SQUEEZING AND ENTANGLEMENT PROPERTIES OF LIGHT PRODUCED BY NON-DEGENERATE THREE-LEVEL LASER WITH DEGENERATE PARAMETRIC AMPLIFIERS AND SQUEEZED VACUUM RESERVOIR

by

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Abstract

The squeezing and entanglement properties of light produced by non degenerate three-level laser whose cavity contains two degenerate parametric amplifiers and coupled to squeezed vacuum reservoir is analyzed using c-number langevin equation . Employing the c-number Langevin equation associated with the normal ordering along the correlation properties of the noise forces.We determined the quadrature squeezing (the quadrature variances of the cavity and output modes as well as the squeezing spectrum of the output modes). Furthermore,using the criterion developed by Duan et al, the quantum entanglement of the cavity modes and output modes are determined. The light produced by the system under consideration exhibits **squeezing** and **entanglement**. It is observed that the degree of squeezing and entanglement for the system under consideration increase with the amplitude of the parametric amplifiers and with the squeezing parameter of the squeezed vacuum reservoir. Moreover, the degree of squeezing of the cavity modes is found to be greater than that of output modes.

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Introduction

Light has played a special role in our attempts to understand nature both classically and quantum mechanically. Classically, light field consists of waves with well defined amplitude and phase but this is not the case when we treat light quantum mechanically. The quantum properties of light are largely determined by the well known quantum states of light; the number state, the coherent state, the chaotic state and the squeezed state. Recent developments of technology greatly improved our ability to control individual quantum system and realize the quantum properties of light. This has attracted a great deal of interest in studying nonclassical features of light such as Squeezing and entanglement.

In squeezed light the noise in one quadrature is below the coherent-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in low-noise optical communication and weak signal detection and can be generated by optical devices such as parametric oscillator and by three-level laser under certain conditions.

In three level lasers when the three-level atom makes transition from the top to bottom level via the intermediate level two photons are generated. The two photons are highly correlated and this correlation is responsible for the squeezing of the light generated. If the two photons have the same frequencies, the atom is degenerate otherwise is non-degenerate [3; 4, 5].

The squeezing of the light produced by three-level lasers with the atoms initially prepared in a coherent superposition of the top and bottom levels have been studied by many autors [9; 10]. These studies showed that a three-level laser under certain conditions can generate squeezed light.

On the other hand, it is well known that a parametric oscillator is a typical source of squeezed light with a maximum intra cavity squeezing of 50 percent [14]. It has also been predicted that the presence of a parametric amplifier in the cavity of a three-level laser increased the interacavity squeezing [15].

Quantum entanglement is a physical phenomenon which is created usually by direct interaction between sub-atomic particles such as photons, electrons and molecules and then separated. Before interaction each particle is described by its own quantum state. It is believed that the key ingredient of quantum information is entanglement which has been recognized as the essential resource for quantum teleportation, quantum decoding, quantum computation and quantum cryptography.

Research in quantum entanglement was initiated by the 1935 paper [18] by Albert Einstein, Boris Podolsky and Nathan Rosen describing the EPR Paradox and several papers by Erwin Schrdinger shortly. Recently, much attention has been paid to the generation and detection of continuous variable entanglement as it might be easier to manipulate than the discrete counter parts, quantum bits, in order to perform quantum information processing. On the other hand, the efficiency of the quantum information processing highly depends on the degree of entanglement. Hence, it is desirable to generate strongly entangled continuous variable state.

The two-mode squeezed state itself is an entangled state [17]. Recently, it has been shown that the non-degenerate three-level cascade lasers can generate macroscopic entangled state.

Tesfa [19] has studied the squeezing property of the cavity modes produced by a nondegenerate three-level laser applying the solutions of stochastic differential equations. He has found that the two-mode cavity radiation exhibits squeezing if the atoms are initially prepared with more atoms in the bottom level than in the upper level, and the degree of squeezing increases with the linear gain coefficient.

Mekonnen [21] has studied squeezing and entanglement of light by non-degenerate three-level laser with a strong driving coherent light and squeezed vacuum reservoir. He has found that the squeezing and entanglement of the cavity modes will be higher for large values of linear gain coefficient and the degree of entanglement is directly proportional to the degree of squeezing of the two-mode light but the degree of entanglement for the output modes will disappear for large values of linear gain coefficient. He also found that the two-mode squeezed vacuum reservoir considerably increases the degree of squeezing as well as entanglement in the cavity.

Friew [22] has studied the squeezing and statistical properties of the light generated by a non-degenerate three-level laser whose cavity contains two parametric amplifiers coupled to a vacuum reservoir. He has found that the parametric amplifiers increase the degree of squeezing.

In this M.sc thesis, we study the squeezing and entanglement properties of light produced by non-degenerate three-level laser whose cavity contains two parametric amplifiers coupled to a squeezed vacuum reservoir. Our analysis is carried out by deriving the master equation in the linear and adiabatic approximation schemes for the non-degenerate three-level laser whose cavity contains two degenerate parametric oscillators coupled with squeezed vacuum reservoir. Using this master equation, we obtain the c-number Langevin equation associated with the normal ordering and the correlation properties of the noise forces. Employing these equations, we determine the quadrature squeezing (the quadrature variances of the cavity and output modes as well as the squeezing spectrum of the output modes). Furthermore, using the criterion developed by Duan et al. [25] the quantum entanglement of the cavity modes and output modes are determined.

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The Hamiltonian and Master Equation

In this thesis, we consider the non degenerate three-level laser whose cavity contains two degenerate parametric oscillators with different modes. We first obtain the time evolution of the density operator for the interaction between an atom and cavity modes, the down conversion process and interaction between cavity modes and squeezed vacuum reservoir.

2.1 Non-degenerate Three-Level Laser

Here we consider a non-degenerate three-level cascade laser coupled to a two-mode squeezed vacuum reservoir. In this system, three level atoms in a cascade configuration; where the top, intermediate and bottom levels are denoted respectively by $|a\rangle$, $|b\rangle$, and $|c\rangle$ as shown in fig 2.1. In addition, we assume the two modes a and b to be at resonance with the two transitions $|a\rangle \longrightarrow |b\rangle$, and $|b\rangle \longrightarrow |c\rangle$, respectively and direct transition between level $|a\rangle$ and level $|c\rangle$ to be dipole forbidden. Moreover, the cavity modes interact resonantly with the two mode squeezed vacuum.This system is outlined in fig.(2.1) the interaction of the system with reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{2.1}$$

where $\hat{H}_S$ is the Hamiltonian of the system in the cavity and $\hat{H}_{SR}$ is the Hamiltonian that describes the interaction of the system inside the cavity and reservoir. In this case, $\hat{H}_S = \hat{H}_{S_1} + \hat{H}_{S_2}$ with $\hat{H}_{S_1}$ being the Hamiltonian describing the interaction of **a three -level atom with the cavity modes** at resonance with the two transitions $|a\rangle \longrightarrow |b\rangle$, and $|b\rangle \longrightarrow |c\rangle$, which is represented as;

$$\hat{H}_{S_1} = ig[|a\rangle\langle b|\hat{a} - \hat{b}^\dagger|c\rangle\langle b| - \hat{a}^\dagger|b\rangle\langle a| + |b\rangle\langle c|\hat{b}] \tag{2.2}$$

and $\hat{H}_{S_2}$ describes the down conversion processes.

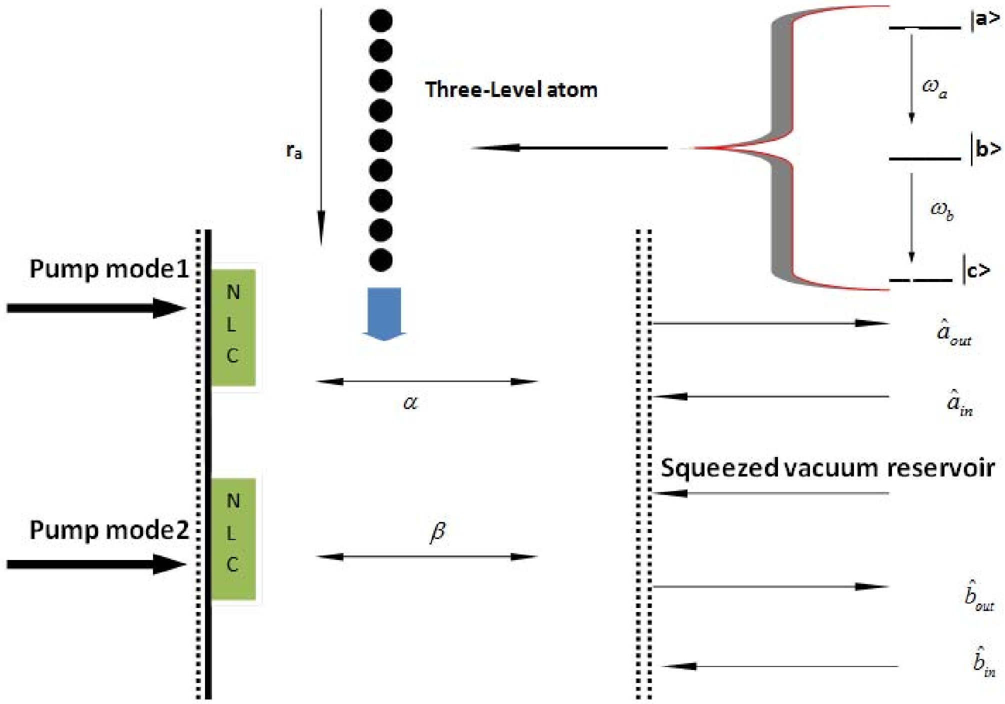

Fig. 2.1: A schematic representation of a non - degenerate three level laser with Parametric Amplifiers and squeezed vacuum reservoir.

We take a three-level atom is initially in the state

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle, \tag{2.3}$$

and the corresponding density operator can be written as

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.4}$$

where

$$\rho_{aa}^{(0)} = |C_a|^2, \rho_{cc}^{(0)} = |C_c|^2 \tag{2.5}$$

are respectively the probabilities for the atom to be initially in the upper and the lower levels and

$$\rho_{ac}^{(0)} = C_a C_c^*, \rho_{ca}^{(0)} = C_c C_a^* \tag{2.6}$$

represents the atomic coherence at the initial time.We note that

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)} \tag{2.7}$$

2.2 Interaction between an atom and cavity modes

Suppose $\hat{\rho}_{AR}(t,t_j)$ is the density operator for single atom plus the cavity mode at time t, with atom injected at time t_j such that $(t-\tau)\le t_j\le t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as

$$\hat{\rho}_{AR}(t)=r_a\sum_j\hat{\rho}_{AR}(t,t_j)\Delta t_j, \tag{2.8}$$

where $r_a\Delta t_j$ represents the number of atoms injected into cavity in a time Δt_j. Now converting the summation into integration in the limit $\Delta t_j\to 0$, we have

$$\hat{\rho}_{AR}(t)=r_a\int_{t-\tau}^{t}\hat{\rho}_{AR}(t,t^{'})dt^{'} \tag{2.9}$$

and on differentiating with respect to t and taking into account the Leibnitz' rule

$$\frac{d}{d_x}\int_{u(x)}^{v(x)}f(x,x^{'})d_{x^{'}}=f(x,v)\frac{d_v(x)}{d_x}-f(x,u)\frac{d_u(x)}{d_x}+\int_{u(x)}^{v(x)}\frac{\partial}{\partial_x}f(x,x^{'})d_{x^{'}}$$

there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t)=r_a[\hat{\rho}_{AR}(t,t)-\hat{\rho}_{AR}(t,t-\tau)]+r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt' \tag{2.10}$$

we observe that $\hat{\rho}_{AR}(t,t)$ is the density operator the cavity modes plus an atom injected at time t. This operator can thus be expressed as

$$\hat{\rho}_{AR}(t,t)=\hat{\rho}_A(t)\hat{\rho}(t) \tag{2.11}$$

with $\hat{\rho}(t)$ being the density operator for the cavity mode alone. We also that $\hat{\rho}_{AR}(t,t-\tau)$ represents the density operator for an atom plus cavity mode at time t, with the atom being removed from the cavity at this time. This operator can also be put in the form of

$$\hat{\rho}_{AR}(t,t-\tau)=\hat{\rho}_A(t,t-\tau)\hat{\rho}(t). \tag{2.12}$$

Now in view of Eqs. (2.12) and (2.11), one can rewrite Eq. (2.10) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t)=r_a[\hat{\rho}_A(t)-\hat{\rho}_A(t,t-\tau)]\hat{\rho}(t)+r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt' \tag{2.13}$$

the density operator involves in time according to the

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')=-i[\hat{H}_I,\hat{\rho}_{AR}(t,t')] \tag{2.14}$$

so that using this relation along with Eqs. (2.9) and (2.14), one can put Eq. (2.13) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t,t')=r_a[\hat{\rho}_A(t)-\hat{\rho}_A(t,t-\tau)]\hat{\rho}(t)-i[\hat{H}_I,\hat{\rho}_{AR}(t)]. \tag{2.15}$$

Furthermore, tracing over the atomic variables and taking into account the damping of cavity modes by a squeezed vacuum reservoir [1], we have

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) =& - iTr_A[\hat{H}_I, \hat{\rho}_{AR}(t,t')] + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) \\ &+ \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}),\end{aligned} \tag{2.16}$$

where we have used the fact that

$$Tr(\hat{\rho}_A(t)) = Tr(\hat{\rho}_A(t-\tau)) = 1, \tag{2.17}$$

k is the cavity damping constant. Employing Eqs, (2.2) the equation of evolution of the density operator for the cavity modes can be put in the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) =& [\hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} - \hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{ab}\hat{a}^\dagger + \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{b} - \hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{bc}\hat{b}^\dagger] \\ &+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}),\end{aligned} \tag{2.18}$$

In which the matrix element
$\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.19}$$

with α, β = a, b, c. On the other hand, we next proceed to determine the matrix elements $\hat{\rho}_{\alpha\beta}$ involved in Eq.(2.15)

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_{\alpha\beta} =& [r_a\langle\alpha|\hat{\rho}_A(t)|\beta\rangle - r_a\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle]\hat{\rho}(t) \\ &- i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}(t)|\beta\rangle] + i[\langle\alpha|\hat{\rho}_{AR}(t)\hat{H}_I|\beta\rangle] - \gamma\hat{\rho}_{\alpha\beta},\end{aligned} \tag{2.20}$$

where γ is considered to be the same for all the three levels, is the atomic decay rate and the last term in the above expression indicates the decay of atom due to spontaneous emission. We assume that the atoms are removed from the cavity after they have decayed to a level other than the intermediate or the bottom level [1] . Then we see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0 \tag{2.21}$$

and hence Eq. (2.20) reduces to

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta} = r_a\langle\alpha|\hat{\rho}_A(t)|\beta\rangle\hat{\rho}(t) - i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}(t)|\beta\rangle] + i[\langle\alpha|\hat{\rho}_{AR}(t)\hat{H}_I|\beta\rangle] - \gamma\hat{\rho}_{\alpha\beta}. \tag{2.22}$$

Applying Eq. (2.22) and taking into account Eqs. (2.2) and (2.4) one readily obtains

$$\frac{d}{dt}\hat{\rho}_{ab} = g(\hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a} + \hat{\rho}_{ac}\hat{b}^\dagger) - \gamma\hat{\rho}_{ab}, \tag{2.23}$$

$$\frac{d}{dt}\hat{\rho}_{bc} = g(\hat{b}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{b} - \hat{a}^{\dagger}\hat{\rho}_{ac}) - \gamma\hat{\rho}_{bc}, \tag{2.24}$$

$$\frac{d}{dt}\hat{\rho}_{aa} = r_a\rho_{aa}^{(0)}\hat{\rho} + g(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^{\dagger}) - \gamma\hat{\rho}_{aa}, \tag{2.25}$$

$$\frac{d}{dt}\hat{\rho}_{cc} = r_a\rho_{cc}^{(0)}\hat{\rho} - g(\hat{b}^{\dagger}\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}) - \gamma\hat{\rho}_{cc}, \tag{2.26}$$

$$\frac{d}{dt}\hat{\rho}_{ac} = r_a\rho_{ac}^{(0)}\hat{\rho} + g(\hat{b}^{\dagger}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{b}) - \gamma\hat{\rho}_{ac} \tag{2.27}$$

$$\frac{d}{dt}\hat{\rho}_{bb} = -g(\hat{a}^{\dagger}\hat{\rho}_{ab} - \hat{b}\hat{\rho}_{cb} + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{b}^{\dagger}) - \gamma\hat{\rho}_{bb} \tag{2.28}$$

we confine ourselves to linear analysis and this can be achieved by dropping the g terms in Eqs. (2.25), (2.26), (2.27) and (2.28) and applying the large time adiabatic approximation scheme, we get

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}, \tag{2.29}$$

$$\hat{\rho}_{cc} = \frac{r_a\rho_{cc}^{(0)}}{\gamma}\hat{\rho}, \tag{2.30}$$

$$\hat{\rho}_{ac} = \frac{r_a\rho_{ac}^{(0)}}{\gamma}\hat{\rho}, \tag{2.31}$$

$$\hat{\rho}_{bb} = 0. \tag{2.32}$$

Substituting the above results into Eqs. (2.23) and (2.24), we have

$$\frac{d}{dt}\hat{\rho}_{ab} = \frac{gr_a}{\gamma}(\rho_{ac}^{(0)}\rho\hat{b}^{\dagger} - \rho_{aa}^{(0)}\rho\hat{a}) - \gamma\hat{\rho}_{ab}, \tag{2.33}$$

$$\frac{d}{dt}\hat{\rho}_{bc} = \frac{gr_a}{\gamma}(\hat{b}\rho_{cc}^{(0)}\hat{\rho} - \hat{a}^{\dagger}\rho_{ac}^{(0)}\hat{\rho}) - \gamma\hat{\rho}_{bc}, \tag{2.34}$$

Using once more the large-time adiabatic approximation scheme, we easily find the

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}(\rho_{ac}^{(0)}\rho\hat{b}^{\dagger} + \rho_{aa}^{(0)}\rho\hat{a}), \tag{2.35}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}(\rho_{cc}^{(0)}\hat{b}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^{\dagger}\hat{\rho}). \tag{2.36}$$

Finally, on account of Eqs. (2.35), and (2.36), along with (2.18) the equation of evolution of the density operator for the cavity interacted with the atom turns out to be

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) =& \frac{1}{2}A\rho_{cc}^{(0)}(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}) + \frac{1}{2}A\rho_{aa}^{(0)}(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}) \\ &+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}) \\ &+ \frac{1}{2}A\rho_{ac}^{(0)^*}(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{b}\hat{\rho}\hat{a}) + \frac{1}{2}A\rho_{ac}^{(0)}(\hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - 2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}).\end{aligned} \tag{2.37}$$

Where

$$A = \frac{2g^2 r_a}{\gamma^2}$$

is the linear gain coefficient with γ being the spontaneous atomic decay rate assumed to be the same for the three-levels.

2.3 The down conversion processes

The pump modes treated classically, the two different parametric Amplifiers can be described by the Hamiltonian

$$\hat{H}_{S2} = i\varepsilon_a(\hat{a}^2 - \hat{a}^{\dagger^2}) + i\varepsilon_b(\hat{b}^2 - \hat{b}^{\dagger^2}), \tag{2.38}$$

where ε_a and ε_b are the amplitudes proportional to the pump modes and $\hat{a}$, $\hat{a}^{\dagger}$ and $\hat{b}$,$\hat{b}^{\dagger}$ are the annihilation and creation operators respectively. Using (2.38) the time evolution of the density operator of the cavity modes can be written as

$$\frac{d}{dt}\hat{\rho}(t) = \varepsilon_a[\hat{a}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^2 - \hat{a}^{\dagger^2}\hat{\rho}(t) + \hat{\rho}(t)\hat{a}^{\dagger^2}] + \varepsilon_b[\hat{b}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^2 - \hat{b}^{\dagger^2}\hat{\rho}(t) + \hat{\rho}(t)\hat{b}^{\dagger^2}] \tag{2.39}$$

This represents the master equation corresponding to the down conversion processes.

2.4 Interaction between cavity modes and squeezed vacuum reservoir

Here we wish to determine the master equation describing the interaction of the cavity modes to the two-mode squeezed vacuum reservoir through the lossy single-port mirror. Now let $\hat{\rho}_{SR}$ be the density operator for the system plus the reservoir in the interaction picture. We can then write the time evolution of this operator as

$$\frac{d\hat{\rho}_{SR}(t)}{dt} = -i[\hat{H}_S(t) + \hat{H}_{SR}(t), \hat{\rho}_{SR}(t)]. \tag{2.40}$$

In order to obtain the time evolution of the system alone, which we are interested in, we trace $\hat{\rho}_{SR}(t)$ over the reservoir variables, that is, the density operator for the system

$$\hat{\rho}(t) = Tr_R(\hat{\rho}_{SR}(t)). \tag{2.41}$$

In view of this, Eq. (2.40) takes the form

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S(t), \hat{\rho}(t) - iTr_R[\hat{H}_{SR}(t), \hat{\rho}_{SR}(t)]. \tag{2.42}$$

Furthermore, a formal solution of Eq. (2.40) can be written as

$$\hat{\rho}_{SR}(t) = \hat{\rho}_{SR}(0) - i\int_0^t [\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\rho}_{SR}(t')]dt. \tag{2.43}$$

So as to proceed further, we need to introduce a certain approximation scheme. To this end, assuming here is a weak interaction between the system and reservoir, we can write approximately valid relation $\hat{\rho}_{SR}(t') = \hat{\rho}(t')\hat{R}$, where $\hat{R}$ is the density operator for the reservoir assumed to be constant in time. Now applying this approximation, Eq. (2.43) becomes

$$\hat{\rho}_{SR}(t) = \hat{\rho}(0)\hat{R} - i\int_0^t [\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}]dt. \tag{2.44}$$

In substituting this result into Eq. (2. 42), we get

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} =& - i[\hat{H}_S(t), \hat{\rho}(t)] - i[\langle\hat{H}_{SR}(t)\rangle_R, \hat{\rho}(0)]\\ &- \int_0^t [\langle\hat{H}_{SR}(t)\rangle_R, [\hat{H}_S(t'), \hat{\rho}(t')]]dt'\\ &- \int_0^t Tr_R[\hat{H}_{SR}(t), [\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}]]dt\end{aligned} \tag{2.45}$$

now consider a two-mode squeezed vacuum reservoir interacting with two cavity modes. This interaction can be described in the interaction picture by the Hamiltonian

$$\hat{H}_{SR}(t) = i\sum\lambda_m(\hat{a}^\dagger\hat{C}_m e^{i(\omega_a-\omega_m)t} - \hat{a}\hat{C}_m^\dagger e^{-i(\omega_a-\omega_m)t} + i\sum\lambda_n\hat{b}^\dagger\hat{D}_n e^{i(\omega_b-\omega_n)t} - \hat{b}\hat{D}_n^\dagger e^{-i(\omega_b-\omega_n)t}). \tag{2.46}$$

In which $\hat{a}$ and $\hat{b}$ are annihilation operators for the cavity modes with frequencies ω_a and ω_b, $\hat{C}_m$ and $\hat{D}_n$ are annihilation operators for the reservoir modes having frequencies ω_m and ω_n . λ_m and λ_n are coupling constants between the cavity modes and the reservoir modes. The reservoir mode operators satisfy the following properties:[21]

$$\langle\hat{C}_m\rangle_R = \langle\hat{D}_n\rangle_R = 0 \tag{2.47}$$

$$\langle\hat{C}_m^\dagger\hat{C}_n\rangle_R = \langle\hat{D}_m^\dagger\hat{D}_n\rangle_R = N\delta_{mn}. \tag{2.48}$$

$$\langle\hat{C}_m\hat{C}_n^\dagger\rangle_R = \langle\hat{D}_m\hat{D}_n^\dagger\rangle_R = (N+1)\delta_{mn}. \tag{2.49}$$

$$\langle \hat{C}_m \hat{C}_n \rangle_R = \langle \hat{C}_m^\dagger \hat{C}_n^\dagger \rangle_R = 0 \tag{2.50}$$

$$\langle \hat{D}_m \hat{D}_n \rangle_R = \langle \hat{D}_m^\dagger \hat{D}_n^\dagger \rangle_R = 0 \tag{2.51}$$

$$\langle \hat{C}_m \hat{D}_n \rangle_R = M\delta_m, 2_{no-n:} \langle \hat{C}_m^\dagger \hat{D}_n^\dagger \rangle_R = M^*\delta_m, 2_{no-n} \tag{2.52}$$

$$\langle \hat{C}_m \hat{D}_n^\dagger \rangle_R = \langle \hat{C}_m^\dagger \hat{D}_n \rangle_R = 0 \tag{2.53}$$

where N and M characterize the two-mode squeezed vacuum, such that $|M|^2 \leq N(N+1)$. N represents the mean photon number of each squeezed vacuum reservoir that constitute the two-mode reservoir and $|M|$ signifies the correlation between the two modes. The equality $|M|^2 = N(N+1)$ holds for minimum uncertainty squeezed states, which is the case considered in this thesis. On account of Eq. (2.47), we easily see that $\langle \hat{H}_{SR}(t) \rangle_R = 0$. Hence, in view of this result, Eq. (2. 45) reduces to

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} =& -i[\hat{H}_S(t), \hat{\rho}(t)] - \int_0^t Tr_R(\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{R}\hat{\rho}(t')dt' \\ &- \int_0^t \hat{\rho}(t')Tr_R(\hat{R}\hat{H}_{SR}(t')\hat{H}_{SR}(t)dt' \\ &+ \int_0^t Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t'))dt' \\ &+ \int_0^t Tr_R(\hat{H}_{SR}(t)\hat{R}\hat{\rho}(t')\hat{H}_{SR}(t'))dt' \end{aligned} \tag{2.54}$$

applying the Hamiltonian, Eq. (2.46), and Eqs. (2.48), (2.49), (2.50), (2.51), (2.52) and (2.53), we readily obtain

$$\begin{aligned}-iTr_R(\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{R} =& [p_1\hat{a}^\dagger\hat{a} + p_2\hat{a}\hat{a}^\dagger + p_3\hat{a}\hat{b} \\ &+ p_3^*\hat{a}^\dagger\hat{b}^\dagger + p_4\hat{b}\hat{a} + p_4^*\hat{b}^\dagger\hat{a}^\dagger + p_5\hat{b}^\dagger\hat{b} + p_6\hat{b}\hat{b}^\dagger],\end{aligned} \tag{2.55}$$

where,

$$p_1 = -N\sum_m \lambda_m^2 e^{i(\omega_a-\omega_m)(t-t')}. \tag{2.56}$$

$$p_2 = -(N+1)\sum_m \lambda_m^2 e^{-i(\omega_a-\omega_m)(t-t')} \tag{2.57}$$

$$p_3 = M\sum_n \lambda_n\lambda_{2no-n} e^{-i(\omega_b-\omega_m)t'-i(\omega_a-\omega_{2no-n})t} \tag{2.58}$$

$$p_4 = M\sum_m \lambda_m \lambda_{2mo-m} e^{-i(\omega_a-\omega_m)t'-i(\omega_b-\omega_{2mo-m})t} \tag{2.59}$$

$$p_5 = -N\sum_n \lambda_n^2 e^{i(\omega_b-\omega_a)(t-t')} \tag{2.60}$$

$$p_6 = -(N+1)\sum_n \lambda_n^2 e^{-i(\omega_b-\omega_a)(t-t')} \tag{2.61}$$

assuming the frequencies of the reservoir modes to be closely spaced, the summation can be changed to integration over frequency. Then applying the Markov's approximation [23] to the resulting integral we find the following result:

$$\sum_m \lambda_m^2 e^{\pm i(\omega_a-\omega_m)(t-t')} = k_j\delta(t-t'), \tag{2.62}$$

in which $k_j = 2\pi g(\omega_j)\lambda^2(\omega_j), (j = a, b)$ is defined to be the damping constant for cavity mode j with $g(\omega_j)$ being the density operator for the reservoir modes which is assumed to be at resonance with the cavity mode j. In our analysis, we assume that $k_a = k_b = k$ for convenience. Therefore, on account of these results, Eqs. (2.56), (2.57), (2.60), (2.61) take the form

$$p_1 = p_5 = -\kappa N\delta(t-t'), \tag{2.63}$$

$$p_2 = p_6 = -\kappa(N+1)\delta(t-t'), \tag{2.64}$$

following the same procedure, it can also be established that

$$p_3 = p_4 = \kappa M\delta(t-t'). \tag{2.65}$$

Substitution of Eqs. (2.63), (2.64)(2.65) into (2.55) yields

$$-iTr_R(\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{R} = \kappa[-N(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})-(N+1)(\hat{a}\hat{a}^\dagger+\hat{b}\hat{b}^\dagger)+M(\hat{a}\hat{b}+\hat{b}\hat{a})+M^*(\hat{a}^\dagger\hat{b}^\dagger+\hat{b}^\dagger\hat{a}^\dagger)]\delta(t-t'), \tag{2.66}$$

it then follows that

$$-\int_t^0 Tr_R(\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{R}\hat{\rho}(t')dt' = \frac{\kappa}{2}[-N(\hat{a}^\dagger\hat{a}\hat{\rho}+\hat{b}^\dagger\hat{b}\hat{\rho})-(N+1)(\hat{a}\hat{a}^\dagger\hat{\rho}+\hat{b}\hat{b}^\dagger\hat{\rho}) + M(\hat{a}\hat{b}\hat{\rho}+\hat{b}\hat{a}\hat{\rho})+M^*(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho}+\hat{b}^\dagger\hat{a}^\dagger\hat{\rho})], \tag{2.67}$$

in a similar procedure, one can easily verify that

$$-\int_t^0 \hat{\rho}(t')Tr_R(\hat{R}\hat{H}_{SR}(t')\hat{H}_{SR}(t)dt' = \frac{\kappa}{2}[-N(\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{b}^\dagger\hat{b})-(N+1)(\hat{\rho}\hat{a}\hat{a}^\dagger+\hat{\rho}\hat{b}\hat{b}^\dagger) + M(\hat{\rho}\hat{a}\hat{b}+\hat{\rho}\hat{b}\hat{a})+M^*(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger+\hat{\rho}\hat{b}^\dagger\hat{a}^\dagger)], \tag{2.68}$$

furthermore, applying Eq. (2.46), we readily get

$$\begin{aligned} Tr_R\hat{H}_{SR}(t)\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t) = &- [p_1\hat{a}\hat{\rho}(t')\hat{a}^\dagger + p_2\hat{a}^\dagger\hat{\rho}(t')\hat{a} + p_3\hat{b}\hat{\rho}(t')\hat{a}^\dagger \\ &+ p_3^*\hat{b}^\dagger\hat{\rho}(t')\hat{a}^\dagger + p_4\hat{a}\hat{\rho}(t')\hat{b} + p_4^*\hat{a}^\dagger\hat{\rho}(t')\hat{b}^\dagger \\ &+ p_5\hat{b}\hat{\rho}(t')\hat{b}^\dagger + p_6\hat{b}^\dagger\hat{\rho}(t')\hat{b}], \end{aligned} \tag{2.69}$$

with the aid of Eqs. (2.63), (2.64) and (2.65), we find

$$\begin{aligned} \int_t^0 Tr_R\hat{H}_{SR}(t)\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t)dt' =& \frac{\kappa}{2}(N+1)(\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{b}\hat{\rho}\hat{b}^\dagger) + N(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{b}) \\ &- M(\hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b}) - M^*(\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger)], \end{aligned} \tag{2.70}$$

it can be established in a similar way that

$$\begin{aligned} \int_t^0 Tr_R[\hat{H}_{SR}(t)\hat{R}\hat{\rho}(t')\hat{H}_{SR}(t')dt' =& \frac{\kappa}{2}(N+1)(\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{b}\hat{\rho}\hat{b}^\dagger) + N(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{b}) \\ &- M(\hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b}) - M^*(\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger)]. \end{aligned} \tag{2.71}$$

Upon substituting Eqs. (2.67), (2.68), (2.70), and (2.71) into Eq. (2.54), we obtain the master equation for the cavity modes coupled to a two-mode squeezed vacuum reservoir to be

$$\begin{aligned} \frac{d\hat{\rho}}{dt} =& - i[\hat{H}_S(t), \hat{\rho}(t)] + \frac{\kappa}{2}N(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{\hat{a}}) + \frac{\kappa}{2}N(2\hat{b}^\dagger\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\ &+ \frac{\kappa}{2}(N+1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) + \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ &+ \kappa M(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - \hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{b}) + \kappa M^*(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a^\dagger}\hat{b}^\dagger\hat{\rho} - \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger). \end{aligned} \tag{2.72}$$

To this end, on account of eq .(2.37), (2.39) and (2.72) the master equation for a non-degenerate three-level laser, with two mode parametric amplifiers coupled to squeezed vacuum reservoir can be written as:

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) =& \varepsilon_a[\hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 - \hat{a}^{\dagger^2}\hat{\rho} + \hat{\rho}\hat{a}^{\dagger^2}] + \varepsilon_b[\hat{b}^2\hat{\rho} - \hat{\rho}\hat{b}^2 - \hat{b}^{\dagger^2}\hat{\rho} + \hat{\rho}\hat{b}^{\dagger^2}] \\ &+ \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{1}{2}[A\rho_{cc}^{(0)} + \kappa(N+1)](2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\ &+ \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa N)(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \frac{1}{2}\kappa N(2\hat{b}^\dagger\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\ &+ \frac{1}{2}(A\rho_{ac}^{(0)^*} + \kappa M)(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{b}\hat{\rho}\hat{a}) + \frac{\kappa M}{2}(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{b}) \\ &+ \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M^*)(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger) + \frac{\kappa M^*}{2}(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - 2\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger) \end{aligned} \tag{2.73}$$

where N = $\sinh^2(r)$ and M = $e^{i\theta}\cosh(r)$ with r and θ being respectively the squeeze parameter and phase of the squeezed vacuum, characterize the two-mode squeezed vacuum reservoir. On the other hand, k is the cavity damping constant, N and M are the reservoir parameters are defined as

$$N = \sinh^2 r = \frac{1}{4}[e^{2r} + e^{-2r} - 2] \tag{2.74}$$

$$M = \cosh r \sinh r = [\frac{e^r + e^{-r}}{2}][\frac{e^r - e^{-r}}{2}] = \frac{1}{4}[e^{2r} - e^{-2r}], \tag{2.75}$$

and the squeeze parameter r is taken for convenience to be real and positive [1] But for $\theta = 0$ and $\rho_{ca}^{(0)} = \rho_{ac}^{(0)^*}$ the above equation is simplified to

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}(t) &= \varepsilon_a[\hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 - \hat{a}^{\dagger^2}\hat{\rho} + \hat{\rho}\hat{a}^{\dagger^2}] + \varepsilon_b[\hat{b}^2\hat{\rho} - \hat{\rho}\hat{b}^2 - \hat{b}^{\dagger^2}\hat{\rho} + \hat{\rho}\hat{b}^{\dagger^2}] \\
&+ \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{1}{2}[A\rho_{cc}^{(0)} + \kappa(N+1)](2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\
&+ \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa N)(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \frac{1}{2}\kappa N(2\hat{b}^\dagger\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\
&+ \frac{1}{2}(A\rho_{ca}^{(0)} + \kappa M)(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{b}\hat{\rho}\hat{a}) + \frac{\kappa M}{2}(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{b}) \\
&+ \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M)(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger) + \frac{\kappa M}{2}(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - 2\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger)
\end{aligned} \tag{2.76}$$

3

c-number Langevin Equations

We next proceed to obtain applying the master equation, the c-number Langevin equations associated with the normal ordering. The time evolution of the expectation value of an operator $\hat{A}$ in the schrodinger picture can be expressed as

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}A). \tag{3.1}$$

Now taking into account (2. 76) along with (3. 1), one can write

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = Tr\Bigg(&\varepsilon_a[\hat{a}^2\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2\hat{a} - \hat{a}^{\dagger^2}\hat{\rho}\hat{a} + \hat{\rho}\hat{a}^{\dagger^2}\hat{a}] + \varepsilon_b[\hat{b}^2\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^2\hat{a} - \hat{b}^{\dagger^2}\hat{\rho}\hat{a} + \hat{\rho}\hat{b}^{\dagger^2}\hat{a}]\\ &+ \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}) + \frac{1}{2}[A\rho_{cc}^{(0)} + \kappa(N+1)](2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a})\\ &+ \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa N)(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) + \frac{1}{2}\kappa N(2\hat{b}^\dagger\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a})\\ &+ \frac{1}{2}(A\rho_{ca}^{(0)} + \kappa M)(\hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{a}\hat{b}\hat{\rho}\hat{a} - 2\hat{b}\hat{\rho}\hat{a}\hat{a}) + \frac{\kappa M}{2}(\hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{a}\hat{b}\hat{\rho}\hat{a} - 2\hat{a}\hat{\rho}\hat{b}\hat{a})\\ &+ \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M)(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - 2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a}) + \frac{\kappa M}{2}(\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - 2\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a})\Bigg).\end{aligned} \tag{3.2}$$

Applying the cyclic property of trace operation together with the commutation relations

$$[\hat{a},\hat{a}^\dagger] = [\hat{b},\hat{b}^\dagger] = 1, \tag{3.3}$$

$$[\hat{a},\hat{b}] = [\hat{a}^\dagger,\hat{b}^\dagger] = [\hat{a},\hat{b}^\dagger] = [\hat{a}^\dagger,\hat{b}] = 0 \tag{3.4}$$

and

$$[\hat{a}^2,\hat{a}^\dagger] = 2\hat{a} \tag{3.5}$$

we readily find

$$\frac{d}{dt}\langle\hat{a}\rangle = -2\varepsilon_a\langle\hat{a}^\dagger\rangle - \frac{\mu_a}{2}\langle\hat{a}\rangle + \frac{V_-}{2}\langle\hat{b}^\dagger\rangle \tag{3.6}$$

where,

$$\mu_a = k - A\rho_{aa}^{(0)} \tag{3.7}$$

and

$$V_- = -A\rho_{ac}^{(0)}. \tag{3.8}$$

It can also be established in a similar procedure that

$$\frac{d}{dt}\langle\hat{b}\rangle = -2\varepsilon_b\langle\hat{b}^\dagger\rangle - \frac{\mu_c}{2}\langle\hat{b}\rangle + \frac{V_+}{2}\langle\hat{a}^\dagger\rangle, \tag{3.9}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -2\varepsilon_a - \mu_a\langle\hat{a}^2\rangle + V_-\langle\hat{b}^\dagger\hat{a}\rangle - 4\varepsilon_a\langle\hat{a}^\dagger\hat{a}\rangle, \tag{3.10}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -2\varepsilon_b - \mu_c\langle\hat{b}^2\rangle + V_+\langle\hat{a}^\dagger\hat{b}\rangle - 4\varepsilon_b\langle\hat{b}^\dagger\hat{b}\rangle, \tag{3.11}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -2\varepsilon_a(\langle\hat{a}^{\dagger^2}\rangle + \langle\hat{a}^2\rangle) - \mu_a\langle\hat{a}^\dagger\hat{a}\rangle + \frac{V_-^*}{2}\langle\hat{a}\hat{b}\rangle + \frac{V_-}{2}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + A\rho_{aa}^{(0)} + \kappa N, \tag{3.12}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -2\varepsilon_b(\langle\hat{b}^{\dagger^2}\rangle + \langle\hat{b}^2\rangle) - \mu_c\langle\hat{b}^\dagger\hat{b}\rangle + \frac{V_+^*}{2}\langle\hat{a}\hat{b}\rangle + \frac{V_+}{2}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \kappa N, \tag{3.13}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -2\varepsilon_a\langle\hat{a}\hat{b}\rangle - 2\varepsilon_b\langle\hat{a}^\dagger\hat{b}^\dagger\rangle - \frac{1}{2}\mu\langle\hat{a}^\dagger\hat{b}\rangle + \frac{1}{2}V_-^*\langle\hat{b}^2\rangle + \frac{1}{2}V_+\langle\hat{a}^{\dagger^2}\rangle, \tag{3.14}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -2\varepsilon_a\langle\hat{a}^\dagger\hat{b}\rangle - 2\varepsilon_b\langle\hat{a}\hat{b}^\dagger\rangle + \frac{1}{2}V_+ - \frac{1}{2}\mu\langle\hat{a}\hat{b}\rangle + \frac{1}{2}V_+\langle\hat{a}^+\hat{a}\rangle + \frac{1}{2}V_-\langle\hat{b}^\dagger\hat{b}\rangle + \kappa M. \tag{3.15}$$

In Which

$$\mu_a = k - A\rho_{aa}^{(0)}, \mu_c = k + A\rho_{cc}^{(0)}, V_- = -A\rho_{ac}^{(0)}, V_+ = A\rho_{ac}^{(0)},$$
$$\mu = \frac{1}{2}[2k + A(\rho_{cc}^{(0)} - \rho_{aa}^{(0)})] \tag{3.16}$$

We see that Eqs (3.6), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15) are in normal order. By taking $\hat{a} \longrightarrow \alpha$, $\hat{a}^\dagger \longrightarrow \alpha^*$, $\hat{b} \longrightarrow \beta$ and $\hat{b}^\dagger \longrightarrow \beta^*$, the corresponding c-number equations are

$$\frac{d}{dt}\langle\alpha\rangle = -2\varepsilon_a\langle\alpha^*\rangle - \frac{1}{2}\mu_a\langle\alpha\rangle + \frac{1}{2}V_-\langle\beta^*\rangle \tag{3.17}$$

$$\frac{d}{dt}\langle\beta\rangle = -2\varepsilon_b\langle\beta^*\rangle - \frac{1}{2}\mu_c\langle\beta\rangle + \frac{1}{2}V_+\langle\alpha^*\rangle \tag{3.18}$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -2\varepsilon_a - \mu_a\langle\alpha^2\rangle + V_-\langle\beta^*\alpha\rangle - 4\varepsilon_a\langle\alpha^*\alpha\rangle \tag{3.19}$$

$$\frac{d}{dt}\langle\beta^2\rangle = -2\varepsilon_b - \mu_c\langle\beta^2\rangle + V_+\langle\alpha^*\beta\rangle - 4\varepsilon_b\langle\beta^*\beta\rangle \tag{3.20}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -2\varepsilon_a(\langle{\alpha^*}^2\rangle + \langle\alpha^2\rangle) - \mu_a\langle\alpha^*\alpha\rangle + \frac{V_-^*}{2}\langle\alpha\beta\rangle + \frac{V_-}{2}\langle\alpha^*\beta^*\rangle + A\rho_{aa}^{(0)} + \kappa N \tag{3.21}$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -2\varepsilon_b(\langle{\beta^*}^2\rangle + \langle\beta^2\rangle) - \mu_c\langle\beta^*\beta\rangle + \frac{V_+^*}{2}\langle\alpha\beta\rangle + \frac{V_+}{2}\langle\alpha^*\beta^*\rangle + \kappa N \tag{3.22}$$

$$\frac{d}{dt}\langle\alpha^*\beta\rangle = -2\varepsilon_a\langle\alpha\beta\rangle - 2\varepsilon_b\langle\alpha^*\beta^*\rangle - \frac{1}{2}\mu\langle\alpha^*\beta\rangle + \frac{1}{2}V_-^*\langle\beta^2\rangle + \frac{1}{2}V_+\langle{\alpha^*}^2\rangle \tag{3.23}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -2\varepsilon_a\langle\alpha^*\beta\rangle - 2\varepsilon_b\langle\alpha\beta^*\rangle + \frac{1}{2}V_+ - \frac{1}{2}\mu\langle\alpha\beta\rangle + \frac{1}{2}V_+\langle\alpha^*\alpha\rangle + \frac{1}{2}V_-\langle\beta^*\beta\rangle + \kappa M \tag{3.24}$$

On account of Eqs. (3.17) and (3.18), One can write the c-number Langevin equations as

$$\frac{d}{dt}\alpha(t) = \frac{-1}{2}\mu_a\alpha(t) + \frac{V_-}{2}\beta^*(t) + f_\alpha(t) - 2\varepsilon_a\alpha^*(t) \tag{3.25}$$

$$\frac{d}{dt}\beta(t) = \frac{-1}{2}\mu_c\beta(t) + \frac{V_+}{2}\alpha^*(t) - 2\varepsilon_b\beta^*(t) + f_\beta(t) \tag{3.26}$$

and the complex conjugate of Eq. (3.26)

$$\frac{d}{dt}\beta^*(t) = \frac{-1}{2}\mu_c\beta^*(t) + \frac{V_+}{2}\alpha(t) - 2\varepsilon_b\beta(t) + f_\beta^*(t), \tag{3.27}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are the noise forces. The formal solution of these equations can be put in the form

$$\alpha(t) = \alpha(0)e^{-\mu_a t/2} + \int_0^t dt' e^{-\mu_a(t-t')/2}\frac{1}{2}V_-\beta^*(t') - 2\varepsilon_a\alpha^*(t) + f_\alpha(t'), \tag{3.28}$$

$$\beta(t) = \beta(0)e^{-\mu_c t/2} + \int_0^t dt' e^{-\mu_c(t-t')/2}\left[\frac{1}{2}V_+\alpha^*(t') - 2\varepsilon_b\beta^*(t) + f_\beta(t')\right], \tag{3.29}$$

$$\beta^*(t) = \beta^*(0)e^{-\mu_c t/2} + \int_0^t dt' e^{-\mu_c(t-t')/2}\left[\frac{1}{2}V_+^*\alpha(t') - 2\varepsilon_b\beta^*(t) + f_\beta^*(t')\right]. \tag{3.30}$$

We now proceed to determine the properties of the noise forces. We note that Eqs. (3.17) and the expectation value of Eq. (3.25) as well as Eq. (3.18) and expectation value of (3.26) will have the same form provided that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.31}$$

Furthermore, using Eqs. (3.25) and (3.26) together with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\frac{d\alpha(t)}{dt}\beta(t)\rangle + \langle\alpha(t)\frac{d\beta(t)}{dt}\rangle \tag{3.32}$$

it can be verified that

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2\rangle + V_-\langle\beta^*\alpha\rangle - 4\varepsilon_a\langle\alpha^*\alpha\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle \tag{3.33}$$

comparison of this equation with Eq. (3.19) leads us to:

$$2\langle\alpha(t)f_\alpha(t)\rangle = -2\varepsilon_a \tag{3.34}$$

$$\langle\alpha(t)f_\alpha(t)\rangle = -\varepsilon_a \tag{3.35}$$

on account of Eq. (3.28) together with Eq. (3.35), we see that

$$-\varepsilon_a = \langle\alpha(0)f_\alpha(t)e^{\frac{-\mu_a}{2}t} + \int_0^t dt' e^{\frac{-\mu_a}{2}(t-t')}[\frac{1}{2}V_-\langle\beta^*(t')f_\alpha(t)\rangle + \langle f_\alpha(t')f_\alpha(t)\rangle + 2\varepsilon_a\alpha^*(t)\langle f_\alpha(t)\rangle] \tag{3.36}$$

taking in to account Eq. (3.31) along with the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, we have

$$\int_0^t dt' e^{\frac{-\mu_a}{2}(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle = -\varepsilon_a \tag{3.37}$$

now on the basis of the relation [1]

$$\int_0^t dt' e^{\frac{-\mu_a}{2}(t-t')}\langle\langle f(t')g(t')\rangle\rangle = D \tag{3.38}$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'), \tag{3.39}$$

where a is a constant and D is a constant or some function of time t.It then follows

$$\langle f_\alpha(t')f_\alpha(t)\rangle = -2\varepsilon_a\delta(t-t'). \tag{3.40}$$

This can be written as

$$\int_0^t dt' e^{-\mu_a(t-t')/2}[f_\alpha(t')f_\alpha(t)] = -2\int_0^t dt' e^{-\mu_a(t-t')/2}\varepsilon_a\delta(t-t')dt'. \tag{3.41}$$

Similarly, we can easily establish that

$$\langle f_\beta(t')f_\beta(t)\rangle = \langle f^*_\beta(t')f^*_\beta(t)\rangle = -2\varepsilon_b\delta(t-t'). \tag{3.42}$$

Furthermore,using Eq. (3.25) and its complex conjugate, we have

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu_a\langle\alpha^*\alpha\rangle + \frac{1}{2}V_-^*\langle\alpha\beta\rangle + \frac{1}{2}V_-^*\langle\alpha^*\beta^*\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle + \langle f_\alpha^*(t)\alpha(t)\rangle - 2\varepsilon_a(\langle\alpha^{*2}\rangle + \langle\alpha^2\rangle) \tag{3.43}$$

comparison of Eq. (3.21) with Eq. (3.43) show that

$$\langle f_\alpha^*(t)\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle = A\rho_{aa}^{(0)} + \kappa N \tag{3.44}$$

now taking into account Eq. (3.28), (3.29) and the complex conjugate of (3.28), we find

$$\int_0^t dt' e^{\frac{-\mu_a}{2}(t-t')}[\langle f_\alpha^*(t')f_\alpha(t)\rangle + \langle f_\alpha^*(t)f_\alpha(t')\rangle] = A\rho_{aa}^{(0)} + \kappa N. \tag{3.45}$$

Assuming that

$$\langle f_\alpha^*(t')f_\alpha(t)\rangle = \langle f_\alpha^*(t)f_\alpha(t')\rangle \tag{3.46}$$

, we have

$$\int_0^t dt' e^{\frac{-\mu_a}{2}(t-t')}\langle f_\alpha^*(t)f_\alpha(t')\rangle = \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa N). \tag{3.47}$$

In view of the property of the Dirac delta function, we have

$$\int_0^t f(t')\delta(t-t')dt' = \frac{1}{2}f(t). \tag{3.48}$$

Employing Eq. (3.48) in to Eq. (3.47) can be written as

$$\int_0^t dt' e^{\frac{-\mu_a}{2}(t-t')}\langle f_\alpha^*(t)f_\alpha(t')\rangle = \int_0^t dt' e^{\frac{-\mu_a}{2}(t-t')}(A\rho_{aa}^{(0)} + \kappa N)\delta(t-t'). \tag{3.49}$$

Then follows that

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = \langle f_\alpha^*(t')f_\alpha(t)\rangle = (A\rho_{aa}^{(0)} + \kappa N)\delta(t-t'). \tag{3.50}$$

It can also be established in a similar fashion that

$$\langle f_\beta^*(t)f_\beta(t')\rangle = \langle f_\beta^*(t')f_\beta(t)\rangle = \kappa N\delta(t-t') \tag{3.51}$$

$$\langle f_\alpha(t)f_\beta(t')\rangle = \langle f_\alpha(t')f_\beta(t)\rangle = (\frac{1}{2}V_+ + \kappa M)\delta(t-t') \tag{3.52}$$

$$\langle f_\alpha(t)f_\beta^*(t')\rangle = \langle f_\alpha(t')f_\beta^*(t)\rangle = 0 \tag{3.53}$$

$$\langle f_\beta^*(t')f_\alpha(t)\rangle = \langle f_\beta^*(t)f_\alpha(t')\rangle = 0 \tag{3.54}$$

$$\langle f_\alpha(t) f_\alpha^*(t')\rangle = \langle f_\alpha(t') f_\alpha^*(t)\rangle = \kappa(N+1)\delta(t-t') \tag{3.55}$$

this results described by Eqs. (3.31), (3.40), (3.42), (3.50) and (3.51)-(3.55) represents the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering. We next proceed to obtain the solutions of the coupled differential equations Eqs. (3. 25) and (3. 26). Employing Eqs. (3. 25) and (3. 26), we can write

$$\frac{d}{dt}B_\pm(t) = \frac{-1}{2}(\mu_a \pm 4\varepsilon_a)B_\pm(t) \mp \frac{1}{2}V(t)C_\pm(t) + F_{\alpha_\pm}(t \tag{3.56}$$

$$\frac{d}{dt}C_\pm(t) = \frac{-1}{2}(\mu_c \pm 4\varepsilon_b)C(t) \pm \frac{1}{2}V(t)B_\pm(t) + F_{\beta_\pm}(t). \tag{3.57}$$

In which we have taken

$$V = V^* \tag{3.58}$$

and

$$B_\pm = \alpha \pm \alpha^* \tag{3.59}$$

$$C_\pm = \beta \pm \beta^* \tag{3.60}$$

$$F_{\alpha_\pm} = f_\alpha \pm f_{\alpha^*} \tag{3.61}$$

$$F_{\beta_\pm} = f_\beta \pm f_{\beta^*} \tag{3.62}$$

we write Eqs. (3.56) and (3.57) in a matrix form as

$$\frac{d}{dt}Z_\pm(t) = \frac{1}{2}MZ_\pm(t) + F_\pm(t) \tag{3.63}$$

$$Z_\pm(t) = \begin{pmatrix} B_\pm(t) \\ C_\pm(t) \end{pmatrix} \tag{3.64}$$

$$M = \begin{pmatrix} \mu_a \pm 4\varepsilon_a & \pm V \\ \mp V & \mu_c \pm 4\varepsilon_b \end{pmatrix} \tag{3.65}$$

$$F_\pm(t) = \begin{pmatrix} f_{\alpha_\pm}(t) \\ f_{\beta_\pm}(t) \end{pmatrix} \tag{3.66}$$

introducing a matrix defined by

$$V_{\pm} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \tag{3.67}$$

with

$$V_{1\pm} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{3.68}$$

and

$$V_{2\pm} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \tag{3.69}$$

being the eigen vectors of the matrix M, Eq (3.63) can be written as

$$\frac{d}{dt}Z_{\pm}(t) = \frac{-1}{2}V_{\pm}V_{\pm}^{-1}MV_{\pm}V_{\pm}^{-1}Z_{\pm}(t) + F_{\pm}(t) \tag{3.70}$$

multiplying both sides from the left by $V_{\pm}^{-1}$, we see that

$$\frac{d}{dt}(V_{\pm}^{1}Z_{\pm}(t)) = \frac{-1}{2}R(V_{\pm}^{-1}Z_{\pm}(t) + V_{\pm}^{-1}F_{\pm}(t) \tag{3.71}$$

$$R = V_{\pm}^{-1}MV_{\pm} = \begin{pmatrix} \lambda_{1\pm} & 0 \\ 0 & \lambda_{2\pm} \end{pmatrix} \tag{3.72}$$

in which $\lambda_{1\pm}$ and $\lambda_{2\pm}$ are the eigen values of the matrix M. We note that Eq. (3.71) has a well defined solution for $\lambda_{1\pm} > 0$ and $\lambda_{2\pm} > 0$. The solution of this equation can be written as

$$V_{\pm}^{-1}Z_{\pm}(t) = V_{\pm}e^{\frac{-1}{2}R(t)}V_{\pm}^{-}Z_{\pm}(0) + \int_0^t e^{\frac{-1}{2}R(t-t')}V_{\pm}^{-}F_{\pm}(t')dt' \tag{3.73}$$

from which follows

$$Z_{\pm}(t+\tau) = V_{\pm}e^{\frac{-1}{2}R(\tau)}V_{\pm}^{-}Z_{\pm}(t) + \int_0^t V_{\pm}e^{\frac{-1}{2}R(\tau-\tau')}V_{\pm}^{-1}F_{\pm}(t+\tau')d\tau' \tag{3.74}$$

we next proceed to find the eigen values and eigen vectors of the matrix M. Applying the eigen value equation.

$$MV_{i\pm} = \lambda_{i\pm}V_{i\pm} \tag{3.75}$$

along with Eq. (3.65), we find the characteristic equation.

$$\lambda_{\pm}^2 - \lambda_{\pm}(\mu_a + \mu_c \pm 4[\varepsilon_a + \varepsilon_b]) + (\mu_a \pm 4\varepsilon_a)(\mu_c \pm 4\varepsilon_b + V^2 = 0 \tag{3.76}$$

the roots of this quadratic equation are found to be

$$\lambda_{1\pm} = \frac{1}{2}\left(\mu_a + \mu_c \pm 4(\varepsilon_a + \varepsilon_b) + \sqrt{\mu_a - \mu_c \pm 4(\varepsilon_a - \varepsilon_b)^2 - 4V^2}\right) \tag{3.77}$$

$$\lambda_{2\pm} = \frac{1}{2}\left(\mu_a + \mu_c \pm 4(\varepsilon_a + \varepsilon_b) - \sqrt{\mu_a - \mu_c \pm 4(\varepsilon_a - \varepsilon_b)^2 - 4V^2}\right) \tag{3.78}$$

Taking in to account Eqs. (3.7), (3.8), (3.16), and the relation

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1 \tag{3.79}$$

we have

$$\lambda_{1\pm} = \frac{1}{2}(2\kappa + A\eta \pm 4\varepsilon_+ + \chi_\pm) \tag{3.80}$$

$$\lambda_{2\pm} = \frac{1}{2}(2\kappa + A\eta \pm 4\varepsilon_+ - \chi_\pm) \tag{3.81}$$

where

$$\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)} \tag{3.82}$$

$$|\rho_{ac}^{(0)}|^2 = |\rho_{aa}^{(0)}||\rho_{cc}^{(0)}|$$

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}$$

,

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}$$

and

$$|\rho_{ac}^{(0)}| = \frac{\sqrt{1-\eta^2}}{2}$$

$$\varepsilon_\pm = \varepsilon_b \pm \varepsilon_a \tag{3.83}$$

$$\chi_\pm = \sqrt{A^2\eta^2 + \varepsilon_-^2 \pm 8A\varepsilon_-}. \tag{3.84}$$

With the aid of Eqs. (3.7), (3.8), (3.65) and (3.80), we have

$$(A \pm 4\varepsilon_- + \chi_\pm)x_1 \mp 2Vy_1 = 0 \tag{3.85}$$

and taking into account the normalization condition:

$$x_1^2 + y_1^2 = 1 \tag{3.86}$$

we get

$$x_1 = \frac{2V}{\sqrt{(A \pm 4\varepsilon_- + \chi_\pm)^2 + 4V^2}} \tag{3.87}$$

$$y_1 = \mp \frac{(A \pm 4\varepsilon_- + \chi_\pm)}{\sqrt{(A \pm 4\varepsilon_- + \chi_\pm)^2 + 4V^2}} \tag{3.88}$$

similarly we can also easily show that the elements of the eigenvector corresponding to $\lambda_{2\pm}$ to be

$$x_2 = \mp \frac{2V}{\sqrt{(A \pm 4\varepsilon_- - \chi_\pm)^2 + 4V^2}} \tag{3.89}$$

$$y_2 = (\frac{(A \pm 4\varepsilon_- - \chi_\pm)}{\sqrt{(A \pm 4\varepsilon_- - \chi_\pm)^2 + 4V^2}} \tag{3.90}$$

now substitution of Eqs. (3.87), (3.88), (3.89) and (3.90) into Eq. (3.67) yields

$$V_\pm = \begin{pmatrix} \frac{2V}{\sqrt{(A \pm 4\varepsilon_- + \chi_\pm)^2 + 4V^2}}, & \mp \frac{2V}{\sqrt{(A \pm 4\varepsilon_- - \chi_\pm)^2 + 4V^2}} \\ \mp \frac{(A \pm 4\varepsilon_- + \chi_\pm)}{\sqrt{(A \pm 4\varepsilon_- + \chi_\pm)^2 + 4V^2}}, & \frac{(A \pm 4\varepsilon_- - \chi_\pm)}{\sqrt{(A \pm 4\varepsilon_- - \chi_\pm)^2 + 4V^2}} \end{pmatrix} \tag{3.91}$$

in which

$$K_\pm = A \pm 4\varepsilon_- + \chi_\pm$$

and

$$N_\pm = A \pm 4\varepsilon_- - \chi_\pm$$

So,

$$V_\pm = \begin{pmatrix} \frac{2V}{\sqrt{K_\pm^2 + 4V^2}}, & \mp \frac{2V}{\sqrt{N_\pm^2 + 4V^2}} \\ \mp \frac{K_\pm}{\sqrt{K_\pm^2 + 4V^2}}, & \frac{N_\pm}{\sqrt{N_\pm^2 + 4V^2}} \end{pmatrix} \tag{3.92}$$

and the inverse of the matrix $V_\pm$ is found to be

$$V_\pm^{-1} = \frac{1}{4V\chi_\pm} \begin{pmatrix} N_\pm \sqrt{K_\pm^2 + 4V^2} & \pm 2V \sqrt{K_\pm^2 + 4V^2} \\ \pm K_\pm \sqrt{N_\pm^2 + 4V^2} & 2V \sqrt{N_\pm^2 + 4V^2} \end{pmatrix} \tag{3.93}$$

since Eq.(3.72) is diagonal matrix, we can write as

$$e^{\frac{-1}{2} R(\tau)} = \begin{pmatrix} e^{\frac{-1}{2} \lambda_1 \pm (\tau)} & 0 \\ 0 & e^{\frac{-1}{2} \lambda_2 \pm (\tau)} \end{pmatrix} \tag{3.94}$$

$$e^{\frac{-1}{2} R(\tau - \tau')} = \begin{pmatrix} e^{\frac{-1}{2} \lambda_1 \pm (\tau - \tau')} & 0 \\ 0 & e^{\frac{-1}{2} \lambda_2 \pm (\tau - \tau')} \end{pmatrix} \tag{3.95}$$

it then follows that

$$V_\pm e^{\frac{-1}{2} R\tau} V_\pm^{-1} = \begin{pmatrix} Y_{1\pm}\tau & W_{1_\pm}\tau \\ W_{2_\pm}(\tau) & Y_{2\pm}(\tau) \end{pmatrix} \tag{3.96}$$

and

$$V_{\pm}e^{\frac{-1}{2}R(\tau-\tau')}V_{\pm}^{-1} = \begin{pmatrix} Y_{1\pm}(\tau-\tau') & W_{1_{\pm}}(\tau-\tau') \\ W_{2_{\pm}}(\tau-\tau') & Y_{2\pm}(\tau-\tau') \end{pmatrix} \tag{3.97}$$

$$Y_{1\pm}(\tau) = \frac{1}{2\chi_{\pm}}(K_{\pm}e^{\frac{-1}{2}\lambda_{2}\pm\tau} - N_{\pm}e^{\frac{-1}{2}\lambda_{1}\pm\tau}) \tag{3.98}$$

$$Y_{2\pm}(\tau) = \frac{1}{2\chi_{\pm}}(N_{\pm}e^{\frac{-1}{2}\lambda_{2}\pm\tau} - K_{\pm}e^{\frac{-1}{2}\lambda_{1}\pm\tau}) \tag{3.99}$$

$$W_{1\pm}(\tau) = \pm\frac{2V}{2\chi_{\pm}}(e^{\frac{-1}{2}\lambda_{2}\pm\tau} - e^{\frac{-1}{2}\lambda_{1}\pm\tau}) \tag{3.100}$$

$$W_{2\pm}(\tau) = \mp\frac{2V}{2\chi_{\pm}}(e^{\frac{-1}{2}\lambda_{2}\pm\tau} - e^{\frac{-1}{2}\lambda_{1}\pm\tau}) \tag{3.101}$$

$$Y_{1\pm}(\tau-\tau') = \frac{1}{2\chi_{\pm}}(K_{\pm}e^{\frac{-1}{2}\lambda_{2}\pm(\tau-\tau')} - N_{\pm}e^{\frac{-1}{2}\lambda_{1}\pm(\tau-\tau')}) \tag{3.102}$$

$$Y_{2\pm}(\tau-\tau') = \frac{1}{2\chi_{\pm}}(N_{\pm}e^{\frac{-1}{2}\lambda_{2}\pm(\tau-\tau')} - K_{\pm}e^{\frac{-1}{2}\lambda_{1}\pm(\tau-\tau')}) \tag{3.103}$$

$$W_{1\pm}(\tau-\tau') = \pm\frac{2V_{-}}{2\chi_{\pm}}(e^{\frac{-1}{2}\lambda_{2}\pm(\tau-\tau')} - e^{\frac{-1}{2}\lambda_{1}\pm(\tau-\tau')}) \tag{3.104}$$

$$W_{2\pm}(\tau-\tau') = \mp\frac{2V_{-}}{2\chi_{\pm}}(e^{\frac{-1}{2}\lambda_{2}\pm(\tau-\tau)} - e^{\frac{-1}{2}\lambda_{1}\pm(\tau-\tau')}) \tag{3.105}$$

with the aid of Eqs (3.64) and (3.74) we see that

$$B_{\pm}(t) = Y_{1\pm}(t)B_{\pm}(0) + W_{1\pm}(t)C_{\pm}(0) + \int_{0^{t}}(Y_{1\pm}(t-t')F_{\alpha_{\pm}}(t') + W_{1\pm}(t-t')F_{\beta_{\pm}}(t'))dt' \tag{3.106}$$

$$C_{\pm}(t) = Y_{2\pm}(t)C_{\pm}(0) + W_{2\pm}(t)B_{\pm}(0) + \int_{0^{t}}(Y_{2\pm}(t-t')F_{\beta_{\pm}}(t') + W_{2\pm}(t-t')F_{\alpha_{\pm}}(t'))dt' \tag{3.107}$$

the explicit expression for α and β is obtained by subtracting B_{-} from $B_{+}$and $C_{-}$from C_{+} respectively.Hence,

$$\alpha(t) = \frac{1}{2}\Big(Y_{1}^{+}(t)\alpha(0) + Y_{1}^{-}(t)\alpha^{*}(0) + W_{1}^{+}(t)\beta(0) + W_{1}^{-}(t)\beta^{*}(0) + H_{1}(t)\Big) \tag{3.108}$$

$$\beta(t) = \frac{1}{2}\left(Y_2^+(t)\beta(0) + Y_2^-(t)\beta^*(0) + W_2^+(t)\alpha(0) + W_2^-(t)\alpha^*(0) + H_2(t)\right) \tag{3.109}$$

where

$$H_1(t) = \int_0^t \left(Y_1^+(t-t')f_\alpha(t') + Y_1^-(t-t')f_\alpha^*(t') + W_1^+(t)f_\beta(t') + W_1^-(t-t')f\beta^*(t)\right)dt' \tag{3.110}$$

$$H_2(t) = \int_0^t \left(Y_2^+(t-t')f_\beta(t') + Y_2^-(t-t')f_\beta^*(t') + W_2^+(t-t')f_\alpha(t') + W_2^-(t-t')f_\alpha^*(t')\right)dz' \tag{3.111}$$

where

$$Y_1^\pm = Y_{1+} \pm Y_{1-} \tag{3.112}$$

$$Y_2^\pm = Y_{2+} \pm Y_{2-} \tag{3.113}$$

$$W_1^\pm = W_{1+} \pm W_{1-} \tag{3.114}$$

$$W_2^\pm = W_{2+} \pm W_{2-} \tag{3.115}$$

4

Quadrature Fluctuations

In this chapter, we wish to determine the quadrature variances of the cavity modes as well as the squeezing spectrum of the output modes employing the solutions of the c-number Langevin equations and the correlation properties of the noise forces.

4.1 Quadrature variances of the cavity modes

Here we seek to calculate the quadrature variances of the cavity modes produced by the system under consideration coupled to the squeezing vacuum reservoir. The quadrature operators for two-mode light are defined as [1]

$$\hat{c}_{\pm} = \sqrt{\pm 1}(\hat{c}^{+} \pm \hat{c}), \tag{4.1}$$

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{4.2}$$

Using the commutation relation

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 1 \tag{4.3}$$

it can easily be shown that

$$[\hat{c}, \hat{c}^{+}] = 1. \tag{4.4}$$

The squeezing properties of a two- mode light are described by the quadrature operators

$$\hat{c}_{+} = \hat{c} + \hat{c}^{+}, \tag{4.5}$$

$$\hat{c}_{-} = i(\hat{c}^{+} - \hat{c}). \tag{4.6}$$

With the aid of Eqs. (4.3), (4.4) and (4.5), one can readily verify that

$$[\hat{c}_{+}, \hat{c}_{-}] = 2i \tag{4.7}$$

a two mode light is said to be in a squeezed state if either$\triangle c_+ < 1$ or $\triangle c_- > 1$ such that $\triangle c_+ \triangle c_- \geqslant 1$.
The variance of the quadrature operators are defined by:

$$\triangle \hat{c}^2_{\pm}(t) = \langle \hat{c}^2_{\pm}(t)\rangle - \langle \hat{c}_{\pm}(t)\rangle^2 \tag{4.8}$$

applying Eqs. (4.4) and (4.5), one can write (4.8) in the normal order as

$$\triangle \hat{c}^2_{\pm}(t) = 1 \pm \langle \hat{c}^{\dagger 2}(t) + \hat{c}^2(t) \pm 2\hat{c}^{\dagger}(t)\hat{c}(t) \mp \langle \hat{c}^{\dagger}(t) \pm \hat{c}(t)\rangle^2. \tag{4.9}$$

This can be expressed in terms of c-number variables associated with normal ordering as

$$\triangle \hat{c}^2_{\pm}(t) = 1 \pm \langle \gamma^{*2}(t) + \gamma^2(t) \pm 2\gamma^*(t)\gamma(t)\rangle \mp \langle\langle \gamma^*(t) \pm \gamma(t)\rangle^2 \tag{4.10}$$

where $\gamma(t)$ is the c-number corresponding to the operator $\hat{c}(t)$.
Introducing a new variable defined by

$$\gamma_{\pm}(t) = \gamma^*(t) \pm \gamma(t) \tag{4.11}$$

Eq. (4.9) can be rewritten as

$$\triangle \hat{c}^2_{\pm}(t) = 1 \pm (\langle \gamma^2_{\pm}(t)\rangle - \langle \gamma_{\pm}(t)\rangle^2) \tag{4.12}$$

the variance of the quadrature represented by the operators defined by (4.1) can be expressed in terms of c-number variables associated with the normal ordering as

$$\triangle \hat{c}^2_{\pm}(t) = 1 \pm (\langle \gamma_{\pm}(t), \gamma_{\pm}(t)\rangle \tag{4.13}$$

where

$$\gamma_{\pm}(t) = \frac{1}{\sqrt{2}}\Big(\alpha^*(t) + \beta^*(t) \pm \alpha(t) \pm \beta(t)\Big) \tag{4.14}$$

with

$$\gamma(t) = \frac{1}{\sqrt{2}}(\alpha(t) + \beta(t)), \tag{4.15}$$

0n account of Eq. (4.14), we see that

$$\begin{aligned}\langle \gamma_{\pm}(t), \gamma_{\pm}(t)\rangle =& \frac{1}{2}[\langle \alpha(t), \alpha(t)\rangle + \langle \beta^*(t), \beta^*(t)\rangle + 2\langle \alpha(t), \beta(t)\rangle \\ &\pm \langle \alpha^*(t), \alpha(t)\rangle \pm \langle \beta^*, \beta(t)\rangle \pm 2\langle \beta^*(t), \alpha(t)\rangle] + c.c,\end{aligned} \tag{4.16}$$

in which c.c stands for complex conjugate. Using Eq. (3.31) and assuming that the cavity modes are initially in a vacuum state along with the fact that the noise force at a certain time doesn't affect the cavity mode variables at earlier time, we easily find

$$\langle \alpha, \alpha \rangle = \frac{1}{4}\langle H_1(t)H_2(t)\rangle. \tag{4.17}$$

Further employing Eq. (3.109) along with its complex conjugate, we have

$$\begin{aligned}\langle \alpha(t), \alpha(t)\rangle =& \frac{1}{4}\int_0^t \Big[Y_1^+(t-t')Y_1^+(t-t'')\langle f_\alpha(t')f_\alpha(t'')\rangle + Y_1^+(t-t'')Y_1^-(t-t'')\langle f_\alpha(t')f_\alpha^*(t'')\rangle \\ &+ Y_1^+(t-t'')W_1^+(t-t'')\langle f_\alpha(t')f_\beta(t'')\rangle + Y_1^+(t-t'')W_1^-(t-t'')\langle f_\alpha(t')f_\beta^*(t'')\rangle \\ &+ Y_1^-(t-t'')Y_1^+(t-t'')\langle f_\alpha^*(t')f_\alpha(t'')\rangle + Y_1^-(t-t'')Y_1^-(t-t'')\langle f_\alpha^*(t')f_\alpha^*(t'')\rangle \\ &+ Y_1^-(t-t'')W_1^+(t-t'')\langle f_\alpha^*(t')f_\beta(t'')\rangle + Y_1^-(t-t'')W_1^-(t-t'')\langle f_\alpha^*(t')f_\beta^*(t'')\rangle \\ &+ W_1^+(t-t'')Y_1^+(t-t'')\langle f_\beta(t')f_\alpha(t'')\rangle + W_1^+(t-t'')Y_1^-(t-t'')\langle f_\beta(t')f_\alpha^*(t'')\rangle \\ &+ W_1^+(t-t'')W_1^+(t-t'')\langle f_\beta(t')f_\beta(t'')\rangle + W_1^+(t-t'')W_1^-(t-t'')\langle f_\beta(t')f_\beta^*(t'')\rangle \\ &+ W_1^-(t-t'')Y_1^+(t-t'')\langle f_\beta^*(t')f\alpha(t'')\rangle + W_1^-(t-t'')Y_1^-(t-t'')\langle f_\beta^*(t')f_\alpha^*(t'')\rangle \\ &+ W_1^-(t-t'')W_1^+(t-t'')\langle f_\beta^*(t')f_\beta(t'')\rangle + W_1^-(t-t'')W_1^-(t-t'')\langle f_\beta^*(t')f_\beta^*(t'')\rangle\Big] dt'dt''.\end{aligned} \tag{4.18}$$

With the aid of Eqs. (3.31), (3.40), (3.42), (3.50), (3.51), (3.52), (3.53)-(3.56), Eq. (4.18) can be written as

$$\begin{aligned}\langle \alpha, \alpha \rangle =& \frac{1}{4}\int_0^t \bigg(f_{\alpha\alpha}(Y_1^+(t-t')Y_1^+(t-t'') + Y_1^-(t-t'')Y_1^-(t-t'')) \\ &+ f_{\beta\beta}(W_1^+(t-t')W_1^+(t-t'') + W_1^-(t-t'')W_1^-(t-t'')) \\ &+ f_{\alpha^*\alpha}(Y_1^+(t-t')Y_1^-(t-t'') + Y_1^-(t-t'')Y_1^+(t-t'')) \\ &+ f_{\beta^*\beta}(W_1^+(t-t')W_1^-(t-t'') + W_1^-(t-t'')W_1^+(t-t'')) \\ &+ \frac{V}{2}(Y_1^+(t-t')W_1^+(t-t'') + W_1^+(t-t'')Y_1^+(t-t'')) \\ &+ (Y_1^-(t-t')W_1^-(t-t'') + W_1^-(t-t'')Y_1^-(t-t''))\bigg)\delta(t'-t'')dt'dt'',\end{aligned} \tag{4.19}$$

where

$$f_{\alpha\alpha} = -2\varepsilon_a \tag{4.20}$$

$$f_{\beta\beta} = -2\varepsilon_b \tag{4.21}$$

$$f_{\alpha^*\alpha} = A_{\hat{\rho}_{aa}}^{(0)} + kN \tag{4.22}$$

$$f_{\beta^*\beta} = kN \tag{4.23}$$

$$f_{\alpha\beta} = \frac{V}{2} + kM \tag{4.24}$$

applying the properties of delta function, we see that

$$\begin{aligned}\langle\alpha,\alpha\rangle =& \frac{f_{\alpha\alpha}}{4}\int_0^t \Big(Y_1^{+2}(t-t') + Y_1^{-2}(t-t''))dt' \\ &+ \frac{f_{\beta\beta}}{4}\int_0^t (W_1^{+2}(t-t') + W_1^{-2}(t-t''))dt' \\ &+ \frac{2f_{\alpha^*\alpha}}{4}\int_0^t (Y_1^{+}(t-t')Y_1^{-}(t-t''))dt' \\ &+ \frac{2f_{\beta^*\beta}}{4}\int_0^t (W_1^{+}(t-t')W_1^{-}(t-t''))dt' \\ &+ \frac{2f_{\alpha\beta}}{4}\int_0^t (Y_1^{+}(t-t')W_1^{+}(t-t'') + Y_1^{-}(t-t'')W_1^{-}(t-t''))\Big)dt'\end{aligned} \tag{4.25}$$

again, we can rewrite Eq. (4.25) as

$$\begin{aligned}\langle\alpha,\alpha\rangle =& \frac{f_{\alpha\alpha}+f_{\alpha^*\alpha}}{2}\int_0^t Y_{1+}^2(t-t')dt' + \frac{f_{\alpha\alpha}-f_{\alpha^*\alpha}}{2}\int_0^t Y_{1-}^2(t-t')dt' \\ &+ \frac{f_{\beta\beta}+f_{\beta^*\beta}}{2}\int_0^t W_{1+}^2(t-t')dt' + \frac{f_{\beta\beta}-f_{\beta^*\beta}}{2}\int_0^t W_{1-}^2(t-t')dt' \\ &+ f_{\alpha\beta}\int_0^t (Y_{1+}(t-t')W_{1+}(t-t') + Y_{1-}(t-t')W_{1-}(t-t'))dt'\end{aligned} \tag{4.26}$$

employing Eqs. (3.103) and (3.105) and then carrying out the integration, we get

$$\int_0^t Y_{1\pm}^2(t-t')dt' = \frac{N_\pm^2(1-e^{\lambda_{1\pm}t})}{4\chi_\pm^2\lambda_{1\pm}} + \frac{K_\pm^2(1-e^{-\lambda_{2\pm}}t)}{4\chi_\pm^2\lambda_{2\pm}} - \frac{4K_\pm N_\pm(1-e^{\frac{1}{2}(\lambda_{1\pm}+\lambda_{2\pm})t})}{4\chi_\pm^2(\lambda_{1\pm}+\lambda_{2\pm})} \tag{4.27}$$

$$\int_0^t W_{1\pm}^2(t-t')dt' = \frac{V^2(1-e^{\lambda_{1\pm}t})}{\chi_\pm^2\lambda_{1\pm}} + \frac{V^2(1-e^{-\lambda_{2\pm}}t)}{\chi_\pm^2\lambda_{2\pm}} - \frac{4V^2(1-e^{\frac{1}{2}(\lambda_{1\pm}+\lambda_{2\pm})t})}{\chi_\pm^2(\lambda_{1\pm}+\lambda_{2\pm})} \tag{4.28}$$

$$\begin{aligned}\int_0^t Y_{1\pm}(t-t')W_{1\pm}(t-t')dt' =& \pm\Big(\frac{VN_\pm(1-e^{\lambda_{1\pm}t})}{2\chi_\pm^2\lambda_{1\pm}} + \frac{VK_\pm(1-e^{-\lambda_{2\pm}}t)}{2\chi_\pm^2\lambda_{2\pm}} \\ &- \frac{2V(K_\pm+N_\pm)(1-e^{\frac{-1}{2}(\lambda_{1\pm}+\lambda_{2\pm})t})}{2\chi_\pm^2(\lambda_{1\pm}+\lambda_{2\pm})}\Big)\end{aligned} \tag{4.29}$$

on account of Eqs. (4.27), (4.28), (4.29) and (4.26) takes the form

$$
\begin{aligned}
\langle \alpha, \alpha \rangle =& \frac{K_+^2(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} + f_{\beta^*\beta}) + 4K_+ V f_{\alpha\beta}}{8\chi_+^2 \lambda_{2+}}(1 - e^{\lambda_{2+}t}) \\
&+ \frac{N_+^2(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} + f_{\beta^*\beta}) + 4N_+ V f_{\alpha\beta}}{8\chi_+^2 \lambda_{1+}}(1 - e^{\lambda_{1+}t}) \\
&- \frac{4K_+N_+(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + 16V^2(f_{\beta\beta} + f_{\beta^*\beta}) + 8(K_+ + N_+)V f_{\alpha\beta}}{8\chi_+^2 (\lambda_{1+} + \lambda_{2+})}(1 - e^{\frac{1}{2}(\lambda_{1+}+\lambda_{2+})t}) \\
&+ \frac{K_-^2(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} - f_{\beta^*\beta}) - 4K_- V f_{\alpha\beta}}{8\chi_-^2 \lambda_{2-}}(1 - e^{\lambda_{2-}t}) \\
&+ \frac{K_-^2(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} - f_{\beta^*\beta}) - 4N_- V f_{\alpha\beta}}{8\chi_-^2 \lambda_{1-}}(1 - e^{\lambda_{1-}t}) \\
&- \frac{4K_-N_-(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + 16V^2(f_{\beta\beta} - f_{\beta^*\beta}) - 8(K_- + N_-)V f_{\alpha\beta}}{8\chi_-^2 (\lambda_{1-} + \lambda_{2-})}(1 - e^{\frac{1}{2}(\lambda_{1-}+\lambda_{2-})t})
\end{aligned} \tag{4.30}
$$

following a similar procedure, we also find

$$
\begin{aligned}
\langle \beta, \beta \rangle =& \frac{4V^2(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + N_+^2(f_{\beta\beta} + f_{\beta^*\beta}) + 4N_+ V f_{\alpha\beta}}{8\chi_+^2 \lambda_{2+}}(1 - e^{-\lambda_{2+}t}) \\
&+ \frac{4V^2(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + K_+^2(f_{\beta\beta} + f_{\beta^*\beta}) + 4N_+ V f_{\alpha\beta}}{8\chi_+^2 \lambda_{1+}}(1 - e^{-\lambda_{1+}t}) \\
&- \frac{16V^2(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + 4K_+N_+(f_{\beta\beta} + f_{\beta^*\beta}) + 8(K_+ + N_+)V f_{\alpha\beta}}{8\chi_+^2 (\lambda_{1+} + \lambda_{2+})}(1 - e^{\frac{-1}{2}(\lambda_{1+}+\lambda_{2+})t}) \\
&+ \frac{4V^2(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + N_-^2(f_{\beta\beta} - f_{\beta^*\beta}) - 4N_- V f_{\alpha\beta}}{8\chi_-^2 (\lambda_{2-})}(1 - e^{(-\lambda_{2-})t}) \\
&+ \frac{4V^2(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + K_-^2(f_{\beta\beta} - f_{\beta^*\beta}) - 4K_- V f_{\alpha\beta}}{8\chi_-^2 \lambda_{1-}}(1 - e^{-\lambda_{1-}t}) \\
&- \frac{16V^2(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + 4K_-N_-(f_{\beta\beta} - f_{\beta^*\beta}) - 8(K_- + N_-)V f_{\alpha\beta}}{8\chi_-^2 (\lambda_{1-} + \lambda_{2-})}(1 - e^{\frac{-1}{2}(\lambda_{1-}+\lambda_{2-})t})
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
\langle \alpha, \beta \rangle =& - \frac{2VK_+(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + 2VN_+(f_{\beta\beta} + f_{\beta^*\beta}) + (K_+N_+ + 4V^2)f_{\alpha\beta}}{8\chi_+^2 \lambda_{2+}}(1 - e^{-\lambda_{2+}t}) \\
&- \frac{2VN_+(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + 2VK_+(f_{\beta\beta} + f_{\beta^*\beta}) + (K_+N_+ + 4V^2)f_{\alpha\beta}}{8\chi_+^2 \lambda_{1+}}(1 - e^{-\lambda_{1+}t}) \\
&+ \frac{4V(K_+ + N_+)(f_{\alpha\alpha} + f_{\alpha^*\alpha}) + 4V(K_+ + N_+)(f_{\beta\beta} + f_{\beta^*\beta}) + 2(K_+^2 + N_+^2 + 8V^2)f_{\alpha\beta}}{8\chi_+^2 (\lambda_{1+} + \lambda_{2+})}(1 - e^{\frac{-1}{2}(\lambda_{1+}+\lambda_{2+} \\
&+ \frac{2VK_-(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + 2VN_-(f_{\beta\beta} - f_{\beta^*\beta}) - (K_-N_- + 4V^2)f_{\alpha\beta}}{8\chi_+^2 \lambda_{2-}}(1 - e^{-\lambda_{2-}t}) \\
&+ \frac{2VN_-(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + 2VK_-(f_{\beta\beta} - f_{\beta^*\beta}) - (K_-N_- + 4V^2)f_{\alpha\beta}}{8\chi_-^2 \lambda_{1-}}(1 - e^{-\lambda_{1-}t}) \\
&- \frac{4V(K_- + N_-)(f_{\alpha\alpha} - f_{\alpha^*\alpha}) + 4V(K_- + N_-)(f_{\beta\beta} - f_{\beta^*\beta}) - 2(K_-^2 + N_-^2 + 8V^2)f_{\alpha\beta}}{8\chi_-^2 (\lambda_{1-} + \lambda_{2-})}(1 - e^{\frac{-1}{2}(\lambda_{1-}+\lambda_{2-}
\end{aligned} \tag{4.32}
$$

$$\begin{aligned}
\langle\alpha^*,\alpha\rangle =&\frac{2|K_+|^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}+f_{\beta^*\beta})+4(K_+^*V+N_+V^*)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{2+}+\lambda_{2+}^*)}(1-e^{\frac{-1}{2}(\lambda_{2+}+\lambda_{2+}^*)t})\\
&+\frac{2|N_+|^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}+f_{\beta^*\beta})+4(K_+V^*+N_+^*V)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}+\lambda_{1+}^*)}(1-e^{\frac{-1}{2}(\lambda_{1+}+\lambda_{1+}^*)t})\\
&-\frac{2K_+^*N_+(f_{\alpha\alpha}+f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}+f_{\beta^*\beta})+4(K_+^*V+N_+V^*)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}+\lambda_{2+}^*)}(1-e^{\frac{-1}{2}(\lambda_{1+}+\lambda_{2+}^*)t})\\
&-\frac{2K_+N_+^*(f_{\alpha\alpha}+f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}+f_{\beta^*\beta})+4(N_+^*V+K_+V^*)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}^*+\lambda_{2+})}(1-e^{\frac{-1}{2}(\lambda_{1+}^*+\lambda_{2+})t})\\
&-\frac{2|K_-|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}-f_{\beta^*\beta})-4(K_-^*V+K_-V^*)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{2-}+\lambda_{2-}^*)}(1-e^{\frac{-1}{2}(\lambda_{2-}+\lambda_{2-}^*)t})\\
&-\frac{2|N_-|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}-f_{\beta^*\beta})-4(N_-^*V+N_-V^*)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}^*+\lambda_{1-})}(1-e^{\frac{-1}{2}(\lambda_{1-}^*+\lambda_{1-})t})\\
&+\frac{2K_-^*N_-(f_{\alpha\alpha}-f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}-f_{\beta^*\beta})-4(K_-^*V+N_-V^*)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}+\lambda_{2-}^*)}(1-e^{\frac{-1}{2}(\lambda_{1-}+\lambda_{2-}^*)t})\\
&+\frac{2K_-N_-^*(f_{\alpha\alpha}-f_{\alpha^*\alpha})+8|V|^2(f_{\beta\beta}+f_{\beta^*\beta})-4(N_-^*V+K_-V^*)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}^*+\lambda_{2-})}(1-e^{\frac{-1}{2}(\lambda_{1-}^*+\lambda_{2-})t})
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\langle\beta^*,\beta\rangle =&\frac{8|V^2|(f_{\alpha\alpha}+f_{\alpha^*\alpha})+2|N_+|^2(f_{\beta\beta}+f_{\beta^*\beta})+2(N_+^*V+N_+V^*)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{2+}+\lambda_{2+}^*)}(1-e^{\frac{-1}{2}(\lambda_{2+}+\lambda_{2+}^*)t})\\
&+\frac{8|V|^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+2|K_+|^2(f_{\beta\beta}+f_{\beta^*\beta})+2(K_+^*V+K_+V^*)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}+\lambda_{1+}^*)}(1-e^{\frac{-1}{2}(\lambda_{1+}+\lambda_{1+}^*)t})\\
&-\frac{8|V|^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+2K_+N_+^*(f_{\beta\beta}+f_{\beta^*\beta})+2(K_+V^*+N_+^*V)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}+\lambda_{2+}^*)}(1-e^{\frac{-1}{2}(\lambda_{1+}+\lambda_{2+}^*)t})\\
&-\frac{8|V|^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+2K_+^*N_+(f_{\beta\beta}+f_{\beta^*\beta})+2(K_+^*V+N_+V^*)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}^*+\lambda_{2+})}(1-e^{\frac{-1}{2}(\lambda_{1+}^*+\lambda_{2+})t})\\
&-\frac{8|V|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+2|N_-|^2(f_{\beta\beta}-f_{\beta^*\beta})-2(N_-^*V+N_-V^*)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{2-}+\lambda_{2-}^*)}(1-e^{\frac{-1}{2}(\lambda_{2-}+\lambda_{2-}^*)t})\\
&-\frac{8|V|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+2|K_-|^2(f_{\beta\beta}-f_{\beta^*\beta})-2(K_-^*V+K_-V^*)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}+\lambda_{1-}^*)}(1-e^{\frac{-1}{2}(\lambda_{1-}+\lambda_{1-}^*)t})\\
&+\frac{8|V|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+2K_-N_-^*(f_{\beta\beta}-f_{\beta^*\beta})-2(K_-V^*+N_-^*V)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}+\lambda_{2-}^*)}(1-e^{\frac{-1}{2}(\lambda_{1-}+\lambda_{2-}^*)t})\\
&+\frac{8|V|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+2K_-^*N_-(f_{\beta\beta}-f_{\beta^*\beta})-2(K_-^*V+N_-V^*)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}^*+\lambda_{2-})}(1-e^{\frac{-1}{2}(\lambda_{1-}^*+\lambda_{2-})t})
\end{aligned} \tag{4.34}$$

$$
\begin{aligned}
\langle\alpha^*,\beta\rangle = &- \frac{4K_+^* V(f_{\alpha\alpha}+f_{\alpha^*\alpha})+4N_+V^*(f_{\beta\beta}+f_{\beta^*\beta})+2(K_+^*N_+ + 4|V|^2)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{2+}+\lambda_{2+}^*)}(1-e^{\frac{-1}{2}(\lambda_{2+}^*+\lambda_{2+})t}) \\
&- \frac{4N_+^* V(f_{\alpha\alpha}+f_{\alpha^*\alpha})+4K_+V^*(f_{\beta\beta}+f_{\beta^*\beta})+2(K_+N_+^* + 4|V|^2)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}+\lambda_{1+}^*)}(1-e^{\frac{-1}{2}(\lambda_{1+}^*+\lambda_{1+})t}) \\
&+ \frac{4K_+^* V(f_{\alpha\alpha}+f_{\alpha^*\alpha})+4K_+V^*(f_{\beta\beta}+f_{\beta^*\beta})+2(|K_+|^2 + 4|V|^2)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}+\lambda_{2+}^*)}(1-e^{\frac{-1}{2}(\lambda_{1+}+\lambda_{2+}^*)t}) \\
&+ \frac{4N_+^* V(f_{\alpha\alpha}+f_{\alpha^*\alpha})+4N_+V^*(f_{\beta\beta}+f_{\beta^*\beta})+2(|N_+|^2 + 4|V|^2)f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}^*+\lambda_{2+})}(1-e^{\frac{-1}{2}(\lambda_{1+}^*+\lambda_{2+})t}) \\
&- \frac{4K_-^* V(f_{\alpha\alpha}-f_{\alpha^*\alpha})+4N_-V^*(f_{\beta\beta}-f_{\beta^*\beta})-2(K_-^*N_- + 4|V|^2)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{-}+\lambda_{2-}^*)}(1-e^{\frac{-1}{2}(\lambda_{2+}^*+\lambda_{2+})t}) \\
&- \frac{4N_-^* V(f_{\alpha\alpha}-f_{\alpha^*\alpha})-4K_-V^*(f_{\beta\beta}-f_{\beta^*\beta})-2(K_-N_-^* + 4|V|^2)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}+\lambda_{1-}^*)}(1-e^{\frac{-1}{2}(\lambda_{1-}^*+\lambda_{1-})t}) \\
&+ \frac{4K_-^* V(f_{\alpha\alpha}-f_{\alpha^*\alpha})+4K_-V^*(f_{\beta\beta}-f_{\beta^*\beta})-2(|K_-|^2 + 4|V|^2f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}+\lambda_{2-}^*)}(1-e^{\frac{-1}{2}(\lambda_{1-}+\lambda_{2-}^*)t}) \\
&+ \frac{4N_-^* V(f_{\alpha\alpha}-f_{\alpha^*\alpha})+4N_-V^*(f_{\beta\beta}-f_{\beta^*\beta})-2(|N_-|^2 + 4|V|^2)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}^*+\lambda_{2-})}(1-e^{\frac{-1}{2}(\lambda_{1-}^*+\lambda_{2-})t})
\end{aligned}
\tag{4.35}
$$

now substitution of Eqs. (4.30), (4.31), (4.32), (4.33), (4.34) and (4.35), and the complex conjugate of Eqs. (4.32) and (4.35) into Eq. (4.16) at steady state leads to

$$
\langle\gamma(t)\pm,\gamma(t)\pm\rangle = \left(
\begin{aligned}
&\frac{(K_+-2V)^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(N_+-2V)^2(f_{\beta\beta}+f_{\beta^*\beta})-2(K_+-2V)(N_+-2V)f_{\alpha\beta}}{16\chi_+^2\lambda_{2+}} \\
&+\frac{(N_+-2V)^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(K_+-2V)^2(f_{\beta\beta}+f_{\beta^*\beta})-2(K_+-2V)(N_+-2V)f_{\alpha\beta}}{16\chi_+^2\lambda_{1+}} \\
&-4\frac{(K_+-2V)(N_+-2V)((f_{\alpha\alpha}+f_{\alpha^*\alpha})+(f_{\beta\beta}+f_{\beta^*\beta}))-((K_+-2V)^2+(N_+-2V)^2)f_{\alpha\beta}}{16\chi_+^2(\lambda_{1+}+\lambda_{2+})} \\
&+\frac{(k_-+2V)^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(K_-+2V)^2(f_{\beta\beta}-f_{\beta^*\beta})-2(K_-+2V)(N_-+2V)f_{\alpha\beta}}{16\chi_-^2\lambda_{2-}} \\
&+\frac{(N_-+2V)^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(N_-+2V)^2(f_{\beta\beta}-f_{\beta^*\beta})-2(K_-+2V)(N_-+2V)f_{\alpha\beta}}{16\chi_-^2\lambda_{1-}} \\
&-4\frac{(K_-+2V)(N_-+2V)((f_{\alpha\alpha}-f_{\alpha^*\alpha})+(f_{\beta\beta}-f_{\beta^*\beta}))-((K_-+2V)^2+(N_-+2V)^2)f_{\alpha\beta}}{16\chi_-^2(\lambda_{1-}+\lambda_{2-})} \\
&\pm\frac{|K_+-2V|^2(f_{\alpha\alpha}+f_{\alpha^*\alpha}+|N_+-2V|^2(f_{\beta\beta}+f_{\beta^*\beta})-((K_+^*-2V^*)(N_+-2V)+(N_+^*-2V^*)(K_+-2V)(f_{\alpha\beta}}{8\chi_+|^2(\lambda_{2+}^*+\lambda_{2+})} \\
&\pm\frac{|N_+-2V|^2(f_{\alpha\alpha}+f_{\alpha^*\alpha}+|K_+-2V|^2(f_{\beta\beta}+f_{\beta^*\beta})-((K_+^*-2V^*)(N_+-2V)+(N_+^*-2V^*)(K_+-2V)(f_{\alpha\beta}}{8\chi_+|^2(\lambda_{1+}^*+\lambda_{1+})} \\
&\mp2\frac{(N_+^*-2V^*)(K_+-2V)((f_{\alpha\alpha}+f_{\alpha^*\alpha})+(f_{\beta\beta}+f_{\beta^*\beta}))-(|K_+-2V|^2)+(|N_+-2V|^2))f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}^*+\lambda_{2+})} \\
&\mp2\frac{(K_+^*-2V^*)(N_+-2V)((f_{\alpha\alpha}+f_{\alpha^*\alpha})+(f_{\beta\beta}+f_{\beta^*\beta}))-(|K_+-2V|^2)+(|N_+-2V|^2))f_{\alpha\beta}}{8|\chi_+|^2(\lambda_{1+}+\lambda_{2+}^*)} \\
&\mp\frac{|K_-+2V|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+|N_-+2V|^2(f_{\beta\beta}-f_{\beta^*\beta})-(K_-^*+2V^*)(N_-+2V)+(N_-^*+2V^*)(K_-+2V)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{2-}^*+\lambda_{2-})} \\
&\mp\frac{|N_-+2V|^2(f_{\alpha\alpha}-f_{\alpha^*\alpha}+2|K_-+2V|^2(f_{\beta\beta}-f_{\beta^*\beta})-((K_-^*+2V^*)(N_-+2V)+(N_-^*+2V^*)(K_-+2V)(f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}^*+\lambda_{1-})} \\
&\pm2\frac{(K_-^*+2V^*)(N_-+2V)((f_{\alpha\alpha}-f_{\alpha^*\alpha}+(f_{\beta\beta}-f_{\beta^*\beta}))-((K_-+2V)^2+(N_-+2V)^2)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}^*+\lambda_2-)} \\
&\pm2\frac{(N_-^*+2V^*)(K_-+2V)(f_{\alpha\alpha}-f_{\alpha^*\alpha}+2(f_{\beta\beta}-f_{\beta^*\beta})-((K_-+2V)^2+(N_-+2V)^2)f_{\alpha\beta}}{8|\chi_-|^2(\lambda_{1-}+\lambda_2^*)} + C.C.
\end{aligned}
\right)
\tag{4.36}
$$

Eq. (3.72) can have a well defined solution if $\lambda_{1\pm} > 0$ and $\lambda_{2\pm} > 0$. Taking into this condition, we have

$$K_{\pm} = K_{\pm}^{*}$$

$$N_{\pm} = N_{\pm}^{*}$$

$$\chi_{\pm} = \chi_{\pm}^{*}$$

in view of this Eq. (4.36) takes the form

$$\langle \gamma(t)\pm, \gamma(t)\pm \rangle = \begin{pmatrix} (1\pm 1)\frac{(K_{+}-2V)^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(N_{+}-2V)^2(f_{\beta\beta}+f_{\beta^*\beta})+2(K_{+}-2V)(N_{+}-2V)f_{\alpha\beta}}{8|\chi_{+}|^2\lambda_{2+}} \\ +(1\pm 1)\frac{(N_{+}-2V)^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(K_{+}-2V)^2(f_{\beta\beta}+f_{\beta^*\beta})+2(K_{+}-2V)(N_{+}-2V)f_{\alpha\beta}}{8|\chi_{+}|^2\lambda_{1+}} \\ -(1\pm 1)4\frac{(K_{+}-2V)(N_{+}-2V)[(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(f_{\beta\beta}+f_{\beta^*\beta})]+[(K_{+}-2V)^2+(N_{+}-2V)^2]f_{\alpha\beta}}{8|\chi_{+}|^2(\lambda_{1+}+\lambda_{2+}}) \\ +(1\mp 1)\frac{(K_{-}+2V)^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(N_{-}+2V)^2(f_{\beta\beta}-f_{\beta^*\beta})-2(K_{-}+2V)(N_{-}+2V)f_{\alpha\beta}}{8|\chi_{-}|^2\lambda_{2-}} \\ +(1\mp 1)\frac{(N_{-}+2V)^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(K_{-}+2V)^2(f_{\beta\beta}-f_{\beta^*\beta})-2(K_{-}+2V)(N_{-}+2V)f_{\alpha\beta}}{8|\chi_{-}|^2\lambda_{1-}} \\ -(1\mp 1)4\frac{(K_{-}+2V)(N_{-}+2V)[(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(f_{\beta\beta}-f_{\beta^*\beta})]-[(K_{-}+2V)^2+(N_{-}+2V)^2]f_{\alpha\beta}}{8|\chi_{-}|^2(\lambda_{1-}+\lambda_{2-})} \end{pmatrix} \tag{4.37}$$

hence on account of Eq. (4.37), (4.13) takes the form

$$\triangle c_{\pm}^2 = 1 + \begin{pmatrix} (1\pm 1)\frac{(K_{+}-2V)^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(N_{+}-2V)^2(f_{\beta\beta}+f_{\beta^*\beta})+2(K_{+}-2V)(N_{+}-2V)f_{\alpha\beta}}{8|\chi_{+}|^2\lambda_{2+}} \\ +(1\pm 1)\frac{(N_{+}-2V)^2(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(K_{+}-2V)^2(f_{\beta\beta}+f_{\beta^*\beta})+2(K_{+}-2V)(N_{+}-2V)f_{\alpha\beta}}{8|\chi_{+}|^2\lambda_{1+}} \\ -(1\pm 1)4\frac{(K_{+}-2V)(N_{+}-2V)[(f_{\alpha\alpha}+f_{\alpha^*\alpha})+(f_{\beta\beta}+f_{\beta^*\beta})]+[(K_{+}-2V)^2+(N_{+}-2V)^2]f_{\alpha\beta}}{8|\chi_{+}|^2(\lambda_{1+}+\lambda_{2+})} \\ +(1\mp 1)\frac{(K_{-}+2V)^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(N_{-}+2V)^2(f_{\beta\beta}-f_{\beta^*\beta})-2(K_{-}+2V)(N_{-}+2V)f_{\alpha\beta}}{8|\chi_{-}|^2\lambda_{2-}} \\ +(1\mp 1)\frac{(N_{-}+2V)^2(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(K_{-}+2V)^2(f_{\beta\beta}-f_{\beta^*\beta})-2(K_{-}+2V)(N_{-}+2V)f_{\alpha\beta}}{8|\chi_{-}|^2\lambda_{1-}} \\ -(1\mp 1)4\frac{(K_{-}+2V)(N_{-}+2V)[(f_{\alpha\alpha}-f_{\alpha^*\alpha})+(f_{\beta\beta}-f_{\beta^*\beta})]-[(K_{-}+2V)^2+(N_{-}+2V)^2]f_{\alpha\beta}}{8|\chi_{-}|^2(\lambda_{1-}+\lambda_{2-})} \end{pmatrix} \tag{4.38}$$

using eqs. (3.80)-(3.85) we obtain

$$X_{\pm} = \sqrt{A^2\eta^2 + (\varepsilon_a - \varepsilon_b)^2 \pm 8A(\varepsilon_a - \varepsilon_b)}$$

$$\lambda_{1\pm} = \frac{1}{2}(2\kappa + A\eta \pm 4(\varepsilon_a + \varepsilon_b) + \sqrt{A^2\eta^2 + (\varepsilon_a - \varepsilon_b)^2 \pm 8A(\varepsilon_a - \varepsilon_b)})$$

$$\lambda_{2\pm} = \frac{1}{2}(2\kappa + A\eta \pm 4(\varepsilon_a + \varepsilon_b) - \sqrt{A^2\eta^2 + (\varepsilon_a - \varepsilon_b)^2 \pm 8A(\varepsilon_a - \varepsilon_b)})$$

again from Eqs. (3.7), (3.8) and (3.16), we get

$$\mu_a + \mu_c = \kappa - A\rho_{aa}^{(0)} + \kappa + A\rho_{cc}^{(0)} = 2\kappa + A(\rho_{cc}^{(0)} - \rho_{aa}^{(0)}) = 2\kappa + A\eta$$

$$\mu_a - \mu_c = \kappa - A\rho_{aa}^{(0)} - (\kappa + A\rho_{cc}^{(0)}) - A(\rho_{aa}^{(0)} + \rho_{cc}^{(0)}) = -A$$

$$V = A\rho_{ac}^{0} = \frac{A\sqrt{1-\eta^2}}{2}$$

and

$$\frac{V}{2} = \frac{A\sqrt{1-\eta^2}}{4}$$

similarly using Eqs. (4.20)-(4.24), we obtain

$$f_{\beta\beta} \pm f_{\beta^*\beta} = -2\varepsilon_b \pm \kappa N$$

and

$$f_{\alpha\beta} = \frac{A\sqrt{1-\eta^2}}{4} + \kappa M$$

$$f_{\alpha\alpha} \pm f_{\alpha^*\alpha} = -2\varepsilon_a \pm \frac{A}{2}(1-\eta) \pm \kappa N$$

using the expression between Eqs. (3.92)-(3.93), we obtain

$$K_{\pm} = A \pm 4(\varepsilon_a - \varepsilon_b) + \sqrt{A^2\eta^2 + (\varepsilon_a - \varepsilon_b)^2 + 8A(\varepsilon_a - \varepsilon_b)})$$

$$N_{\pm} = A \pm 4(\varepsilon_a - \varepsilon_b) - \sqrt{A^2\eta^2 + (\varepsilon_a - \varepsilon_b)^2 - 8A(\varepsilon_a - \varepsilon_b)})$$

on account of this we readily obtain

$$(K_{\pm} \pm 2V) = A \pm 4(\varepsilon_a - \varepsilon_b) + \sqrt{A^2\eta^2 + (\varepsilon_a - \varepsilon_b)^2 \pm 8A(\varepsilon_a - \varepsilon_b)} \pm A\sqrt{1-\eta^2})$$

$$(N_{\pm} \pm 2V) = A \pm 4(\varepsilon_a - \varepsilon_b) - \sqrt{A^2\eta^2 + (\varepsilon_a - \varepsilon_b)^2 \pm 8A(\varepsilon_a - \varepsilon_b)} \pm A\sqrt{1-\eta^2}$$

using the above relations, the simplified form of eq. (4.38) for the plus quadrature when $\varepsilon_a = \varepsilon_b$ can be written as;

$$\begin{aligned}\triangle c_+^2 =& 1 + \frac{1}{4\eta^2}\Bigg[(-2\varepsilon_a + \frac{A}{2}(1-\eta) + \kappa N)\Bigg(\frac{2(1+\eta)[1-\sqrt{1-\eta^2}]}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{2(1-\eta)[1-\sqrt{1-\eta^2}]}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)}\Bigg) \\ &+ (-2\varepsilon_b + \kappa N)\Bigg(\frac{2(1-\eta)[1-\sqrt{1-\eta^2}]}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{2(1+\eta)[1-\sqrt{1-\eta^2}]}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)}\Bigg) \\ &+ 4(1-\eta^2 - \sqrt{1-\eta^2})(\frac{A\sqrt{1-\eta^2}}{4} + \kappa M)\Bigg(\frac{1}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{1}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)}\Bigg) \\ &- \frac{16}{2\kappa + A\eta + 4(\varepsilon_a + \varepsilon_b)}\Bigg((1-\eta^2 - \sqrt{1-\eta^2})(-\varepsilon_a - \varepsilon_b + \frac{A}{4}(1-\eta) + \kappa N) \\ &+ (1 - \sqrt{1-\eta^2})(\frac{A\sqrt{1-\eta^2}}{4} + \kappa M)\Bigg)\Bigg]\end{aligned} \quad (4.39)$$

this represents the the quadrature variances of the cavity modes for a non-degenerate three-level laser whose cavity contains two parametric amplifiers coupled to a two-mode squeezed vacuum reservoir.

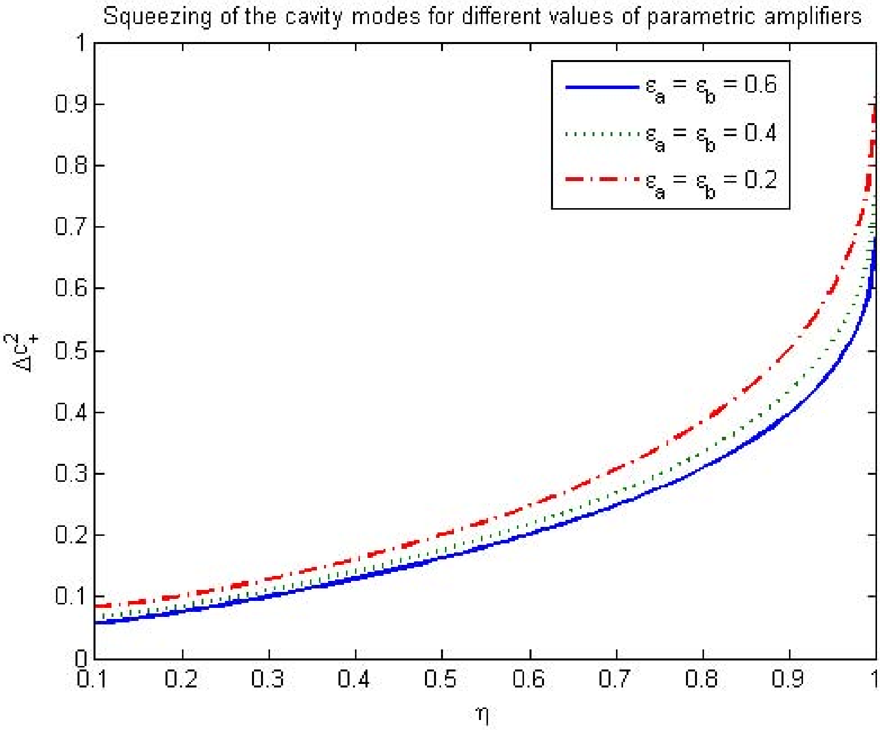

Fig. 4.1: Plots of the quadrature variances $[Eq.(4.39)]$ versus η for A=100, $\kappa = 0.8$, r= 0.6 and for, $\varepsilon_a = \varepsilon_b = 0.6$ (solid curve), $\varepsilon_a = \varepsilon_b = 0.4$ (dotted curve), and $\varepsilon_a = \varepsilon_b = 0.2$ (dash-dotted curve).

From fig 4.1 we easily see that squeezed light can be produced by the system under consideration. Moreover, the minimum value of the quadrature variance is found to be $\Delta c_+^2 = 0.05731$ and occurs at $\eta = 0.1$. This result implies that the maximum intracavity for the given values is 94.27% below the coherent state level.The degree of squeezing for the system under consideration increases with the amplitude of the parametric amplifiers.

Moreover, the squeezing of the cavity modes increases as the parameter r in the squeezed vacuum reservoir has increased.

We next consider the special case in which the parametric amplifier is removed from the cavity. Thus up on setting $\varepsilon_a = \varepsilon_b = 0$ we get from Eq. (4.39) that

$$\begin{aligned}\triangle c_\pm^2 =& 1 + \frac{A(1-\eta)(2\kappa + 2A\eta + A) - 2A^2\eta^2 N}{2(\kappa + A\eta)(2\kappa + A\eta)} \pm \frac{A\sqrt{1-\eta^2}[2\kappa + A\eta + A + 2A(N \mp M)]}{2(\kappa + A\eta)(2\kappa + A\eta)} \\ &+ \frac{2[(2\kappa + A\eta)^2(N \pm M) + A^2(N \mp M)]}{2(\kappa + A\eta)(2\kappa + A\eta)} \end{aligned} \tag{4.40}$$

this is quadrature variances of the cavity modes for a non-degenerate three-level laser whose cavity is coupled with a two-mode squeezed vacuum reservoir.

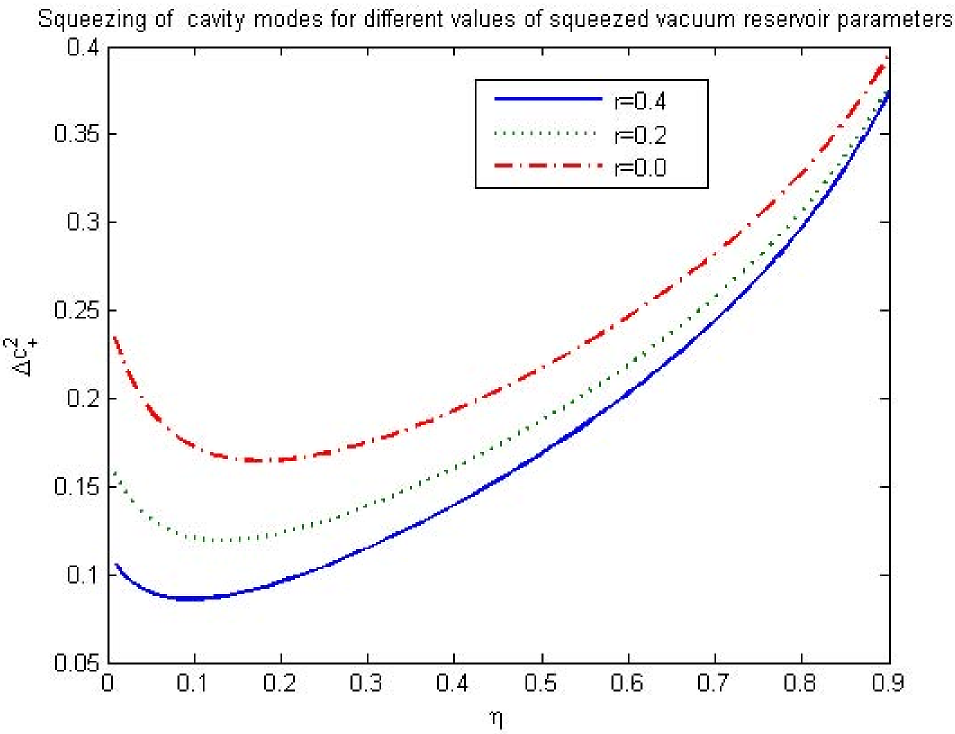

Fig. 4.2: Plots of the quadrature variances $[Eq.(4.39)$ versus η for A=50, $\kappa = 0.8$, $\varepsilon_a = \varepsilon_b = 0.6$, and for, $r = 0.4$ (solid curve), $r = 0.2$ (dotted curve), and $r = 0$ (dash-dotted curve)

4.2 Quadrature variances of the output modes

The squeezing properties of the output modes are described by quadrature operators:

$$\hat{c}_{\pm}^{out} = \sqrt{\pm 1}(\hat{c}_{out}^{+} \pm \hat{c}_{out}), \tag{4.41}$$

$$\hat{c}_{out} = \frac{1}{\sqrt{2}}(\hat{a}_{out} + \hat{b}_{out}) \tag{4.42}$$

From Eqs. (4.41) and (4.42) one can obtain

$$\triangle \hat{c}_{\pm out}^{2} = 1 \pm \langle : \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out} : \rangle. \tag{4.43}$$

SO, Eq. (4.43)can be written in the normal ordering as

$$\triangle \hat{c}_{\pm out}^{2} = 1 \pm \langle : \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out} : \rangle. \tag{4.44}$$

This can be expressed in terms of c-number variables associated with normal ordering as

$$\triangle \hat{c}_{\pm out}^{2}(t) = 1 \pm \langle \gamma_{\pm}^{out}, \gamma_{\pm}^{out} \rangle \tag{4.45}$$

where $\gamma(t)$ is the c-number corresponding to the operator $\hat{c}(t)$.
Introducing anew variable defined by

$$\gamma_{\pm}^{out}(t) = \sqrt{k}\gamma_{\pm} - \gamma_{\pm}^{in} \tag{4.46}$$

Eq.(4.45) can be rewritten as

$$\triangle\hat{c}^2_{\pm out}(t) = 1 + k\triangle\hat{c}^2_{\pm} \pm \langle\gamma_{\pm}^{in}, \gamma_{\pm}^{in}\rangle - k(1 \pm \frac{2}{\sqrt{k}}\langle\gamma_{\pm}, \gamma_{\pm}^{in}\rangle) \tag{4.47}$$

where

$$\langle\gamma_{\pm}, \gamma_{\pm}^{in}\rangle_{ss} = \sqrt{k}(M \pm N), \langle\gamma_{\pm}^{in}, \gamma_{\pm}^{in}\rangle = 2(N \pm M) \tag{4.48}$$

hence on account of these two equations, we obtain

$$\triangle\hat{c}^2_{\pm out} = k\triangle\hat{c}^2_{\pm} + (1-k)(1 + 2N \pm 2M) \tag{4.49}$$

the first and second terms on the right side of Eq. (4.49) represent the quadrature variances of the transmitted cavity modes and reflected input modes. Similar to the cavity modes, the output modes are squeezed for the plus quadrature for which the simplified form of Eq. (4.49) will take the form,

$$\begin{aligned}
\triangle c^2_{+out} =& \kappa + \frac{\kappa}{4\eta^2}\Bigg[(-2\varepsilon_a + \frac{A}{2}(1-\eta) + \kappa N)\Bigg(\frac{2(1+\eta)[1-\sqrt{1-\eta^2}]}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{2(1-\eta)[1-\sqrt{1-\eta^2}]}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)}\Bigg) \\
&+ (-2\varepsilon_b + \kappa N)\Bigg(\frac{2(1-\eta)[1-\sqrt{1-\eta^2}]}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{2(1+\eta)[1-\sqrt{1-\eta^2}]}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)}\Bigg) \\
&+ 4(1-\eta^2 - \sqrt{1-\eta^2})(\frac{A\sqrt{1-\eta^2}}{4} + \kappa M)\Bigg(\frac{1}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{1}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)}\Bigg) \\
&- \frac{16}{2\kappa + A\eta + 4(\varepsilon_a + \varepsilon_b)}\Bigg((1-\eta^2 - \sqrt{1-\eta^2})(-\varepsilon_a - \varepsilon_b + \frac{A}{4}(1-\eta) + \kappa N) \\
&+ (1-\sqrt{1-\eta^2})(\frac{A\sqrt{1-\eta^2}}{4} + \kappa M)\Bigg)\Bigg] + (1-k)(1 + 2N + 2M)
\end{aligned} \tag{4.50}$$

From this we easily see that the squeezing of the cavity modes is greater than the squeezing of output modes.

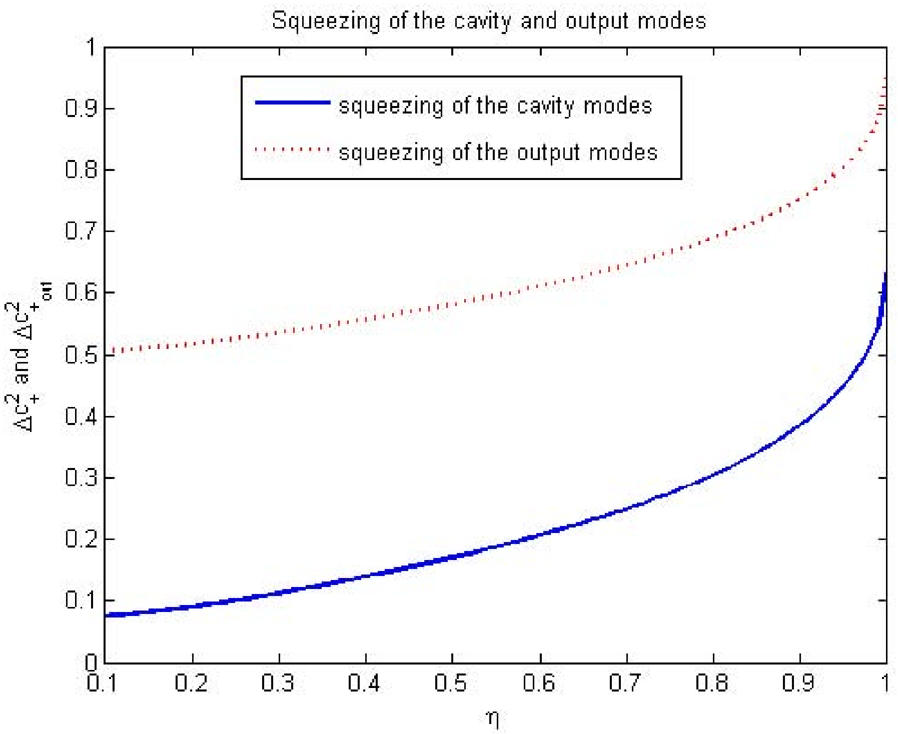

Fig. 4.3: The graph of the quadrature variances for the **cavity modes** $[Eq.(4.39)]$ and **output modes**$[Eq.(4.50)]$versus η for A= 100, $\kappa = 0.8$, $r = 0.4$ and for $\varepsilon_a = \varepsilon_b = 0.6$ (solid curve) and (dotted curve) respectively.

4.3 Squeezing spectrum of the out put modes

The squeezing spectrum of the out put modes can be expressed in terms of c-number variables associated with the normal ordering as

$$S_{\pm}^{out}(\omega) = 1 \pm 2R_e \int d\tau e^{i(\omega-\omega_o)\tau} \langle \gamma_{\pm}^{out}, \gamma_{\pm}^{out}(t+\tau)\rangle_{ss}, \tag{4.51}$$

where the subscript "ss" stands for steady state. Taking into account the input-output relation

$$\gamma_{\pm}^{out}(t) = \sqrt{\kappa}\gamma_{\pm}(t) - \gamma_{\pm}^{in}(t), \tag{4.52}$$

and the fact that

$$\langle \gamma_{\pm}^{in}(t)\rangle_{ss} = \langle \gamma_{\pm}^{in}(t+\tau)\rangle_{ss} = \langle \gamma_{\pm}(t)\gamma_{\pm}(t)\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} = 0, \tag{4.53}$$

we can write

$$S_{\pm}^{out}(\omega) = 1 \pm 2\kappa R_e \int d\tau e^{i(\omega-\omega_o)\tau} \langle \gamma_{\pm}, \gamma_{\pm}(t+\tau)\rangle_{ss}, \tag{4.54}$$

We next proceed to obtain the explicit forms of the two time correlation functions involved in Eq.(3.54).On account of Eq.(4.2)

$$\begin{aligned}\langle\gamma_\pm(t),\gamma_\pm(t+\tau)\rangle_{ss} =&\frac{1}{2}(\langle\alpha(t),\alpha(t+\tau)\rangle_{ss}+\langle\alpha(t),\beta(t+\tau)\rangle_{ss}+\langle\beta(t),\alpha(t+\tau)\rangle_{ss}\\&+\langle\beta(t),\beta(t+\tau)\rangle_{ss}\pm\langle\alpha^*(t),\alpha(t+\tau)\rangle_{ss}\pm\langle\beta^*(t),\alpha(t+\tau)\rangle_{ss}\\&\pm\langle\alpha(t),\beta^*(t+\tau)\rangle_{ss}+\langle\beta(t),\beta^*(t+\tau)\rangle_{ss})+C.C\end{aligned} \tag{4.55}$$

taking into account $\lambda_1=\lambda_1^*$, Eq. (3.55) turns out to be

$$\begin{aligned}\langle\gamma_\pm,\gamma_\pm(t+\tau)\rangle_{ss} =&(\langle\alpha(t),\alpha(t+\tau)\rangle_{ss}+\langle\alpha(t),\beta(t+\tau)\rangle_{ss}+\langle\beta(t),\alpha(t+\tau)\rangle_{ss}\\&+\langle\beta(t),\beta(t+\tau)\rangle_{ss}\pm\langle\alpha^*(t),\alpha(t+\tau)\rangle_{ss}\pm\langle\beta^*(t),\alpha(t+\tau)\rangle_{ss}\\&\pm\langle\alpha(t),\beta^*(t+\tau)\rangle_{ss}+\langle\beta(t),\beta^*(t+\tau)\rangle_{ss}).\end{aligned} \tag{4.56}$$

Moreover, in view of Eqs. (3.109) and (3.110), we see that

$$\begin{aligned}\alpha(t+\tau) =&\frac{1}{2}\{Y_1^+(t+\tau)\alpha(t)+Y_1^-(t+\tau)\alpha^*(t)+W_1^+(t+\tau)\beta(t)+W_1^-(t+\tau)\beta^*(t)\\&+H_1(t+\tau)\}\end{aligned} \tag{4.57}$$

$$\begin{aligned}&\beta(t+\tau)=\frac{1}{2}\{Y_2^+(t+\tau)\beta(t)+Y_2^-(t+\tau)\beta^*(t)+W_2^+(t+\tau)\alpha(t)+W_2^-(t+\tau)\alpha^*(t)\\&+H_2(t+\tau)\},\end{aligned} \tag{4.58}$$

where

$$\begin{aligned}H_1(t+\tau)=&\int_0^\tau\{Y_1^+(\tau-\tau')f_\alpha(t+\tau')+Y_1^-(\tau-\tau')f_\alpha^*(t+\tau)\\&+W_1^+(\tau-\tau')f_\beta(t+\tau')+W_1^-(\tau-\tau')f_\beta^*(t+\tau')\}dt'\end{aligned} \tag{4.59}$$

$$\begin{aligned}H_2(t+\tau)=&\int_0^\tau\{Y_2^+(\tau-\tau')f_\beta(t+\tau')+Y_2^-(\tau-\tau')f_\beta^*(t+\tau)\\&+W_2^+(\tau-\tau')f_\alpha(t+\tau')+W_2^-(\tau-\tau')f_\alpha^*(t+\tau')\}dt'\end{aligned} \tag{4.60}$$

with the aid of (4.57) and (4.58), we have

$$\begin{aligned}\langle\alpha(t),\alpha(t+\tau)\rangle_{ss} =&\frac{1}{2}\{Y_1^+(\tau)\langle\alpha(t)\alpha(t)\rangle_{ss}+Y_1^-(\tau)\langle\alpha^*(t)\alpha(t)\rangle_{ss}+W_1^+(\tau)\langle\alpha(t)\beta(t)\rangle_{ss}\\&+W_1^-(\tau)\langle\alpha(t)\beta^*(t)\rangle_{ss}+\langle\alpha(t)H_1(t+\tau)\rangle_{ss}\},\end{aligned} \tag{4.61}$$

$$\begin{aligned}\langle\alpha(t),\beta(t+\tau)\rangle_{ss} =&\frac{1}{2}\{Y_2^+(\tau)\langle\alpha(t)\beta(t)\rangle_{ss}+Y_2^-(\tau)\langle\alpha(t)\beta^*(t)\rangle_{ss}+W_2^+(\tau)\langle\alpha(t)\alpha(t)\rangle_{ss}\\&+W_2^-(\tau)\langle\alpha(t)\alpha^*(t)\rangle_{ss}+\langle\alpha(t)H_2(t+\tau)\rangle_{ss}\},\end{aligned} \tag{4.62}$$

$$\langle\beta(t),\alpha(t+\tau)\rangle_{ss} =\frac{1}{2}\{Y_1^{+}(\tau)\langle\beta(t)\alpha(t)\rangle_{ss}+Y_1^{-}(\tau)\langle\beta(t)\alpha^*(t)\rangle_{ss}+W_1^{+}(\tau)\langle\beta(t)\beta(t)\rangle_{ss} + W_1^{-}(\tau)\langle\beta(t)\beta^*(t)\rangle_{ss}+\langle\beta(t)H_1(t+\tau)\rangle_{ss}\}, \tag{4.63}$$

$$\langle\beta(t),\beta(t+\tau)\rangle_{ss} =\frac{1}{2}\{Y_2^{+}(\tau)\langle\beta(t)\beta(t)\rangle_{ss}+Y_2^{-}(\tau)\langle\beta(t)\beta^*(t)\rangle_{ss}+W_2^{+}(\tau)\langle\beta(t)\alpha(t)\rangle_{ss} + W_2^{-}(\tau)\langle\beta(t)\alpha^*(t)\rangle_{ss}+\langle\beta(t)H_2(t+\tau)\rangle_{ss}\}, \tag{4.64}$$

$$\langle\alpha^*(t),\alpha(t+\tau)\rangle_{ss} =\frac{1}{2}\{Y_1^{+}(\tau)\langle\alpha^*(t)\alpha(t)\rangle_{ss}+Y_1^{-}(\tau)\langle\alpha^*(t)\alpha^*(t)\rangle_{ss}+W_1^{+}(\tau)\langle\alpha^*(t)\beta(t)\rangle_{ss} + W_1^{-}(\tau)\langle\alpha^*(t)\beta^*(t)\rangle_{ss}+\langle\alpha^*(t)H_1(t+\tau)\rangle_{ss}\}, \tag{4.65}$$

$$\langle\beta^*(t),\alpha(t+\tau)\rangle_{ss} =\frac{1}{2}\{Y_1^{+}(\tau)\langle\beta^*(t)\alpha(t)\rangle_{ss}+Y_1^{-}(\tau)\langle\beta^*(t)\alpha^*(t)\rangle_{ss}+W_1^{+}(\tau)\langle\beta^*(t)\beta(t)\rangle_{ss} + W_1^{-}(\tau)\langle\beta^*(t)\beta^*(t)\rangle_{ss}+\langle\beta^*(t)H_1(t+\tau)\rangle_{ss}\}, \tag{4.66}$$

$$\langle\alpha(t),\beta^*(t+\tau)\rangle_{ss} =\frac{1}{2}\{Y_2^{+*}(\tau)\langle\alpha(t)\beta^*(t)\rangle_{ss}+Y_2^{-*}(\tau)\langle\alpha(t)\beta(t)\rangle_{ss}+W_2^{+*}(\tau)\langle\alpha^*(t)\alpha(t)\rangle_{ss} + W_2^{-*}(\tau)\langle\alpha(t)\alpha(t)\rangle_{ss}+\langle\alpha(t)H_2^*(t+\tau)\rangle_{ss}\}, \tag{4.67}$$

$$\langle\beta(t),\beta^*(t+\tau)\rangle_{ss} =\frac{1}{2}\{Y_2^{+*}(\tau)\langle\beta^*(t)\beta(t)\rangle_{ss}+Y_2^{-*}(\tau)\langle\beta(t)\beta(t)\rangle_{ss}+W_2^{+*}(\tau)\langle\beta(t)\alpha^*(t)\rangle_{ss} + W_2^{-*}(\tau)\langle\beta(t)\alpha(t)\rangle_{ss}+\langle\beta(t)H_2^*(t+\tau)\rangle_{ss}\}, \tag{4.68}$$

so that combination of Eqs. (4.56) and (4.61)-(4.68) yields

$$\begin{aligned}\langle\gamma_\pm(t),\gamma_\pm(t+\tau)\rangle_{ss} =&\frac{1}{2}x_\pm(Y_1^{+}(\tau)+W_2^{+}(\tau)\pm Y_1^{-}(\tau)\pm W_2^{-}(\tau))\\ &+\frac{1}{2}y_\pm(W_1^{+}(\tau)+Y_2^{+}(\tau)\pm W_1^{-}(\tau)\pm Y_2^{-}(\tau))\\ &+\frac{1}{2}z_\pm(Y_2^{+}(\tau)+Y_1^{+}(\tau)+W_2^{+}(\tau)+W_1^{+}(\tau))\\ &\pm\frac{1}{2}y_\pm(Y_2^{-}(\tau)+Y_1^{-}(\tau)+W_2^{-}(\tau)+W_1^{-}(\tau))\end{aligned} \tag{4.69}$$

where

$$x_\pm=\langle\alpha(t)\alpha(t)\rangle_{ss}\pm\langle\alpha^*(t)\alpha(t)\rangle_{ss}, \tag{4.70}$$

$$y_\pm=\langle\beta(t)\beta(t)\rangle_{ss}\pm\langle\beta^*(t)\beta(t)\rangle_{ss}, \tag{4.71}$$

$$z_\pm=\langle\alpha(t)\beta(t)\rangle_{ss}\pm\langle\alpha^*(t)\beta(t)\rangle_{ss}. \tag{4.72}$$

On account of Eqs. (4.30), we can write as

$$\begin{aligned} x_{\pm} =& \frac{K_{\pm}^2(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} \pm f_{\beta^*\beta}) + 4K_{\pm}Vf_{\alpha\beta}}{4\chi_{\pm}^2\lambda_{2\pm}} + \frac{N_{\pm}^2(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} \pm f_{\beta^*\beta}) + 4N_{\pm}Vf_{\alpha\beta}}{4\chi_{\pm}^2\lambda_{1+}} \\ &- \frac{4K_{\pm}N_{\pm}(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + 16V^2(f_{\beta\beta} \pm f_{\beta^*\beta}) + 8(K_{\pm} + N_{\pm})Vf_{\alpha\beta}}{\chi_{\pm}^2 * (\lambda_{2\pm} + \lambda_{1\pm})} \end{aligned} \quad (4.73)$$

$$\begin{aligned} y_{\pm} =& \frac{4V^2(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + N_{\pm}^2(f_{\beta\beta} \pm f_{\beta^*\beta}) + 4N_{\pm}Vf_{\alpha\beta}}{4\chi_{\pm}^2\lambda_{2\pm}} + \frac{4V^2(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + K_{\pm}^2(f_{\beta\beta} \pm f_{\beta^*\beta}) + 4K_{\pm}Vf_{\alpha\beta}}{4\chi_{\pm}^2\lambda_{1\pm}} \\ &- \frac{16V^2(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + 4K_{\pm}N_{\pm}(f_{\beta\beta} \pm f_{\beta^*\beta}) \pm 8(K_{\pm} + N_{\pm})Vf_{\alpha\beta}}{\chi_{\pm}^2(\lambda_{2\pm} + \lambda_{1\pm})} \end{aligned} \quad (4.74)$$

$$\begin{aligned} z_{\pm} =& \frac{K_{\pm}^2(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} \pm f_{\beta^*\beta}) + 4K_{\pm}Vf_{\alpha\beta}}{4\chi_{\pm}^2\lambda_{2\pm}} + \frac{N_{\pm}^2(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + 4V^2(f_{\beta\beta} \pm f_{\beta^*\beta}) + 4N_{\pm}Vf_{\alpha\beta}}{4\chi_{\pm}^2\lambda_{1\pm}} \\ &- \frac{4K_{\pm}N_{\pm}(f_{\alpha\alpha} \pm f_{\alpha^*\alpha}) + 16V^2(f_{\beta\beta} \pm f_{\beta^*\beta}) \pm 8(K_{\pm} + N_{\pm})Vf_{\alpha\beta}}{\chi_{\pm}^2(\lambda_{2\pm} + \lambda_{1\pm})} \end{aligned} \quad (4.75)$$

so that combination of Eqs. (4.73), (4.74) and (4.75) in Eq. (4.69) we can write as

$$\begin{aligned} \langle \gamma_{\pm}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} =& x_{\pm}[(1 \pm 1)(Y_{1+}(\tau) + W_{2+}(\tau)) + (1 \mp 1)(Y_{1-}(\tau) + W_{2-}(\tau))] \\ &+ y_{\pm}[(1 \pm 1)(W_{1+}(\tau) + Y_{2+}(\tau)) + (1 \mp 1)(W_{1-}(\tau)Y_{2-}(\tau))] \\ &+ z_{\pm}[(1 \pm 1)(e^{-\frac{1}{2}\lambda_{2+}\tau} + e^{-\frac{1}{2}\lambda_{1+}\tau}) + (1 \mp 1)(e^{-\frac{1}{2}\lambda_{2-}\tau} + e^{-\frac{1}{2}\lambda_{1-}\tau})] \end{aligned} \quad (4.76)$$

in view of Eq. (3.99), (3.100), (3.101) and (3.102), Eq. (4.76) becomes

$$\begin{aligned} \langle \gamma_{\pm}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} =& (1 \pm 1)\frac{(x_+(K_+ - 2V) - y_+(N_+ - 2V) + 2y_+\chi_+)}{2\chi_+}e^{-\frac{1}{2}\lambda_{2+}\tau} \\ &- (1 \pm 1)\frac{(x_+(N_+ - 2V) - y_+(K_+ - 2V) + 2y_+\chi_+)}{2\chi_+}e^{-\frac{1}{2}\lambda_{1+}\tau} \\ &- (1 \mp 1)\frac{(x_-(K_- + 2V) - y_-(N_- + 2V) + 2y_-\chi_+)}{2\chi_-}e^{-\frac{1}{2}\lambda_{2-}\tau} \\ &- (1 \mp 1)\frac{(x_-(N_- + 2V) - y_-(K_- + 2V) + 2y_-\chi_+)}{2\chi_-}e^{-\frac{1}{2}\lambda_{1-}\tau} \end{aligned} \quad (4.77)$$

substituting Eq. (4.77) into Eq. (4.54),we have

$$\begin{aligned} S_{\pm}^{out}(\omega) =& 1 + 2\kappa R_e\int_0^{\infty} d\tau\Bigg((1 \pm 1)\frac{(x_+(K_+ - 2V) - y_+(N_+ - 2V) + 2y_+\chi_+)}{2\chi_+}e^{\frac{-(\lambda_{2+} - 2i(\omega-\omega0))\tau}{2}} \\ &- (1 \pm 1)\frac{(x_+(N_+ - 2V) - y_+(K_+ - 2V) + 2y_+\chi_+)}{2\chi_+}e^{\frac{-(\lambda_{1+} - 2i(\omega-\omega0))\tau}{2}} \\ &- (1 \mp 1)\frac{(x_-(K_- + 2V) - y_-(N_- + 2V) + 2y_-\chi_+)}{2\chi_-}e^{\frac{-(\lambda_{2-} - 2i(\omega-\omega0))\tau}{2}} \\ &- (1 \mp 1)\frac{(x_-(N_- + 2V) - y_-(K_- + 2V) + 2y_-\chi_+)}{2\chi_-}e^{\frac{-(\lambda_{1-} - 2i(\omega-\omega0))\tau}{2}}\Bigg) \end{aligned} \quad (4.78)$$

performing the integrating and taking real of the above equation gives

$$
\begin{aligned}
S_{\pm}^{out}(\omega) =& 1 + (1 \pm 1)2\kappa\lambda_{2+}\frac{x_+(K_+ - 2V) - y_+(N_+ - 2V) + 2y_+\chi_+}{\chi_+(\lambda_{2+}^2 + 4(\omega - \omega_0)^2)} \\
&- (1 \pm 1)2\kappa\lambda_{1+}\frac{x_+(N_+ - 2V) - y_+(K_+ - 2V) + 2y_+\chi_+}{\chi_+(\lambda_{1+}^2 + 4(\omega - \omega_0)^2)} \\
&- (1 \mp 1)2\kappa\lambda_{2-}\frac{x_-(K_- + 2V) - y_-(N_- + 2V) + 2y_-\chi_+}{\chi_-(\lambda_{2-}^2 + 4(\omega - \omega_0)^2)} \\
&+ (1 \mp 1)2\kappa\lambda_{1-}\frac{x_-(N_- + 2V) - y_-(K_- + 2V) + 2y_-\chi_+}{\chi_-(\lambda_{1-}^2 + 4(\omega - \omega_0)^2)}
\end{aligned} \tag{4.79}
$$

for $\omega = \omega_0$, Eq. (4.79)turns out to be

$$
\begin{aligned}
S_{\pm}^{out}(\omega) =& 1 + 4(1 \pm 1)\frac{x_+(K_+ - 2V) - y_+(N_+ - 2V) + 2y_+\chi_+}{2\chi_+\lambda_{2+}} \\
&- (1 \pm 1)\frac{x_+(N_+ - 2V) - y_+(K_+ - 2V) + 2y_+\chi_+}{2\chi_+\lambda_{1+}} \\
&- (1 \mp 1)\frac{x_-(K_- + 2V) - y_-(N_- + 2V) + 2y_-\chi_+}{2\chi_-\lambda_{2-}} \\
&+ (1 \mp 1)\frac{x_-(N_- + 2V) - y_-(K_- + 2V) + 2y_-\chi_+}{2\chi_-\lambda_{1-}}
\end{aligned} \tag{4.80}
$$

following the same procedure, as Eq. (4.39) the squeezing spectrum of the output modes for the system under consideration when $\varepsilon_a = \varepsilon_b$ at steady state can be written as

$$
\begin{aligned}
S_{\pm}^{out}(\omega) =& 1+ \\
&\frac{1}{\eta^3}\Bigg[(-2\varepsilon_a + \frac{A}{2}(1+\eta) + \kappa N)(\frac{2\eta}{\kappa + 2(\varepsilon_a + \varepsilon_b)} - \frac{(1 + \eta + \sqrt{1-\eta^2})}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)}) \\
&\left(\frac{1+\eta}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{1-\eta}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)} - \frac{16(1-\eta^2)}{2\kappa + A\eta + 4(\varepsilon_a + \varepsilon_b)}\right) \\
&+ (-2\varepsilon_b + \kappa N)(\frac{1 + \eta + \sqrt{1-\eta^2}}{\kappa + 2(\varepsilon_a + \varepsilon_b)})\left(\frac{1-\eta}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{1+\eta}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)} + \frac{40\sqrt{1-\eta^2}}{\kappa + A\eta + 4(\varepsilon_a + \varepsilon_b)}\right) \\
&+ (\frac{A\sqrt{1-\eta^2}}{4} + \kappa M)(\frac{1 + \eta + \sqrt{1-\eta^2}}{\kappa + 2(\varepsilon_a + \varepsilon_b)})\Bigg(\frac{\sqrt{1-\eta^2}}{\kappa + 2(\varepsilon_a + \varepsilon_b)} + \frac{2\sqrt{1-\eta^2}}{\kappa + A\eta + 2(\varepsilon_a + \varepsilon_b)} \\
&+ \frac{32\sqrt{1-\eta^2}}{\kappa + A\eta + 4(\varepsilon_a + \varepsilon_b)}\Bigg)\Bigg]
\end{aligned} \tag{4.81}
$$

this is the expression describing the squeezing spectrum of the output modes for a three level laser whose cavity contains parametric amplifiers coupled with a squeezed vacuum reservoir.

From fig.4.5 we see that the squeezing spectrum increases with the amplitude of the parametric amplifiers for the system under consideration.

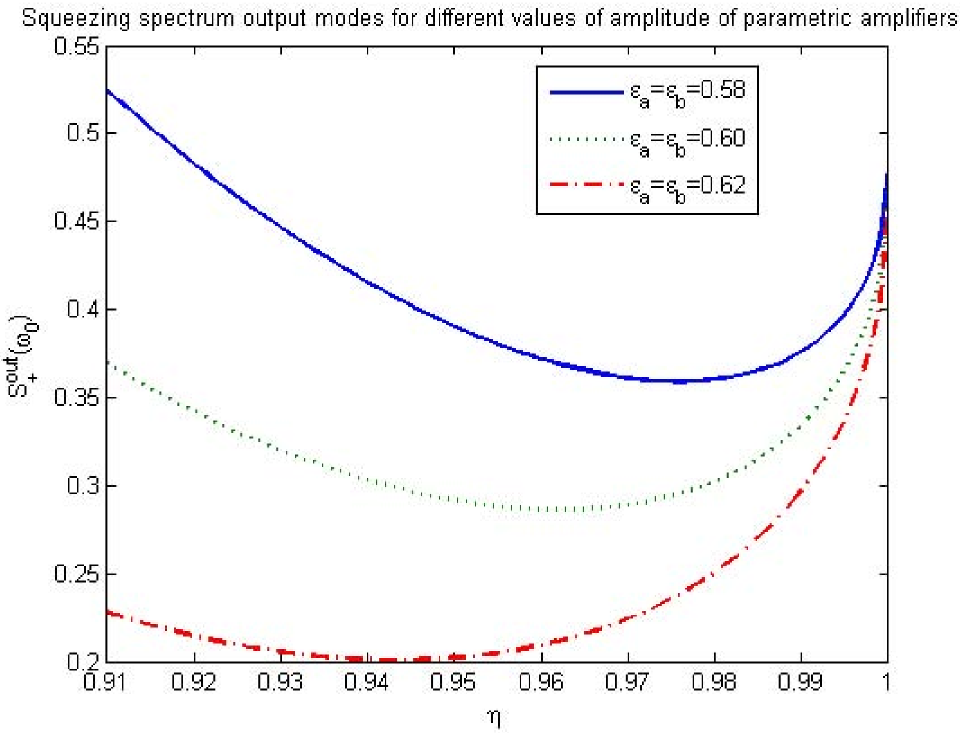

Fig. 4.4: Plots of the squeezing spectrum $[Eq.(4.81)]$ versus η for the values of $A = 1$, $\kappa = 0.75$, $r = 0.6$ and for, $\varepsilon_a = \varepsilon_b = 0.58$ (solid curve), $\varepsilon_a = \varepsilon_b = 0.60$ (dotted curve), and $\varepsilon_a = \varepsilon_b = 0.62$ (dash-dotted curve).

5

Quadrature Entanglement

In this chapter we seek to study the entanglement properties of the cavity as well as output modes using the criterion developed by Duan et al.[25]

5.1 Entanglement of the cavity modes

A quantum state of a system, ρ, of two modes a and b is said to be separable if and only if

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i^a \bigotimes \hat{\rho}_i^b \tag{5.1}$$

suppose $\hat{\rho}_i^a$ and $\hat{\rho}_i^b$ to be normalized density operators of mode a and b, respectively with $p_i \geq 0$ and $\sum p_i = 1$. Otherwise it is said to be **entangled**. Based on the inseparability of the system density matrix, entanglement criteria for continuous variable has been proposed by some authors. One of the criteria to verify the entanglement between two modes in a cavity is the Duan et al. criterion. A maximally entangled continuous variable states can be expressed as a co-eigenstate of a pair of EPR (Einstein, Podosky and Rosen) like operators such as $\hat{x_a} - \hat{x_b}$ and $\hat{p_a} + \hat{p_b}$. Therefore the sum of the variance of these two operators reduces to zero for maximally entangled continuous variable states [26]. Moreover, according to the criterion set by Duan et al. [25], quantum states of a system are said to be entangled, provided that the sum of the the variances of a pair of EPR-like operators $\hat{u}$ and $\hat{v}$ satisfy the inequality

$$\Delta u^2 + \Delta v^2 < 2. \tag{5.2}$$

where

$$\hat{u} = \hat{x_a} - \hat{x_b} \tag{5.3}$$

and

$$\hat{v} = \hat{p_a} + \hat{p_b} \tag{5.4}$$

with

$$\hat{x}_a = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \tag{5.5}$$

$$\hat{x}_b = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger), \tag{5.6}$$

$$\hat{p}_a = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}), \tag{5.7}$$

$$\hat{p}_b = \frac{i}{\sqrt{2}}(\hat{b}^\dagger - \hat{b}), \tag{5.8}$$

being the quadrature operators for the cavity modes $\hat{a}$ and $\hat{b}$.
The variances of the EPR like operators described by Eqs. (5.3) and (5.4) are

$$\Delta u^2 = \langle \hat{u}^2 \rangle - \langle \hat{u} \rangle^2, \tag{5.9}$$

$$\Delta v^2 = \langle \hat{v}^2 \rangle - \langle \hat{v} \rangle^2. \tag{5.10}$$

Then from the combination of Eqs. (5.9) and (5.10), the sum of the variances of these operators will take the form

$$\Delta u^2 + \Delta v^2 = (\langle \hat{u}^2 \rangle + \langle \hat{v}^2 \rangle) - (\langle \hat{u} \rangle^2 + \langle \hat{v} \rangle^2). \tag{5.11}$$

In terms of Eqs. (5.5), (5.6), (5.7) and (5.8), the EPR-like operators can be rewritten as

$$\hat{u} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger - \hat{b}^\dagger + \hat{a} - \hat{b}) \tag{5.12}$$

and

$$\hat{v} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger + \hat{b}^\dagger - \hat{a} - \hat{b}). \tag{5.13}$$

As eqs. (5.12) and (5.13) are in normal ordering, the corresponding c-number equations will be

$$\hat{u} = \frac{1}{\sqrt{2}}\left[(\hat{\alpha}^* - \hat{\beta}^*) + (\hat{\alpha} - \hat{\beta})\right] \tag{5.14}$$

and

$$\hat{v} = \frac{i}{\sqrt{2}}\left[(\hat{\alpha}^* + \hat{\beta}^*) - (\hat{\alpha} + \hat{\beta})\right] \tag{5.15}$$

we recall that equations (3.17) and (3.18) are expressed as

$$\frac{d}{dt}\langle\alpha\rangle = -2\varepsilon_a\langle\alpha^*\rangle - \frac{1}{2}\mu_a\langle\alpha\rangle + \frac{1}{2}V_-\langle\beta^*\rangle \tag{5.16}$$

$$\frac{d}{dt}\langle\beta\rangle = -2\varepsilon_b\langle\beta^*\rangle - \frac{1}{2}\mu_c\langle\beta\rangle + \frac{1}{2}V_+\langle\alpha^*\rangle \tag{5.17}$$

and the complex conjugate of Eq. (5.17) takes the form

$$\frac{d}{dt}\langle\hat{\beta}^*(t)\rangle = -2\varepsilon_b\langle\beta\rangle - \frac{1}{2}\mu_c\langle\beta^*\rangle + \frac{1}{2}V_+^*\langle\alpha\rangle \tag{5.18}$$

the above Eqs. (5.16) and (5.18) are linear differential equations for $\alpha(t)$and $\beta(t)$. On the other hand, taking the assumption that the cavity modes are initially in a vacuum state, we observe that $\alpha(t)$ and $\beta(t)$ are Gaussian variables with a non-vanishing means.

Considering Eqs. (5.14) and (5.15) together with the property that $\alpha(t)$ and $\beta(t)$ are Gaussian variables with a non-vanishing means; consequently, the EPR-like operators u and v are Gaussian variables with non-vanishing means.i.e.

$$\langle u(t)\rangle = \langle v(t)\rangle \neq 0 \tag{5.19}$$

using Eqs. (5.12) and (5.13) we obtain

$$\begin{aligned}\langle\hat{u}^2\rangle &= 1 + \frac{1}{2}(\langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}^2\rangle + \langle\hat{b}^{\dagger 2}\rangle + \langle\hat{b}^2\rangle) + (\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}\rangle \\ &\quad + \langle\hat{b}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle),\end{aligned} \tag{5.20}$$

$$\begin{aligned}\langle\hat{u}\rangle^2 &= \frac{1}{2}(\langle\hat{a}^\dagger\rangle^2 + \langle\hat{a}\rangle^2 + \langle\hat{b}^\dagger\rangle^2 + \langle\hat{b}\rangle^2) + (\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle + \langle\hat{b}^\dagger\rangle\langle\hat{b}\rangle + \langle\hat{a}^\dagger\rangle\langle\hat{b}\rangle \\ &\quad + \langle\hat{b}^\dagger\rangle\langle\hat{a}\rangle + \langle\hat{a}\rangle\langle\hat{b}\rangle + \langle\hat{a}^\dagger\rangle\langle\hat{b}^\dagger\rangle),\end{aligned} \tag{5.21}$$

$$\begin{aligned}\langle\hat{v}^2\rangle &= 1 - \frac{1}{2}(\langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}^2\rangle + \langle\hat{b}^{\dagger 2}\rangle + \langle\hat{b}^2\rangle) + (\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle - \langle\hat{a}^\dagger\hat{b}\rangle \\ &\quad - \langle\hat{b}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle)\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}\langle\hat{v}\rangle^2 &= \frac{1}{2}(-\langle\hat{a}^\dagger\rangle^2 - \langle\hat{a}\rangle^2 - \langle\hat{b}^\dagger\rangle^2 - \langle\hat{b}\rangle^2) + (\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle + \langle\hat{b}^\dagger\rangle\langle\hat{b}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{b}\rangle \\ &\quad - \langle\hat{b}^\dagger\rangle\langle\hat{a}\rangle + \langle\hat{a}\rangle\langle\hat{b}\rangle + \langle\hat{a}^\dagger\rangle\langle\hat{b}^\dagger\rangle).\end{aligned} \tag{5.23}$$

Substituting Eqs.(5.20), (5.21), (5.22) and (5.23) into Eq. (5.11), we obtain

$$\begin{aligned}\Delta u^2 + \Delta v^2 &= 2 + 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{b}^\dagger\hat{b}\rangle - 2\langle\hat{a}^\dagger\hat{b}^\dagger\rangle - 2\langle\hat{a}\hat{b}\rangle - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle - 2\langle\hat{b}^\dagger\rangle\langle\hat{b}\rangle \\ &\quad + 2\langle\hat{a}^\dagger\rangle\langle\hat{b}^\dagger\rangle + 2\langle\hat{a}\rangle\langle\hat{b}\rangle.\end{aligned} \tag{5.24}$$

This can be written in terms of c-number variables associated with normal ordering as

$$\begin{aligned}\Delta u^2 + \Delta v^2 \quad &= \quad 2 + 2\langle\alpha^*\alpha\rangle + 2\langle\beta^*\beta\rangle - 2\langle\alpha^*\beta^*\rangle - 2\langle\alpha\beta\rangle - 2\langle\alpha^*\rangle\langle\alpha\rangle - 2\langle\beta^*\rangle\langle\beta\rangle \\ &+ 2\langle\alpha^*\rangle\langle\beta^*\rangle + 2\langle\alpha\rangle\langle\beta\rangle. \end{aligned} \tag{5.25}$$

which can be simplified to

$$\Delta u^2 + \Delta v^2 \quad = \quad 2 + 2\langle\alpha^*, \alpha\rangle + 2\langle\beta^*, \beta\rangle - 2\langle\alpha^*, \beta^*\rangle - 2\langle\alpha, \beta\rangle. \tag{5.26}$$

and more specifically, the entanglement of cavity modes at steady state can be expressed as

$$[\Delta u^2 + \Delta v^2]_{ss} \quad = \quad 2\big[1 + \langle\alpha^*, \alpha\rangle_{ss} + \langle\beta^*, \beta\rangle_{ss} - 2\langle\alpha, \beta\rangle_{ss}\big]. \tag{5.27}$$

With similar procedure for the plus quadrature in eq. (4.39) and when $\varepsilon_a = \varepsilon_b$ at steady state we obtain

$$\begin{aligned}
\langle\alpha,\alpha\rangle_{ss} =& \frac{(-2\varepsilon_a + \frac{A}{2}(1-\eta) + \kappa N)}{8A^2\eta^2}[\frac{(A+A\eta)^2}{\kappa + 2(\varepsilon_a+\varepsilon_b)} + \frac{A^2(1-\eta^2)}{\kappa + A\eta + 2(\varepsilon_a+\varepsilon_b)} - \frac{4A^2(1-\eta^2)}{2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b)}] \\
&- \frac{(-2\varepsilon_b + \frac{A}{2}(1-\eta) + \kappa N)}{8A^2\eta^2}[\frac{A^2(1+\eta)^2}{\kappa - 2(\varepsilon_a+\varepsilon_b)} + \frac{A^2(1+\eta)^2}{\kappa + A\eta - 2\varepsilon_a + \varepsilon_b)} - \frac{4A^2(1-\eta^2)}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}] \\
&+ \frac{(-2\varepsilon_b + \kappa N)}{A^2\eta^2}[\frac{A^2(1-\eta^2)}{4(\kappa + 2(\varepsilon_a+\varepsilon_b))} + \frac{A^2(1-\eta^2)}{4(\kappa + A\eta + 2(\varepsilon_a+\varepsilon_b))} - \frac{4A^2(1-\eta^2)}{4(2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b))}] \\
&- \frac{(2\varepsilon_b + \kappa N)A^2(1-\eta^2)}{4A^2\eta^2}[\frac{1}{\kappa - 2(\varepsilon_a+\varepsilon_b)} + \frac{1}{\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b)} - \frac{4}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}] \\
&+ \frac{(A\frac{\sqrt{1-\eta^2}}{4} + \kappa M)}{2A^2\eta^2}[\frac{A^2(1+\eta)\sqrt{1-\eta^2}}{2(\kappa + 2(\varepsilon_a+\varepsilon_b))} + \frac{A^2(1-\eta)\sqrt{1-\eta^2}}{2(\kappa + 2(\varepsilon_a+\varepsilon_b))} - \frac{4(A\sqrt{1-\eta^2}}{2(2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b))}] \\
&- (\frac{A\sqrt{1-\eta^2} + \kappa M}{2A^2\eta^2})[\frac{A^2(1+\eta)\sqrt{1-\eta^2}}{2(\kappa - 2(\varepsilon_a+\varepsilon_b))} + \frac{A^2(1-\eta)\sqrt{1-\eta^2}}{2(\kappa - 2(\varepsilon_a+\varepsilon_b))} - \frac{4(A\sqrt{1-\eta^2}}{2(2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b))}]
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
&\langle\beta,\beta\rangle_{ss} = \\
&\frac{(-2\varepsilon_a + \frac{A}{2}(1-\eta) + \kappa N)}{8A^2\eta^2}\left[\frac{1}{\kappa + 2(\varepsilon_a+\varepsilon_b)} + \frac{1}{\kappa + A\eta + 2(\varepsilon_a+\varepsilon_b)}\right. \\
&\left.-\frac{4}{2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b)}\right] + \frac{2(-2\varepsilon_b + \kappa N)}{8A^2\eta^2}\left[\frac{A^2(1-\eta)^2}{\kappa + 2(\varepsilon_a+\varepsilon_b)}\right. \\
&\left.+\frac{A^2(1+\eta)^2}{\kappa + A\eta + 2(\varepsilon_a+\varepsilon_b)} - \frac{4A^2(1-\eta^2)}{2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b)}\right] \\
&-\frac{A^2(1-\eta^2)(2\varepsilon_a + \frac{A}{2}(1-\eta) + \kappa N)}{8A^2\eta^2}\left[\frac{1}{\kappa - 2(\varepsilon_a+\varepsilon_b)} + \frac{1}{\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b)} -\right. \\
&\left.\frac{4}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}\right] - \frac{(2\varepsilon_b + \kappa N)}{4A^2\eta^2}\left[\frac{A^2(1-\eta)^2}{\kappa - 2(\varepsilon_a+\varepsilon_b)}\right. \\
&\left.+\frac{A^2(1+\eta)^2}{\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b)} - \frac{4A^2(1-\eta^2)}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}\right] \\
&+\frac{A\sqrt{1-\eta^2}\frac{(A\sqrt{1-\eta^2}}{4} + \kappa M)}{4A^2\eta^2}\left[\frac{A(1-\eta)}{\kappa - 2(\varepsilon_a+\varepsilon_b)} + \frac{A(1+\eta)}{\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b)}\right. \\
&\left.-\frac{4A}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}\right]
\end{aligned} \tag{5.29}$$

$$
\begin{aligned}
\langle\alpha^*,\alpha\rangle_{ss} =& \frac{(-2\varepsilon_a+\frac{A}{2}(1+\eta)+\kappa N)}{4A^2\eta^2}[\frac{A^2(1-\eta)^2}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))}+\frac{A^2(1-\eta)^2}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))}\\
&-\frac{2A^2(1-\eta^2)}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}]\\
&+\frac{A^2(1-\eta^2)(-2\varepsilon_b+\kappa N)}{4A^2\eta^2}[\frac{1}{2(\kappa+2(\varepsilon_a+\varepsilon_b))}+\frac{1}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))}\\
&-\frac{2}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}]\\
&+\frac{(2\varepsilon_a+\frac{A}{2}(1+\eta)+\kappa N)}{4A^2\eta^2}[\frac{A^2(1+\eta)^2}{2(\kappa-2(\varepsilon_a+\varepsilon_b))}+\frac{A^2(1-\eta)^2}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}\\
&-\frac{2A^2(1-\eta^2)}{(2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}]\\
&+\frac{\frac{A\sqrt{1-\eta^2}}{\kappa}M)}{4A^2\eta^2}[\frac{2A^2\sqrt{1-\eta^2}}{2(\kappa+2(\varepsilon_a+\varepsilon_b))}+\frac{2A^2\sqrt{1-\eta^2}}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))}\\
&-\frac{2A^2\sqrt{1-\eta^2}}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}+\frac{2A^2(1+\eta)\sqrt{1-\eta^2}}{2(\kappa-2(\varepsilon_a+\varepsilon_b))}\\
&+\frac{2A^2(1-\eta)\sqrt{1-\eta^2}}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}-\frac{4A^2\sqrt{1-\eta^2}}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}]
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
&\langle\beta^*,\beta\rangle_{ss} =\\
&\frac{A^2(1-\eta^2)(-2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)}{4A^2\eta^2}\left[\frac{1}{2(\kappa+2(\varepsilon_a+\varepsilon_b))}+\frac{1}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))}\right.\\
&\left.-\frac{2}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\right]+\frac{(-2\varepsilon_b+\kappa N)}{4A^2\eta^2}\left[\frac{A^2(1-\eta)^2}{2(\kappa+2(\varepsilon_a+\varepsilon_b))}\right.\\
&\left.+\frac{A^2(1+\eta)^2}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))}-\frac{2A^2(1-\eta^2)}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\right]\\
&+\frac{2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)A^2(1-\eta^2)}{4A^2\eta^2}\left[\frac{1}{2(\kappa-2(\varepsilon_a+\varepsilon_b))}+\frac{1}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}\right.\\
&\left.-\frac{2}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\right]+\frac{2\varepsilon_b+\kappa N)}{4A^2\eta^2}\left[\frac{A^2(1-\eta)^2}{2(\kappa-2(\varepsilon_a+\varepsilon_b))}\right.\\
&\left.+\frac{A^2(1+\eta)^2}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}-\frac{2A^2(1-\eta^2)}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\right]\\
&+\frac{\frac{A\sqrt{1-\eta^2}}{4}+\kappa M}{4A^2\eta^2}\left[\frac{A^2(1-\eta)\sqrt{1-\eta^2}}{2(\kappa+2(\varepsilon_a+\varepsilon_b)}+\frac{A^2(1+\eta)\sqrt{1-\eta^2}}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)}\right.\\
&-\frac{2A^2\sqrt{1-\eta^2}}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}+\frac{A^2(1-\eta)\sqrt{1-\eta^2}}{2(\kappa-2(\varepsilon_a+\varepsilon_b)}+\frac{A^2(1+\eta)\sqrt{1-\eta^2}}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}\\
&\left.-\frac{2A^2\sqrt{1-\eta^2}}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\right]
\end{aligned}
\tag{5.31}
$$

$$\langle\alpha,\beta\rangle_{ss} = -\frac{A\sqrt{1-\eta^2}(-2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)}{8A^2\eta^2}\Bigg[\frac{A(1+\eta)}{\kappa+2(\varepsilon_a+\varepsilon_b)}+\frac{A(1-\eta)}{\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)} -\frac{4A}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Bigg]+\frac{A\sqrt{1-\eta^2}(-2\varepsilon_b+\kappa N)}{8A^2\eta^2}\Bigg[\frac{A(1-\eta)}{\kappa+2(\varepsilon_a+\varepsilon_b)} -\frac{A(1+\eta)}{\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)}+\frac{4A}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Bigg] -\frac{A\sqrt{1-\eta^2}(2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)}{8A^2\eta^2}\Bigg[\frac{A(1+\eta)}{\kappa-2(\varepsilon_a+\varepsilon_b)}+\frac{A(1-\eta)}{\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}- \frac{4A}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Bigg]-\frac{A\sqrt{1-\eta^2}(2\varepsilon_b+\kappa N)}{8A^2\eta^2}\Bigg[\frac{A(1-\eta)}{\kappa-2(\varepsilon_a+\varepsilon_b)} +\frac{A(1+\eta)}{\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}-\frac{4A}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Bigg] +\frac{\frac{A\sqrt{1-\eta^2}}{4}+\kappa M}{8A^2\eta^2}\Bigg[\frac{2A^2(1-\eta^2)}{\kappa+2(\varepsilon_a+\varepsilon_b)}-\frac{2A^2(1-\eta^2)}{\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)} +\frac{2A^2((1-\eta)^2+(1+\eta)^2+2(1-\eta^2))}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Bigg]-\frac{\frac{A\sqrt{1-\eta^2}}{4}+\kappa M}{8A^2\eta^2}\Bigg[\frac{2A^2(1-\eta^2)}{\kappa-2(\varepsilon_a+\varepsilon_b)} +\frac{2A^2(1-\eta^2)}{\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}-\frac{2A^2((1-\eta)^2+(1+\eta)^2+2(1-\eta^2))}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Bigg] \tag{5.32}$$

$$\langle\alpha^*,\beta\rangle_{ss} = \frac{-A\sqrt{1-\eta^2}(-2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)}{4A^2\eta^2}\Bigg[\frac{A(1+\eta)}{2(\kappa+2(\varepsilon_a+\varepsilon_b))}+\frac{A(1-\eta)}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))} -\frac{2A}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Bigg]-\frac{A\sqrt{1-\eta^2}(-2\varepsilon_b+\kappa N)}{2A^2\eta^2}\Bigg[\frac{A(1-\eta)}{2(\kappa+2(\varepsilon_a+\varepsilon_b))} +\frac{A(1+\eta)}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b))}-\frac{2A}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Bigg] +\frac{A\sqrt{1-\eta^2}(2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)}{4A^2\eta^2}\Bigg[\frac{A(1+\eta)}{2(\kappa-2(\varepsilon_a+\varepsilon_b))}+\frac{A(1-\eta)}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}- \frac{2A}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Bigg]-\frac{A\sqrt{1-\eta^2}(2\varepsilon_b+\kappa N)}{2A^2\eta^2}\Bigg[\frac{A(1-\eta)}{2(\kappa-2(\varepsilon_a+\varepsilon_b))} +\frac{A(1+\eta)}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}+\frac{2A}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}\Bigg] -\frac{(A\sqrt{1-\eta^2}+\kappa M)}{4A^2\eta^2}\Bigg[\frac{A^2(1-\eta^2)}{2(\kappa+2(\varepsilon_a+\varepsilon_b)}+\frac{A^2(1-\eta^2)}{2(\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)} -\frac{A^2}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}-\frac{A^2(1-\eta^2)}{2(\kappa-2(\varepsilon_a+\varepsilon_b)}-\frac{A^2(1-\eta^2)}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)} +\frac{A^2(2(1-\eta))}{2(\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Bigg] \tag{5.33}$$

with the aid of the simplified Eqs. (5.30)-(5.32) at steady state we obtain

$$
\begin{aligned}
&[\langle\alpha^*,\alpha\rangle_{ss}+\langle\beta^*,\beta\rangle_{ss}-2\langle\alpha,\beta\rangle_{ss}] = \\
&\frac{1}{4\eta^2}\Bigg((-2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)\Big(\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa+2(\varepsilon_a+\varepsilon_b)}+\frac{(1-\eta)(1+\sqrt{1-\eta^2})}{\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)}-\frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(-2\varepsilon_b+\kappa N)\Big(\frac{(1-\eta)(1-\sqrt{1-\eta^2})}{\kappa+2(\varepsilon_a+\varepsilon_b)}+\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)}-\frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)\Big(\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa-2(\varepsilon_a+\varepsilon_b)}+\frac{(1-\eta)(1+\sqrt{1-\eta^2})}{\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}-\frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(2\varepsilon_b+\kappa N)\Big(\frac{(1-\eta)\sqrt{1-\eta^2}}{\kappa-2(\varepsilon_a+\varepsilon_b)}+\frac{(1+\eta)\sqrt{1-\eta^2}}{\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}-\frac{4\sqrt{1-\eta^2}}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(\frac{A\sqrt{1-\eta^2}}{4}+\kappa M)\Bigg((3-\eta)\sqrt{1-\eta^2}+4\sqrt{1-\eta^2}[\frac{1}{2(\kappa+2(\varepsilon_a+\varepsilon_b))}+\frac{1}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}]\\
&+(3+\eta)\sqrt{1-\eta^2}+4\sqrt{1-\eta^2})[\frac{1}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}+\frac{1}{2(\kappa-2(\varepsilon_a+\varepsilon_b))}]\\
&-2(4+3\sqrt{1-\eta^2})[\frac{1}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}+\frac{1}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}]\Bigg)
\end{aligned}
\tag{5.34}
$$

employing Eq. (5.34) into Eq. (5.27), one can easily obtain;

$$
\begin{aligned}
&[\Delta u^2+\Delta v^2]_{ss} = 2\Bigg[1+\\
&\frac{1}{4\eta^2}\Bigg((-2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)\Big(\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa+2(\varepsilon_a+\varepsilon_b)}+\frac{(1-\eta)(1+\sqrt{1-\eta^2})}{\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)}-\frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(-2\varepsilon_b+\kappa N)\Big(\frac{(1-\eta)(1-\sqrt{1-\eta^2})}{\kappa+2(\varepsilon_a+\varepsilon_b)}+\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa+A\eta+2(\varepsilon_a+\varepsilon_b)}-\frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(2\varepsilon_a+\frac{A}{2}(1-\eta)+\kappa N)\Big(\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa-2(\varepsilon_a+\varepsilon_b)}+\frac{(1-\eta)(1+\sqrt{1-\eta^2})}{\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}-\frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(2\varepsilon_b+\kappa N)\Big(\frac{(1-\eta)\sqrt{1-\eta^2}}{\kappa-2(\varepsilon_a+\varepsilon_b)}+\frac{(1+\eta)\sqrt{1-\eta^2}}{\kappa+A\eta-2(\varepsilon_a+\varepsilon_b)}-\frac{4\sqrt{1-\eta^2}}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}\Big)\\
&+(\frac{A\sqrt{1-\eta^2}}{4}+\kappa M)\Bigg((3-\eta)\sqrt{1-\eta^2}+4\sqrt{1-\eta^2}[\frac{1}{2(\kappa+2(\varepsilon_a+\varepsilon_b))}+\frac{1}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}]\\
&+(3+\eta)\sqrt{1-\eta^2}+4\sqrt{1-\eta^2})[\frac{1}{2(\kappa+A\eta-2(\varepsilon_a+\varepsilon_b))}+\frac{1}{2(\kappa-2(\varepsilon_a+\varepsilon_b))}]\\
&-2(4+3\sqrt{1-\eta^2})[\frac{1}{2\kappa+A\eta+4(\varepsilon_a+\varepsilon_b)}+\frac{1}{2\kappa+A\eta-4(\varepsilon_a+\varepsilon_b)}]\Bigg)\Bigg]
\end{aligned}
\tag{5.35}
$$

The following figure indicates that the degree of entanglement of the cavity modes for the system under consideration increases with the amplitude of the parametric amplifiers.

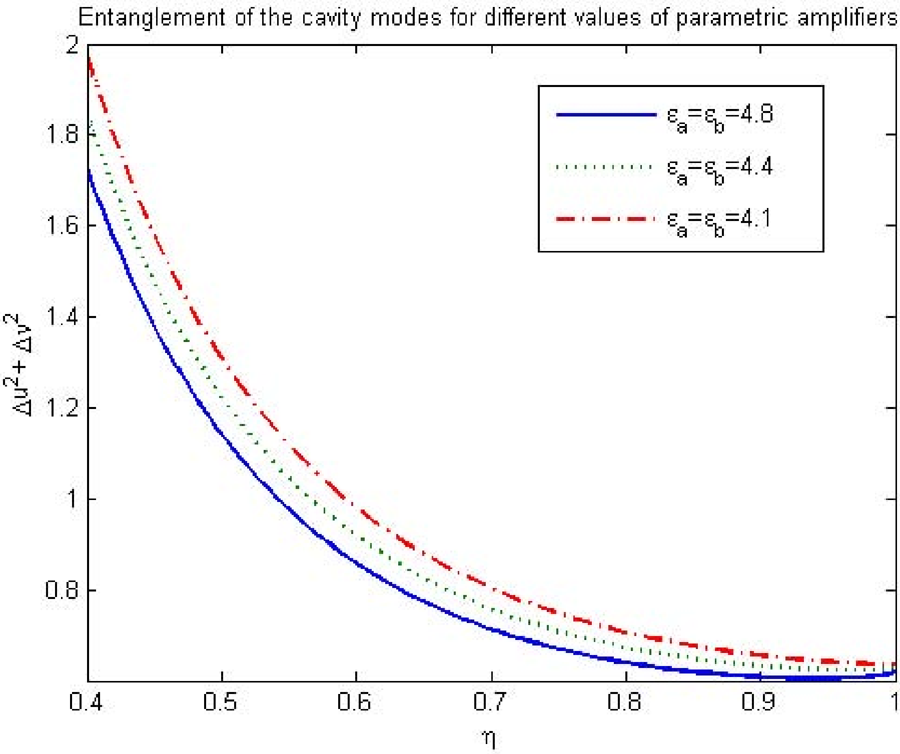

Fig. 5.1: Plots of the entanglement of the cavity modes $[Eq.(5.35)]$ versus η for the values of A=1, $\kappa = 0.8$, r= 1.2 and for, $\varepsilon_a = \varepsilon_b = 4.8$ (solid curve), $\varepsilon_a = \varepsilon_b = 4.4$ (dotted curve), and $\varepsilon_a = \varepsilon_b = 4.1$ (dash-dotted curve).

From this fig 5.1 we see that the light produced by the system under consideration is **entangled state** and the entanglement criterion given by Eq. (5.2) is satisfied as $\Delta u^2 + \Delta v^2 < 2$.

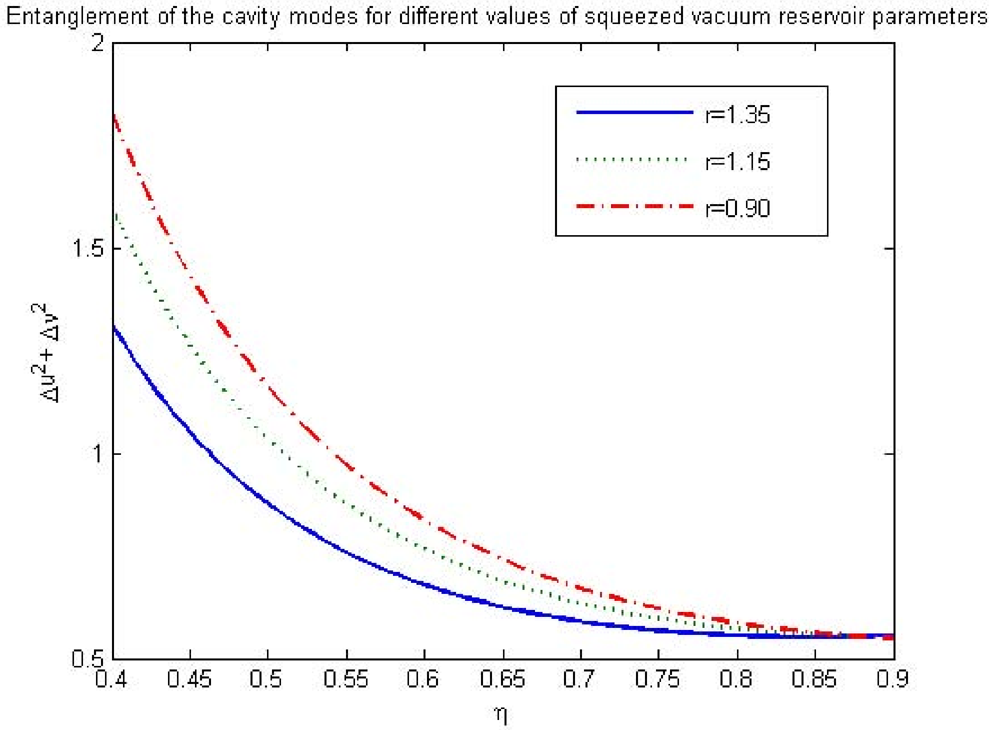

Fig. 5.2: Plots of the entanglement of the cavity modes $[Eq.(5.35)$ versus η for the values of A=1, $\kappa = 0.75$, $\varepsilon_a = \varepsilon_b = 6.4$, and for, $r = 1.35$ (solid curve), $r = 1.15$ (dotted curve), and $r = 0.90$ (dash-dotted curve)

Moreover, the entanglement of cavity modes increases as the parameter r in the squeezed vacuum reservoir has increased.

5.2 Entanglement of the output modes

The EPR-like operators for the output modes have the form

$$\hat{u}^{out} = \hat{x_a}^{out} - \hat{x_b}^{out} \tag{5.36}$$

and

$$\hat{v}^{out} = \hat{p_a}^{out} + \hat{p_b}^{out} \tag{5.37}$$

with

$$\hat{x}_a^{out} = \frac{\sqrt{k}}{\sqrt{2}}(\hat{a}^{\dagger} + \hat{a}), \tag{5.38}$$

$$\hat{x}_b^{out} = \frac{\sqrt{k}}{\sqrt{2}}(\hat{b}^{\dagger} + \hat{b}), \tag{5.39}$$

$$\hat{p}_a^{out} = \frac{i\sqrt{k}}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}), \tag{5.40}$$

and

$$\hat{p}_b^{out} = \frac{i\sqrt{k}}{\sqrt{2}}(\hat{b}^\dagger - \hat{b}), \tag{5.41}$$

being the quadrature operators for modes $\hat{a}$ and $\hat{b}$.

Thus the entanglement criterion for the output modes can be written as

$$\Delta u_{out}^2 + \Delta v_{out}^2 < 2. \tag{5.42}$$

With the aid of the input-output relation one can readily show that

$$\begin{aligned}\Delta u_{out}^2 + \Delta v_{out}^2 &= 2 + k\Big[\langle 2\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{b}^\dagger\hat{b}\rangle - 2\langle\hat{a}^\dagger\hat{b}^\dagger\rangle - 2\langle\hat{a}\hat{b}\rangle - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle - 2\langle\hat{b}^\dagger\rangle\langle\hat{b}\rangle \\ &\quad + 2\langle\hat{a}^\dagger\rangle\langle\hat{b}^\dagger\rangle + 2\langle\hat{a}\rangle\langle\hat{b}\rangle\Big].\end{aligned} \tag{5.43}$$

This can be written in terms of c-number variables associated with normal ordering at steady state as

$$[\Delta u_{out}^2 + \Delta v_{out}^2]_{ss} = 2\Big[1 + k\Big(\langle\alpha^*,\alpha\rangle_{ss} + \langle\beta^*,\beta\rangle_{ss} - 2\langle\alpha,\beta\rangle_{ss}\Big)\Big]. \tag{5.44}$$

With the aid of the simplified Eqs. (5.34) along with Eq. (5.44) at steady state we obtain

$$\begin{aligned}&[\Delta u_{out}^2 + \Delta v_{out}^2]_{ss} = 2k\Bigg[\frac{1}{k} + \\ &\frac{1}{4\eta^2}\Bigg((-2\varepsilon_a + \frac{A}{2}(1-\eta) + \kappa N)\Bigg(\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa + 2(\varepsilon_a+\varepsilon_b)} + \frac{(1-\eta)(1+\sqrt{1-\eta^2})}{\kappa + A\eta + 2(\varepsilon_a+\varepsilon_b)} - \frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b)}\Bigg) \\ &+ (-2\varepsilon_b + \kappa N)\Bigg(\frac{(1-\eta)(1-\sqrt{1-\eta^2})}{\kappa + 2(\varepsilon_a+\varepsilon_b)} + \frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa + A\eta + 2(\varepsilon_a+\varepsilon_b)} - \frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b)}\Bigg) \\ &+ (2\varepsilon_a + \frac{A}{2}(1-\eta) + \kappa N)\Bigg(\frac{(1+\eta)(1+\sqrt{1-\eta^2})}{\kappa - 2(\varepsilon_a+\varepsilon_b)} + \frac{(1-\eta)(1+\sqrt{1-\eta^2})}{\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b)} - \frac{4((1-\eta^2)+\sqrt{1-\eta^2})}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}\Bigg) \\ &+ (2\varepsilon_b + \kappa N)\Bigg(\frac{(1-\eta)\sqrt{1-\eta^2}}{\kappa - 2(\varepsilon_a+\varepsilon_b)} + \frac{(1+\eta)\sqrt{1-\eta^2}}{\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b)} - \frac{4\sqrt{1-\eta^2}}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}\Bigg) \\ &+ (\frac{A\sqrt{1-\eta^2}}{4} + \kappa M)\Bigg((3-\eta)\sqrt{1-\eta^2} + 4\sqrt{1-\eta^2}[\frac{1}{2(\kappa + 2(\varepsilon_a+\varepsilon_b))} + \frac{1}{2(\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b))}] \\ &+ (3+\eta)\sqrt{1-\eta^2} + 4\sqrt{1-\eta^2})[\frac{1}{2(\kappa + A\eta - 2(\varepsilon_a+\varepsilon_b))} + \frac{1}{2(\kappa - 2(\varepsilon_a+\varepsilon_b))}] \\ &- 2(4 + 3\sqrt{1-\eta^2})[\frac{1}{2\kappa + A\eta + 4(\varepsilon_a+\varepsilon_b)} + \frac{1}{2\kappa + A\eta - 4(\varepsilon_a+\varepsilon_b)}]\Bigg)\Bigg]\end{aligned} \tag{5.45}$$

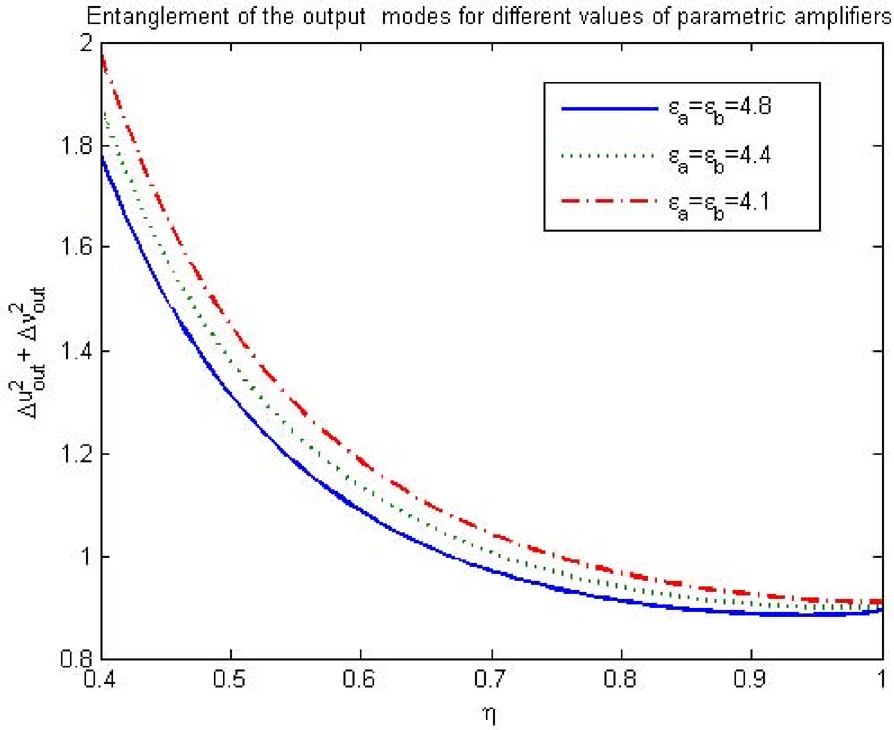

Fig. 5.3: Plots of the entanglement of output modes $[Eq.(5.45)]$ versus η for the values of A=1, $\kappa = 0.8$, r= 1.2 and for, $\varepsilon_a = \varepsilon_b = 4.8$ (solid curve), $\varepsilon_a = \varepsilon_b = 4.4$ (dotted curve), and $\varepsilon_a = \varepsilon_b = 4.1$ (dash-dotted curve).

Fig 5.3 indicates that the degree of entanglement of output modes for the system under consideration increases with the amplitude of the parametric amplifiers.

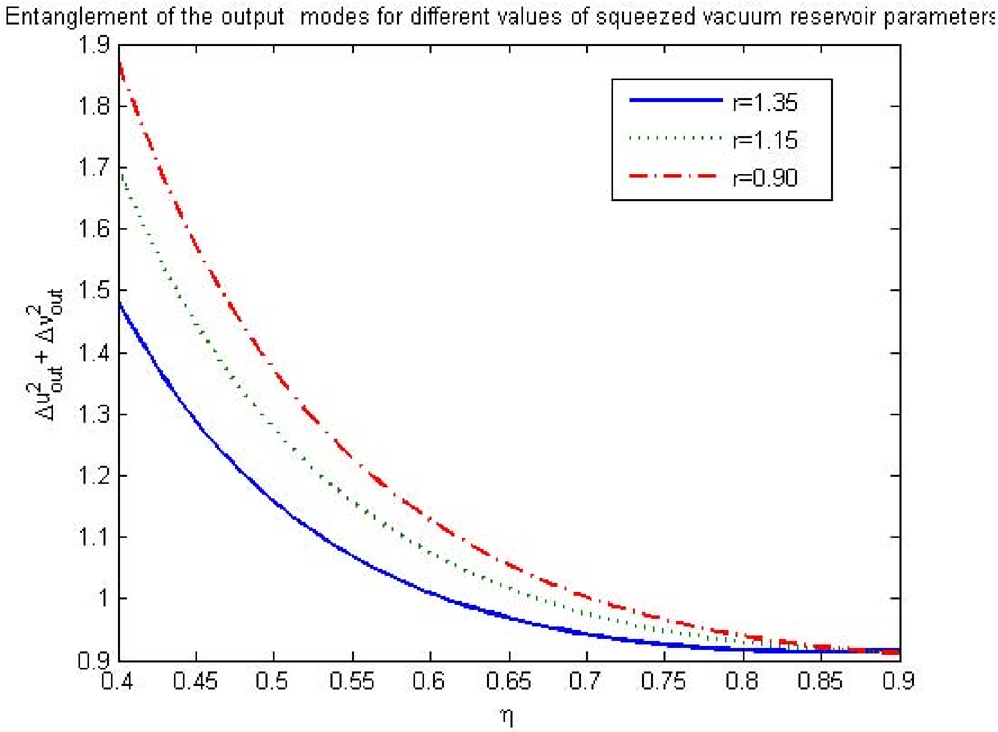

Fig. 5.4: Plots of the entanglement of the output modes $[Eq.(5.45)]$ versus η for the values of A=1, $\kappa = 0.75$, $\varepsilon_a = \varepsilon_b = 6.4$, and for, $r = 1.35$ (solid curve), $r = 1.15$ (dotted curve), and $r = 0.90$ (dash-dotted curve).

From fig 5.4 we see that the entanglement of output modes increases as the parameter r in the squeezed vacuum reservoir has increased.

6

Conclusion

In this thesis, we analyzed the squeezing and entanglement properties of light produced by non-degenerate three-level laser whose cavity contains two degenerate parametric amplifiers and coupled to a squeezed vacuum reservoir. We first obtained the master equation for the system under consideration. Using the master equation, we obtained c-number langiven equation associated with the normal ordering and the correlation properties of the noise forces . Applying the solutions of the resulting c-number langiven equations, we determined the quadrature squeezing (the variances of the cavity and out put modes as well as the squeezing spectrum of the output modes).In addition,applying the criterion developed by Duan et al. the quantum entanglement of the cavity and output modes are studied.

We observed that the light produced by the system under consideration exhibits **squeezing** and **entanglement**. It is found that the degree of squeezing for the system under consideration increases with the amplitude of the parametric amplifiers . This implies that the presence of the parametric amplifiers enhances the squeezing of light generated by the system under consideration and the minimum value of the quadrature variances described by Eq. (4.39) for $A = 100$, $\kappa = 0.8$, $r = 0.6$ and $\varepsilon_a = \varepsilon_b = 0.6$, is found to be $\Delta c_+^2 = 0.05731$ and occurs at $\eta = 0.1$. This indicates that the maximum intracavity squeezing is 94.27% below the coherent state level. It is also observed that the degree of squeezing for the system under consideration increases with the squeezing parameter of the squeezed vacuum reservoir. It so turns out that the squeezed vacuum reservoir increases the degree of squeezing. Moreover, the squeezing of the cavity modes is found to be greater than that of output modes.

It is also observed that the amplitudes of parametric amplifiers as well as the squeezed vacuum reservoir considerably increase the degree of entanglement of the cavity and output modes.

References

[1] K. Fesseha, Fundamentals of Quantum Optics, (Lulu, North Carolina, 2008).

[2] K. Fesseha, Phys. Rev. A**63**, 033811 (2001).

[3] Wubshet Mekonnen, M.sc Thesis, Addis Ababa University (2007).

[4] Misrak Getahun, Ph.D Dissertation, Addis Ababa University (2009).

[5] Mesfin Ayele ,MSc Thesis ,Addis Abeba University,2012.

[6] N.A.Ansari, Phys. Rev. A **48**,4686 (1993).

[7] Solomon Getachew, M.sc Thesis, Addis Ababa University (2011).

[8] Beyene Abiti, M.Sc Thesis, Addis Ababa University (2011).

[9] M.O.Scully and M.S.Zubairy,Quantum Optics (Cambridge University Press,Cambridge,1997).

[10] J.Anwar,and M.S.Zubairy,Phys.Rev.A **49**,481(1994).

[11] K. Fesseha, Opt. Commun. 156, 145 (1998)

[12] Tewodros Yirgashew, M.sc Thesis, Addis Ababa University (2006).

[13] E. Alebachew and K.Fesseha Opt. Communication, 265, 314 (2006).

[14] M.O.Scully, K.Wodkiewicz, M.s.Zubairy, J.Bergou, N, Lu, and J. Meyerte Vehn, Phys.Rev.Latt.60,1832 (1988).

[15] Adanh Samuel, M.sc Thesis, Addis Ababa University (2003).

[16] C.A. Blockly and D.F. Walls Phys. Rev. A **43**, 5049 (1991).

[17] X.Fen, China, Phys, Socie, 15, 0235 (2006).

[18] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).

[19] S. Tesfa, Phys. Rev A **74**, 043816 (2006).

[20] B. Teklu, Opt. Commun. 261, 310 (2006).

[21] Mekonnen Bekele, M.sc Thesis, Adama Science and Technology University (2014).

[22] Friew Gebremedhin, M.sc Thesis, Dilla University (2011).

[23] D.F. Walls, G.J. Milburn, Quantum Optics, Springer Verlag, Berlin, (1994).

[24] Tewodros Yirgashewa, PhD Dissertation Addis Abeba university,2010.

[25] Lu.Ming Duan, G.Giedke, J.I.Cirak, and P.Zoller, Phys.Rev.84, 12 (2000).

[26] S.Tesfa,e-print:quant-ph/07073684.

[27] C.M.Savage and D.F.Walls,Phys.Rev.A **33**,3282 (1986).

[28] B. Daniel and K. Fesseha, Opt. Commun. 151, 384 (1998).

[29] N.Lu and S.Y.Zhu,Phys.Rev.A **40**,5735 (1989).

[30] G.J.Milburn and D.F.Walls,Phys.Rev.A **27**,392 (1983).

[31] N.Lu.F. X Zhao, and J.Bergou, Phys.Rev.A **39**, 5189 (1989).

[32] Horodecki R, Horodecki M, Horodecki K, ”Quantum entanglemt”, Rev. Mod. Phys. **81(2)**: 865-942(2007)

[33] Jeffery Bub,Quantum Entanglement and information,(Revised) Oxford University press (2010).

DECLARATION

I hereby declare that this Thesis is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the Thesis have been duly acknowledged.

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This Thesis has been submitted to for examination with my approval as university advisor.

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