Analysis of the dynamics of two preys and one predator model with prey refuge.

By

Yoseph Legesse Tilahun

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

AT

ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

ADAMA, ETHIOPIA

© Copyright by Adama Science and Technology University, November 2016

Declaration

I declare that this work is a result of my efforts. All information in this work have been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY
APPLIED MATHEMATICS PROGRAM

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance of a thesis entitled "**Analysis of the dynamics of two preys and one predator model with prey refuge.** " by **Yoseph Legesse Tilahun** in partial fulfillment of the requirements for the degree of Master's of Science, **MSc.** Degree.

Date: November, 2016

Examining Committee

External Examiner: ______________________

Advisors:

____

Internal Examiner:

____

Chair Person:

____

ADAM SCIENCE AND TECHNOLOGY UNIVERSITY

Date: November, 2016

Author: Yoseph Legesse Tilahun
Title: **Analysis of the dynamics of two preys and one predator model with prey refuge.**
Program: Applied Mathematics
Degree: MSc. Convocation: November Year: 2016

Author's Signature

To my family

Acknowledgements

First of all, Thanks to the Almighty God for giving me the endurance and patience in accomplishing this piece of work.
I would like to express my heart felt thanks to my advisor, Dr. Legesse Lemecha for his guidance, precious time and encouragement throughout the course of the thesis. The success of this work has been due to the amazing intellectual visionary ideas of my advisor.I would like to give a lot of thanks to Dr. Sheferaw Feyisa for his assistance during proposal. I would like to thank Ato. Yadeta Chimdesa for his cooperation in downloading Latex software. I also wish to thank my colleague Ato. Mekonnen H/Mariam and Ato. Aynachew Gashaw for their support of my work. Finally, I would like to thank my parents for their love and understanding have allowed me to carry out this thesis. My work is dedicated to my beloved wife, W/ro Lakech Yilma , and to our children Soliana and Kidus, for all the joy and hope they bring to my life.

Abstract

In this thesis, a three component mathematical model consisting of two prey and one predator incorporating a prey refuge is proposed and analysed where the predator shows a Holling Type I response to one of the prey, and a Holling Type II response to the other prey. The purpose of this work is to present some mathematical analysis of the dynamics of a two prey and one predator in a given ecological system. The stability of positive constant equilibrium are investigated. We derived criteria which guarantee the persistence of the three species and the global dynamics of the model system. Results of the analysis of the model show that the 3 species would co-exist.Other major observation from the analysis is that the predator population density increases significantly when the intrinsic growth rate of both preys increases. This can imply that a high intrinsic growth rate of the prey initially increases their population density which increases the predator′s chance of capturing the prey and so the predator′s population density increases.

Keywords: Prey, predator, ecosystem, extinction, local and global stability, and persistence.

Contents

Lists of abbreviations, symbols and the like are easily formatted with the help of the Springer-enhanced `description` environment.

Chapter 1
Introduction and Background

1.1 Introduction

An ecosystem or an ecological system is the whole biotic community in a given area and its abiotic environment. It can be any size: from an area as small as a pinhead to the whole biosphere. There are many examples of ecosystems: a pond, a forest, an estuary, a grassland. The boundaries are not fixed in any objective way, although sometimes they seem obvious, as with the shoreline of a small pond. Usually the boundaries of an ecosystem are chosen for practical reasons having to do with goals of the particular study. Ecosystems are characterized by the interaction between different species and natural environment. There exist varying degrees of positive, negative and even neutral interactions among organisms at both inter- and intra-specific levels. Some important ecological interactions are Competition, Mutualism and Predation. These interactions have been studied extensively using mathematical models by several researchers. There are good reasons for mathematical models to be so widely used in ecology. Ecosystems tend to be very complex and governed by many intricate and usually non-linear systematic interactions. Mathematics is ideally suited not only to express these complex relationships in a succinct way, but also to force one to be careful in his or her statements of a system. Moreover, once a model has been formulated, mathematics offers the appropriate tools to analyse its consequences. Hence mathematics can be viewed as a language that is most appropriate for logical reasoning and logical analysis problems.

Mathematical modeling in population dynamics has gained a lot of attention and appreciation during the last few decades, and among these models predator-prey systems play an important role. The dynamic relationship between predator and their prey has long been and will continue to be one of the dominant topics in both applied mathematics and theoretical ecology due to its universal existence and importance. The predator-prey models are the basis which help to see the life and ecosystems in which the predator and prey have certain roles. This model explains how the predators interact with their prey. It explains the sustenance, evaluation and alternative dispersion of some species in the case of failure to complete in the life in which the stronger has the advantage of domination role. The predator-prey model is like the survival of the fittest-theory. The fittest are the stronger species targeting the weaker species and win lives for themselves and this evaluation of life for one species results in the numerical and sometime general extinction of other weaker species. The weaker species remain in constraint struggle to achieve their security in the diaspora where general fear of life remains ever present. The weaker species which become prey adopt many measures to trick the predator to avoid being hunted. Briefly predator-prey interaction is like a win-loss interaction, it is the victory for one and loss for the other; and life for one and elimination of the other.

Some of the aspects that need to be critically considered in a realistic and plausible prey-predator mathematical model includes carrying capacity which is the maximum number of prey that the ecosystem can sustain in absence of predator, competition among prey and predators which can be intra specific, and functional responses of predator. The study of the consequence of prey refuge on the dynamics of predator-prey interactions can be recognized as a major but rather challenging issue in applied mathematics and theoretical ecology. Some of the empirical and theoretical works based on prey refuge has concluded that the refuge used by prey has stabilizing effect on predator-prey interactions.

In this thesis, we intend to use Holing type I functional response to represent the interaction between one of the prey and predator. The interaction between the other prey and the predator is assumed to be governed by Holing type II response function. Such different choice of functional response holds because of the difference of handling time of the preys.

1.2 Functional Responses

Functional response is a function that describes the consumption of one organism (prey) by another (predator). Selective pressures has led to an evolutionary arms race between prey and predator, resulting in various anti-predator adaptations. Predators lower the fitness of their prey, and thus reduce the prey's chances of survival and reproduction and it is the most important element in predator-prey models. Hence understanding and clearly quantifying functional response is at the heart of ecological modeling. Generally we categorized functional response as:

(i) prey density-dependent f(N)
(ii) ratio-dependent $f(\frac{N}{P})$ and
(iii) prey-predator density-dependent f(N,P).

,where N and P are prey and predator population densities respectively.
For prey density-dependent responses, the consumption rate of the predator varies with the prey density alone. Holling categorized the prey-dependent responses into three types;

1.2.1 Holling Type I functional response

Holling Type I functional response is

$$f(N) = HN, H > 0. \tag{1.1}$$

and the logistic ODE with proportional or constant rate harvesting is

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - HNP. \tag{1.2}$$

,K and L are environmental carrying capacities of the prey species, For predators with a Type I functional response, the rate of prey consumption increases linearly with the prey population size. If the number of prey quadruple, the predators will eat four times as much per day. Predators with such unlimited appetites are rarely found in nature. This type of response is found in passive predators like spiders. The number of flies caught in the net is proportional to fly density. Prey mortality due to predation is constant.

1.2.2 Holling Type II functional response

The Holling Type II functional response is

$$f(N) = \frac{AN}{B+N}. \tag{1.3}$$

where A,B>0 and the logistic ODE with Holling Type II response is

$$\frac{\mathrm{dN}}{\mathrm{dt}} = \mathrm{rN}(1 - \frac{\mathrm{N}}{\mathrm{K}}) - \frac{\mathrm{AN}}{\mathrm{B} + \mathrm{N}}\mathrm{P}. \tag{1.4}$$

For predators with a Type II functional response, the rate of prey consumption increases with the prey population size, but saturates at some maximum level A. This functional response seems to be the most common and is well documented in empirical studies. Murdoch and Oaten (1975). Other name for this functional response is the Monod response or Michaelis-Merten response. This response is characteristic of organisms that require non-trivial amounts of time to capture and ingest their prey. Holling gave a simple mechanistic explanation of this functional response. Predation involves two tasks: searching for prey and consuming the prey (chasing, killing, eating, and digesting). At low prey densities, predators spend most of their time searching for prey, and at high prey densities, predators spend most of their time on handling prey. Predators of this type cause maximum mortality at low prey density.

1.2.3 Holling Type III functional response

The Holling Type III functional response is

$$f(N) = \frac{dN^{\gamma}}{F + N^{\gamma}} \tag{1.5}$$

and the logistic ODE with Holling Type III response is

$$\frac{\mathrm{dN}}{\mathrm{dt}} = \mathrm{rN}(1 - \frac{\mathrm{N}}{\mathrm{K}}) - \frac{\mathrm{dN}^{\gamma}}{\mathrm{F} + \mathrm{N}^{\gamma}}\mathrm{P} \tag{1.6}$$

where $\gamma > 1$ is the encounter rate between predator and prey before the predator reaches maximum efficiency. According to Sharov (1996), Holling Type III functional response occurs in predators which increase their search activity with increasing prey density. For example, many predators respond to kairomones (chemicals emitted by prey) and increase their activity. Polyphagous vertebrate predators (e.g.birds) can switch to the most abundant prey species by learning to recognize it visually. Mortality first increases with prey increasing density, and then declines. If predator density is constant (e.g.birds, small mammals) then they can regulate prey density only if they have a Holling Type III functional response because this is the only type of functional response for which prey mortality can increase with increasing prey density. However, regulating effect of predators is limited to the interval of prey density where mortality increases. If prey density exceeds the threshold value of this interval, then mortality due to predation starts declining, and predation will cause a positive feedback. As a result, the number of prey will get out of control. They will grow in numbers until some other factors (diseases or food shortage) will stop their reproduction.

1.2.4 Prey refuge

In nature, prey populations often have access to area where they are safe from their predators. Such refuge are usually playing two significant roles, serving both to reduce the chance of extinction due to predation and to damp prey-predator oscillations. These are therefore a potentially important means of increasing species richness in natural communities and of stabilizing population sizes, biomass and productivity. It is well known that many more attentions have paid on the effects of a prey refuge for predator-prey system. Predator-prey interactions often exhibit spatial refuge which afford the prey some degree of protection from predation and reduce the chance of extinction due to predation.

1.2.5 The Logistic equation

The exponential growth law for population size is unrealistic over long times. Eventually, growth will be checked by the over-consumption of resources. We assume that the environment has an intrinsic carrying capacity K, and populations larger than this size experience heightened death rates. To model population growth with an environmental carrying capacity K, we look for a nonlinear equation of the form

$$\frac{dN}{dt} = rNF(N) \tag{1.7}$$

where F(N) provides a model for environmental regulation. This function should satisfy F(0)= 1(the population grows exponentially with growth rate r when N is small), F(K)= 0 (the population stops growing at the carrying capacity) and $F(N) < 0$ when $N > K$ (the population decays when it is larger than the carrying capacity). The simplest function F (N) satisfying these conditions is linear and given by

$$F(N) = 1 - \frac{N}{K} \tag{1.8}$$

The resulting model is the well-known logistic equation,

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \tag{1.9}$$

1.3 Literature Review

Mathematical population model have been used to study the dynamics of prey-predator system since Lotka (1925) and Volterra (1927) proposed a simple model of prey-predator interaction now called Lotka-Volterra model. Since then many mathematical model have been constructed based on more realistic, explicit and implicit biological assumptions. Over the past decades, mathematics has made a considerable impact as a tool to model and understand biological phenomena. In return, biologists have confronted the mathematicians with variety of challenging problems, which have stimulated developments in the theory of nonlinear differential equations. Such differential equations have long played important role in the field of theoretical population dynamics, and they will, no doubt, continue to serve as indispensable tools in future investigations. Differential equation models for interactions between species are one of the classical applications of mathematics to biology. The development and use of analytical techniques and the growth of computer power have progressively improved our understanding of these types of models. Although the predator-prey theory has seen much progress, many long standing mathematical and ecological problems remain exist. Theoretical ecology remained silent about the astonishing array of dynamical behaviors of three species models for a long time. Of course, the increasing number of differential equations and the increasing dimensionality raise considerable additional problems both for the experimentalist and theoretician.

Kar and Chaudhuri (2004) considered a two-prey one-predator harvesting model with interference. The model is based on Lotka-Volterra dynamics with two competing species which are affected not only by harvesting but also by the presence of a predator, the third species. Optimal harvesting policy and the possibility of existence of a bio-economic equilibrium is discussed.Though the preys are affected by harvesting and predation,the mechanism of prey protection is not mentioned, which is considered in our model.

Dubey and Upadhyay (2004) proposed a two predator one prey system with ratio dependent predator growth rate. Criteria for local stability, instability and global stability of the non-negative equilibria were obtained. They also discussed about the permanent coexistence of the three species. However, ratio dependent predator growth rate is time consuming one it may not give guarantee for prey extinction.

Braza (2008) considered a two predator- one prey model in which one predator interferes significantly with the other predator is analyzed. Zhang et al.(2006) studied the stability of three species population model consisting of an endemic prey (bird), an alien prey (rabbit) and an alien predator(cat). It may be pointed out here that all the above studies are based on the traditional prey dependent models. Recently, it has been observed that in some situations, especially when a predator have to search for food and have two different choice of food, a more suitable predator-prey theory should be based on the so-called ratio-dependent theory, in which the per-capital growth rate should be function of the ratio of prey to predator abundance, and should be the so-called predator functional response (Abrams and Ginzburg, 2000 ;Akcakaya et al., 1995 ;Arditi et al., 1991 ;Arditi and Saiah, 1992 ,Kar and Batabyal 2010).

Kesh et al. (2000) proposed and analyzed a mathematical model of two competing prey and one predator species where the prey species follow Lotka-volterra dynamics and predator uptake functions are ratio dependent. They derived conditions for the existence of different boundary equilibria and discussed their global stability. They also obtain sufficient conditions for the permanence of the system.

Hsu et al. (2001) studied the qualitative properties of a ratio dependent predator-prey model. They showed that the dynamics outcome of interactions depend upon parameter values and initial data. N.Daga , B.Singh , S.Jain and G.Ujjainkar (2014) studied a two prey one predator model with a non-homogeneous transmission functional responses.Criteria for local and global stability of non-negative eqilibria are obtained.

N.Daga , B.Singh , S.Jain and G.Ujjainkar (2014) considered a two prey one predator system with Holling type III functional responses.They also discussed the local and global stability of the system. A. George Maria Selvam , R. Dhineshbabu and P. Rathinavel (2015) discussed the stability of equilibrium points of a discrete prey-predator system with three species. The dynamical analysis of the system is performed with numerical simulation which supports the theoretical findings.

M.ReniSagaya Raj, A.George Maria Selvam and R.Janagaraj (2013) proposed and studied Ecological model with interspecies interactions in three species food chain with prey - predator and scavenger. some dynamical behaviors are investigated. Dynamical behavior of three species food chain discrete model is investigated at equilibrium points.

Prabir Panja , Shyamal Kumar Mondal (2015) proposed a prey-predator model

for the study of dynamical behaviors of three species such as toxin-producing Phytoplankton, Zooplankton and Fish in a fishery system. The stability and existence condition of equilibrium have been established. Holling type II functional response function has been considered to analysis of the proposed model. All equilibriums of the proposed system are determined, and the behavior of the system is also investigated near the positive equilibrium point.
A.George Maria Selvam , R. Dhineshbabu and V.Sathish (2014) proposed and investigated the dynamical behavior of a discrete prey-predator system describing the interactions among three species. Stability analysis is performed.
In general, the literature review focused on mathematical models of three species with emphasis on two prey-one predator mathematical models. In some of the given models the prey become victim of harvesting and predation. As a result the prey population may face the challenge of extinction. Further more the models in the review give little attention to prey protection mechanism. Thus in our thesis we tried to gives plausible emphasis to the matter.

1.4 Statement of the problem

The use of refuge has a great role in prey-predator interaction whenever there is predation pressure. So that many researchers studied prey refuge mechanism to solve the problem of extinction and facilitate the co-existence of predators and their preys.

Therefore, in this thesis, we have to formulate a mathematical model which clearly describes the existing reality between two prey and one predator system with one prey abundant and the other prey constant refuge in maintaining both populations in stable ecological system. Hence, the thesis raise the following leading research questions:

(i) What are the dynamic properties of two preys and one predator with one abundant and the other constant prey refuge?
(ii) Is the system both locally and globally stable?
(iii) What are the impacts of one prey refuge on the local stability of the predator-prey dynamics with one prey abundant ?
(iv) Under what condition two preys and one predator dynamical system coexists ?

1.5 Objectives of the research

1.5.1 General objectives

The general objective of this study is to investigate the effect of constant prey refuge on the dynamics of two prey one predator interactions in a given habitat.

1.5.2 Specific objectives

The specific objectives for this study are to:

(i) Modify a mathematical model which describes the dynamical properties of two preys and one predator with one abundant and the other prey refuge.
(ii) analyze local and global stability of the predator-prey dynamics governed by the formulated model.
(iii) study the effect of two prey and one predator system with prey refuge on the local stability of the predator-prey dynamics.
(iv) give detail of mathematical analysis on the coexistence of two prey and one predator dynamical system with one abundant and one prey refuge.

1.6 Scope of the study

To carry out a research on prey-predator interaction it is vital to visit National Parks and other protected areas frequently to collect data from the concerned body through different methods to make the study complete.But,this is not be possible due to different reasons such as shortage of time, finance and so on.Hence study emphasis theoretical analysis without model validation with real data.

1.7 Significance of the study

It is hoped that, if the formulated mathematical model nicely reflect the effect of prey refuge on stable co-existence of these population,the findings of the proposed study would be benefit different bodies differently: Hence, it:

(i) provides some information about prey refuge
(ii) Initiates other researchers to undertake further extension and rigorous mathematical analysis.

Chapter 2
Mathematical Preliminaries

2.1 Basic definitions,Theorems and notations

For main results, definition and concepts we refer to (John W. Cain and Angela Reynolds(2010),Walter G. Kelley • Allan C. Peterson (2010)).

2.1.1 Phase portrait, Trace and Determinant

A more compact way of classifying phase portraits of planar systems can be stated in terms of the trace and determinant of the coefficient matrix.

Consider the system.

$$X' = AX \tag{2.1}$$

where

$$X' = \begin{bmatrix} x \\ y \end{bmatrix}', A = \begin{bmatrix} a\ b \\ c\ d \end{bmatrix} and X = \begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$\begin{cases} x' = ax + by = f(x,y) \\ y' = cx + dy = g(x,y) \end{cases}$$

Definition 2.1.1. The system (2.1) is called a planar autonomous system. The term autonomous means self-governing, justified by the absence of the time variable t in the functions f (x, y) and g(x, y). It is assumed that f and g are continuously differentiable in some region D in the xy plane. A graph which contains all the equilibria and the typical trajectories or orbits of a planar autonomous system (2.1) is called a phase portrait.

Definition 2.1.2 The trace of a square matrix A is the sum of the entries on its main diagonal and is denoted by trA.
The eigenvalues of a 2×2 matrix A can be expressed in terms of trA and det A. Then the matrix

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

has determinant $(a-\lambda)(d-\lambda) - bc$.
Equivalently, the characteristic equation is

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

Since trA = a + d and det A = ad - bc, the characteristic equation can be written as

$$\lambda^2 - (trA)\lambda + detA = 0. \tag{2.2}$$

The roots of (2.2) are

$$\lambda = \frac{(trA) \pm \sqrt{(trA)^2 - 4detA}}{2} \tag{2.3}$$

Note that the sum of these eigenvalues is trA. This is true of all square matrices, not just 2×2 matrices. We also know that det A is the product of the eigenvalues. For 2×2 matrices, we have either two real eigenvalues or a complex conjugate pair of eigenvalues, $\alpha \pm \beta$. Thus, for planar systems $x' = Ax$, we can use trA and det A to classify the origin as a saddle, unstable node, stable node, unstable focus, stable focus or center:

Case 1: If det A < 0, we claim that the origin is a saddle. To see why, we must show that the eigenvalues of A are real and have opposite sign. Suppose indirectly that A has complex conjugate eigenvalues $\alpha \pm \beta i$. Then the product of the eigenvalues (which equals det A) would be positive, contradicting our assumption that det A < 0. It follows that the eigenvalues must be real, and they must have opposite sign in order for det A < 0. Therefore, the origin is a saddle,as claimed.

Case 2: Next, suppose that det A > 0 and $(trA)^2 - 4detA \geq 0$. From equation (2.2) we know that the eigenvalues are real because the discriminant is positive. Since det A > 0, the eigenvalues have the same sign, and it follows that the origin is a node. Whether the origin is stable or unstable depends upon trA (the sum of the eigenvalues):

(i) If trA > 0, then both eigenvalues are positive and the node is unstable.
(ii) If trA < 0, then both eigenvalues are negative and the node is stable.

Case 3: Finally, suppose that det A > 0 and $(trA)^2 - 4detA < 0$. The discriminant in equation (2.2) is negative, implying that the eigenvalues are complex conjugate. The origin is either a focus or a center depending upon the trace of A.
The sum of the eigenvalues $\alpha \pm \beta i$ is 2α, or equivalently trA = 2α.

(i) If trA > 0, the real part of the eigenvalues is positive and the origin is an unstable focus.
(ii) If trA < 0, the real part of the eigenvalues is negative and the origin is a stable focus.
(iii) If trA = 0, the real part of the eigenvalues is zero and the origin is a center.

2.1.2 Equilibria and Linearization

When approximating nonlinear systems with linear ones, one typically works in the vicinity of equilibrium solutions.
We begin this section with some definitions which will be used frequently throughout the this thesis.

Definition 2.1.3. An equilibrium solution of

$$x' = f(x)$$

is any constant vector x^* such that $f(x^*) = 0$.

Definition 2.1.4. Let ε be a fixed, positive number and suppose $x \in R^n$. The open ball of radius ε centered at x is the set of all points whose distance from x is less than ε. We will use the notation.

B(x,ε) ={y $\in R^n$ such that | |x-y| | < ε}

Definition 2.1.5. An equilibrium x^* of x′ =f(x) is called isolated if there exists a positive number ε such that the open ball $B(x^*, \varepsilon)$ contains no equilibria other than x^*.
Definition 2.1.6. The system

$$x' = f(x^*) + Jf(x^*)(x - x^*) \tag{2.4}$$

is called the linearization of the system x′ = f (x) at the point x =x^*.
If x^* happens to be an equilibrium of the system, then f (x^*) = 0 and the linearization takes the particularly convenient form

$$x' = Jf(x^*)(x - x^*) \tag{2.5}$$

,where $Jf(x^*)$ is Jacobian of f at $x = x^*$

2.1.3 Global Stability

Classifying non-hyperbolic equilibria x^* as stable, asymptotically stable, or unstable can be incredibly difficult (and often impossible). We now describe a classification technique that was originally proposed by Russian mathematician A.M. Ly apunov in his (1892) doctoral dissertation.

Consider the ODE $x' = f(x)$, where f is continuously differentiable. A solution x (t) of this equation can be written in terms of its component functions

$$x(t) = [x_1(t), x_2(t), ...x_n(t)].$$

Now suppose that $V : R^n \longrightarrow R$ is a continuously differentiable scalar-valued function. Then by the chain rule,where f is continuously differentiable. A solution x(t) of this equation can be written in terms of its component functions,

$$x(t) = [x_1(t), x_2(t), ...x_n(t)]$$

Now suppose that $V : R^n \longrightarrow R$ is a continuously differentiable scalar-valued function. Then by the chain rule,

$$\begin{gathered}
\frac{d}{dt}V(x(t)) = \frac{d}{dt}V(x_1(t), x_2(t), ..., x_n(t)) \\
= \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + ... + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} \\
= \tfrac{\partial V}{\partial x_1}, \tfrac{\partial V}{\partial x_2}, \ldots, \tfrac{\partial V}{\partial x_n} \cdot \left[\tfrac{dx_1}{dt}, \tfrac{dx_2}{dt}, \ldots, \tfrac{dx_n}{dt}\right] \\
\nabla V(x) \cdot x'(t) = \nabla V(x) \cdot f(x)
\end{gathered}$$

Observation. This calculation tells us how the function V changes as we move along a solution curve x (t). In particular, if we find that

$$\frac{d}{dt}V(x(t)) < 0$$

inside some set $E \subset R^n$, then the function V decreases as we move along solution curves in E in the direction of increasing t. Lyapunov exploited this observation to provide a creative but intuitive way for analyzing stability of equilibria x^*. The idea is to define a function V on a set E containing x^*, where V is chosen in such a way that we can tell whether the flow in E is towards or away from the equilibrium. In what follows, we have in mind a system $x' = f(x)$ with an isolated equilibrium x^*. We assume that f is continuously differentiable in some open ball E = $B(x^*, \varepsilon)$ of radius $\varepsilon > 0$ centered at the equilibrium.

Theorem 2.1. *(Lyapunov). suppose there is a function* $V : R^n \longrightarrow R$ *which is*

(i) defined and continuously differentiable on the set E = $B(x^*, \varepsilon)$*;*
(ii) $V(x^*) = 0$*; and*
(iii) $V(x) > 0$ *if* $x \neq x^*$.

then the equilibrium x^* is

(1) stable if

$$\frac{d}{dt}V(x(t)) = \nabla V(x) \cdot f(x) \leq 0$$

for all $x \in E$.

(2) Asymptotically stable if

$$\frac{d}{dt}V(x(t)) = \nabla V(x) \cdot f(x) < 0$$

for all $x \in E$, except possibly at x^* itself.

(3) unstable if

$$\frac{d}{dt}V(x(t)) = \nabla V(x) \cdot f(x) \geq 0$$

for all $x \in E$, except possibly at x^* itself.

Proof: Given any $\varepsilon > 0$, choose $r \in (0, \varepsilon)$ such that $B_r = \{x \in R_n, ||x|| \leq r\} \subset D$. Let $\alpha = min_{|x| \longrightarrow 1} V(x)$. Choose $\beta \in (0, \alpha)$ and define $\Omega_\beta = \{\mathrm{x} \in \mathrm{B_r}, \mathrm{V(x)} \leq \beta\}$

It holds that if $x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \forall t$ because

$$\dot{\mathrm{V}} \leq 0 \Rightarrow \mathrm{V(x(t))} \leq \mathrm{V(x(0))} \leq \beta$$

Further $\exists \delta > 0$ such that $||x|| < \delta \Rightarrow V(x) < \beta$.
Therefore, we have that

$$\beta_\delta \subset \Omega_\beta \subset \beta_r$$

and furthermore

$$x(0) \in \beta_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega \Rightarrow x(t) \in \beta_r$$

Finally, it follows that

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < r \leq \varepsilon, \forall t > 0$$

In order to show asymptotic stability, we need to to show that $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$. In this case, it turns out that it is sufficient to show that $V(x(t)) \longrightarrow 0$ as $t \longrightarrow \infty$. Since V is monotonically decreasing and bounded from below by 0, then

$$\mathrm{V(x) \longrightarrow c \geq 0ast \longrightarrow \infty}$$

Finally, it can be further shown by contradiction that the limit c is actually equal to 0.

Definition 2.1.7. Any function $V : R^n \longrightarrow R$ satisfying the conditions of (Theorem 2.1) is called a Lyapunov function.

Theorem 2.2 (Existence and uniqueness).

(a) *If f is continuous on an open rectangle* $R : \{a < x < b,\ c < \ y \ < d\}$ *containing* (x_0, y_0), *then the initial value problem* $x' = f(x,y), y(x_0) = y_0$ *has at least one solution on some open subinterval (a,b) of* x_0.

(b) *If both f and* f_y *are continuous on R then* $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ *has a unique solution on some open subinterval (a,b) of containing* x_0

Definition 2.1.8. Bounded function: Let f be a real valued function defined on a domain D.The function f is said to be bounded on D if and only if there is a positive number M such that $f(x,y) \leq M$ for all $(x,y) \in D$.

Before we state (Bendixson-Dulac Theorem), we need a couple of definitions.
Definition 2.1.9. We define the divergence of the vector field

$$F(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

by

$$divF(x,y) = f_x(x,y) + g_y(x,y).$$

Definition 2.1.10. A domain $D \subset R_2$ is said to be a simply connected domain provided it is connected and for any simple closed curve C in D the interior of C is a subset of D.

Consider the system of equations

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \tag{2.6}$$

Theorem 2.3. *(Bendixson-Dulac Theorem) Assume there is a continuously differentiable function $\alpha(x,y)$ on a simply connected domain $D \subset R_2$ such the*

$$div[\alpha(x,y)F(x,y)]$$

is either always positive or always negative on D. Then the system (2.6), does not have a cycle in D.

Proof Assume that the system (2.6), does have a cycle. Then there is a nonconstant periodic solution x, y in D. Assume this periodic solution has minimum period ω, let C be the corresponding simple closed curve oriented by increasing t, $0 \le t \le \omega$, and let E be the region interiorto C. Since $div[\alpha(x,y)F(x,y)]$ is of one sign on $E \subset D$, we have that the double integral

$$\int_E div[\alpha(x,y)F(x,y)]dA \neq 0.$$

On the other hand, we get, using Green's theorem,

$$\int_E div[\alpha(x,y)F(x,y)]dA$$

$$= \int_E \left\{ \frac{\delta}{\delta x}[\alpha(x,y)f(x,y) + \frac{\delta}{\delta y}[\alpha(x,y)g(x,y)]dA \right.$$

$$= \pm \int_C [\alpha(x,y)f(x,y)dy - \alpha(x,y)g(x,y)dx].$$

$$= \pm \int_0^\omega \alpha(x(t),y(t))[f(x(t),y(t))\dot{y}(t) - g(x(t),y(t))\dot{x}(t))]dt$$

$$= \pm \int_0^\omega \alpha(x(t),y(t))[f(x(t),y(t))g(x(t),y(t)) - g(x(t),y(t))f(x(t),y(t))]dt$$

=0
which is a contradiction.

Chapter 3
The Mathematical Model

3.1 Introduction

In this chapter, we present, basic assumptions model formulation. Consider a prey-predator model which describes the interaction between two prey species and one predator specie with constant prey refuge. Let x(t) and y(t) represent the population of the first and second prey species respectively and z(t) represent the predator specie at any time t. The main feature of the model is that two different functional responses of the predator are incorporated in the model to represent the difference in the way the predator feeds on each of the prey species. Terms representing logistic growth of the prey species in absence of the predator are included in the prey equations. The model has three non-linear autonomous ordinary differential equations describing how the population densities of the three species would vary with time.

3.2 Assumptions and parameters of the model

3.2.1 The assumptions

The following assumptions are made in order to construct the model:

(i) The species live in an ecosystem where external factors such as droughts, fires, epidemics are stable or have a similar effect on the interacting species.

(ii) Due to unlimited food supply there is no competitions among prey.

(iii) To over come the risk of extinction some portion of the second prey could be reserved.

(iv) There is logistic growth of the prey in absence of the predator. That is the population of the prey would increase (or decrease) exponentially until it reaches the carrying capacity of the given ecosystem.

(v) The rate of increase of the predator population depends on the amount of prey biomass it converts as food.

3.2.2 The parameters

The following are the parameters used in the model:

(i) r and s are the intrinsic growth rate of prey x and y respectively.

(ii) K and L are carrying capacities for prey x and y respectively.

(iii) ω_1 and ω_2 are the first and the second prey specie's searching efficiency of the predator.

(iv) q is rate of reserve.

(v) b_1 and b_2 are the conversion factors denoting the number of newly born predators for each captured of first and second prey respectively.

(vi) c is natural mortality rate of predator z.

(vii) m is half saturation co-efficient.

3.3 Model Formulation

From the model description and assumptions the equations to represent the dynamics of the predator and the two prey species ecosystem are formulated as given in (T. K. Kar and Ashim Batabyal (2010)).

$$\begin{cases} \frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{rx}(1-\frac{\mathrm{x}}{\mathrm{K}}) - \omega_1 \mathrm{xz} \\ \frac{\mathrm{dy}}{\mathrm{dt}} = \mathrm{sy}(1-\frac{\mathrm{y}}{\mathrm{L}}) - \omega_2 \frac{\mathrm{yz}}{\mathrm{m+y}} \\ \frac{\mathrm{dz}}{\mathrm{dt}} = \mathrm{b}_1 \omega_1 \mathrm{xz} + \mathrm{b}_2 \omega_2 \frac{\mathrm{yz}}{\mathrm{m+y}} - \mathrm{cz} \end{cases} \tag{3.1}$$

$x(0) = x_0 > 0, y(0) = y_0 > 0, z(0) = z_0 > 0$

Now we assume that the population of the second prey is few and we protect from continuous predation. If we refuge qy, $q \in (0,1)$ of the prey population and let (1-q)y for predation, then the above model is formulated as follows:

$$\begin{cases} \frac{dx}{dt} = rx(1-\frac{x}{K}) - \omega_1 xz \\ \frac{dy}{dt} = sy(1-\frac{y}{L}) - \omega_2 \frac{(1-q)yz}{m+(1-q)y} \\ \frac{dz}{dt} = b_1\omega_1 xz + b_2\omega_2 \frac{(1-q)yz}{m+(1-q)y} - cz \end{cases} \tag{3.2}$$

$x(0)=x_0>0, y(0)>y_0, z(0)=z_0>0$
where all parameters in the model are positive.

3.3.1 Boundedness

Boundedness implies that the system is biologically well-behaved.
The next proposition ensure the boundedness of system (3.2).

Proposition 3.1. *The prey population is always bounded from above.*

Proof. The following inequality is found from the first sub equation of (3.2)

$$\frac{dx}{dt} = rx(1-\frac{x}{K}) - \omega_1 xz \leq rx(1-\frac{x}{K})$$

This implies

$$\frac{dx}{dt} \leq rx(1-\frac{x}{K}).$$

Rearranging and separating variables we have

$$\frac{K}{rx(K-x)}dx \leq dt.$$

By partial fraction decomposition we have

$$\left(\frac{1}{x} - \frac{1}{x-K}\right)dx \leq rdt.$$

Integrating both sides yields

$$\ln\left(\frac{x}{K-x}\right) \leq rt + c.$$

Rearranging and simplifying we have

$$x(t) \leq \frac{Ke^{rt+c}}{1+e^{rt+c}}.$$

Taking limit and employing L'Hospital's one easily obtain that

$$\limsup_{x\to\infty} x(t) \leq K.$$

The proof for the second prey can be done in the same way. Thus the system is bounded.

3.3.2 Existence of Equilibrium Points

In this section, we establish conditions for the existence of the equilibrium points of the system. By equating (3.2) to zero, we find that the system has 7 possible non-negative equilibria.
Namely $E_0(0,0,0), E_1(x,0,0), E_2(0,y,0), E_3(x,y,0), E_4(0,y,z), E_5(x,0,z)$, and the co-existence equilibrium $E_6(x^*,y^*,z^*)$.

The existence of E_0(0,0,0) is trivial. We show the other equilibria as follows:

(i) **Existence of E_1(x,0,0)**
Let y=0 and z=0.Then the first sub equation of equation (3.2) gives:

$$rx(1-\frac{x}{K}) - \omega_1 xz = 0 \tag{3.3}$$

substituting z=0 in (3.3) we have

$$r(1-\frac{x}{K}) = 0 \tag{3.4}$$

solving (3.4) for x, we get

$$x = K.$$

Thus, the first equilibrium point is

$$E_1(x,0,0) = (K,0,0).$$

Hence in absence of prey y and predator z the density of prey x will increase or decrease until it reaches the carrying capacity K of the ecosystem.

(ii) **Existence of E_2(0,y,0)**
Let x=0 and z=0.Then the second sub equation of equation (3.2) gives:

$$sy(1-\frac{y}{L}) - \omega_2 \frac{(1-q)yz}{m+(1-q)y} = 0 \tag{3.5}$$

substituting z=0 in (3.5) yields

$$(1-\frac{y}{L}) = 0 \tag{3.6}$$

Solving (3.6) for y yields

$$y = L$$

Therefore, the second equilibrium point is

$$E_2(0, y, 0) = (0, L, 0).$$

This implies that in the absence of prey x and predator z, the density of prey y will increase or decrease until it reaches the carrying capacity L of the ecosystem.

(iii) **Existence of E_3(x,y,0)**

Let z=0.Then the first and second sub equation of equation (3.2) gives:

$$\begin{cases} rx(1-\frac{x}{K}) - \omega_1 xz = 0 \\ sy(1-\frac{y}{L}) - \omega_2 \frac{(1-q)yz}{m+(1-q)y} = 0 \end{cases} \tag{3.7}$$

substituting z=0 in (3.7) we have

$$\begin{cases} r(1-\frac{x}{K}) = 0 \\ s(1-\frac{y}{L}) = 0 \end{cases} \tag{3.8}$$

solving (3.8) we have

$$x = K$$

and

$$y = L$$

Hence the third equilibrium point is

$$E_3(x, y, 0) = (K, L, 0).$$

This condition implies that,in absence of the predator,the vital parameters for existence of the two prey species are; per capital intrinsic growth rates of the preys,carrying capacities of the preys and inter specific competition among the preys.But by our assumption there is no inter specific competition among the prey species.

Therefore the two prey species will co-exist.

(iv) **Existence of E_4(0,y,z)**

Here we note that, in the absence of the first prey the only food resource for the predator is the second prey. So that the searching efficiency and conversion factor of the predator ω_2 and b_2 respectively should be adjusted with the death rate of the predator c.That is $b_2{}^* \omega_2{}^* > c$. This implies that the predator density is controlled by the portion of the prey that are reserved and let for predation. Let x=0.Then the second and third sub equation of equation (3.2) gives:

$$\begin{cases} \mathrm{sy}(1-\frac{\mathrm{y}}{\mathrm{L}})-\omega_2\frac{(1-\mathrm{q})\mathrm{yz}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}}=0 \\ \mathrm{b}_1\omega_1\mathrm{xz}+\mathrm{b}_2^*\omega_2^*\frac{(1-\mathrm{q})\mathrm{yz}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}}-\mathrm{cz}=0 \end{cases} \tag{3.9}$$

substituting x=0 in (3.9) yields

$$\begin{cases} \mathrm{s}(1-\frac{\mathrm{y}}{\mathrm{L}})-\omega_2\dfrac{(1-\mathrm{q})\mathrm{z}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}}=0 \\ \mathrm{b_2}^*\omega_2{}^*\dfrac{(1-\mathrm{q})\mathrm{y}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}}-\mathrm{c}=0 \end{cases} \tag{3.10}$$

Solving the second equation of (3.10) for y we get

$$\mathrm{y}=\frac{\mathrm{cm}}{(1-\mathrm{q})(\mathrm{b}_2^*\omega_2^*-\mathrm{c})}$$

substituting the value of y in the first equation of (3.10) we have

$$z=b_2ms\left[\frac{L(b_2^*\omega_2^*-c)(1-q)-cm}{L((1-q)(b_2^*\omega_2^*-c))^2}\right]$$

Therefore the fourth equilibrium point is

$$\mathrm{E}_4(0,\mathrm{y},\mathrm{z})=\left(0,\frac{\mathrm{cm}}{(1-\mathrm{q})(\mathrm{b}_2^*\omega_2^*-\mathrm{c})},\mathrm{b}_2\mathrm{ms}\left[\frac{\mathrm{L}(\mathrm{b}_2^*\omega_2^*-\mathrm{c})(1-\mathrm{q})-\mathrm{cm}}{\mathrm{L}((1-\mathrm{q})(\mathrm{b}_2^*\omega_2^*-\mathrm{c}))^2}\right]\right)$$

which implies that having refuge serves both species to reduce the chance of extinction.

(v) **Existence of E_5(x,0,z)**

Let y=0.Then the first and third sub equation of equation (3.2) gives:

$$\begin{cases} rx(1-\frac{x}{K})-\omega_1xz=0 \\ b_1\omega_1xz+b_2\omega_2\frac{(1-q)yz}{m+(1-q)y}-cz=0 \end{cases} \tag{3.11}$$

substituting y=0 in (3.11) yields

$$\begin{cases} r(1-\frac{x}{K})-\omega_1z=0 \\ b_1\omega_1x-c=0 \end{cases} \tag{3.12}$$

solving the second equation of (3.12) we get

$$x = \frac{c}{b_1 \omega_1}$$

substituting the value of x in the first equation of (3.12) we have ,

$$z = \frac{r}{\omega_1}(1 - \frac{c}{b_1 \omega_1 k})$$

Hence ,the fifth equilibrium point is

$$E_5(x,0,z) = \left(\frac{c}{b_1 \omega_1}, 0, \frac{r}{\omega_1}(1 - \frac{c}{b_1 \omega_1 k})\right)$$

,for

$$\mathrm{b}_1 > \frac{\mathrm{c}}{\omega_1 \mathrm{K}}$$

This implies that the conversion rate and the searching efficiency of the predator are inversely proportional. Since the prey population x is abundant, the searching efficiency of the predator is less, as the result the conversion rate of the predator increases consequently the newly born predator increase. Again, due to this, the density of prey population decrease which implies the searching efficiency of the predator increase so that the conversion rate decrease as the result the newly born predator decrease and visversal.

(vi) **Existence of** $E_6(x^*, y^*, z^*)$
Equate equations (3.2) to zero and from this we find two functions f(x,y) and g(x,y) which intersect at the equilibrium point $E_6(x^*, y^*, z^*)$ Equating equations (3.2) to zero gives,

$$\begin{cases} \mathrm{r}(1 - \frac{\mathrm{x}}{\mathrm{K}}) - \omega_1 \mathrm{z} = 0 \\ \mathrm{s}(1 - \frac{\mathrm{y}}{\mathrm{L}}) - \omega_2 \frac{(1-\mathrm{q})\mathrm{z}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}} = 0 \\ \mathrm{b}_1 \omega_1 \mathrm{x} + \mathrm{b}_2 \omega_2 \frac{(1-\mathrm{q})\mathrm{y}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}} - \mathrm{c} = 0 \end{cases} \tag{3.13}$$

From the third equation in (3.13) we get

$$f(x,y) = 0 \tag{3.14}$$

where

$$f(x,y) = b_1 \omega_1 xm + b_1 \omega_1 xy + b_2 \omega_2 y + cqy - [b_1 \omega_1 qxy + b_2 \omega_2 qy + cm + cy]$$

From the first and the second equation in (3.13) we have

$$g(x,y) = 0 \tag{3.15}$$

where

$$g(x,y) = \omega_1 s(L-y)[m+(1-q)y] - Lr\omega_2(1-q)(K-x)$$

Equations (3.14) and (3.15) are two functions of x and y. To prove the existence of $E_6(x^*,y^*,z^*)$ conditions under which f(x,y) and g(x,y) meet in the interior of the positive (x,y) plane,at a point (x^*,y^*) , are found. Knowing (x^*,y^*), the value of z^* can be obtained from.

$$r(1-\frac{x}{K}) - \omega_1 z = 0 \tag{3.16}$$

solving for z_6^* in (3.16) we get

$$z_6^* = \frac{r}{\omega_1}(K-x_6^*)$$

From (3.14) we note the following
when $x \to 0$,then $y \to y_a$ where

$$y_a = \frac{cm}{(1-q)(b_2\omega_2 - c)}$$

we note that $y_a > 0$ if $b_2\omega_2 > c$
Also from (3.14) we have

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{A}_1}{\mathrm{B}_1} \tag{3.17}$$

where

$$\mathrm{A}_1 = \mathrm{b}_1\omega_1[\mathrm{m}+(1-\mathrm{q})\mathrm{y}] > 0$$
$$\mathrm{B}_1 = (1-\mathrm{q})[\mathrm{c}-(\mathrm{b}_1\omega_1\mathrm{x}+\mathrm{b}_2\omega_2)]$$

It is clear that

$$\frac{\mathrm{dy}}{\mathrm{dx}} > 0$$

if

$$B_1 > 0$$

Again from (3.15), we note that when $x \to 0$,then $y \to y_b$,
where

$$\mathrm{y_b} = \frac{-\mathrm{B}_2 + \sqrt{\mathrm{B}_2^2 - 4\mathrm{A}_2\mathrm{C}_2}}{2\mathrm{A}_2} \tag{3.18}$$
$$\mathrm{A}_2 = -\omega_1\mathrm{Ks}(1-\mathrm{q})$$

$$B_2 = \omega_1 Ks[L(1-q)-m]$$
$$C_2 = \omega_1 KsmL - KLr\omega_2(1-q)$$

Clearly $A_2 < 0$ and $C_2 > 0$
if

$$\omega_1 KsmL > KLr\omega_2(1-q)$$

We have

$$\frac{dy}{dx} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} \tag{3.19}$$

Where

$$\frac{\partial g}{\partial x} = Lr\omega_2(1-q) > 0$$
$$\frac{\partial g}{\partial y} = \omega_1 Ks[(L-2y)(1-q)-m]$$

we note that

$$\frac{dy}{dx} < 0$$

If

$$\frac{\partial g}{\partial y} > 0$$

From the above analysis we note that two isoclines (3.14) and (3.15) intersect at a unique point (x^*, y^*).
Knowing the value of x^* and y^* the value of z^* can be calculated from

$$z^* = \frac{r}{\omega_1} K(K - x^*)$$

It may be noted that for z^* to be positive, we must have $K > x^*$
This completes the existence of $E_6(x^*, y^*, z^*)$
Hence the three species co-exists in the ecosystem.

3.3.3 Stability Analysis

In this part of the thesis we present the local and global stability of non-negative equlibria in (3.2)

3.3.3.1 Local Stability Analysis

We present local stability of each equilibrium points by calculating the Jacobian matrix.

$$J(E_i) = \begin{pmatrix} \mathrm{r} - \frac{2\mathrm{rx}}{\mathrm{K}} - \omega_1 \mathrm{z} & 0 & -\omega_1 \mathrm{x} \\ 0 & \mathrm{s} - \frac{2\mathrm{sy}}{\mathrm{L}} - \omega_2 \frac{(1-\mathrm{q})\mathrm{mz}}{[\mathrm{m}+(1-\mathrm{q})\mathrm{y}]^2} & \omega_2 \frac{(1-\mathrm{q})\mathrm{y}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}} \\ \mathrm{b}_1 \omega_1 \mathrm{z} & \mathrm{b}_2 \omega_2 \frac{(1-\mathrm{q})\mathrm{mz}}{[\mathrm{m}+(1-\mathrm{q})\mathrm{y}]^2} & \mathrm{b}_1 \omega_1 \mathrm{x} + \mathrm{b}_2 \omega_2 \frac{(1-\mathrm{q})\mathrm{y}}{\mathrm{m}+(1-\mathrm{q})\mathrm{y}} - \mathrm{c} \end{pmatrix}$$

Then we compute the eigenvalues of the Jacobian matrix at there equilibrium points. Based on the sign of the eigenvalue we identify the stability of the point.

(i) $E_0(0,0,0)$
The Jacobian matrix evaluated at E_0 gives

$$J(E_0) = \begin{pmatrix} r-\lambda & 0 & 0 \\ 0 & s-\lambda & 0 \\ 0 & 0 & -c-\lambda \end{pmatrix}$$

The eigenvalues of $J(E_0)$ are r , s and -c,
We see that $r > 0$ and $s > 0$.
so $E_0(0,0,0)$ is unstable
This implies that E_0 is always a saddle node and there can not be total extinction of the system (3.2) for positive initial conditions.

(ii) $E_1(x,0,0) = (K,0,0)$
The Jacobian matrix evaluated at E_1 gives

$$J(E_1) = \begin{pmatrix} -r-\lambda & 0 & 0 \\ 0 & s-\lambda & 0 \\ 0 & 0 & \mathrm{b}_1\omega_1 - \mathrm{c} - \lambda \end{pmatrix}$$

The eigenvalues of $J(E_1)$ are -r , s and $\mathrm{b}_1\omega_1 - \mathrm{c}$.

We see that $s > 0$ and if $\mathrm{b}_1\omega_1 > \mathrm{c}$,then $E_1(K,0,0)$ is unstable
This situation implies us E_1 is a saddle point with locally stable manifold in x directions and with locally unstable manifold in y – z plane.

(iii) $E_2(0,y,0) = (0,L,0)$
The Jacobian matrix evaluated at E_2 gives

$$J(E_2) = \begin{pmatrix} r-\lambda & 0 & 0 \\ 0 & -s-\lambda & \omega_2\frac{(1-\mathrm{q})\mathrm{L}}{\mathrm{m}+(1-\mathrm{q})\mathrm{L}} \\ 0 & 0 & \mathrm{b}_2\omega_2\frac{(1-\mathrm{q})\mathrm{L}}{\mathrm{m}+(1-\mathrm{q})\mathrm{L}}-\mathrm{c}-\lambda \end{pmatrix}$$

The eigenvalues of $J(E_2)$ are r,-s and $b_2\omega_2\frac{(1-q)L}{m+(1-q)L}-c$. We see that $r > 0$ and if $b_2\omega_2\frac{(1-q)L}{m+(1-q)L} > c$,then E_2 is unstable in x-z plane,
but if $b_2\omega_2\frac{(1-q)L}{m+(1-q)L} < c$, then E_2 is locally asymptotically stable in y-z plane.
This situation implies us the second prey exists while there can not be total extinction of the system for positive initial conditions.

(iv) $E_3(x,y,0) = (K,L,0)$
The Jacobian matrix evaluated at E_3 gives

$$J(E_3) = \begin{pmatrix} -r-\lambda & 0 & 0 \\ 0 & -s-\lambda & \omega_2\frac{(1-q)L}{m+(1-q)L} \\ 0 & 0 & b_1\omega_1K+b_2\omega_2\frac{(1-q)L}{m+(1-q)L}-c-\lambda \end{pmatrix}$$

Since the eigenvalues of $J(E_3)$ are $-r < 0$, $-s < 0$ E_3 exists and is asymptotically stable in x – y plane, but if

$$b_1\omega_1K+b_2\omega_2\frac{(1-q)L}{m+(1-q)L} < c$$

,then E_3 is stable in x- y-z space.

(v) $\mathrm{E}_4(0,\mathrm{y},\mathrm{z}) = \left(0, \frac{\mathrm{cm}}{(1-\mathrm{q})(\mathrm{b}_2^*\omega_2^*-\mathrm{c})}, \mathrm{b}_2^*\mathrm{ms}\left[\frac{\mathrm{L}(\mathrm{b}_2^*\omega_2^*-\mathrm{c})(1-\mathrm{q})-\mathrm{cm}}{\mathrm{L}((1-\mathrm{q})(\mathrm{b}_2^*\omega_2^*-\mathrm{c}))^2}\right]\right)$
The Jacobian matrix evaluated at E_4, gives

$$J(E_4) = \begin{pmatrix} r-b_2^*\omega_1\frac{(L-c)}{L(1-q)(b_2^*\omega_2^*-c)}-\lambda & 0 & 0 \\ 0 & A^* & \frac{cm}{mb_2} \\ b_1b_2^*\omega_1ms\frac{(L-cm)}{L(1-q)(b_2^*\omega_2^*-c)} & B^* & -\lambda \end{pmatrix}$$

where
$A^* = s - [\frac{2cms}{L(1-q)(b_2^*\omega_2^*-c)} + ms\frac{(1-q)(b_2^*\omega_2^*-c)(L-cm)}{L}] - \lambda$
$B^* = b_2^*\omega_2^* - c)(b_2^*ms)\frac{(L-cm)}{L}$
If $b_2^*\omega_1\frac{(L-c)}{L(1-q)(b_2^*\omega_2^*-c)} > r$, and
$\frac{2cms}{L(1-q)(b_2^*\omega_2^*-c)} + ms\frac{(1-q)(b_2^*\omega_2^*-c)(L-cm)}{L} > s$
then E_4 it will be asymptotically stable in x – y – z space is stable in x-y-z space.

(vi) $E_5(x,0,z) = (\frac{c}{b_1\omega_1}, 0, \frac{r}{\omega_1}(1 - \frac{c}{b_1\omega_1 k})$
The Jacobian matrix evaluated at E_5, gives

$$J(E_5) = \begin{pmatrix} \frac{\mathrm{c(1-2r)}}{\mathrm{b_1\omega_1K}} - \lambda & 0 & \frac{\mathrm{-c}}{\mathrm{b_1}} \\ 0 & \mathrm{s} + \frac{\mathrm{\omega_2rc}}{\mathrm{mb_1\omega_1^2K}} - \frac{\mathrm{\omega_2r}}{\mathrm{\omega_1m}} - \lambda & 0 \\ \mathrm{r(b_1} - \frac{\mathrm{c}}{\mathrm{\omega_1K}}) & \mathrm{rb_2\omega_2}\frac{\mathrm{(1-q)}}{\mathrm{m\omega_1}}(1 - \frac{\mathrm{c}}{\mathrm{b_1\omega_1K}}) & -\lambda \end{pmatrix}$$

If $1 < 2r$ and $\frac{\omega_2 r}{\omega_1 m} > s + \frac{\omega_2 rc}{mb_1\omega_1^2K}$,then E_5 exists and is asymptotically stable in x – y – z space.
This situation correspond to the co-existance of the first prey and predator.

(vii) $E_6(x^*, y^*, z^*)$.

Now, we investigate the local stability of E_6, we first linearize the system (3.2) using the following transformations.

$$x = x^* + X, y = y^* + Y, z = z^* + Z$$

where X, Y, Z is small perturbation about

$$E_6(x^*, y^*, z^*)$$

and then the linear form of (3.2) is calculated as follows:

$$\begin{cases} \frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{rx}(1 - \frac{\mathrm{x}}{\mathrm{K}}) - \omega_1\mathrm{xz} \\ \frac{\mathrm{dy}}{\mathrm{dt}} = \mathrm{sy}(1 - \frac{\mathrm{y}}{\mathrm{L}}) - \omega_2\frac{\mathrm{(1-q)yz}}{\mathrm{m+(1-q)y}} \\ \frac{\mathrm{dz}}{\mathrm{dt}} = \mathrm{b_1}\omega_1\mathrm{xz} + \mathrm{b_2}\omega_2\frac{\mathrm{(1-q)yz}}{\mathrm{m+(1-q)y}} - \mathrm{cz} \end{cases}$$

To obtain the linearized form we begin by computing the Jacobian of (3.2)

$$\begin{pmatrix} r-\frac{2rx}{K}-\omega_1 z & 0 & -\omega_1 x \\ 0 & s-\frac{2sy}{L}-\omega_2\frac{(1-q)mz}{[m+(1-q)y]^2} & \omega_2\frac{(1-q)y}{m+(1-q)y} \\ b_1\omega_1 z & b_2\omega_2\frac{(1-q)mz}{[m+(1-q)y]^2} & b_1\omega_1 x+b_2\omega_2\frac{(1-q)y}{m+(1-q)y}-c \end{pmatrix}$$

Evaluate the Jacobian at (x^*, y^*, z^*)

$$\begin{pmatrix} r-\frac{2rx^*}{K}-\omega_1 z^* & 0 & -\omega_1 x^* \\ 0 & s-\frac{2sy^*}{L}-\omega_2\frac{(1-q)mz^*}{[m+(1-q)y^*]^2} & \omega_2\frac{(1-q)y^*}{m+(1-q)y^*} \\ b_1\omega_1 z^* & b_2\omega_2\frac{(1-q)mz^*}{[m+(1-q)y^*]^2} & b_1\omega_1 x^*+b_2\omega_2\frac{(1-q)y^*}{m+(1-q)y^*}-c \end{pmatrix}$$

Therefore the linearizion of the system at (x^*, y^*, z^*)

$$\begin{pmatrix} r-\frac{2rx^*}{K}-\omega_1 z^* & 0 & -\omega_1 x^* \\ 0 & s-\frac{2sy^*}{L}-\omega_2\frac{(1-q)mz^*}{[m+(1-q)y^*]^2} & \omega_2\frac{(1-q)y^*}{m+(1-q)y^*} \\ b_1\omega_1 z^* & b_2\omega_2\frac{(1-q)mz^*}{[m+(1-q)y^*]^2} & b_1\omega_1 x^*+b_2\omega_2\frac{(1-q)y^*}{m+(1-q)y^*}-c \end{pmatrix}\begin{pmatrix} x-x^* \\ y-y^* \\ z-z^* \end{pmatrix}$$

$$\dot{X} = rx - \frac{2rx^*x}{K} - \omega_1 z^*x + 0 - \omega_1 x^*z$$

$$\dot{X} = rx - \frac{rx^*x}{K} - \omega_1 z^*x + (-\frac{rx^*x}{K} - \omega_1 x^*z)$$

$$\dot{X} = -\frac{rx^*x}{K} - \omega_1 x^*z$$

$$\dot{Y} = sy - \frac{2sy^*y}{L} - \omega_2\frac{(1-q)mz^*y}{[m+(1-q)y^*]^2} - \omega_2\frac{(1-q)y^*z}{m+(1-q)y^*}$$

$$\dot{Y} = (sy - \frac{sy^*y}{L} - \omega_2\frac{(1-q)z^*y}{m+(1-q)y^*})$$

$$+(\omega_2\frac{(1-q)z^*y}{m+(1-q)y^*} - \mathrm{fracsy}^*\mathrm{yL} - \omega_2\frac{(1-q)mz^*y}{[m+(1-q)y^*]^2} - \omega_2\frac{(1-q)y^*z}{m+(1-q)y^*})$$

$$\dot{Y} = -\frac{sy^*y}{L} + \omega_2\frac{(1-q)z^*y}{m+(1-q)y^*} - \omega_2\frac{(1-q)mz^*y}{[m+(1-q)y^*]^2} - \omega_2\frac{(1-q)y^*z}{m+(1-q)y^*})$$

$$\dot{Y} = -\frac{sy^*y}{L} - \omega_2\frac{(1-q)y^*z}{m+(1-q)y^*} + \omega_2\frac{(1-q)^2z^*y^*y}{(m+(1-q)y^*)^2}$$

$$\dot{z} = b_1\omega_1 z^*x + b_2\omega_2\frac{(1-q)mz^*y}{[m+(1-q)y^*]^2} + (b_1\omega_1 x^*z + b_2\omega_2\frac{(1-q)y^*z}{m+(1-q)y^*} - cz)$$

$$\dot{z} = b_1\omega_1 z^* x + b_2\omega_2\frac{(1-q)z^*y}{m+(1-q)y^*} - b_2\omega_2\frac{(1-q)z^*y}{m+(1-q)y^*} + b_2\omega_2\frac{(1-q)mz^*y}{[m+(1-q)y^*]^2}$$

$$\dot{z} = b_1\omega_1 z^* x + [b_2\omega_2\frac{(1-q)z^*}{m+(1-q)y^*} - b_2\omega_2\frac{(1-q)y^*z^*}{[m+(1-q)y^*]^2}]y$$

$$\begin{cases} \dot{X} = -\frac{rx^*x}{K} - \omega_1 x^* z \\ \dot{Y} = -\frac{sy^*y}{L} - \omega_2\frac{(1-q)y^*z}{m+(1-q)y^*} + \omega_2\frac{(1-q)^2y^*z^*y}{(m+(1-q)y^*)^2} \\ \dot{z} = b_1\omega_1 z^* x + [b_2\omega_2\frac{(1-q)z^*}{m+(1-q)y^*} - b_2\omega_2\frac{(1-q)y^*z^*}{[m+(1-q)y^*]^2}]y \end{cases} \tag{3.20}$$

We now consider the following positive definite function

$$u = \frac{1}{2x^*}x^2 + \frac{d_1}{2y^*}y^2 + \frac{d_2}{2z^*}z^2$$

where d_1, d_2 are positive constants to be choosen later.
Differentiating U with respect to time t of linear model (3.20)

$$\dot{u} = \frac{2x\dot{x}}{2x^*} + \frac{2d_1y\dot{y}}{2y^*} + \frac{2d_2z\dot{z}}{2z^*}$$

substituting solution of model (3.20) and simlifying we have

$$\dot{u} = -\frac{r}{K}x^2 - \omega_1 xz + d_1(-\frac{s}{L}y^2 - \omega_2\frac{(1-q)yz}{m+(1-q)y^*} + \omega_2\frac{(1-q)^2z^*y^2}{(m+(1-q)y^*)^2})$$
$$+d_2(b_1\omega_1 xz + b_2\omega_2\frac{(1-q)myz}{(m+(1-q)y^*)^2})$$

Now choose

$$\begin{cases} d_2 = \frac{1}{b_1} \\ d_1 = \frac{mb_2}{b_1(m+(1-q)y^*)} \end{cases} . \tag{3.21}$$

After some rearengment we have

$$\dot{u} = -\frac{r}{K}x^2 - d_1(\frac{s}{L} - \omega_2\frac{(1-q)^2z^*}{(m+(1-q)y^*)^2})y^2$$

We can see that $\dot{u}$ is negative definte if

$$-\frac{4rd_1}{K}(\frac{s}{L} - \omega_2\frac{(1-q)^2z^*}{(m+(1-q)y^*)^2}) \leq 0$$

Heance we arrive at conclusion

Theorem 3.1. *If*

$$-\frac{4\mathrm{rd}_1}{\mathrm{K}}(\frac{\mathrm{s}}{\mathrm{L}}-\omega_2\frac{(1-\mathrm{q})^2\mathrm{z}^*}{(\mathrm{m}+(1-\mathrm{q})\mathrm{y}^*)^2})\leq 0$$

,then $E_6(x^*,y^*,z^*)$ is locally asymptotically stable.

This equilibrium point is an interior equilibrium point which establishes the coexistence of all species in the system.

3.3.3.2 Global stability and persistence of the system

In this section we shall determine the global stability analysis of the equilibrium points. Since my thesis focus on the co-existence of the species the research left E_0,E_1,E_2. Thus,we have the following theorem.

Theorem 3.2. *The interior equilibrium E_3 is globally asymptotically stable in the interior of the quadrant of the x-y plane.*

Proof:Let $H(x,y)=\frac{1}{xy}$
Clearly $H(x,y)$ is positive in the interior of the positive quadrant of the xy-plane.

$$h_1(x,y)=rx(1-\frac{x}{K})$$

$$h_2(x,y)=sy(1-\frac{y}{L})$$

Then

$$\Delta(x,y)=\frac{\delta}{\delta x}(h_1H)+\frac{\delta}{\delta y}(h_2H)$$

$$\Delta(x,y)=-\frac{r}{Ky}-\frac{s}{Lx}<0$$

From the above equation, we note that $\Delta(x,y)$ does not change sign and is not identically zero in the interior of the positive quadrant of the xy- plane.
Therefore, by Bendixson-Dulac criterion there exists no limit cycle in the positive quadrant of x-y plane. So, if E_3 is locally asymptotically stable then it will be globally asymptotically stable in the interior of positive quadrant of $\mathrm{x}-\mathrm{y}$ plane (Hale, 1969).
Heance E_3 is globally asymptotically stable.

Theorem 3.3. *The interior equilibrium E_4 is globally asymptotically stable in the interior of the quadrant of the y-z plane.*

Proof:Let $H(y,z) = \frac{1}{yz}$.
Clearly $H(y,z)$ is positive in the interior of the positive quadrant of the yz-plane.

$$h_1(y,z) = sy(1-\frac{y}{L}) - \omega_2\frac{(1-q)yz}{m+(1-q)y}$$

$$h_2(y,z) = b_2\omega_2\frac{(1-q)yz}{m+(1-q)y} - cz$$

Then

$$\Delta(y,z) = \frac{\delta}{\delta y}(h_1H) + \frac{\delta}{\delta z}(h_2H)$$

$$\Delta(y,z) = \frac{\omega_2(1-q)^2}{[m+(1-q)y]^2} - \frac{s}{zL} < 0$$

when

$$\frac{\omega_2(1-q)^2}{[m+(1-q)y]^2} < \frac{s}{zL}$$

Hence E_4 is globally asymptotically stable in R^2_+ of yz-plane.
Where

$$\mathrm{R}^2_+ = \{(\mathrm{y},\mathrm{z}) : \mathrm{y} > 0, \mathrm{z} > 0\}$$

and

$$\frac{\omega_2(1-q)^2}{[m+(1-q)y]^2} < \frac{s}{zL}$$

Theorem 3.4. *The interior equilibrium E_5 is globally asymptotically stable in the interior of the quadrant of the x-z plane.*

Proof:Let $H(x,z) = \frac{1}{xz}$.
Clearly $H(x,z)$ is positive in the interior of the positive quadrant of the xz-plane.

$$h_1(x,z) = rx(1-\frac{x}{K}) - \omega_1 xz$$

$$h_2(x,z) = b_1\omega_1 xz - cz$$

$$\Delta(x,z) = \frac{\delta}{\delta x}(h_1H) + \frac{\delta}{\delta z}(h_2H)$$

$$\Delta(x,z) = -\frac{r}{zK} < 0$$

Hence E_5 is globally asymptotically stable in R^2_+ of xz-plane.
Where

$$\mathrm{R}^2_+ = \{(\mathrm{x},\mathrm{z}) : \mathrm{x} > 0, \mathrm{z} > 0\}$$

Theorem 3.5. *Let the following hold*

$$-\frac{4rd_1}{K}(\frac{s}{L}-\omega_2\frac{(1-q)^2z^*}{(m+(1-q)y^*)^2})\leq 0$$

where d_1, d_2 are same as defined in (3.21), then the positive equilibrium E_6 is globally asymptotically stable with respect to all solutions initiating in the interior of R^3_+.

Proof:Consider the following positive definite function about E_6,

$$V(t)=(x-x^*-x^*\ln(\frac{x}{x^*}))+d_1(y-y^*-y^*\ln(\frac{y}{y^*}))+d_2(z-z^*-z^*\ln(\frac{z}{z^*}))$$

$$V(t)=(x-x^*-x^*(\ln x-\ln x^*))+d_1(y-y^*-y^*(\ln y-\ln y^*))+d_2(z-z^*-z^*(\ln z-\ln z^*))$$

Differentiating V with respect to t,we have

$$\frac{dV}{dt}=(\frac{dx}{dt}-0-x^*(\frac{1}{x}\frac{dx}{dt}-0))+d_1(\frac{dy}{dt}-0-y^*(\frac{1}{y}\frac{dy}{dt}-0))+d_2(\frac{dz}{dt}-0-z^*(\frac{1}{z}\frac{dz}{dt}-0))$$

Rearranging gives;

$$\frac{dV}{dt}=(x-x^*)\frac{1}{x}\frac{dx}{dt}+(y-y^*)\frac{1}{y}\frac{dy}{dt}+(z-z^*)\frac{1}{z}\frac{dz}{dt}$$

Using the system of equation (3.2) along with the solution of E_6 given below

$$\begin{cases}\frac{dx}{dt}=-r\frac{x^2}{K}-\omega_1 xz\\ \frac{dy}{dt}=-s\frac{y^2}{L}-\omega_2\frac{(1-q)yz}{m+(1-q)y}\\ \frac{dz}{dt}=b_1\omega_1 xz+b_2\omega_2\frac{(1-q)yz}{m+(1-q)y}\end{cases}$$

we have

$$\frac{dV}{dt}=(x-x^*)\frac{1}{x}(x)\left[-r\frac{(x-x^*)}{K}-\omega_1(z-z^*)\right]+d_1(y-y^*)\frac{1}{y}(y)\left[-s\frac{(y-y^*)}{L}-\omega_2\frac{(1-q)(z-z*)}{m+(1-q)y}\right]$$
$$+d_2(z-z^*)\frac{1}{z}(z)\left[b_1\omega_1(x-x*)+b_2\omega_2\frac{(1-q)(y-y*)}{m+(1-q)y*}\right]$$

Simplification yields

$$\frac{dV}{dt}=(x-x^*)\left[-r\frac{(x-x^*)}{K}-\omega_1(z-z^*)\right]+d_1(y-y^*)\left[-s\frac{(y-y^*)}{L}-\omega_2\frac{(1-q)(z-z^*)}{m+(1-q)y}\right]$$
$$+d_2(z-z^*)\left[b_1\omega_1(x-x^*)+b_2\omega_2\frac{(1-q)(y-y^*)}{m+(1-q)y^*}\right]$$

Expanding we have

$$\frac{dV}{dt} = -r\frac{(x-x^*)^2}{K} - \omega_1(x-x^*)(z-z^*) - d_1 s\frac{(y-y^*)^2}{L} - d_1\omega_2\frac{(1-q)(y-y^*)(z-z^*)}{m+(1-q)y}$$
$$+d_2 b_1\omega_1(x-x^*)(z-z^*) + d_2 b_2\omega_2\frac{(1-q)(y-y^*)(z-z^*)}{m+(1-q)y^*}$$

Then

$$\frac{dV}{dt} = -\frac{r}{K}(x-x^*)^2 - (\omega_1 - d_2 b_1\omega_1)(x-x^*)(z-z^*) - \frac{d_1 s}{L}(y-y^*)^2$$
$$-d_1\omega_2\frac{(1-q)(y-y^*)(z-z^*)}{m+(1-q)y} + d_2 b_2\omega_2\frac{(1-q)(y-y^*)(z-z^*)}{m+(1-q)y^*}$$

Further simplified to

$$\frac{dV}{dt} = -\frac{r}{K}(x-x^*)^2 - (w_1 - d_2 b_1\omega_1)(x-x^*)(z-z^*) - \frac{d_1 s}{L}(y-y^*)^2$$
$$+\frac{(m\omega_2(1-q)(b_2 d_2 - d_1) - d_1\omega_2(1-q)^2 y^*)}{(m+(1-q)y)(m+(1-q)y^*)}(y-y^*)(z-z^*)$$
$$+\frac{d_1\omega_2 z^*}{(m+(1-q)y)(m+(1-q)y^*)}(y-y^*)^2$$

Rearranging we have

$$\frac{dV}{dt} = -\frac{r}{K}(x-x^*)^2 - (\omega_1 - d_2 b_1\omega_1)(x-x^*)(z-z^*)$$
$$-\frac{(m\omega_2(1-q)(d_1 - b_2 d_2) - d_1\omega_2(1-q)^2 y^*)}{(m+(1-q)y)(m+(1-q)y^*)}(y-y^*)(z-z^*)$$
$$-(\frac{d_1 s}{L} - \frac{d_1\omega_2 z^*}{(m+(1-q)y)(m+(1-q)y^*)})(y-y^*)^2$$

The above equation can further be written as

$$\frac{dV}{dt} = -\left[a_{11}(x-x^*)^2 + a_{22}(y-y^*)^2 + a_{12}(x-x^*)(z-z^*) + a_{23}(y-y^*)(z-z^*)\right]$$

where

$$a_{11} = \frac{r}{K}, a_{12} = \omega_1(1 - d_2 b_1)$$
$$a_{22} = \frac{d_1 s}{L} - \frac{d_1\omega_2 z^*}{(m+(1-q)y)(m+(1-q)y^*)}$$
$$a_{23} = \frac{m\omega_2(1-q)(d_1 - b_2 d_2) - d_1\omega_2(1-q)^2 y^*}{(m+(1-q)y)(m+(1-q)y^*)}$$

This implies

$$\frac{dV}{dt}<0$$

Therefore V is a Lypunov function with respect to E_6.
Hence E_6 is globally asymptotically stable in R^3_+ of xyz-space.
Where

$$R^3_+=\{(x,y,z):x>0,y>0,z>0\}$$

To examine the permanence of the system (3.2) we shall use the method of "average lyapunov function", (Gard and Hallam, 1997; Hafbauer, 1981). This method was first applied by Hutson and Vickers (1983) to ecological problems.

Let the average Lyapunov function for the system (3.2) be $\sigma(X)=x^py^p_1z^p_2$, where p, p_1 and p_2 are positive constants. Clearly in the interior of R^3_+ we have

$$\dot{\Psi}(x)=\frac{\dot{\sigma}(x)}{\sigma(x)}=\frac{px^{p-1}\dot{x}y^{p_1}z^{p_2}+x^p(p_1)y^{p_1-1}\dot{y}z^{p_2}+x^py^{p_1}(p_2)z^{p_2-1}\dot{z}}{x^py^{p_1}z^{p_2}}$$

$$\dot{\Psi}(x)=\frac{px^{p-1}\dot{x}y^{p_1}z^{p_2}}{x^py^{p_1}z^{p_2}}+\frac{x^pp_1y^{p_1-1}\dot{y}z^{p_2}}{x^py^{p_1}z^{p_2}}+\frac{x^py^{p_1}p_2z^{p_2-1}\dot{z}}{x^py^{p_1}z^{p_2}}$$

After simplifcation we have

$$\dot{\Psi}(x)=p\frac{\dot{x}}{x}+p_1\frac{\dot{y}}{y}+p_2\frac{\dot{z}}{z} \tag{3.22}$$

Substituting (3.2) in (3.22) we get

$$\dot{\Psi}(x)=p\frac{1}{x}(x)\left[r(1-\frac{x}{K})-\omega_1z\right]+p_1\frac{1}{y}(y)\left[s(1-\frac{y}{L})-\frac{\omega_{2(1-q)z}}{m+(1-q)y}\right]$$
$$+p_2\frac{1}{z}(z)\left[b_1\omega_1x+\frac{b_2\omega_2(1-q)y}{m+(1-q)y}-c\right]$$

Simplification gives us

$$\dot{\Psi}(x)=p\left[r(1-\frac{x}{K})-\omega_1z\right]+p_1\left[s(1-\frac{y}{L})-\frac{\omega_{2(1-q)z}}{m+(1-q)y}\right]$$
$$+p_2\left[b_1\omega_1x+\frac{b_2\omega_2(1-q)y}{m+(1-q)y}-c\right]$$

Then E_3,E_4 and E_5 exist, further there are no periodic orbits in the interior of x – y plane, x – z plane and in the region R^2_+ of y – z plane.
Thus to prove the uniform persistence of the system, it is enough to show that

$\Psi(x) > 0$ in the domain D of R^3_+ where

$$D = \{(x,y,z) : x > 0, y > 0, z > 0\}$$

for a suitable choice of p, p_1 and $p_2 > 0$.
That is the following conditions must be satisfied for the system to be uniformly persistent.

(i) $E_0(0,0,0)$

$$\dot{\Psi}(E_0) = p\left[r(1-\frac{(0)}{K}) - \omega_1(0)\right] + p_1\left[s(1-\frac{(0)}{L}) - \frac{\omega_2(1-q)(0)}{m+(1-q)(0)}\right] + p_2\left[b_1\omega_1(0) + \frac{b_2\omega_2(1-q)(0)}{m+(1-q)(0)} - c\right]$$

Which is simplified to the form

$$\dot{\Psi}(E_0) = pr + p_1 s - p_2 c$$

Here we note that by increasing p to sufficiently large value, $\Psi(E_0)$ can be made positive.
Hence

$$\dot{\Psi}(E_0) = pr + p_1 s - p_2 c > 0 \tag{3.23}$$

(ii) $E_1(K,0,0)$

$$\dot{\Psi}(E_1) = p\left[r(1-\frac{K}{K}) - \omega_1(0)\right] + p_1\left[s(1-\frac{(0)}{L}) - \frac{\omega_2(1-q)(0)}{m+(1-q)(0)}\right] + p_2\left[b_1\omega_1 K + \frac{b_2\omega_2(1-q)(0)}{m+(1-q)(0)} - c\right]$$

After simplification we have

$$\dot{\Psi}(E_1) = p_1 s + p_2(b_1\omega_1 K - c) > 0 \tag{3.24}$$

Whenever

$$b_1\omega_1 K > c$$

(iii) $E_2(0,L,0)$

$$\dot{\Psi}(E_2) = p\left[r(1-\frac{0}{K}) - \omega_1(0)\right] + p_1\left[s(1-\frac{(L)}{L}) - \frac{\omega_2(1-q)(0)}{m+(1-q)(L)}\right] + p_2\left[b_1\omega_1 0 + \frac{b_2\omega_2(1-q)(L)}{m+(1-q)(L)} - c\right]$$

Simplifying we get

$$\dot{\Psi}(E_2) = pr + p_2\left[\frac{b_2\omega_2(1-q)(L)}{m+(1-q)(L)} - c\right]$$

$$\dot{\Psi}(E_2) = pr + p_2\left[\frac{b_2\omega_2(1-q)(L) - c(1-q)L_c m}{m+(1-q)(L)}\right]$$

$$\dot{\Psi}(E_2) = pr + p_2\left[\frac{L(1-q)(b_2\omega_2 - c) - cm}{m+(1-q)(L)}\right] > 0 \qquad (3.25)$$

If

$$L(1-q)(b_2\omega_2 - c) > cm$$

(iv) $E_3(K,L,0)$

$$\dot{\Psi}(E_3) = p\left[r(1-\frac{K}{K}) - \omega_1(0)\right] + p_1\left[s(1-\frac{L}{L}) - \frac{\omega_2(1-q)0}{m+(1-q)L}\right] +$$
$$p_2\left[b_1\omega_1 K + \frac{b_2\omega_2(1-q)L}{m+(1-q)L} - c\right]$$

Which is simplified to the form

$$\dot{\Psi}(E_3) = p_2 b_1\omega_1 K + p_2\left[\frac{b_2\omega_2(1-q)L}{m+(1-q)L} - c\right]$$

$$\dot{\Psi}(E_3) = p_2 b_1\omega_1 K + p_2\left[\frac{L(1-q)(b_2\omega_2 - c) - cm}{m+(1-q)(L)}\right] > 0 \qquad (3.26)$$

(v) $E_4(0,y,z) = \left(0, \frac{cm}{(1-q)(b_2^*\omega_2^*-c)}, b_2 ms\left[\frac{L(b_2^*\omega_2^*-c)(1-q)-cm}{L((1-q)(b_2^*\omega_2^*-c))^2}\right]\right)$

$\dot{\Psi}(E_4) = p\left[r(1-\frac{x}{K}) - \omega_1 z\right] + p_1\left[s(1-\frac{y}{L}) - \frac{\omega_2^*(1-q)z}{m+(1-q)y}\right] + p_2\left[b_1\omega_1 x + \frac{b_2^*\omega_2^*(1-q)y}{m+(1-q)y} - c\right]$

$$\dot{\Psi}(E_4) = p\left[r(1-\frac{0}{K}) - \omega_1 z\right] + p_1\left[s(1-\frac{cm}{L(1-q)(b_2^*\omega_2^*-c)}) - \frac{\omega_2^*(1-q)z}{m+(1-q)\frac{cm}{(1-q)(b_2^*\omega_2^*-c)}}\right] +$$
$$p_2\left[b_1\omega_1(0) + \frac{b_2^*\omega_2^*(1-q)\frac{cm}{(1-q)(b_2^*\omega_2^*-c)}}{m+(1-q)\frac{cm}{(1-q)(b_2^*\omega_2^*-c)}} - c\right]$$

$$\dot{\Psi}(E_4) = p\left[r(1-\frac{0}{K}) - \omega_1 b_2 ms\left[\frac{L(b_2^*\omega_2^*-c)(1-q)-cm}{L((1-q)(b_2^*\omega_2^*-c))^2}\right]\right] +$$

$$p_1\left[s(1-\frac{cm}{L(1-q)(b_2^*\omega_2^*-c)})-\frac{\omega_2^*(1-q)b_2ms\left[\frac{L(b_2^*\omega_2^*-c)(1-q)-cm}{L((1-q)(b_2^*\omega_2^*-c))^2}\right]}{m+(1-q)\frac{cm}{(1-q)(b_2^*\omega_2^*-c)}}\right]+$$

$$p_2\left[b_1\omega_1(0)+\frac{b_2^*\omega_2^*(1-q)\frac{cm}{(1-q)(b_2^*\omega_2^*-c)}}{m+(1-q)\frac{cm}{(1-q)(b_2^*\omega_2^*-c)}}-c\right]$$

Simplifying and rearenging we have

$$\dot{\Psi}(E_4)=p\left[r-\omega_1 b_2ms\left[\frac{L(1-q)(b_2^*\omega_2^*-c)-cm}{L((1-q)(b_2^*\omega_2^*-c))^2}\right]\right]>0 \qquad (3.27)$$

(vi) $E_5(x,0,z)=(\frac{c}{b_1\omega_1},0,\frac{r}{\omega_1}(1-\frac{c}{b_1\omega_1 k}))$

$$\dot{\Psi}(E_5)=p\left[r(1-\frac{c}{b_1\omega_1 K})-\omega_1\frac{r}{\omega_1}(1-\frac{c}{b_1\omega_1 k})\right]+$$

$$p_1\left[s(1-\frac{0}{L})-\frac{\omega_2(1-q)\frac{r}{\omega_1}(1-\frac{c}{b_1\omega_1 k})}{m+(1-q)(0)}\right]+$$

$$p_2\left[b_1\omega_1\frac{c}{b_1\omega_1}+\frac{b_2\omega_2(1-q)(0)}{m+(1-q)(0)}-c\right]$$

After simplification we get

$$p_1\left[s-\frac{\omega_2 r(1-q)}{m\omega_1}(1-\frac{c}{b_1\omega_1 k})>0\right] \qquad (3.28)$$

Theorem 3.6. *In addition to inequality*

$$b_1>\frac{c}{\omega_1 K}$$

let the hypotheses of theorem (3.2),(3.3,(3.5)) hold, and then the system (3.2) is uniformly persistent if the following inequalities hold

$$b_1\omega_1 K+b_2\omega_2\frac{(1-q)L}{m+(1-q)L}>c$$

$$s>\frac{\omega_2 r(1-q)}{m\omega_1}(1-\frac{c}{b_1\omega_1 K})$$

$$r>\frac{b_2\omega_1 ms}{(1-q)(b_2\omega_2-c)}(1-\frac{cm}{L(b_2\omega_2-c)})$$

Chapter 4
Conclusion and future work

In this chapter, conditions for existence of all the seven possible equilibrium points (steady states) were established. It is found out that the first prey can exist on its own or in the presence of the predator. The two prey species would co-exist in the absence of the predator. The existence of the predator with either the first prey alone or the second prey alone required that the proportion of biomass of each prey specie converted into fertility (reproductivity rate) by the predator must be greater than the quotient of natural mortality rate and the capturing rate of the predator. The co-existence of all three species were discussed.

Furthermore, conditions for the local and global asymptotic stability of the steady states were established , by choosing a suitable Lyapunov function and differentiating with respect to time along with each solutions. Conditions for the global asymptotic stability of the steady states E_3 and E_5 were established. These steady states would be globally asymptotically stable if the existing conditions prevailed. The steady state E_4 , which is the existence of the predator with the portion of, [(1-q)y], second prey alone, would be globally asymptotically stable if the product of capturing rate and the conversion factor of the predator is greater than the death rate of the predator.The global stability of the co-existence steady state E_6 was stated in (3.5).Numerical investigation and validation of the model with the real data would be considered in the future work.

Bibliography

[1] Abrams P.A and Ginzburg L.R. (2000). The nature of predation, *:prey dependent,ratio dependent or neither.Trends in Ecology and Evolution*,Vol.15,pp.337-341.

[2] Akcakaya H.R., Arditi R. and Ginzburg L.R. (1995). Ratio-dependent predition, *:an abstraction that works.Ecology* Vol.76,pp.995-1004.

[3] Arditi R., Ginzburg L.R. and Akcakaya H.R. (1991). Variation in plankton density among lakes*:a case of ratio dependent models.American Naturalist*,Vol.138, pp.1287–1296.

[4] Arditi R. and Saiah H. (1992). Empirical evidence of the role of heterogeneity in ratio dependent consumption. *Ecology*,Vol.73,pp.1544-1551.

[5] A.George Maria Selvam , R. Dhineshbabu and V.Sathish (2014). Dynamical Behavior in a Discrete Three Species Prey-Predator System.International Journal of Innovative Science, Engineering Technology, Vol. 1 Issue 8

[6] A.George Maria Selvam , R. Dhineshbabu and V.Sathish (2015). Stability of Equilibria in a Discrete Model of Interacting Species.International Journal of Emerging Research in Management Technology ISSN:(Volume-4, Issue-3),pp. 2278-9359

[7] Braza P.A.A. (1995). Dominant predator,a predator,and a prey.*Mathematical Biosciences and Engineering*,Vol.5,No.1,pp.61-73.

[8] Dubey B., Upadhyay R.K. (2004). Persistence and extinction of one prey and two predator system, *J. Nonlinear Analysis. Modelling and Control*,Vol.9, No.4,pp.307-329.

[9] Hafbauer J., (1981). A general cooperation theorem for hyper cycles. Monatsh Math ., Vol. 91, pp. 233-240.

[10] Gard T. C. and Hallam T. G., (1997). Persistence in food webs -1, Lotka -Volterra food chains. Bull. Math. Biol., Vol. 41, pp. 877- 891.

[11] Hale J. K., (1969). Ordinary Differential Equations, Johan Wiley Sons, New York.

[12] Hutson V. and Vickers G. T., (1983). A criterion for permanent coexistence of species with an application to two prey, one predator system. Math. Biosci ., Vol.63, pp. 253-269.

[13] John W.Cain and Angela Reynolds (2010). Ordinary and Partial Differential Equations*:An Introduction to Dynamical Systems.Virginia Commonwealth University*.Richmond,Virginia.

[14] Kar and Batabyal (2010). International Journal of Engineering,Science and Technology,Vol.2,No.2 pp.174-190.

[15] Kar T.K.and Chaudhury K.S. (2004). Harvesting in a two-prey one-predator fishery. ANZIAM J.,Vol.45, pp.443-456.

[16] Kesh D., Sarkar A.Kand., Roy A.B. (2000). Persistence of two prey one predator system with ratio dependent predator influence. Math.Appl.Sci., Vol.23,pp. 347-356.

[17] M.ReniSagaya Raj , A.George Maria Selvam and R.Janagaraj (2013).Dynamical Behavior of Discrete Prey-Predator system with Scavenger. International Journal of Emerging Research in Management Technology ISSN:(Volume-2, Issue-11) pp. 2278-9359

[18] Murdoch, W.W., Oaten (1975) A.: Predation and population stability. Adv. Ecol. Res. 9, 1–31

[19] Neelima Daga , Bijendra Singh , Suman Jain ad Gajendra Ujjainkar (2014). Prey-predator Model with Persistence of Stability.International Journal of Latest Research in Science and Technology Volume 3, Issue 4: Page No.171-175.

[20] Neelima Daga , Bijendra Singh , Suman Jain ad Gajendra Ujjainkar (2014). Analysis of a Prey-Predator System with Modified Transmission Function. American Journal of Engineering Research (AJER) : 2320-0847 p-ISSN : 2320-0936. Volume-3, Issue-9, pp-194-202

[21] Prabir Panja · Shyamal Kumar Mondal. (2015) Stability analysis of coexistence of three species prey–predator model. Nonlinear Dyn 81:373–382.

[22] Sharov,A.(1996)(http://home.comcast.net/ sharov/PopEcol/lec10/funcresp.html)

[23] Walter G. Kelley and Allan C. Peterson (2010) The Theory of Differential Equations*:Classical and Qualitative. University of Oklahoma*

[24] Zhang J.Fan M.and Kuang Y. (2006). Rabbits Killing birds revisited.*Mathematical Biosciences*,Vol.203, pp.100–123.