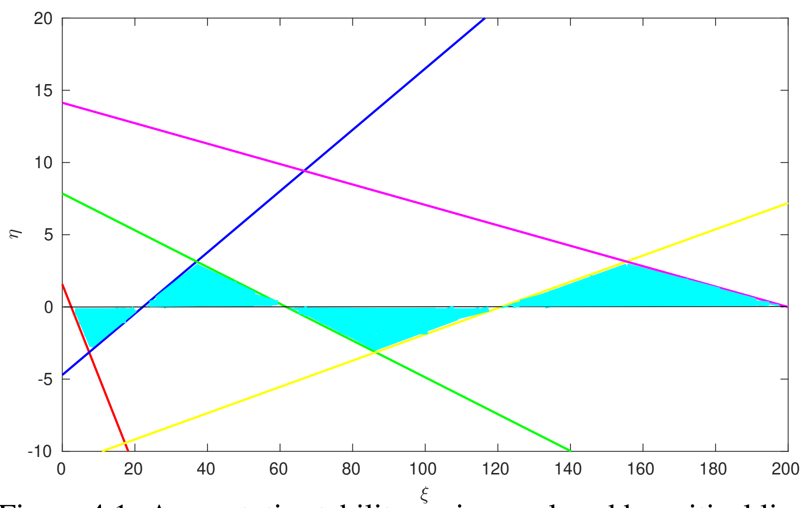

Figure 4.1: Asymptotic stability region enclosed by critical lines.

For arbitrary delay such that $\tau \neq 1$ of equation (4.10), the first step is rewriting it in the form of (4.11) . This can be done by a coordinate transformation $\mu = \frac{t}{\tau}$. Now define Y(t) by $Y(t) = \overline{Y}(\mu(t))$.

Then

$$\dot{Y}(t) = \frac{d}{d\mu}\overline{Y}(\mu(t)).\frac{d\mu(t)}{dt} = \frac{1}{\tau}\dot{\overline{Y}}(\mu),$$
$$Y(t-\tau) = \overline{Y}\left(\frac{t-\tau}{\tau}\right).$$

Hence equation (4.10) equivalent to

$$\dot{\overline{Y}}(\mu) = \tau A\overline{Y}(\mu) + \tau B\overline{Y}(\mu - 1), \mu > 1. \tag{4.18}$$

Thus, the procedures to analyze the stability of equation (4.18) is the same with what we have seen above.

Remark 4.1.1. *We have* $\lim_{t\to\infty} Y(t) = 0$ *if and only if* $\lim_{\mu\to\infty} \overline{Y}(\mu) = 0$*, because* $t \to \infty$ *if and only* $\mu \to \infty$*. Hence,* $\overline{Y}(\mu) = 0$ *is asymptotically stable for (4.18) if and only if* $Y(t) = 0$ *is asymptotically stable for (4.10) (Li, 2019).*

4.2 Application of steps method to find analytical solution

In this section we are going to drive general formula to solve

$$\begin{aligned} \ddot{x}(t) + \xi x(t) + \eta \dot{x}(t-\tau) = 0, \forall t \in [t_0, t_f] \\ x(t) = \phi(t), \forall t \in [-\tau, t_0]. \end{aligned} \tag{4.19}$$

Hence after reduction of order equation (4.19) becomes:

$$\dot{Y}(t) = AY(t) + BY(t-\tau), \forall t_0 \le t \le t_f, \tag{4.20}$$

where $A, B, \dot{Y}(t)$ and $Y(t-\tau)$ are as we defined in section 1.2. Then equation (4.20) with history function defined on $-\tau \le t \le 0$ becomes:

$$\begin{cases} \dot{Y}(t) = AY(t) + BY(t-\tau), \ \ t_0 \le t \le t_f \\ Y(t) = \phi_Y(t) = \left\{ \begin{bmatrix} \phi_x(t) & \phi_y(t) \end{bmatrix}^T = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T \right\}, \ \ t < t_0. \end{cases} \tag{4.21}$$

Now by taking initial history function on $t_0 - \tau \le t \le t_0$, then start from the $t_0 \le t \le t_0 + \tau$ to solve equation (4.21) .

Hence,

$$\begin{aligned} t_0 \le t \le t_0 + \tau \Longrightarrow t_0 - \tau \le t - \tau \le t_0 \\ Y(t-\tau) = \phi_{Y_0}(t-\tau). \end{aligned} \tag{4.22}$$

Thus, from equation (4.21) and (4.22) we get

$$\dot{Y}(t) = AY(t) + B\phi_{Y_0}(t-\tau), \ \ t_0 \le t \le t_0 + \tau, \tag{4.23}$$

with the initial conditions

$$Y(t_0) = \phi_{Y_0}(t_0) = \left\{ \begin{bmatrix} \phi_{x_0}(t_0) & \phi_{y_0}(t_0) \end{bmatrix}^T = \begin{bmatrix} x_0(t_0) & y_0(t_0) \end{bmatrix}^T \right\}, \ \ t < t_0.$$

Equation (4.23) can be written as

$$\dot{Y}(t) - AY(t) = B\phi_{Y_0}(t-\tau), \ \ t_0 \le t \le t_0 + \tau. \tag{4.24}$$

Equation (4.24) is nonhomogeneous ODE and it's solution form is

$$Y(t) = \varphi(t)\varphi^{-1}(t_0)\phi_{Y_0}(t_0) + \varphi(t) \int_{t_0}^{t} \left(\varphi^{-1}(\alpha) B \phi_{Y_0}(\alpha - \tau) \right) d\alpha, \tag{4.25}$$

where $\varphi(t)$ is fundamental matrix. For second interval such that $t_0 + \tau \le t \le t_0 + 2\tau$, we use $\phi_{Y_1}(t)$ is history function, where $\phi_{Y_1}(t)$ is equal with equation (4.25).

Equation (4.20) with history function (4.25) to gather can be written as

$$\begin{cases} \dot{Y}(t) = AY(t) + BY(t-\tau), \;\; t_0 \le t \le t_f \\ Y(t) = \phi_{Y_1}(t) = \left\{ \begin{bmatrix} \phi_{x_1}(t) & \phi_{y_1}(t) \end{bmatrix}^T = \begin{bmatrix} x_1(t) & y_1(t) \end{bmatrix}^T \right\}, \;\; t < t_0 + \tau. \end{cases} \tag{4.26}$$

To omit delay part from (4.26), if $t_0 + \tau \le t \le t_0 + 2\tau$, then $t_0 \le t - \tau \le t_0 + \tau$ and we have also $Y(t-\tau) = \phi_{Y_1}(t-\tau)$. Hence,on the second interval (4.20) is

$$\dot{Y}(t) = AY(t) + B\phi_{Y_1}(t-\tau), \;\; t_0 + \tau \le t \le t_0 + 2\tau \tag{4.27}$$

with the initial conditions

$$Y(t_0+\tau) = \phi_{Y_1}(t_0+\tau) = \left\{ \begin{bmatrix} \phi_{x_1}(t_0+\tau) & \phi_{y_1}(t_0+\tau) \end{bmatrix}^T = \begin{bmatrix} x_1(t_0+\tau) & y_1(t_0+\tau) \end{bmatrix}^T \right\}.$$

Equation (4.27) is nonhomogeneous ODE and it's solution has a form of

$$Y(t) = \varphi(t)\varphi^{-1}(t_0+\tau)\phi_{Y_1}(t_0+\tau) + \varphi(t)\int_{t_0+\tau}^{t} \left(\varphi^{-1}(\alpha)B\phi_{Y_1}(\alpha-\tau)\right) d\alpha. \tag{4.28}$$

Continuing with similar fashion, for final time interval $t_0 + (k-1)\tau \le t \le t_0 + k\tau$, we use history function $Y(t) = \phi_{Y_{k-1}}(t)$ defined on $t_0 + (k-2)\tau \le t \le (t_0 + k - 1)\tau$

with the initial condition,

$$\begin{aligned} Y(t_0 + (k-1)\tau) &= \phi_{Y_{t_0(k-1)\tau}}(t_0 + (k-1)\tau) \\ &= \begin{bmatrix} \phi_{x_{t_0+(k-1)\tau}}(t_0 + (k-1)\tau) & \phi_{y_{t_0+(k-1)\tau}}(t_0 + (k-1)\tau) \end{bmatrix}^T \\ &= \begin{bmatrix} x_{t_0+(k-1)\tau}(t_0 + (k-1)\tau) & y_{t+0+(k-1)\tau}(t_0 + (k-1)\tau) \end{bmatrix}^T. \end{aligned}$$

Hence, for final time interval, $t_0 + (k-1)\tau \le t \le t_0 + k\tau$, we get a solution of the form

$$Y(t) = \varphi(t)\varphi^{-1}(t_0+(k-1)\tau)\phi_{Y_{k-1}}(t_0+(k-1)\tau) + \varphi(t)\int_{t_0+(k-1)\tau}^{t} \left(\varphi^{-1}(\alpha)B\phi_{Y_{k-1}}(\alpha-\tau)\right) d\alpha,$$

where $k \in \mathbb{N}$ and $\phi_{Y_{k-1}}(t)$ is solution of (4.20) for $t \in [t_0 + (k-2)\tau, t_0 + (k-1)\tau]$.

4.3 Illustrated Examples

Example 4.3.1. *Solve*

$$\begin{cases} \ddot{x}(t) = 4x(t) + 2\dot{x}(t-1), \;\; 0 \le t \le 3, \\ x(t) = \phi(t) = t+1, \;\; t \le 0, \end{cases} \tag{4.29}$$

by steps method for 0≤ t ≤ 3, where $\phi(t)$ is history functions and compare the solution with mathlab solver DDE23.

Solution 1. *The reduced form of (4.29) is*

$$\dot{Y}(t) = AY(t) + BY(t-1), \;\; 0 \le t \le 3, \tag{4.30}$$

where

$$\dot{Y}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \; Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \; Y(t-1) = \begin{bmatrix} x(t-1) \\ y(t-1) \end{bmatrix}.$$

(4.30) with history functions is

$$\begin{cases} \dot{Y}(t) = AY(t) + BY(t-1), \;\; 0 \le t \le 3, \tau = 1 \\ Y(t) = \phi_{Y_0}(t) = \left\{ \begin{bmatrix} \phi_{x_0}(t) & \phi_{y_0}(t) \end{bmatrix}^T = \begin{bmatrix} t+1 & 1 \end{bmatrix}^T \right\}, \;\; t \le 0. \end{cases} \tag{4.31}$$

Since we have initial history function $\phi_{Y_0}(t)$ on $-1 \le t \le 0$, we will start from $0 \le t \le 1$ to solve equation (4.31).

Hence,

$$\begin{aligned} 0 \le t \le 1 &\Longrightarrow -1 \le t-1 \le 0, \\ Y(t-\tau) &= \phi_{Y_0}(t-1). \end{aligned} \tag{4.32}$$

Thus, from equation (4.31) and (4.32) we get nonhomogeneous ODE

$$\dot{Y}(t) = AY(t) + B\phi_{Y_0}(t-1), \;\; 0 \le t \le 1 \tag{4.33}$$

with the initial conditions

$$Y(0) = \phi_{Y_0}(0) = \left\{ \begin{bmatrix} \phi_{x_0}(0) & \phi_{y_0}(0) \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \right\}, \;\; t < 0.$$

Equation (4.33) can be written as

$$\dot{Y}(t) - AY(t) = B\phi_{Y_0}(t-1), \;\; 0 \le t \le 1. \tag{4.34}$$

Now fundamental matrix $\varphi(t)$, it's inverse $\varphi^{-1}(t)$ and $\varphi^{-1}(t_0 = 0)$ are

$$\varphi(t) = \begin{bmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{bmatrix}, \; \varphi^{-1}(t) = \frac{-1}{4}\begin{bmatrix} -2e^{-2t} & -e^{-2t} \\ -2e^{2t} & e^{2t} \end{bmatrix}, \; \varphi^{-1}(0) = \frac{-1}{4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}$$

respectively. Then, solution of (4.31) becomes

$$\begin{aligned} Y(t) &= \varphi(t)\varphi^{-1}(0)\phi_{Y_0}(0) + \varphi(t)\int_0^t \varphi^{-1}(\alpha)B\phi_{Y_0}(\alpha-1)d\alpha \\ &= \begin{cases} x_1(t) = e^{2t} + \frac{1}{2}e^{-2t} - \frac{1}{8} \\ y_1(t) = e^{-2t}(0.5e^{2t} - 1) - e^{2t}(0.5e^{-2t} - 2) \end{cases}, 0 \le t \le 1. \end{aligned} \tag{4.35}$$

where $\tau = 1$, we use $\phi_{Y_1}(t)$ is history function, where $\phi_{Y_1}(t)$ is equal with equation [4.35]. Thus equation (4.30) with history function (4.35) can be written as

$$\begin{cases} \dot{Y}(t) = AY(t) + BY(t-1), \;\; 0 \le t \le 3, \\ Y(t) = \phi_{Y_1}(t) = \left\{ \begin{bmatrix} \phi_{x_1}(t) & \phi_{y_1}(t) \end{bmatrix}^T = \begin{bmatrix} x_1(t) & y_1(t) \end{bmatrix}^T \right\}, \;\; t < 1. \end{cases} \tag{4.36}$$

To omit delay part from (4.36), if $1 \le t \le 2$, then $0 \le t-1 \le 1$ and we have also $Y(t-1) = \phi_{Y_1}(t-1)$. Hence, on the second interval equation (4.30) becomes

$$\dot{Y}(t) = AY(t) + B\phi_{Y_1}(t-1), \;\; 1 \le t \le 2, \tag{4.37}$$

where:

$$\phi_{Y_1}(t-1) = \begin{cases} x_1(t-1) = e^{2(t-1)} + \frac{1}{2}e^{-2(t-1)} - \frac{1}{8} \\ y_1(t-1) = e^{-2(t-1)}(0.5e^{2(t-1)} - 1) - e^{2(t-1)}(0.5e^{-2(t-1)} - 2). \end{cases}$$

Here we need to find $\varphi^{-1}(1)$ to solve (4.37).

Hence,

$$\varphi^{-1}(1) = \frac{-1}{4}\begin{bmatrix} -2e^2 & -e^{-2} \\ -2e^{-2} & e^{-2} \end{bmatrix}.$$

Thus solution of (4.37) for $1 \le t \le 2$ becomes:

$$x_2(t) = \frac{1}{2}e^{-2t}\left(\frac{1}{4}e^{2t} + \frac{t}{4}e - \frac{7}{16}e^{4t-2} - 0.41\right) + \frac{1}{2}e^{2t}\left(\frac{1}{4}e^{-2t} + \frac{7t}{4}e^{-2} + \frac{1}{16}e^{-4t+1} - 11.76\right)$$
$$y_2(t) = e^{2t}\left(\frac{1}{4}e^{-2t} + \frac{7t}{4}e^{-2} + \frac{1}{16}e^{-4t+1} - 11.76\right) - \frac{1}{4} - e^{-2t}\left(\frac{t}{4}e - \frac{7}{16}e^{4t-2} - 0.4068\right) \tag{4.38}$$

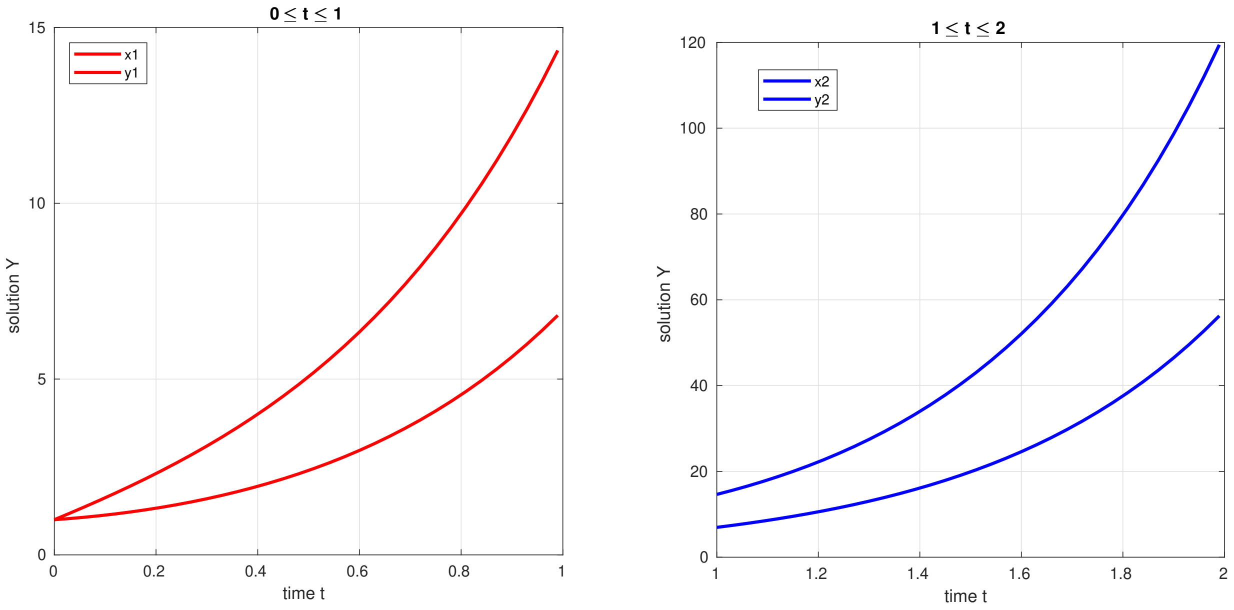

Figure 4.2: Graph of Equation [4.35] and [4.38] respectively.

And, by the same to above procedure, for third interval i.e $2 \le t \le 3$, we used $\phi_{Y_2}(t)$ as history function where, $\phi_{Y_2}(t)$ is equal with equation (4.38).

Hence , if $2 \leq t \leq 3$*, then* $x_3(t)$ *and* $y_3(t)$ *are*

$$Y_3(t) = \begin{cases} x_3(t) = \frac{1}{2}e^{-2t}\left(\frac{1}{32}te^{2t} + 0.68t^2 - 1.88t^3 + 0.008e^{4t}\right) + \frac{1}{2}e^{2t}\left(369.51 + 0.627t^4\right) - \\ \quad \frac{1}{2}e^{2t}\left(0.029t^3e^{4t-2} + +0.01t^2e^{4t} - 0.0033te^{4t}\right) - \frac{1}{t}e^{2t}\left(e^{-4t} - 0.628t^3 + 0.942t^2\right) \\ \quad +\frac{1}{t}e^{2t}\left(0.1282t + 0.0321\right) - 1.5832- \\ \quad \frac{1}{t}e^{2t}\left(0.0004t^4 + e^{-2t}\left(\frac{1}{16}t + \frac{1}{32} - 0.00554t^2 + 0.0093t^3\right)\right) \\ y_3(t) = y_3(t) = -e^{2t}\left(e^{-4t} - 1.57t^3 + 0.1569t^2 + .128t + .0321\right) - 1.58 - 0.004t^4 + \\ \quad \left(\frac{t}{16} + \frac{1}{32}\right) - 0.055t^2 - 0.009t^3 - e^{-2t}\left(\frac{1}{32}e^{2t} - \frac{t}{16}e^{2t}\right) - e^{-2t}\left(0.68t^2 + 1.88t^3\right) + \\ \quad e^{-2t}\left(0.008e^{4t} + 369.51 + 0.628t^4\right) - e^{-2t}\left(-0.029te^{4t-2} + 0.01t^2e^{4t} - 0.0327te^{4t}\right) \end{cases} \tag{4.39}$$

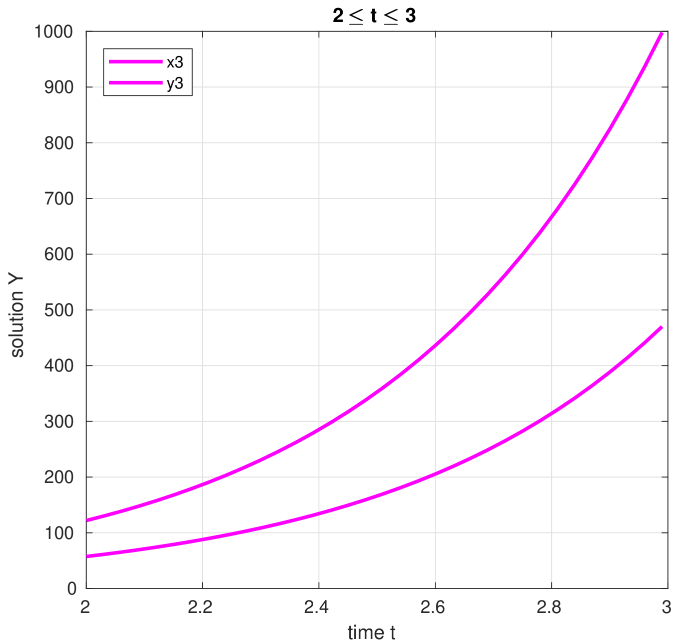

Figure 4.3: Graph of Equation (4.39)

Thus, for $0 \leq t \leq 3$, *solution of (4.29) can be written as*

$$Y(t) = \begin{cases} Y_1(t) = \begin{cases} x_1(t) = e^{2t} + \frac{1}{2}e^{-2t} - \frac{1}{8} \\ y_1(t) = e^{-2t}(0.5e^{2t} - 1) - e^{2t}(0.5e^{-2t} - 2) \end{cases} & 0 \leq t \leq 1, \\ Y_2(t) = \begin{cases} x_2(t) = \frac{1}{2}e^{-2t}\left(\frac{1}{4}e^{2t} + \frac{t}{4}e - \frac{7}{16}e^{4t-2} - 0.4068\right) + \\ \quad \frac{1}{2}e^{2t}\left(\frac{1}{4}e^{-2t} + \frac{7t}{4}e^{-2} + \frac{1}{16}e^{-4t+1} - 11.7601\right) \\ y_2(t) = e^{2t}\left(\frac{1}{4}e^{-2t} + \frac{7t}{4}e^{-2} + \frac{1}{16}e^{-4t+1} - 11.7601\right) - \\ \quad e^{-2t}\left(\frac{1}{4}e^{2t} + \frac{t}{4}e - \frac{7}{16}e^{4t-2} - 0.4068\right) \end{cases} & 1 \leq t \leq 2, \\ Y_3(t) = \begin{cases} x_3(t) = x(t) = \frac{1}{2}e^{-2t}\left(\frac{1}{32}te^{2t} + 0.685t^2 - 1.883t^3 + 0.0082e^{4t}\right) + \\ \quad \frac{1}{2}e^{-2t}(369.5076 + 0.6274t^4 - 0.0296t^3e^{4t-2}) + \\ \quad \frac{1}{2}e^{-2t}(+0.01t^2e^{4t} - 0.00327te^{4t} - 06277t^2e^{-2t}) - \\ \quad \frac{1}{t}e^{2t}(e^{-4t}(-0.6277t^3 + 0.9415t^2 + 0.1282t + 0.0321)) - \\ \quad \frac{1.6}{t}e^{2t} - \frac{1}{t}e^{2t}\left(e^{-2t}\left(\frac{1}{16}t + \frac{1}{32} - 0.00554t^2 + 0.0093t^3\right)\right) \\ y_3(t) = -e^{2t}(e^{-4t} - 1.57t^3 + 0.17t^2 + .128t + .032) - 1.5832 - \\ \quad 0.004t^4 + \left(\frac{t}{16} + \frac{1}{32}\right) - 0.0554t^2 - 0.0093t^3 - e^{-2t}\left(\frac{1}{32}e^{2t} - \frac{t}{16}e^{2t}\right) \\ \quad -e^{-2t}(0.685t^2 + 1.883t^3 + 0.0082e^{4t} + 369.5076 + 0.6277t^4) - \\ \quad e^{-2t}(-0.0296te^{4t-2} + 0.01t^2e^{4t} - 0.0327te^{4t}) \end{cases} & 2 \leq t \leq 3. \end{cases} \tag{4.40}$$

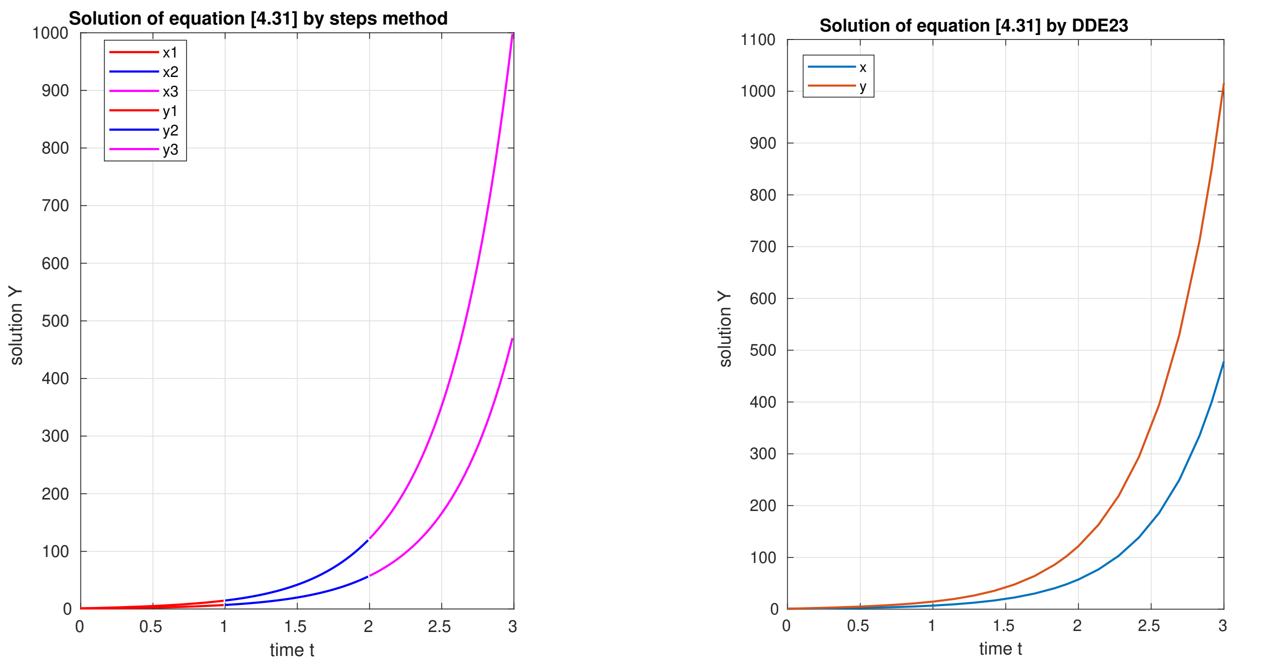

Figure 4.4: Graph of equation (4.31) by steps method and DDE23 respectively.

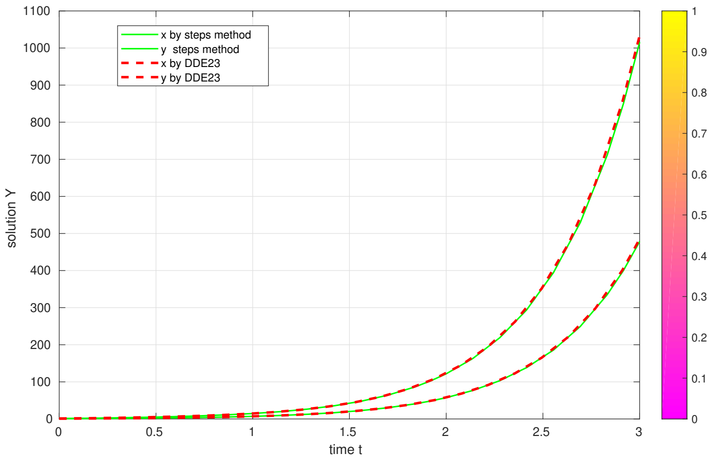

Figure 4.5: Comparing the two solutions exact(solid green line) and approximate solution using DDE23 (the dash) of equation (4.31).

Hence from figure 4.5, we conclude that, as $t \to \infty$, solution of equation (4.31) is increase exponentially.

Time t	**values of (x, y) by steps method**	**Values of (x, y) by DDE23**	**Absolute error**
0.0000	(1.0000,1.0000)	(1.0000,1.0000)	(0.0000,0.000)
0.3333	(1.7000 ,3.491)	(1.700 ,3.4913)	(0.000,0.0003)
0.6667	(3.4232 ,7.385)	(3.4223 ,7.373)	(0.0009,0.012)
1.0000	(6.923 , 14.697)	(6.921 ,14.548)	(0.002,0.149)
1.3333	(14.023 , 29.465)	(14.0,29.100)	(0.023,0.365)
1.6667	(28.346 ,59.912)	(28.322 ,60.23)	(0.024,0.318)
2.0000	(57.300 , 121.765)	(57.6, 122.500)	(0.300,0.735)
2.3333	(116.370,246.896)	(117.0 ,248.9)	(0.6300,2.004)
2.6667	(235.8133,500.500)	(236.600 ,502.6)	(0.7867,2.100)
3.0000	(478.367 ,1015.623)	(480.27, 1019.3)	(1.903,3.377)

Table 4.1: Exact value , approximate value and absolute error of example 4.3.1

Example 4.3.2. *Solve*

$$\begin{cases} \ddot{x}(t) = -30x(t) - \dot{x}(t-1), \;\; 0 \leq t \leq 4, \\ x(t) = \phi(t) = t, \;\; t \leq 0, \end{cases} \tag{4.41}$$

if ξ and η are in the asymptotic stability region of figure 2.

Solution 2. *Now by following all the procedures as the previous example 4.3.1,our solutions are as below.*

For $0 \leq t \leq 1$;

$$Y_1(t) = \begin{cases} x_1(t) = 0.0333(5.4772\cos(\tan^{-1}(5.4772) - 5.4772t)) - 0.0333, \\ y_1(t) = 0.0333(30.4959\cos(\tan^{-1}(0.1826) + 5.4772t)). \end{cases}$$

For $1 \leq t \leq 2$,

$$Y_2(t) = \begin{cases} x_2(t) = (0.0102\cos(5t - 5.1974)) - (0.0102\cos(5t - 4.8026)) + \\ (0.1020\sin(5t - 4.8026)) - (0.2401\cos(5t + \tan^{-1}(0.4312) - 5)) - \\ (0.1020t\sin(5t - 4.8026)), \\ y_2(t) = (0.5099\cos(5t - 4.8026)) - (0.0510\sin(5t - 5.1974)) - \\ (0.0510\sin(5t - 4.8026)) + (1.2006\cos(\tan^{-1}(2.3191) - 5t + 5)) - \\ (0.5099t\cos(5t - 4.8026)). \end{cases}$$

For $2 \leq t \leq 3$,

$$Y_3(t) = \begin{cases} x_3(t) = 0.0528\cos(5t - 10.0599) - 0.0528\cos(5t - 9.9401) + \\ 1.0559\sin(5t - 9.9401) + 0.0530\cos(5t - 11.3808) + 0.5322\sin(5t - 8.6192) + \\ 0.0026\sin(5t - 11.3808) - 0.0528t\sin(5t - 9.9401) - \\ 1.1537\cos(\tan^{-1}(0.1296) - 5t + 10) + 0.1324t^2\sin(5t - 8.6192) - \\ (0.0265t\cos(5t - 11.3808)) - (0.5295t\sin(5t - 8.6192)), \\ \\ y_3(t) = 5.2796\cos(5t - 9.9401) - 0.2640\sin(5t - 10.0599) - \\ 0.2640\sin(5t - 9.9401) + 2.6609\cos(5t - 8.6192) - 0.0132\cos(5t - 11.3808) - \\ 0.5295\sin(5t - 8.6192) - 0.2648\sin(5t - 11.3808) - 2.6398t\cos(5t - 9.9401) - \\ 5.7683\cos(5t + \tan^{-1}(7.7179) - 10) + 0.6619t^2\cos(5t - 8.6192) - \\ 2.6476t\cos(5t - 8.6192) + 0.2648t\sin(5t - 8.6192) + 0.1324t\sin(5t - 11.3808). \end{cases}$$

For $3 \leq t \leq 4$

$$Y_4(t) = \begin{cases} x_4(t) = 0.0026\sin(5t-14.9401) + (0.0577\cos(5t-13.5581)) - \\ (0.03\cos(5t-13.62)) + (0.03\cos(5t-16.3808)) - (0.0577\cos(5t-16.45)) - \\ (1.7305\sin(5t-13.5581)) + (0.7983\sin(5t-13.6192)) + \\ (1.1879\sin(5t-14.94)) + (0.079\cos(5t-15.06)) + (0.003\sin(5t-15.0599)) + \\ 0.0859\cos(5t-13.6192) + 0.0332\cos(5t-16.3808) - 0.1946\sin(5t-13.6192) + \\ 0.5768t\sin(5t-13.5581) - (0.2661t\sin(5t-13.6192)) - \\ 0.7919t\sin(5t-14.9401) - 0.0264t\cos(5t-15.0599) + \\ (0.1320t^2\sin(5t-14.9401)) + 1.1411\cos(5t+tan^{-1}(3.1688)-15) + \\ 0.0066t^2\cos(5t-13.6192) + 0.0066t^2\cos(5t-16.3808) + \\ 0.1986t^2\sin(5t-13.6192) - 0.0221t^3\sin(5t-13.6192) - \\ (0.0397t\cos(5t-13.619)) - 0.0397t\cos(5t-16.381) - 0.3323t\sin(5t-13.62), \\ \\ y_4(t) = 0.0132\cos(5t-14.9401) - 8.6525\cos(5t-13.5581) + \\ 3.9913\cos(5t-13.62) + 5.94\cos(5t-14.9401) + (0.2884\sin(5t-13.5581)) - \\ 0.1330\sin(5t-13.6192) - (0.133\sin(5t-16.381)) - (0.78\sin(5t-14.94)) + \\ (0.29\sin(5t-16.4419)) - (0.0132\cos(5t-15.06)) - (0.3960\sin(5t-15.06)) - \\ 1.1\cos(5t-13.6) - 0.04\cos(5t-16.4) - 0.8\sin(5t-13.6) - 0.2\sin(5t-16.38) + \\ (2.9t\cos(5t-13.561)) - (1.3304t\cos(5t-13.62)) - (3.96t\cos(5t-14.94)) + \\ (0.2640t\sin(5t-14.94)) + (0.1320t\sin(5t-15.06)) + (0.66t^2\cos(5t-14.9401)) - \\ 5.71\cos(\tan^{-1}(0.32)-5t+15) + 0.99t^2\cos(5t-13.62) - 0.11t^3\cos(5t-13.62) - \\ 0.0993t^2\sin(5t-13.6192) - 0.033t^2\sin(5t-16.381) - 1.6482t\cos(5t-13.6192) + \\ 0.0132t\cos(5t-16.3808) + 0.5957t\sin(5t-13.62) + 0.1986t\sin(5t-16.3808). \end{cases}$$

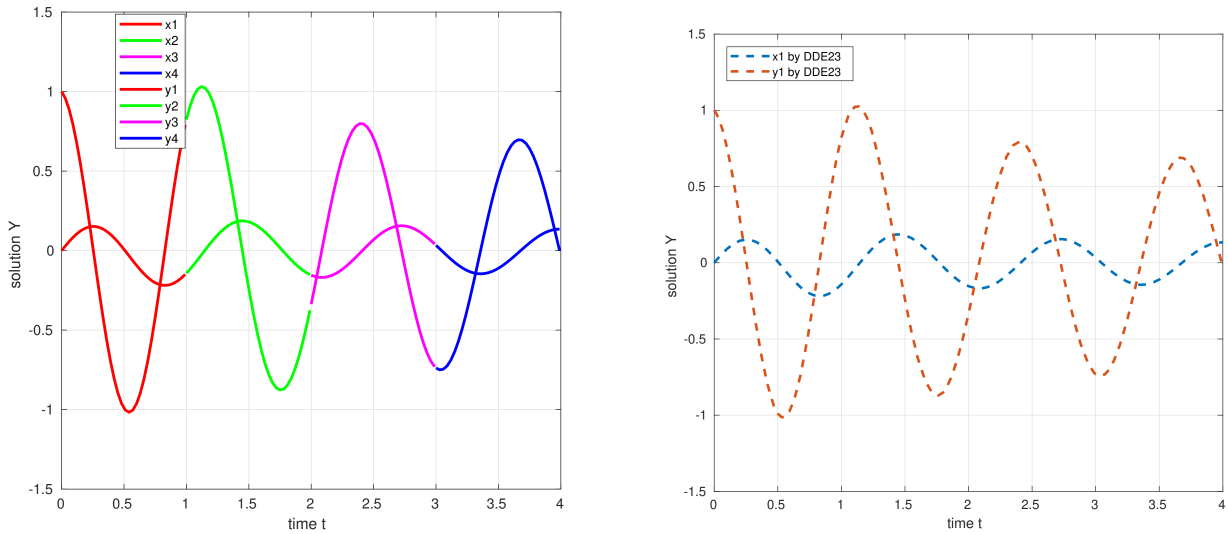

Figure 4.6: Graph of example 4.4.2 by steps method and DDE23 respectively

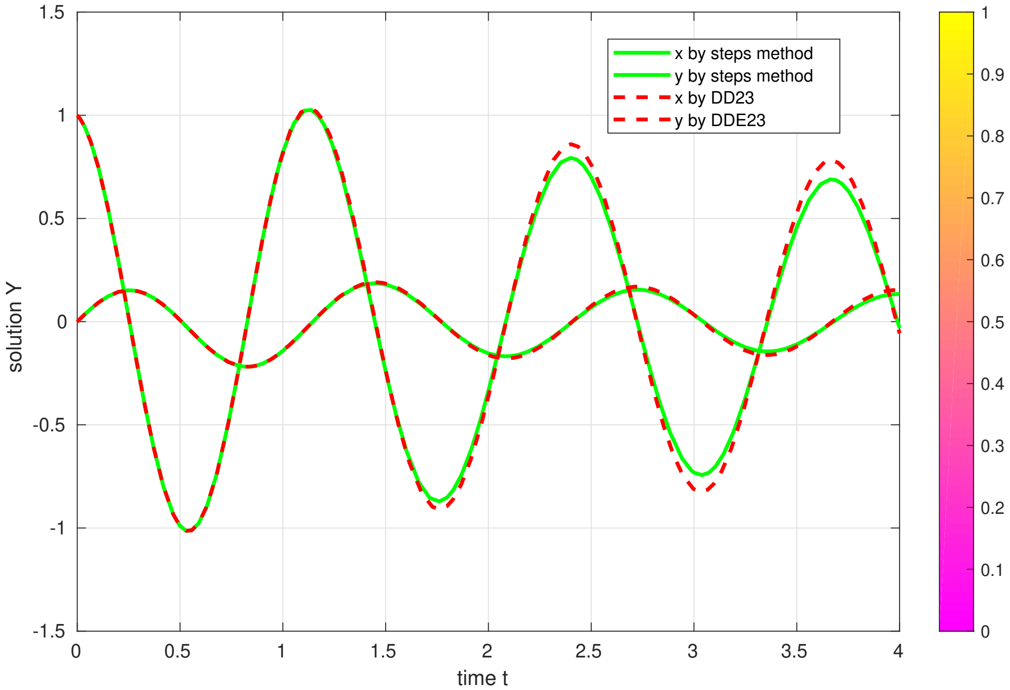

Figure 4.7: Comparing the two solutions steps(solid green line) and approximate solution using dde23 (the dash) of example 4.3.2.

Time t	**Steps value of** (x, y)	**Values of** (x, y) **by DDE23**	**Absolute error**
0.0000	(0.0000,1.0000)	(0.0000,1.0000)	(0.0000,0.000)
0.4444	(0.0598 , -0.8779)	(0.0598 ,-0.8779)	(0.000,0.0000)
0.8889	(-0.2081 ,0.3356)	(-0.2081, 0.3356)	(0.0009,0.0000)
1.3333	(0.1557 , 0.5258)	(0.1588 ,0.5424)	(0.0031,0.0166)
1.7778	(-0.0064 , -0.8665)	(-0.0063 ,-0.9064)	(0.0001,0.0399)
2.2222	(-0.1316 ,0.5060)	(-0.1400 ,0.5409)	(0.0084,0.0349)
2.6667	(0.1491 , 0.2150)	(0.1634 , 0.2279)	(0.0143,0.0129)
3.1111	(-0.0477,-0.6936)	(-0.0568 ,-0.7657)	(0.0091,0.0721)
3.5556	(-0.0810 ,0.5831)	(-0.0872 ,0.6682)	(0.0062,0.60989)
4.0000	(0.1334 ,-0.0309)	(0.1528, -0.0567)	(0.0194,0.0258)

Table 4.2: Exact value obtained by steps method , approximate value by DDE23 and absolute error of example 4.3.2.

Hence, figure 4.7 indicates that, as t tends to infinity, our solution Y(t) tends to zero. Therefore, it is asymptotically stable.

4.4 Discussions

In this thesis we used steps method to solve a second order delay differential equations. First we reduced second order DDEs to system of first order DDEs. Then by using history function and initial time t_0, we converted DDE into nonhomogeneous ordinary differential equations. We obtained analytical solution from the corresponding ODEs. For illustration purpose, examples were selected to show how to apply steps method to find analytic solution.

Conclusion and Recommendation

Conclusion

In this study, we have introduced steps method to solve linear second order DDEs with single delay and constant coefficients. To apply this method, we used history function and initial condition from the previous interval to find the solution on the next interval. The existence, uniqueness and stability analysis for the time-dependent DDE were presented in this work. Stability of 2^{nd} order DDEs was analyzed and consequence of Pontryagin's Theorem was summarized graphically. To further asses the validity of steps method, we have compared analytic solution with respect to their corresponding approximate solution obtained from the MATLAB DDE23.

Recommendation

This thesis specifically focused on application of steps method to solve analytic solution of linear second order DDEs. Steps method is very interesting and easy to find analytic solution on finite successive intervals. But finding solution on infinite interval is so hard. Hence, we recommended that, one may look for the solution of linear second order DDEs with single delay and constant coefficients.Future researchers should try to compare steps method with other method to generalize the efficiency of this method. Also one can extend the method of steps to nonlinear DDEs, SDDDEs and NDDEs.

Future Work

We will work on how to find analytic solution to the following cases:

Case 1: If delays $\tau_i, i = 1, 2, ..., n$ are non-single, with constant coefficients.

- Linear second order delay differential equation with non-single non constant delay, and the delays are not equal, i.e. $\tau_1 \neq \tau_2 \neq ... \neq \tau_n$.
- Linear second order delay differential equation with non-single non constant delay, and the delays are equal, i.e.$\tau_1 = \tau_2 = ... = \tau_n$.

Case 2:If delays $\tau_i, i = 1, 2, ..., n$ are non-single, with variable coefficients such that:

- Linear second order delay differential equation with non-single non constant delay, and the delays are not equal, i.e. $\tau_1 \neq \tau_2 \neq ... \neq \tau_n$.
- Linear second order delay differential equation with non-single non constant delay, and the delays are equal, i.e.$\tau_1 = \tau_2 = ... = \tau_n$.

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References

Aliprantis, C. D., & Burkinshaw, O. (1998). *Principles of real analysis*. Gulf Professional Publishing.

Asl, F. M., & Ulsoy, A. G. (2003). Analysis of a system of linear delay differential equations. *J. Dyn. Sys., Meas., Control*, *125*(2), 215–223.

Bellman, R., & Cooke, K. L. (1963). Differential-difference equations, acad. *Press, NY*.

Breda, D., Maset, S., & Vermiglio, R. (2014). *Stability of linear delay differential equations: A numerical approach with matlab*. Springer.

Cahlon, B., & Schmidt, D. (2004). stability criteria for certain second order delay differential equations with mixed coefficients. *Jaurnal of Computational and Applied mathematics*, *170*(1), 79–102.

Erneux, T. (2009). *Applied delay differential equations* (Vol. 3). Springer Science & Business Media.

Gimeno i Alquézar, J. (2015). On time delay differential equations.

Hale, J. K. (1993). Verduyn lunel s m. introduction to functional differential equations. *Appl. Math. Sciences*, *99*.

Hastings, S. (1992). Differential equations and dynamical systems (lawrence perko). *SIAM Review*, *34*(1), 129–131.

Holte, J. M. (2009). Discrete gronwall lemma and applications. In *Maa-ncs meeting at the university of north dakota* (Vol. 24, pp. 1–7).

Jaaffar, N. T., Abdul Majid, Z., & Senu, N. (2020). Numerical approach for solving delay differential equations with boundary conditions. *Mathematics*, *8*(7), 1073.

Krasovskii, N. (1963). *Stability of motion: application of lyapunov's second method to differential systems and equations with time-delay.* SU Press, Stanford.

Kuang, Y., & Smith, H. (1993). Global stability for infinite delay lotka-volterra type systems. *Journal of differential equations*, *103*(2), 221–246.

Layek, G. (2015). *An introduction to dynamical systems and chaos*. Springer.

Li, V. (2019). On stability of a harmonic oscillator with a delayed feedback system.

Nelson, B. (2019). The weierstrass function. *Retrieved December*, *31*.

Pontrjagin, L. (1942). On zeros of some transcendental functions. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, *6*(3), 115–134.

Rasdi, N., & Majid, Z. A. (2015). Solving second order delay differential equations by direct extended two and three point implicit one-step block method. In *2015 international conference on research and education in mathematics (icrem7)* (pp. 29–34).

Rasdi, N., Majid, Z. A., Ismail, F., Senu, N., Phang, P., & Radzi, H. M. (2013). Solving second order delay differential equations by direct two and three point one-step block method. *Applied Mathematical Sciences*, *7*(54), 2647–2660.

Richard, J.-P. (2003). Time-delay systems: an overview of some recent advances and open problems. *automatica*, *39*(10), 1667–1694.

Seong, H. Y., & Majid, Z. A. (2017). Solving second order delay differential equations using direct two-point block method. *Ain Shams Engineering Journal*, *8*(1), 59–66.

Shampine, L. F., Thompson, S., & Kierzenka, J. (2000). Solving delay differential equations with dde23. *URL http://www. runet. edu/˜ thompson/webddes/tutorial. pdf*.

Smith, H. L. (2011). *An introduction to delay differential equations with applications to the life sciences* (Vol. 57). Springer New York.

Yi, S., Ulsoy, A. G., & Nelson, P. W. (2006). Solution of systems of linear delay differential equations via laplace transformation. In *Proceedings of the 45th ieee conference on decision and control* (pp. 2535–2540).

Zou, L., Fränzle, M., Zhan, N., & Mosaad, P. N. (n.d.). Automatic stability and safety verification for delay differential equations.

Zulkefli, N. A. I., & Maan, N. (2014). The existence and uniqueness theorems of fuzzy delay differential equations. *Malaysian Journal of Fundamental and Applied Sciences*, *10*(3).

Appendix A

The Matlab code for figure 4.5

```
clc;
clear;
close all
t1=0:.03:1;
x1=-(exp(-2*t1).*(exp(2*t1)/2 - 1))/2 - (exp(2*t1).*(exp(-2*
    t1)/2 - 2))/2;
y1=exp(-2*t1).*(0.5*exp(2*t1) - 1) - exp(2*t1).*(0.5*exp(-2*
    t1) - 2);
t2=1:.03:2;
x2=(exp(2*t2).*((120281*exp(-2))/10000 + 2*t2.*exp(-2) + (exp
    (-4*t2).*exp(2))/4))/2 - (exp(-2*t2).*((8647*exp(2))/10000
     - t2.*exp(2) + (exp(4*t2).*exp(-2))/2))/2;
y2=exp(-2*t2).*((8647*exp(2))/10000 - t2.*exp(2) + (exp(4*t2)
    .*exp(-2))/2) + exp(2*t2).*((120281*exp(-2))/10000 + 2*t2
    .*exp(-2) + (exp(-4*t2)*exp(2))/4);

t3=2:.03:3;
x3=(exp(2*t3).*((25667336451150323*t3.*exp(-2))
    /18014398509481984 - (exp(-6)*(30543303900386239*exp(4) -
    10670495690719227 52*exp(2) + 2251212747373118))
    /9007199254740992 - (15513068773195325*exp(2 - 4*t3))
    /4503599627370496 + (4159668786720471*t3.*exp(2 - 4*t3))
```

```
/2251799813685248 + (1218991862308979*(t3).^2.*exp(-2))
/9007199254740992))/2 + (exp(-2*t3).*((exp(2)
*(4275034935914147*exp(4) - 3189359184112384*exp(2) +
1483770494898338 96))/9007199254740992 - (exp(-2)
*(6112086147210336*exp(4*t3) + 1218991862308979*t3.*exp(4*
t3) + 2814864506648889 68*t3.*exp(4) - 6655470058752753 6*(
t3).^2.*exp(4)))/18014398509481984))/2;
y3=exp(2*t3).*((2566733645115032 3*t3.*exp(-2))
/18014398509481984 - (exp(-6)*(3054330390038623 9*exp(4) -
1067049569071922752*exp(2) + 225121274737311 8))
/9007199254740992 - (1551306877319532 5*exp(2 - 4*t3))
/4503599627370496 + (4159668786720471*t3.*exp(2 - 4*t3))
/2251799813685248 + (1218991862308979*(t3).^2*exp(-2))
/9007199254740992) - exp(-2*t3).*((exp(2)
*(4275034935914147*exp(4) - 3189359184112384*exp(2) +
1483770494898338 96))/9007199254740992 - (exp(-2)
*(6112086147210336*exp(4*t3) + 1218991862308979*t3.*exp(4*
t3) + 2814864506648889 68*t3.*exp(4) - 6655470058752753 6*(
t3).^2.*exp(4)))/18014398509481984);

t=[t1,t2,t3];
x=[x1,x2,x3];
y=[y1,y2,y3];
plot(t1,x1,'g',t2,x2,'g',t3,x3,'g','LineWidth',1.3);
hold on
plot(t1,y1,'g',t2,y2,'g',t3,y3,'g','LineWidth',1.3);
xlabel('time t');
ylabel('solution Y');
title('Solution of equation [4.28] by steps method');
tspan = [0:3];
lags = [1];
sol = dde23(@ddefun, lags, @history, tspan);
```

```
hold on;

plot(sol.x,sol.y,'*r','LineWidth',2)
hold off;
box on
xlabel('time t');
ylabel('solution Y');
legend('x ','y ','Location','NorthWest');
title('Solution of equation [4.28] by DDE23');
ylim([0 1100]);
 tint = linspace(0,3,1);
Sint = deval(sol,tint)
function dydt = ddefun(t,y,Z)
  ylag1 = Z(:,1);

  dydt =[y(2);
          4*y(1)+2*ylag1(2);
         ];
end

function s = history(t) % history function for t <= 0
  s = [t+1,1];
end
```